Learning Unknown Lagrange Dynamical Systems with Guaranteed Persistency of Excitation

A. Samanis, P. S. Trakas, X. Papageorgiou, K. J. Kyriakopoulos and C. P. Bechlioulis

***Abstract*— In this paper, we present a methodology that ensures a priori that all possible unknown dynamics of the system within a compact set of operation will be excited. A controller is used to make sure that the system with unknown dynamics will follow the reference trajectory and Radial Basis Function (RBF) neural networks are employed to estimate the unknown nonlinearities. The persistency of excitation condition is guaranteed as a prerequisite to achieve accurate estimation of the unknown nonlinear terms and efficient learning. A simulation example clarifies the proposed approach and verifies the aforementioned assertions.**

***Index Terms*— Prescribed Performance Control, System Identification, Persistency of Excitation, RBF Neural Networks.**

I. Introduction

The concept of learning is met for over half a century in the control literature and many related papers focused on finding learning methods motivated by the need to estimate or approximate parameters or unknown nonlinear functions, which are involved in unknown system dynamics. In particular, the development of learning techniques was motivated by the need of identifying/tuning the appropriate controller parameters in an attempt to achieve efficient control under certain specifications and improve closed-loop system performance and robustness. Moreover, learning about environment or learning new design goals and constraints were other usual goals of learning techniques.

Initial approaches towards this direction tried to invent learning techniques by which, the system with unknown dynamics is excited in an open-loop architecture by applying various (mostly arbitrary) inputs [1]. Then, by analyzing system outputs and using common pattern recognition methods an estimation of the unknown parts of the system was established. Even though these efforts showed some significant results in the literature, they are characterized by the weakness to excite the system sufficiently well in the region of interest, as they are open-loop and unable to handle open-loop unstable dynamics. Consequently, the learning is not achieved in a satisfactory way and it is only achieved in a range of values depending on the applied excitement. The adaptive control techniques [2] used later boosted the efficiency of learning methods and improved the performance as the system follows a desired trajectory, or by adjusting control parameters when the operating conditions were changed. Although the introduction of adaptive techniques enhanced the learning methods [3], no novel excitation method was applied till very lately, as a mechanism to produce various possible system dynamics (e.g., a range of possible system states in a certain area of interest). Nevertheless, there are some recent works [4], [5], [6], [7], which tried to produce a kind of orbits, in an attempt to excite as many as possible of the system dynamics in a predefined certain area, without ensuring a priori that all possible system dynamics of this area can be excited, but only a certain percentage of them, resulting in a rather satisfactory result of learning.

Despite the significant learning results that have been achieved in the above-mentioned efforts, the problem of successful learning all possible system dynamics in a certain predefined area is still open in the system identification literature. Further steps are needed to be made in this field of research, posing challenges to control engineering community. Building upon [8], this paper presents a methodology for uncertain MIMO Lagrangian systems (representing almost all mechanical systems) that proposes a reference trajectory involving state representatives of possible system functionality in a predefined compact set inside which, the learning is aimed to be done. Motivated by this issue and also by the prescribed performance control technique [9], we propose a reference trajectory that traverses all neuron centers of possible system functionality in a way (ergodic condition) that the satisfaction of persistency of excitation is verified [10], [11]. Moreover, a controller based on the prescribed performance technique is developed, which guarantees that system states follow accurately the reference signal to satisfy a priori the persistency of excitation condition. Furthermore, an online neural network identifier constructed out of RBFs in its regressor vector is designed, and as the controlled system tracks the desired trajectory it achieves learning of the system nonlinearities successfully. RBF centers coincide with the nodes given in a certain lattice, which represents the desired compact set of system functionality, in which the target of learning is aimed to be achieved. The stability of the controller and the identifier is based on Lyapunov analysis, ensuring that the closed-loop signals are bounded and that the neural network weight estimates are not only bounded, but they converge to their actual optimal with, respect to modeling error, values as well.

II. Problem Formulation and Preliminaries

In this work, we consider MIMO nonlinear Lagrange systems described by the following model:

$$M(x)\ddot{x} + C(x,\dot{x}) + G(x) = u \tag{1}$$

A.S. and K.J.K. are with the Control Systems Lab, School of Mechanical Engineering, National Technical University of Athens, Greece. P.S.T., X.P. and C.P.B. are with the Department of Electrical and Computer Engineering, University of Patras, Greece. Email: chmpechl@upatras.gr.

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where $x = [x_1, \ldots, x_n]^T$, $\dot{x} = [\dot{x}_1, \ldots, \dot{x}_n]^T$ denote the displacement and velocity states respectively, $M(x)$ is the inertia matrix, $C(x, \dot{x})$ is the matrix of centripetal and coriolis torques/forces, $G(x)$ includes the gravitational torques/forces and $u \in \mathbb{R}^n$ denotes the applied torques/forces (i.e., control inputs). Alternatively, the aforementioned dynamics may be rewritten in canonical form as follows:

$$\begin{aligned} \dot{\bar{x}}_1 =& \bar{x}_2 \\ \dot{\bar{x}}_2 =& f(\bar{x}_1, \bar{x}_2) + g(\bar{x}_1)u \end{aligned} \tag{2}$$

where $\bar{x}_1 = [x_1, \ldots, x_n]^T$ denotes the vector of displacements and $\bar{x}_2 = [\dot{x}_1, \ldots, \dot{x}_n]^T$ represents the vector of velocities with $f(\bar{x}_1, \bar{x}_2) = -M^{-1}(\bar{x}_1)[C(\bar{x}_1, \bar{x}_2) + G(\bar{x}_1)]$ and $g(\bar{x}_1) = M^{-1}(\bar{x}_1)$.

The objective of this work is to learn the nonlinear functions $f(\bar{x}_1, \bar{x}_2), g(\bar{x}_1)$ locally in a chosen compact set, by designing a closed-loop system identification scheme that extracts local accurate approximations of the nonlinear functions over the compact set. Finally, to solve the aforementioned problem, we assume that there exists an unknown positive constant g^* such that $\lambda_{min}(g(\bar{x}_1)) \geq g^* > 0, \forall \bar{x}_1 \in \mathbb{R}^n$, which implies that the dynamics are fully actuated.

A. RBF Neural Networks and Persistency of Excitation

RBFs belong to a category of linearly parameterized networks that are represented by:

$$f_{NN}(x) = \sum_{i=1}^{p} w_i z_i(x) = W^T Z(x) \tag{3}$$

where $x \in \mathbb{R}^n$ is the input vector, $f_{NN}(x) \in \mathbb{R}$ is the output, $W = [w_1, \ldots, w_p]^T \in \mathbb{R}^p$ is the weight vector, $p > 1$ is the neural vector node number and $Z(x) = [z_1(x), \ldots, z_p(x)]^T$ is a p-dimensional regressor vector:

$$Z(x) = [z_1(\|x - c_1\|), \ldots, z_p(\|x - c_p\|)]^T \tag{4}$$

where $z_i(\cdot)$ denote the radial basis functions and $c_i, i \in \{1, \ldots, p\}$ are distinct points in state space (termed centers). In this work, the regressor terms involve radial basis functions (RBFs) with fixed centers and widths. A widely used RBF is the Gaussian function $z(x) = \exp\left(-\|\frac{x-c}{s}\|^2\right)$ where c and s are the center and width of the receptive field, respectively. It has been proven in [11] that an RBF network with sufficiently large node number p and appropriately placed node centers and variances can approximate any continuous function $h(x) : \Omega_x \to \mathbb{R}$ over a compact set $\Omega_x \subset \mathbb{R}^n$ to arbitrary accuracy according to:

$$h(x) = W^{*T} Z(x) + \epsilon(x), \ \forall x \in \Omega_x \tag{5}$$

where W^* denotes the ideal constant weights and $\epsilon(x)$ the approximation error in a way that:

$$\sup_{x \in \Omega_x} \{|h(x) - W^{*T} Z(x)|\} \leq \epsilon.$$

The ideal weight vector W^* is an artificial quantity required for analytical purposes and it is defined as the value of W that minimizes ϵ for all $x \in \Omega_x \subset \mathbb{R}^n$, that is:

$$W^* \triangleq \underset{W \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \sup_{x \in \Omega_x} |h(x) - W^T Z(x)| \right\}. \tag{6}$$

Hence, based on the above equations it is concluded that if the number of regressors p is sufficiently large and the regressor terms are appropriately selected, then there exists a weight vector W^* such that the RBF-NN can approximate any sufficiently smooth function $h(x)$ to any degree of accuracy in a given compact set Ω_x.

The property of Persistency of Excitation (PE) is widely used in adaptive systems as it provides sufficient conditions leading to the parameter estimates convergence to their actual values. Especially in the field of RBF networks, persistency of excitation is widely used giving conditions under which the successful identification of nonlinear systems is achieved. It has been shown in [12] that if the inputs to an RBF network coincide with the network neuron centers, then the corresponding regressor vector satisfies the Persistency of Excitation condition. Additionally, for RBF neural networks with neuron centers dispersed on a regular lattice, it was shown that the corresponding regressor vector is persistently excited provided that the input variables to the RBF networks belong to a certain neighborhood of neuron centers [10]. Particularly in the present work, the size of the neighborhood will be restricted, by the prescribed performance control design to be less than a predefined small value, which is smaller than the minimum distance of any two neuron centers. Towards this direction, a class of ideal reference trajectories which ensure the satisfaction of the PE condition will be extracted as ergodic trajectories visiting the limited neighborhoods of all neuron centers of the RBF network.

B. Prescribed Performance Control

The control design in this work is heavily connected to the prescribed performance control technique that was originally employed to design robust controllers for various classes of nonlinear systems, capable of guaranteeing output tracking with prescribed performance. In this paper, prescribed performance is restricted to tracking error convergence to a predefined arbitrarily small residual set. The work in [9] gives the basic theoretical background of prescribed performance, which is summarized as follows.

Consider a generic tracking error $e(t)$. Prescribed performance is achieved if $e(t)$ evolves strictly within a predefined set that is bounded by decaying functions of time. The mathematical expression of prescribed performance is given by the following inequalities:

$$-\rho(t) < e(t) < \rho(t), \ \forall t \geq 0 \tag{7}$$

where $\rho(t)$ is a smooth, bounded, strictly positive and decreasing function of time satisfying the condition $\lim_{t \to \infty} \rho(t) > 0$, called performance function. The aforementioned statements are clearly illustrated in Fig. 1 for an exponentially decreasing performance function $\rho(t) = (\rho_0 - \rho_\infty)e^{-lt} + \rho_\infty$ with ρ_0, ρ_∞, l appropriately chosen strictly positive constants. The constant $\rho_0 = \rho(0)$ is selected such that $\rho_0 > |e(0)|$. The constant $\rho_\infty = \lim_{t \to \infty} \rho(t)$ represents the maximum allowable size of the tracking error $e(t)$ at the steady state. Moreover, the decreasing rate of $\rho(t)$, which is

affected by the constant l in this case, introduces a lower bound on the required speed of convergence of $e(t)$.

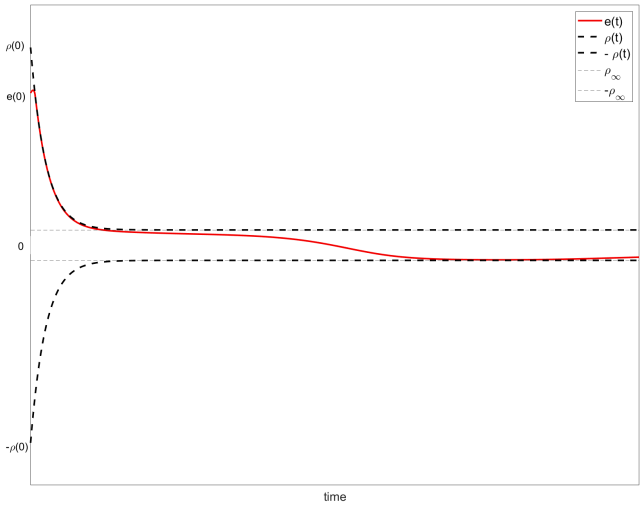

Fig. 1. Graphical illustration of prescribed performance (7).

III. Main Results

In this section, a technique that produces a reference trajectory is analyzed and designed such that it satisfies the persistency of excitation condition a priori. Subsequently, a control scheme is designed based on the prescribed performance control technique that succeeds in forcing the system states follow the desired trajectory and satisfying the persistency of excitation of the RBF neural network regressors that are utilized to achieve local learning of system nonlinear functions. Finally, an identification scheme is proposed to establish learning in a predefined compact set of the system states.

A. Reference Trajectory Design

The reference trajectory is designed in order to satisfy the persistency of excitation condition a priori. As stated in [12], if the inputs to RBFs are periodic and coincide with the neuron centers, placed in a lattice inside a predefined compact set, for a minimum amount of time during every period, then the regressor vector of neural network applied satisfies the persistency of excitation and the estimates of weight vectors converge to their optimal values satisfying the learning purpose. The predefined compact set, is a bounded rectangle for which x-coordinates represent the values of position of the system, and y-coordinates represent the velocity values accordingly. Thus, this rectangle represents all possible states of the system inside this area. Subsequently, this area is defined as the desired compact set inside which, the aim of learning is needed to be achieved. In this compact set a grid is created by placing the nodes of the grid in a way to cover as much as possible the feasible states of system. These nodes are the neuron centers of Radial Basis Functions (RBFs). Increasing the number of nodes improves the result of learning but inevitably increases the computational needs.

Motivated by the above discussion, we aim at designing an elliptical motion profile dictated by:

$$\frac{\left(x_{ref}(t)-A_{i,0}\right)^2}{A_i^2}+\frac{\dot{x}_{ref}(t)^2}{\left(A_i\omega_i\right)^2}=1,\ \forall i\in\{1,\ldots,p\} \quad (8)$$

where each ellipse is centered at $(A_{i,0},0)$ and A_i and $A_i\omega_i$ correspond to the major and minor axes. Hence, the planning of the desired trajectory lies in finding the optimal ellipses that pass through each given node inside a compact set, i.e., each ellipse $i\in\{1,\ldots,p\}$ that passes through the RBF's center c_i and satisfies a certain optimization condition. The goal needed to be satisfied is to cover as long distance as possible within the set, i.e., largest possible perimeter, as possible inside the compact set. So, the center of each ellipse needed to be calculated is in the form $(x_0,0)$. Two kinds of ellipses could pass through each node, the horizontal one whose focal points and major axis a run across $x-$axis, while in the second kind of ellipse, the vertical one, the two focal points are placed bilaterally in symmetric distances from the x-axis and the major axis a is running across the y-axis of the plane. Thus, it is concluded that two optimization problems should be solved, where each time an optimal ellipse from each of the two kinds is calculated. In particular, the optimization problem in case of horizontal ellipse is described by:

$$V_1(a_i,b_i)=\min\{-w_1a_ib_i+w_2f_i\} \quad (9)$$
$$s.t.\ |x_{i,0}|<D\bar{X}-a_i,\ D\geq 1 \quad (10)$$
$$b<D_d\bar{Y},\ D_d\geq 1 \quad (11)$$
$$a<D\bar{X} \quad (12)$$
$$|x_{i,0}|<\bar{X} \quad (13)$$
$$|f|\leq 0.96a \quad (14)$$

for all $i\in\{1,\ldots,p\}$, where a_i,b_i correspond to the major, minor semi-axes respectively, w_1,w_2 are weight factors expressing the importance of optimization goals and f_i denotes the distance between $x_{0,i}$ the focal points. The first term of (9) is utilized to maximize the perimeter of the ellipse lying within the area of the compact set by maximizing the corresponding area of the ellipse. The second term of (9) serves at minimizing the distance between the two focal points as we want to keep as much as possible the trajectory of the ellipse inside the compact set. Furthermore, $\bar{X},-\bar{X}$ are the upper and lower bounds in x-axis of the compact set, and , $\bar{Y},-\bar{Y}$ are the upper and lower bounds in y-axis respectively. The satisfaction of the first two constraints (10), (11) allows a certain percentage of the ellipse's trajectory to be crossed outside the compact set by regulating it with certain parameters D,D_d. Finally, (14) ensures that lines are excluded and the optimization algorithm each time gives an elliptical shape as a solution due to the fact that $f_i=\sqrt{a_i^2-b_i^2}$. Similarly, in case of vertical ellipse the

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optimization problem is described by:

$$V_2(a_i, b_i) = \min\{-w_1 a_i b_i + w_2 f_i\} \tag{15}$$
$$s.t.\ |x_{i,0}| < D\bar{X} - b_i,\ D \geq 1 \tag{16}$$
$$b < D\bar{X} \tag{17}$$
$$a < D_d\bar{Y},\ D_d \geq 1 \tag{18}$$
$$|x_{i,0}| < \bar{X} \tag{19}$$

for all $i \in \{1, \ldots, p\}$. Each optimization problem derives a different ellipse and thus it is chosen the one which has the minimum value of the objective function, i.e., the one having the largest area as well as the largest perimeter inside the compact set.

Subsequently, we assume a route which starts from the top left node following the whole trajectory of the first ellipse two times and next doing the same by moving onto the trajectory of the next consecutive node's ellipse on the right until reaching the last node. The same movement is done from the last node moving again back to the first node. This is the complete route which can be repeated periodically during the learning process. Thus, the desired reference trajectory is defined by the following periodic equations:

$$\begin{aligned} x_{ref}(t) &= A_i \sin(\omega_i t + \phi_i) + A_{i,0} \\ \dot{x}_{ref}(t) &= A_i \omega_i \cos\omega_i + \phi_i \end{aligned} \tag{20}$$

where $A_i, \omega_i, \phi_i, A_{i,0},\ i \in \{1, \ldots, p\}$ are the parameters of each elliptic trajectory of the whole route. So the whole desired reference trajectory for the learning problem is calculated from the above equations (20) by changing parameters $A_i, \omega_i, \phi_i, A_{i,0}$ as moving from one elliptic trajectory to the next. Each elliptic trajectory is the optimal one calculated for each node from which the ellipse passes through. The route to be followed is formed by running the whole ellipses trajectories starting from the trajectory corresponding to the first node till the one corresponding to the final node and then doing the opposite direction as well. Each time, the nodes which correspond to each ellipse's trajectory must be neighboring keeping a consecutive order during the move from the one node's ellipse to the next. As the transition from the one elliptic trajectory to the next one is regarded to be done instantly, the reference trajectory $(x_{ref}, \dot{x}_{ref})$ is filtered in order to smooth the regions of abrupt transition.

B. Control Design

Given the aforementioned reference trajectory, a controller is designed to force the system to follow it, so that learning is achieved. To that end, we propose a robust controller, using the prescribed performance method.

Given the compact set Ω_x, where a locally accurate approximation of the unknown functions $f(\cdot), g(\cdot)$ should be achieved, as well as a smooth bounded trajectory $x_d = [\bar{x}_{ref}(t), \dot{\bar{x}}_{ref}(t)]^T \in \Omega_x, \forall t \geq 0$ as defined in the Section III-A and any initial system condition $x(0) \in \Omega_x$, we define the state errors $e_1 = \bar{x}_1 - \bar{x}_{ref}$, with $\bar{x}_{ref} = [x^1_{ref}, \ldots, x^n_{ref}]^T \in \mathbb{R}^n$ and $e_2 = \bar{x}_2 - \dot{\bar{x}}_{ref}$, with $\dot{\bar{x}}_{ref} = [\dot{x}^1_{ref}, \ldots, \dot{x}^n_{ref}]^T \in \mathbb{R}^n$. The overall tracking error can be written in vector form as $e = [e_1, e_2]^T$. Let us now define the filtered error vector:

$$\sigma(e(t)) = e_2(t) + \Lambda e_1(t) \in \mathbb{R}^n \tag{21}$$

where $\Lambda = diag(\lambda_i),\ \lambda_i > 0,\ i = 1, \ldots, n$ is a gain matrix.

Subsequently, a performance function $\mathcal{R}(t) = \mathrm{diag}(\rho_i(t)),\ i \in \{1, \ldots, n\}$ is defined, with $\rho_i(t)$ selected as described in Section II-B, satisfying the condition $\rho_i(0) > |\sigma^i(e(0))|$. Additionally, select the decay rate l_i less than λ_i and let $\xi = \mathcal{R}^{-1}(t)\sigma(e(t))$ denote the normalized error vector. Taking the aforementioned into account, the control law is designed as:

$$u = -K\mathcal{R}^{-1} J_T(\xi) T(\xi) \tag{22}$$

where $K = \mathrm{diag}(k_i),\ i \in \{1, \ldots, n\}$ is a gain matrix, $T(\star) : \overbrace{(-1,1) \times \cdots \times (-1,1)}^{\text{n times}} \to \mathbb{R}^n$ with $T(0) = 0$ (e.g., $T(\star) = \ln\left(\frac{1+\star}{1-\star}\right)$) and the scaling factor $J_T(\star)$ denotes the Jacobian of mapping $T(\star)$. The properties of the proposed controller are summarized in the following theorem.

Theorem 1: Consider system (2) operating within a compact set Ω_x, any initial system conditions $x(0) \in \Omega_x$ and any smooth bounded trajectory $x_d(t) \in \Omega_x$. The control input (22) guarantees that: i) all signals of the closed-loop system remain bounded for all $t \geq 0$ and ii) the tracking error $e(t) = [e_1(t), e_2(t)]^T$ converges in finite time to the compact sets:

$$\begin{aligned} E_1 &\triangleq \left\{ e_1 = [e^1_1, \ldots, e^n_1]^T \in \mathbb{R}^n : |e^i_1| \leq \frac{\lim\limits_{t\to\infty} \rho_i(t)}{\lambda_i} \right\} \\ E_2 &\triangleq \left\{ e_2 = [e^1_2, \ldots, e^n_2]^T \in \mathbb{R}^n : |e^i_2| \leq 2 \lim_{t\to\infty} \rho_i(t) \right\}. \end{aligned} \tag{23}$$

Proof: Let us first define the generalized error vector $\zeta = [e_1, e_2, \xi]^T$. Differentiating ζ with respect to time and substituting the control input (22) we get:

$$\dot{\zeta} = \begin{bmatrix} e_2 \\ \mathcal{W} \\ \dot{\mathcal{R}}^{-1}\mathcal{R}\xi + \mathcal{R}^{-1}\left(\mathcal{W} + \Lambda e_2\right) \end{bmatrix} \tag{24}$$

where $\mathcal{W} = -M^{-1}\left(C(e_2 + \dot{\bar{x}}_{ref}) + G + K\mathcal{R}^{-1}J_T(\xi)T(\xi)\right) - \ddot{x}_d$ and $\dot{\mathcal{R}} = \mathrm{diag}(\dot{\rho}_i(t)),\ i \in \{1, \ldots, n\}$. Additionally, let $\epsilon = T(\xi) \in \mathbb{R}^n$. Let Ω_ξ be the non-empty set:

$$\Omega_\xi \triangleq \{(\xi_1, ..., \xi_n) \mid \xi_l \in (-1, 1), \forall l \in \{1, ..., n\}\}$$

According to the initial values of the performance functions, it is deduced that $|\xi_l(0)| < 1$, which implies that $\xi(0) \in \Omega_\xi$. Furthermore, (24) is piecewise continuous and locally integrable on t as well as locally $Lipschitz$ on ξ. Thus, as proved in [13], there exists a maximal solution $\xi(t)$ of (24) on a time interval $[0, t_{\max})$ such that $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$. Hence, $\sigma(e(t))$ is absolutely bounded by $\mathcal{R}(t), \forall t \in [0, \tau_{\max})$. As a result, there exists a compact set $\Omega_e \subset \mathbb{R}^{n\times 2}$ the size of which is independent of $\tau_{\max}$ such that $e(t) \in \Omega_e,\ \forall t \in [0, \tau_{\max})$. In this point, consider the radially unbounded

Lyapunov function candidate $V = \frac{1}{2}\epsilon^T K^{-1}\epsilon$. Differentiating with respect to time, we arrive at:

$$\begin{aligned}\dot{V} =&\epsilon^T K^{-1} J_T(\xi)\mathcal{R}^{-1}(\Lambda e_2 - \ddot{x}_d \\ &- M^{-1}\left(C\bar{x}_2 + G - u\right) - \dot{\mathcal{R}}\xi).\end{aligned} \tag{25}$$

Exploiting the fact that $x_d(t) \in \Omega_x,\ e(t) \in \Omega_e,\ \forall t \in [0, \tau_{max})$ and the continuity of $f(\cdot)$ we conclude by the Extreme Value Theorem the boundedness of $f(\cdot),\ \forall t \in [0, \tau_{max})$. Additionally, $\mathcal{R}(t), \dot{\mathcal{R}}(t)$ and $\ddot{x}_d$ are bounded by construction. So, there is a positive constant $\bar{F}$ such that:

$$\|\Lambda e_2 - \ddot{x}_d - M^{-1}\left(C\bar{x}_2 + G\right) - \dot{\mathcal{R}}\xi\| \leq \bar{F} \tag{26}$$

for all $\forall t \in [0, \tau_{max})$. Owing to (26) and substituting the control law (22), equation (25) is transformed into:

$$\begin{aligned}&\dot{V} \leq \left(J_T(\xi)\mathcal{R}^{-1}\epsilon K^{-1}\right)^T\left[\bar{F} + M^{-1}K J_T(\xi)\mathcal{R}^{-1}\epsilon\right] \implies \\ &\dot{V} \leq \left(J_T(\xi)\mathcal{R}^{-1}\epsilon K^{-1}\right)^T M^{-1}K\left(J_T(\xi)\mathcal{R}^{-1}\epsilon\right) \\ &\quad + \left(J_T(\xi)\mathcal{R}^{-1}\epsilon K^{-1}\right)^T \bar{F}.\end{aligned} \tag{27}$$

We showed that $\xi(t) \in \Omega_\xi,\ \forall t \in [0, \tau_{max})$. Thus, invoking the properties of $T(\star), J_T(\star)$ as well as the boundedness of ϵ we arrive at:

$$\dot{V} \leq \|J_T(\xi)\mathcal{R}^{-1}\epsilon\|\left(-\lambda_{min}(M)\|J_T(\xi)\mathcal{R}^{-1}\epsilon\| + k_{max}\bar{F}\right) \tag{28}$$

where k_{max} is the maximum element of K^{-1}. Therefore, $\dot{V} < 0$, when:

$$\|J_T(\xi)\mathcal{R}^{-1}\epsilon\| > \frac{k_{max}\bar{F}}{\lambda_{min}(M)}. \tag{29}$$

By letting $\omega = J_T(\xi)\mathcal{R}^{-1}\epsilon$ we conclude that:

$$\|\omega\| \leq \max\left\{\omega(0), \frac{k_{max}\bar{F}}{\lambda_{min}(M)}\right\} \tag{30}$$

Taking the inverse logarithmic function and combining the boundedness of all elements of $\epsilon(t)$, whose upper bounds are given by $|\epsilon_i| \leq \bar{\epsilon}_i$, we get:

$$-1 < \frac{e^{-\bar{\epsilon}_i} - 1}{e^{-\bar{\epsilon}_i} + 1} = \underline{\xi}_i \leq \xi_i \leq \bar{\xi}_i = \frac{e^{\bar{\epsilon}_i} - 1}{e^{\bar{\epsilon}_i} + 1} < 1 \tag{31}$$

and hence the control signal (22) remains bounded $\forall t \in [0, \tau_{max})$. Since $e(t) \in \Omega_e$ and $x_d(t) \in \Omega_x$ the boundedness of $\bar{x}_1, \bar{x}_2$ can be easily deduced $\forall t \in [0, \tau_{max})$. The rest of the stability analysis regarding the proposed closed-loop system, follows identical steps with Theorem 1 in [14], and thus it is omitted here owing to page limitations.

Finally, owing to the fact that (21) is bounded input bounded output, it is obtained by [15], that the state errors e_1, e_2 converge to the compact sets:

$$\begin{aligned}E_1 &\triangleq \left\{e_1 = [e_1^1, \ldots, e_1^n]^T \in \mathbb{R}^n : |e_1^i| \leq \frac{\lim\limits_{t\to\infty}\rho_i(t)}{\lambda_i}\right\} \\ E_2 &\triangleq \left\{e_2 = [e_2^1, \ldots, e_2^n]^T \in \mathbb{R}^n : |e_2^i| \leq 2\lim_{t\to\infty}\rho_i(t)\right\}.\end{aligned} \tag{32}$$

The proposed control scheme (22) achieves its goals without residing to the need of rendering $\bar{\epsilon} = [\epsilon_i, \ldots, \epsilon_n]^T$ arbitrarily small. Although the unknown system nonlinearities $f(\cdot), g(\cdot)$ affect the size of $\bar{\epsilon}$, they leave unaltered the achieved convergence properties. Finally, the actual tracking performance at the steady state is assessed by the sets E_1, E_2 defined above, which are closely related and depending on the matrix of performance functions $\mathcal{R}(t)$ as well as the control gain matrix Λ, while the convergence rate to steady state is regulated via the factors $l_i, i \in \{1, \ldots, n\}$ ■

C. Identification Scheme

In this section, the controller designed in Section III-B to ensure that the system states follow the desired trajectory, is combined with RBF neural networks to establish local learning of the nonlinear functions $f(\cdot), g(\cdot)$ in a predefined compact set, by imposing the Persistency of Excitation property. In particular, the unknown functions $f(x), g(x)$ are approximated through $f(x) = W_f^{*T}Z_f(x) + \epsilon_f(x),\ g(x) = W_g^{*T}Z_g(x) + \epsilon_g(x)$ respectively, where $Z_f(x) \in \mathbb{R}^{p\times n}, Z_g(x) \in \mathbb{R}^{p\times n}$ are the regressor vectors of selected RBFs with centers $c_i \in \Omega_x,\ i \in \{1, \ldots, p\}$ and W_f^{*T}, W_g^{*T} are the optimal values of RBF weights which minimize the error terms $\bar{e}_f = \sup\limits_{x\in\Omega_x}\{|\epsilon_f(x)|\}, \bar{e}_g = \sup\limits_{x\in\Omega_x}\{|\epsilon_g(x)|\}$. In the following proposition, it is shown how the desired trajectory x_d and the proposed controller (22) should be designed in a way that the regressor vectors $Z_f(x), Z_g(x)$ are persistently excited.

Proposition 1: Consider a periodic, bounded and smooth trajectory $x_d : [0, \infty) \to \Omega_x$ traversing all the centers of the selected RBFs as designed in Section III-A. By designing the performance function matrix $\mathcal{R}(t)$ and the control gain matrix Λ such that:

$$\lim_{t\to\infty}\rho_i(t) < \frac{\min\limits_{j_1\neq j_2}\left\{\frac{1}{2}\|c_{j_1} - c_{j_2}\|\right\}}{2\sqrt{\sum\limits_{i=1}^{n}\left(\frac{2^{i-1}}{\lambda^{n-1}}\right)^2}} \tag{33}$$

where $i \in \{1, \ldots, n\}$ and $j_1, j_2 \in \{1, \ldots, p\}$ then the control law (22) guarantees that the regressor vectors $Z_f(x(t)), Z_g(x(t))$ are persistently exciting.

Proof: Consider any $\epsilon < \min\limits_{i\neq j}\left\{\frac{1}{2}\|c_i - c_j\|\right\}, i, j \in \{1, \ldots, p\}$ and define the ball area $B_t(x_d(t), \frac{\epsilon}{2}) = \left\{x \in \mathbb{R}^n : \|x - x_d(t)\| \leq \frac{\epsilon}{2}\right\}$. If we select $\lim\limits_{t\to\infty}\rho_j(t) < \frac{\epsilon}{2\sqrt{\sum\limits_{i=1}^{2}\left(\frac{2^{i-1}}{\lambda_j}\right)^2}}, j \in \{1, \ldots, n\}$, then according to Theorem 1 it can be verified that there exists a time instant T_0 such that $x(t) \in B_t(x_d(t), \frac{\epsilon}{2}), \forall t \geq T_0$. Subsequently, let us consider the ϵ-neighborhoods of the RBF centers $B_i(c_i, \epsilon) = \{x \in \mathbb{R}^n : \|x - c_i\| \leq \epsilon\}, i \in \{1, \ldots, p\}$ which are non-intersecting. Owing to the smoothness and the boundedness of x_d as well as the fact that $x_d(t_{c_i} + mT) = c_i, \forall i \in \{1, \ldots, p\}$ with $m \in \mathbb{N}$ and T denoting the period of the orbit $x_d(t)$, there exists a positive constant δt such that $B_t(x_d(t), \frac{\epsilon}{2}) \subset B_i(c_i, \epsilon), \forall t \in [t_{c_i} + mT - \frac{\delta t}{2}, t_{c_i} +$

$mT + \frac{\delta t}{2}]$ for all $i \in \{1, \ldots, p\}$. As a consequence, since $x(t) \in B_t(x_d(t), \frac{\epsilon}{2}) \forall t \geq T_0$ it is concluded that $x(t) \in B_i(c_i, \epsilon), \forall t \in [t_{c_i} + mT - \frac{\delta t}{2}, t_{c_i} + mT + \frac{\delta t}{2}] \cap [T_0, \infty)$ for all $i \in \{1, \ldots, p\}$. Thus, invoking Theorem 3.5 in [11], the proof is complete. ■

Having defined the state error vectors $e = [e_1, e_2]^T$ the error dynamics are obtained by $\dot{e}_1 = e_2, \dot{e}_2 = f(x) + g(x)u - \ddot{x}_d$ or in compact form $\dot{e} = Ae + B[f(x) + g(x)u - \ddot{x}_d]$. The identification scheme is given by:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B(\hat{W}_f^T Z_f(x) + \hat{W}_g^T Z_g(x)u + K^T\tilde{x}) \\ \dot{\hat{W}}_f &= \gamma_f[Z_f(x)\tilde{x}^T B - \sigma_f \hat{W}_f] \\ \dot{\hat{W}}_g &= \gamma_g[Z_g(x)\tilde{x}^T B - \sigma_g \hat{W}_g] \end{aligned} \tag{34}$$

where $\tilde{x} = x - \hat{x}$ denotes the state estimation error vector, K is the gain vector selected such that $A - BK^T$ is Hurwitz, $\hat{W}_f, \hat{W}_g$ are the estimates of the unknown optimal weight vectors W_f^*, W_g^* and $\gamma_f, \gamma_g, \sigma_f, \sigma_g$ are positive gains.

Theorem 2: Consider system (2) excited by the control input (22). The identifier (34) guarantees that: i) all signals in the identification loop are bounded, ii) the estimates $\hat{W}_f, \hat{W}_g$ converge to small neighborhoods of their optimal values W_f^*, W_g^* and iii) the state estimation error $\tilde{x}$ converges to a small neighborhood of zero.

Proof: Consider the following Lyapunov function candidate:

$$V = \frac{1}{2}\tilde{x}^T\tilde{x} + \frac{1}{2\gamma_f} tr\{\tilde{W}_f^T\tilde{W}_f\} + \frac{1}{2\gamma_g} tr\{\tilde{W}_g^T\tilde{W}_g\} \tag{35}$$

where $tr\{\cdot\}$ denotes the trace of a square matrix and $\tilde{W}_f = W_f^* - \hat{W}_f, \tilde{W}_g = W_g^* - \hat{W}_g$ represent parametric errors. Differentiating (35) with respect to time, and after some straightforward manipulations we obtain:

$$\begin{aligned} \dot{V} = \tilde{x}^T(A - BK^T)\tilde{x} &- \sigma_f tr\{\tilde{W}_f^T\tilde{W}_f\} - \sigma_g tr\{\tilde{W}_g^T\tilde{W}_g\} \\ &+ \sigma_f tr\{\tilde{W}_f^T W_f^*\} + \sigma_g tr\{\tilde{W}_g^T W_g^*\} \\ &+ \tilde{x}^T B(\epsilon_f(x) + \epsilon_g(x)u) \end{aligned} \tag{36}$$

Note that the quantity $tr\{\tilde{W}_h^T W_h^*\}, h = \{f, g\}$ can be written as $tr\{\tilde{W}_h^T W_h^*\} = \sum_{i=1}^{n}\sum_{j=1}^{p} \tilde{W}_{h_{ji}}^T W_{h_{ji}}^*$. Invoking the Young's inequality as well as substituting the previous equations into (36) we arrive at:

$$\begin{aligned} \dot{V} = \tilde{x}^T(A - BK^T)\tilde{x} + q &- \sigma_f tr\{\tilde{W}_f^T\tilde{W}_f\} \\ &- \sigma_g tr\{\tilde{W}_g^T\tilde{W}_g\} + \tilde{x}^T B(\epsilon_f(x) + \epsilon_g(x)u) \end{aligned} \tag{37}$$

where $q = \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{p} |\tilde{W}_{f_{ji}}|^2 + |W_{f_{ji}}^*|^2 + |\tilde{W}_{g_{ji}}|^2 + |\tilde{W}_{g_{ji}}^*|^2\right)$. Next we substitute $\tilde{x}^T(A - BK^T)\tilde{x} = \tilde{x}^T(\frac{A-BK^T+(A-BK^T)^T}{2})\tilde{x}$ in (37) resulting to:

$$\begin{aligned} \dot{V} = \tilde{x}^T(\frac{A - BK^T + (A - BK^T)^T}{2})\tilde{x} &+ q - \sigma_f tr\{\tilde{W}_f^T\tilde{W}_f\} \\ &- \sigma_g tr\{\tilde{W}_g^T\tilde{W}_g\} + \tilde{x}^T B(\epsilon_f(x) + \epsilon_g(x)u) \end{aligned} \tag{38}$$

Subsequently, by letting $\sigma_{min} = -\frac{\lambda_{min}}{2}\left(A - BK^T + (a - BK^T)^T\right)$ the above equation can be manipulated as:

$$\begin{aligned} \dot{V} \leq -\frac{\sigma_{min}}{2}\|\tilde{x}\|^2 &- \sigma_f tr\{\tilde{W}_f^T\tilde{W}_f\} - \sigma_g tr\{\tilde{W}_g^T\tilde{W}_g\} \\ &+ \tilde{x}^T B(\epsilon_f(x) + \epsilon_g(x)u) + q \end{aligned} \tag{39}$$

Notice that the last term of (39) can be written as $\tilde{x}^T B(\epsilon_f(x) + \epsilon_g(x)u) = \sum_{i=1}^{n} \dot{\tilde{x}}_i \left(\epsilon_{fi}(x) + \sum_{j=1}^{n} \epsilon_{gij}(x)u_j\right)$ owing to the fact that $\tilde{x} = [\tilde{x}_1, \ldots, \tilde{x}_n, \dot{\tilde{x}}_1, \ldots, \dot{\tilde{x}}_n]^T$. Invoking once again the Young's inequality we can arrive at:

$$\begin{aligned} &-\frac{\sigma_{min}}{2n}\|\tilde{x}\|^2 + \dot{\tilde{x}}_1\| \left(\epsilon_{fi}(x) + \sum_{j=1}^{n} \epsilon_{gij}(x)u_j\right) \leq \\ &\frac{n\left(\epsilon_{fi}(x) + \sum_{j=1}^{n} \epsilon_{gij}(x)u_j\right)^2}{2\sigma_{min}} = \mathcal{A}_i \end{aligned} \tag{40}$$

for all $i \in \{1, \ldots, n\}$. Finally by letting $D = q + \sum_{i=1}^{n} \mathcal{A}_i$, (39) becomes:

$$\dot{V} \leq -\frac{\sigma_{min}}{2}\|\tilde{x}\|^2 - \sigma_f tr\{\tilde{W}_f^T\tilde{W}_f\} - \sigma_g tr\{\tilde{W}_g^T\tilde{W}_g\} + D \tag{41}$$

which leads to the uniform ultimate boundness (UUB) of $\tilde{x}, \tilde{W}_f, \tilde{W}_g$ with respect to compact sets:

$$\begin{aligned} \mathcal{X} &\triangleq \left\{\tilde{x} \in \mathbb{R}^{2n} : \|\tilde{x}\| \leq \sqrt{\frac{2D}{\sigma_{min}}}\right\} \\ \mathcal{W}_f &\triangleq \left\{\tilde{W}_f \in \mathbb{R}^p : tr\{\tilde{W}_f^T\tilde{W}_f\} \geq \frac{D}{\sigma_f}\right\} \\ \mathcal{W}_g &\triangleq \left\{\tilde{W}_g \in \mathbb{R}^p : tr\{\tilde{W}_g^T\tilde{W}_g\} \geq \frac{D}{\sigma_g}\right\}\}. \end{aligned}$$

■

Additionally it has been proven in the previous subsection that under the excitation of (22), system (2) generates uniformly bounded state trajectories. Hence, $\hat{x} \in L_\infty$ and therefore all signals of the identification scheme remain bounded. So it can be verified that $\|\tilde{x}(t)\|$ converges to a neighborhood of zero whose size is regulated by permitting σ_{min} to admit sufficiently large values by choosing appropriately the K matrix and putting small values for σ_f, σ_g. Also, according to Proposition 1 the regressor vectors $Z_f(x(t)), Z_g(x(t))$ are persistently exciting. Thus, the parametric errors $\tilde{W}_f, \tilde{W}_g$ converge to neighborhoods of zero, whose size is determined by the value of D which is closely influenced by the values of parameters $\sigma_{min}, \sigma_f, \sigma_g$ for a given RBF network structure which corresponds to specific modeling error bounds $\bar{\epsilon}_f, \bar{\epsilon}_g$.

IV. Simulation Results

A simulation example is provided to assess the proposed identification process. In particular, we consider the following Lagrangian dynamics:

$$I\ddot{q} + ml\sin(q)\dot{q}^2 = u$$

for which the state is expressed as $x_1 = q, x_2 = \dot{q}$ the unknown nonlinearities are given by:

$$f(x) = -\frac{ml}{I}\sin(x_1)x_2^2,\ g(x_1) = \frac{1}{I}$$

The compact set over which we want to learn the system nonlinearities is $\Omega = \{x_1, x_2 : -1.5 \leq x_1, x_2 \leq 1.5\}$. We also selected an RBF neural network with 16 nodes within the set Ω. A reference tractor was then calculated that transverses all nodes and which is depicted in red dashed line in Fig. 2. The evolution of the weight estimates of the

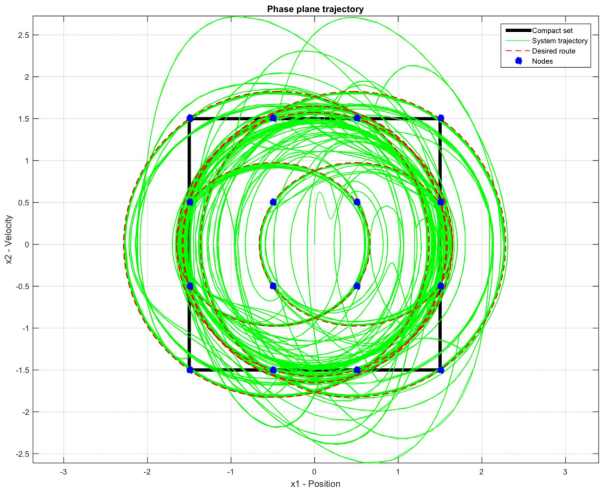

Fig. 2. The reference trajectory in red dashed line that transverses all nodes (blue stars) and the actual trajectory in green solid line.

RBF neural network that was adopted to approximate the unknown function $f(x)$ is illustrated in Fig. 3. Moreover, The

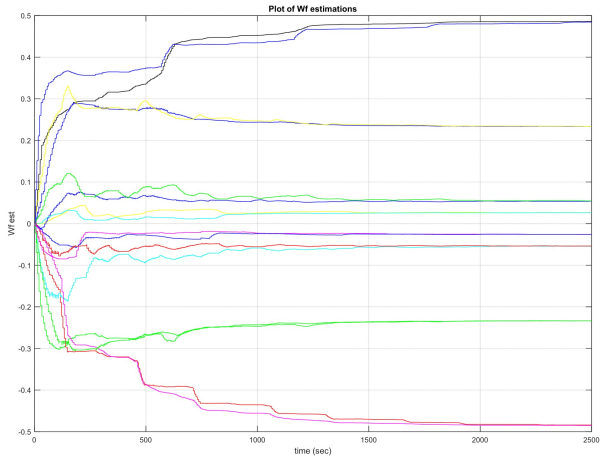

Fig. 3. The evolution of the weight estimates for the function $f(x)$.

approximation quality over the set Ω is given in Fig. 4 for the function $f(x)$. In order to improve further the approximation results, we employed RBF neural networks with 36 and 64 nodes, respectively. The results are summarized in Table I, from which it can be easily deduced that the approximation

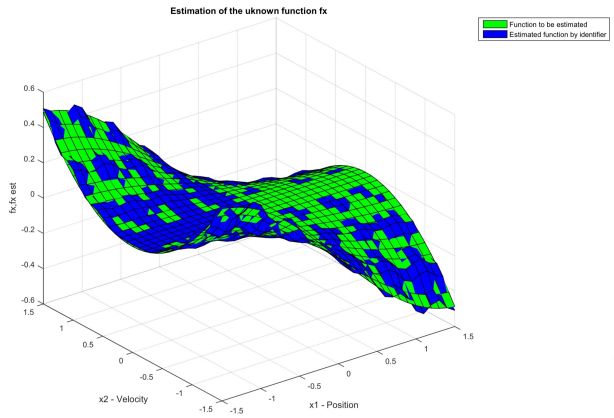

Fig. 4. The unknown function (green) and its approximation (blue).

error reduces as the number of nodes increases, demanding however more time to establish accurate learning as the reference trajectory becomes more complex. Similar results hold for the constant but unknown function $g(x_1) = \frac{1}{I}$.

TABLE I
THE EFFECT OF INCREASING THE NODES NUMBER.

Number of nodes	Maximum error	Average error	Simulation time
16	9.56%	2.18%	250 *sec*
36	7.12%	1.34%	360 *sec*
64	6.07%	1.02%	600 *sec*

V. Conclusions and Future Work

In this work, a system identification methodology was designed for on-line learning the actual nonlinearities of unknown Lagrangian systems. The whole scheme is consisted of an online RBF neural network identifier and a controller that guarantees arbitrarily close tracking of a reference trajectory that satisfies the persistency of excitation condition for the RBF regressors employed in the identification scheme. The nonlinear functions of the system are expressed by RBF neural networks and it has been proven that the estimated weight vector converges close to the optimal weight vector. For future work, these results could be extended for other kind of systems like pure-feedback systems and strict-feedback systems. Another interesting field of application would be fault diagnosis, which plays important role in modern engineering systems.

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