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Robust and efficient methods for stochastic finite element analysis using Monte Carlo simulation

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Abstract

In the present paper the weighted integral method and the Monte Carlo simulation are used together with innovative solution strategies based on the Preconditioned Conjugate Gradient method (PCG) to produce robust and efficient solutions for the stochastic finite element analysis of space frames. The numerical tests presented demonstrate the superiority of the proposed computational strategies compared to the widely used Neumann expansion method both in terms of accuracy and computational efficiency. The superiority is more pronounced in cases where the analysis needs to be performed for large variations of the stochastic parameters.

1. Introduction

The theory and the methods of stochastic analysis have been developed significantly during the last 10 years and have been documented in an increasing number of publications. Considerable progress in applying stochastic process theory to the area of structural engineering has made it possible to achieve higher levels of reliability. This has led to safety measures that design engineers have to take into account due to the inherent probabilistic nature of the design parameters such as material properties, geometry and/or loading conditions. Stochastic analysis involves the estimation of the response variability and/or reliability of a stochastic system defined as a structural system that possesses uncertainties in its material and/or geometric properties. Although from a theoretical point of view the field has reached a stage where the developed methodologies are becoming widespread, from a computational point of view serious obstacles have been encountered in practical implementations. In large and realistic problems the developed methodologies are either cumbersome or computationally intensive, while most of them are incapable of dealing with non-linear structural response.

Analytic solutions to the problem are restricted to simple linear elastic structures under static loads while most of the current research work is focused on obtaining numerical solutions which are more appropriate for handling realistic problems [1–4]. Stochastic Finite Element Methods (SFEM) belong to this category. The most widely used SFEM approaches are based on the representation of stochastic fields as a series of random variables. Among them, the most important methods are the midpoint method [5–9], the nodal point method [10], the interpolation method [5, 11, 12], the local averaging method [5, 13–16], the series expansion methods such as the Karhunen–Loeve expansion method [17, 18], the homogeneous chaos expansion method [18], and the perturbation method [5, 7, 9, 18–21].

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These methods have, however, a limited range of applicability since they can only be accurate for small values of variability of the stochastic properties, while they are not capable of dealing with problems involving non-linearities or dynamic loading [22]. In addition, these methods require, in order to be accurate, fine element meshes so that the element size is a fraction of the correlation distance of the stochastic field involved in the problem. The weighted integral method, developed by Deodatis and Shinozuka [22, 25], does not require discretization of the random field. Its accuracy is, therefore, independent of the chosen mesh, thus avoiding the severe shortcomings of the previously mentioned methods.

The use of Monte Carlo Simulation (MCS) based on SFEM approaches has the major advantage that accurate solutions can be obtained for any problem whose deterministic solution is known either numerically or analytically, since it statistically converges to the correct solution provided that a large number of simulations is employed. In fact, it is the only method available to solve stochastic problems involving non-linearities, dynamic loading, stability effects, parametric excitations etc. The disadvantage of the standard MCS is that it is usually extremely computationally demanding due to the repeated analyses that have to take place. It is for this reason that a Neumann expansion of the inverse of the stiffness matrix [7, 25] has been proposed in order to enhance the computational efficiency of MCS.

In the present paper the weighted integral method in conjunction with the Monte Carlo simulation is used together with innovative solution strategies based on the Preconditioned Conjugate Gradient method (PCG) to produce robust and efficient solutions for the stochastic finite element analysis of space frames. The numerical tests presented demonstrate the superiority of the proposed computational strategies, compared to the widely used Neumann expansion method, both in terms of accuracy and computational efficiency. The superiority is more pronounced in cases where the analysis needs to be performed for large variations of the stochastic parameters.

2. Stochastic finite element analysis of space frames

2.1. The weighted integral method

The weighted integral method has been applied in [22–24] to formulate the stochastic element stiffness matrix for a 2-D beam element. In this section the corresponding stochastic stiffness matrix of a 3-D beam element is derived.

For a 3-D beam element with 6 degrees of freedom per node the modulus of elasticity is assumed to vary randomly along the element length according to

$$E^{(e)}(x) = E_0[1 + f^{(e)}(x)] \tag{1}$$

where E_0 is the mean value of the modulus of elasticity and $f^{(e)}(x)$ is a one-dimensional univariate (1D-1V) zero-mean homogeneous stochastic field.

In order to avoid the possibility of obtaining non-positive values of the elastic modulus, $f^{(e)}(x)$ is assumed to be bounded according to

$$-0.80 < f^{(e)}(x) < 0.80 \tag{2}$$

These bounds are implemented as follows: any generated sample function, that has at least one of its values out of the bounds, is automatically discarded.

The stochastic element stiffness matrix is given by

$$\boldsymbol{K}^{(e)} = \int_0^{L^{(e)}} \boldsymbol{B}^{(e)T} \boldsymbol{D}^{(e)} \boldsymbol{B}^{(e)} \, dx \tag{3}$$

where

$$D^{(e)} = E^{(e)}(x)\begin{bmatrix} A^{(e)} & 0 & 0 & 0 \\ 0 & I_3^{(e)} & 0 & 0 \\ 0 & 0 & I_2^{(e)} & 0 \\ 0 & 0 & 0 & \dfrac{J^{(e)}}{2(1+\nu)} \end{bmatrix} \tag{4}$$

$B^{(e)}$ is the matrix containing the derivatives of the shape functions and $A^{(e)}$, $I_2^{(e)}$, $I_3^{(e)}$, $J^{(e)}$ are the cross-sectional area, the cross-sectional moments of inertia for weak and major axes and the torsional modulus, respectively. $L^{(e)}$ and ν are the length of the element and the Poisson ratio, respectively.

Performing integration with respect to x, the stochastic stiffness matrix $K^{(e)}$ may be expressed as

$$K^{(e)} = K_0^{(e)} + X_0^{(e)}\,\Delta K_0^{(e)} + X_1^{(e)}\,\Delta K_1^{(e)} + X_2^{(e)}\,\Delta K_2^{(e)} \tag{5}$$

or

$$K^{(e)} = K_0^{(e)} + \Delta K^{(e)} \tag{6}$$

$K_0^{(e)}$ and $\Delta K^{(e)}$ are used to denote the stationary part and the fluctuating part of the stochastic element stiffness matrix, respectively. $X_0^{(e)}$, $X_1^{(e)}$ and $X_2^{(e)}$ are the so-called weighted integrals which are random variables defined as

$$X_0^{(e)} = \int_0^{L^{(e)}} f^{(e)}(x)\,\mathrm{d}x \tag{7}$$

$$X_1^{(e)} = \int_0^{L^{(e)}} x f^{(e)}(x)\,\mathrm{d}x \tag{8}$$

$$X_2^{(e)} = \int_0^{L^{(e)}} x^2 f^{(e)}(x)\,\mathrm{d}x \tag{9}$$

K_0 is the mean value of $K^{(e)}$ since the weighted integrals have zero-mean and ΔK_0, ΔK_1 and ΔK_2 are deterministic matrices, the definition of which can be found in Appendix A.

2.2. Representation of the stochastic field

Since the MCS techniques are used to calculate the response variability of the stochastic structural system, it is necessary to digitally generate sample functions of the 1D-1V stationary Gaussian zero-mean homogeneous stochastic field $f(x)$. This is done using the spectral representation method [26, 27], taking advantage of the Fast Fourier Transform (FFT) technique in order to reduce the computational effort of the simulation. This is achieved using the formula

$$f^{(j)}(p\,\Delta x) = \mathrm{Re}\left\{\sum_{n=0}^{M-1} B_n^{(j)}\,\mathrm{e}^{inp(2\pi/M)}\right\}, \quad p = 0, 1, \ldots, M-1; \quad j = 1, 2, \ldots, N_{\mathrm{SAMP}} \tag{10}$$

where Re indicates the real part, M defines the number of points at which $f(x)$ is realised along the element length and N_{SAMP} is the number of samples to be generated. B_n is given by

$$B_n^{(j)} = \sqrt{2} A_n\,\mathrm{e}^{i\phi_n^{(j)}}, \quad n = 0, 1, \ldots, M-1 \tag{11}$$

where $\phi_n^{(j)}$ represents the jth realisations of the independent random phase angles uniformly distributed in the range $[0\text{–}2\pi]$ and A_n is defined as

$$A_n = (2S_{ff}(n\,\Delta k)\,\Delta k)^{1/2}, \quad n = 0, 1, \ldots, M-1 \tag{12}$$

and

$$\Delta k = \frac{k_u}{N} \tag{13}$$

k_u is the upper cut-off wave number and N is the number of intervals in the discretization of the spectrum. S_{ff} is the two-sided power spectral density function defined as

$$S_{ff} = \frac{1}{4}\sigma_f^2 b_f^3 k^2 \mathrm{e}^{-b_f k} \tag{14}$$

where σ_f denotes the standard deviation of the stochastic field, b_f denotes the parameter that influences the shape of the spectrum and hence the scale of the correlation and $k = n\,\Delta k$.

Using Eq. (10), a large number of sample functions, N_{SAMP}, is produced for each element of the structure generating a set of stochastic stiffness matrices. The associated structural problem is solved N_{SAMP} times, while the response variability can finally be calculated by taking the response statistics of the N_{SAMP} simulations [27].

3. Solution procedures

3.1. The Neumann series expansion

The expansion of the inverse of the stochastic global stiffness matrix in a Neumann series within the framework of the Monte Carlo Simulation method has been treated by several researchers [5, 7, 9, 17, 18, 25]. The solution of the stochastic problem

$$(\boldsymbol{K}_0 + \Delta\boldsymbol{K})\boldsymbol{u} = \bar{\boldsymbol{F}} \tag{15}$$

yields

$$\boldsymbol{u} = (\boldsymbol{I} + \boldsymbol{K}_0^{-1}\,\Delta\boldsymbol{K})^{-1}\boldsymbol{K}_0^{-1}\boldsymbol{F} \tag{16}$$

The term in the parentheses can be expressed in a Neumann series expansion giving

$$\boldsymbol{u} = (\boldsymbol{I} - \boldsymbol{P} + \boldsymbol{P}^2 - \boldsymbol{P}^3 + \cdots)\boldsymbol{K}_0^{-1}\boldsymbol{F} \tag{17}$$

with $\boldsymbol{P} = \boldsymbol{K}_0^{-1}\,\Delta\boldsymbol{K}$. The response vector can now be represented by the following series

$$\boldsymbol{u} = \boldsymbol{u}_0 - \boldsymbol{P}\boldsymbol{u}_0 + \boldsymbol{P}^2\boldsymbol{u}_0 - \boldsymbol{P}^3\boldsymbol{u}_0 + \cdots \tag{18}$$

or

$$\boldsymbol{u} = \boldsymbol{u}_0 - \boldsymbol{u}_1 + \boldsymbol{u}_2 - \boldsymbol{u}_3 + \cdots \tag{19}$$

The series solution can also be expressed by the following recursive equation

$$\boldsymbol{K}_0\boldsymbol{u}_i = \Delta\boldsymbol{K}\boldsymbol{u}_{i-1}, \quad i = 1, 2, \ldots \tag{20}$$

This formula has the advantage that only the non-fluctuating part of the stiffness matrix has to be factorized once, while the additive terms $\boldsymbol{u}_i$ to the solution can be computed by successive backward and forward substitutions. The series may be truncated after a fixed number of terms, or according to an error norm given by

$$\|\boldsymbol{u}_i\| \Big/ \left\| \sum_{k=0}^{i} (-1)^k \boldsymbol{u}_k \right\| \leq \varepsilon_1 \tag{21}$$

or

$$\|\boldsymbol{r}_i\| / \|\boldsymbol{F}\| \leq \varepsilon_2 \tag{22}$$

where $\boldsymbol{r}_i = \boldsymbol{b} - (\boldsymbol{K}_0 + \Delta\boldsymbol{K})\boldsymbol{u}_i$. Thus, the final solution is reached when the termination criterion is satisfied. This check is performed after the computation of each additive term $\boldsymbol{u}_i$. The first criterion is most frequently used since it is more computationally efficient. The second criterion requires the evaluation of the residual vector which involves an additional matrix vector multiplication for each Neumann term.

The most interesting feature of this approach, in connection with the stochastic finite element method and the Monte Carlo simulation, is that the matrix factorization is required only once for all samples, while the approximate nature of the solution may be sufficient due to the inherent approximations involved in the Monte Carlo simulation approach. The convergence requirement of the Neumann expansion method, that the absolute values of all eigenvalues of $\boldsymbol{P}$ should be less than 1, can always be satisfied, even for large components of the deviation matrix compared to the corresponding components of $\boldsymbol{K}_0$, after a modification proposed in [7]. According to this modification, a scalar parameter m is introduced satisfying the inequality $m < (\lambda_\kappa + 1)/2$, where λ_κ is the largest positive eigenvalue of $\boldsymbol{P}$. Then the series expansion of Eq. (20) is replaced by

$$\boldsymbol{K}_0\boldsymbol{u}_i^* = \frac{1}{m}\Delta\boldsymbol{K}^*\boldsymbol{u}_{i-1}^* \tag{23}$$

with

$$\boldsymbol{u}_i^* = \frac{1}{m}\boldsymbol{u}_i \quad \text{and} \quad \Delta\boldsymbol{K}^* = (\boldsymbol{K} - m\boldsymbol{K}_0).$$

3.2. The preconditioned conjugate gradient method

The inherent features of the stochastic finite element analysis in conjunction with Monte Carlo simulation makes the Preconditioned Conjugate Gradient method (PCG) very attractive for the solution of the system of linear equations for each Monte Carlo simulation. The PCG is established as the more attractive iterative procedure for solving linear problems resulting from the finite element discretization. An important factor in the success of this method in solving large-scale finite element equations is the preconditioned technique used to improve the ellipticity of the coefficient matrix. This typically consists of replacing the original system $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$ by the equivalent system

$$\boldsymbol{R}^{-1}\boldsymbol{K}\boldsymbol{u} = \boldsymbol{R}^{-1}\boldsymbol{F} \tag{24}$$

where the transformation or preconditioning matrix $\boldsymbol{R}$ is an approximation to $\boldsymbol{K}$ and it is non-singular. The PCG algorithm, based on the most efficient conjugate gradient version in respect to computational labour, storage requirements and accuracy, is defined as follows for the untransformed variables:

$$\begin{aligned}
&\alpha_m = \frac{(\boldsymbol{r}^{(m)}, \boldsymbol{z}^{(m)})}{(\boldsymbol{d}^{(m)}, \boldsymbol{K}\boldsymbol{d}^{(m)})}\\
&\boldsymbol{u}^{(m+1)} = \boldsymbol{u}^{(m)} + \alpha_m\boldsymbol{d}^{(m)}\\
&\boldsymbol{r}^{(m+1)} = \boldsymbol{r}^{(m)} + \alpha_m\boldsymbol{K}\boldsymbol{d}^{(m)}\\
&\text{if } \|\boldsymbol{r}^{(m+1)}\| / \|\boldsymbol{F}\| \leq \varepsilon \text{ then stop}\\
&\boldsymbol{z}^{(m+1)} = \boldsymbol{R}^{-1}\boldsymbol{r}^{(m+1)}\\
&\alpha_m = \frac{(\boldsymbol{r}^{(m+1)}, \boldsymbol{z}^{(m+1)})}{(\boldsymbol{r}^{(m)}, \boldsymbol{z}^{(m)})}\\
&\boldsymbol{d}^{(m+1)} = -\boldsymbol{z}^{(m+1)} + \alpha_m\boldsymbol{d}^{(m)}
\end{aligned} \tag{25}$$

with

$$\boldsymbol{r}^{(0)} = \boldsymbol{K}\boldsymbol{u}^{(0)} - \boldsymbol{F}, \qquad \boldsymbol{z}(0) = \boldsymbol{R}^{-1}\boldsymbol{r}^{(0)}, \qquad \boldsymbol{d}^{(0)} = \boldsymbol{z}^{(0)}$$

At the heart of the PCG iterative procedure for solving $\boldsymbol{Ku}=\boldsymbol{F}$ is the determination of the residual vector and the selection of the preconditioning matrix. The accuracy achieved and the computational labour of the method is largely determined by how these two parameters are selected. A study performed in [28] revealed that the computation of the residual vector from its defining formula $\boldsymbol{r}^{(m)}=\boldsymbol{Ku}^{(m)}-\boldsymbol{F}$ offers no improvement in the accuracy of the computed results. In fact, it was found that, contrary to previous recommendations, the calculation of the residuals by the recursive expression of algorithm in Eq. (25) produces a more stable and well-behaved iterative procedure. Based on this observation, a mixed precision PCG implementation is proposed in which all computations are performed in single precision, except for double precision computation of the matrix vector multiplication occurring during the recursive evaluation of the residual vector. This implementation is a robust and reliable solution procedure, even for handling large and ill-conditioned problems, while it is also computer storage-effective. It was also proved to be more cost-effective, for the same storage demands, than double precision calculations [28, 29].

The preconditioned matrix $\boldsymbol{R}$ has to be selected appropriately so that the eigenvalues of $\boldsymbol{R}^{-1}\boldsymbol{K}$ are spread over a much narrower range than those of K. For the types of problems considered in this study, the non-fluctuating part of the stiffness matrix is taken as the preconditioning matrix. The diagonal factor and the triangular factor of the $\mathrm{LDL}^{\mathrm{T}}$ factorization of K_0 are stored in double or in single precision arithmetic. Thus, the complete factorization of K_0 is performed only once for all Monte Carlo simulations, while the preconditioned vector z of the PCG algorithm is obtained by forward and backward substitutions at each PCG iteration.

3.3. The Neumann-CG method (NCG)

In order to improve the quality of the preconditioning matrix used in the PCG method, a Neumann series expansion is implemented for the calculation of the preconditioned vector z. The preconditioning matrix is now defined as the complete stochastic global stiffness matrix $\boldsymbol{K}=\boldsymbol{K}_0+\Delta\boldsymbol{K}$, but the solution for z is performed approximately using a truncated Neumann series expansion. Thus, the preconditioned vector z of the PCG algorithm is obtained at each iteration by

$$z = z_0 - z_1 + z_2 - z_3 + \cdots \tag{26}$$

z_0 is given by

$$z_0 = \boldsymbol{K}_0^{-1}\boldsymbol{r} \tag{27}$$

and z_i is obtained by

$$\boldsymbol{K}_0 z_i = \Delta\boldsymbol{K} z_{i-1}, \quad i = 1, 2, \ldots \tag{28}$$

while the superscript $(m+1)$ has been dropped for clarity. The solution of Eq. (28) is performed in double or in single precision arithmetic.

The incorporation of the Neumann series expansion in the preconditioned step of the PCG method can be seen from two different perspectives. From the PCG point of view an improvement on the quality of the preconditioning matrix is achieved by computing a better approximation to the solution of $\boldsymbol{u}=(\boldsymbol{K}_0+\Delta\boldsymbol{K})^{-1}\boldsymbol{F}$ than the one provided by the preconditioning matrix K_0. From the Neumann series expansion point of view, the inaccuracy entailed by the truncated series is alleviated by the conjugate gradient iterative procedure. It remains to be seen, however, whether the anticipated improvement on the convergence properties of the PCG method or on the accuracy of the Neumann series expansion implies also a reduction of the overall computational effort by counteracting the additional cost involved at each iteration.

4. Numerical tests

The test example used to validate the performance of the previously described methods is the 20 storey space frame shown in Fig. 1. The loads considered here are deterministic and they consist of vertical forces equivalent to a uniform load of 100 psf (4.788 kN/m^2) and a basic horizontal pressure of 20 psf (0.956 kN/m^2). The modulus of elasticity is considered to be a 1D-1V zero-mean homogeneous stochastic field. Two sets of sample functions were prepared: the first with a standard deviation of 0.15 and the second with a standard deviation of 0.25. The performance of the PCG and the NCG methods compared to the MCS with a direct skyline solver and the truncated Neumann expansion method without correction is examined in terms of both accuracy and computational efficiency. The test were performed on a Silicon Graphics Indigo R4000 workstation.

A compact storage scheme is used for the PCG and the NCG methods to store the stiffness matrix. Non-zero terms are stored in a real vector, while the corresponding column members are stored in an integer vector of equal length. An additional integer vector with length equal to the number of equations is used to record the start of each row inside the compact scheme. The extra storage for the

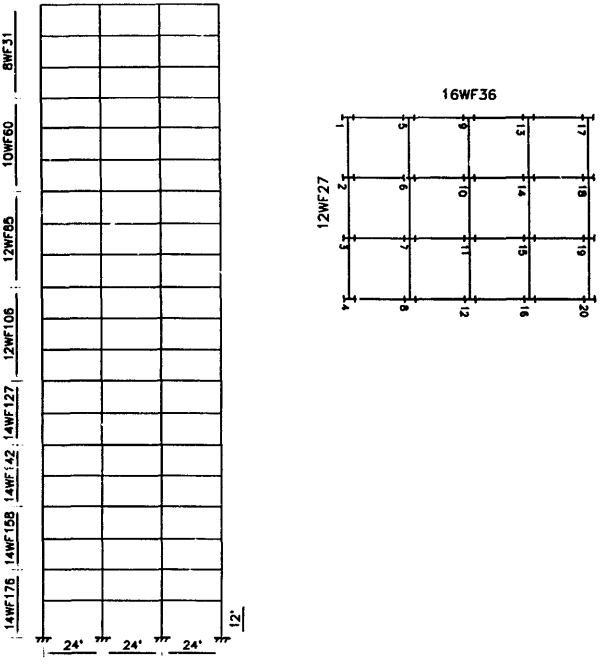

Fig. 1. Twenty-storey space frame.

conjugate gradient method is $5n$ real positions. The direct MCS procedure with the complete Cholesky LDL^T factorization is handled with a skyline storage for L in double precision arithmetic. In estimating the computer storage it is assumed that integers are stored as INTEGER *2 or INTEGER *4 according to their maximum possible values, and the floating point variables as REAL *4 or REAL *8 according to the accuracy of computation, single or double precision, respectively.

Figs. 2 and 3 show the convergence behaviour and the attained accuracy of the methods for predicting the mean value and the standard deviation of the tip deflection of the frame for a standard deviation of the modulus of elasticity of 0.15. The PCG and the NCG methods are implemented with mixed precision arithmetic in which all computations are performed in single precision except for the double precision computation of the matrix-vector multiplication for the calculation of the residual vector. Numbers in parentheses correspond to the value of the termination criterion (ε or ε_2), while the number following the abbreviated name of the Neumann-type method gives the order of expansion of the Neumann series.

Figs. 4 and 5 show the comparisons corresponding to Figs. 2 and 3 for a standard deviation of the modulus of elasticity of 0.25. Fig. 6 depicts the storage requirements for each of the above methods where the abbreviation DP stands for Double Precision arithmetic. Figs. 7 and 8 show the performance of the methods after 100 simulations in terms of computing time, for standard deviations of the modulus of elasticity of 0.15 and 0.25, respectively. Finally, Fig. 9 shows the number of conjugate gradient iterations required for the PCG and the NCG methods in a typical MC simulation.

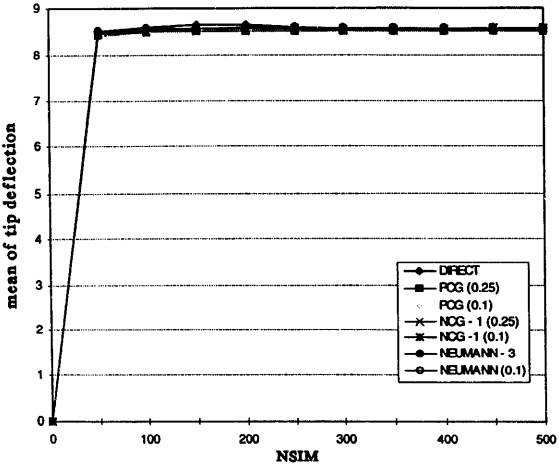

Fig. 2. Mean value of deflection at top storey for $\sigma_f = 0.15$.

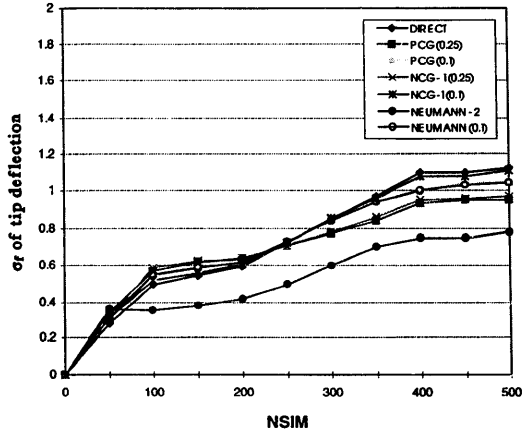

Fig. 3. Standard deviation of deflection at top storey for $\sigma_f = 0.15$.

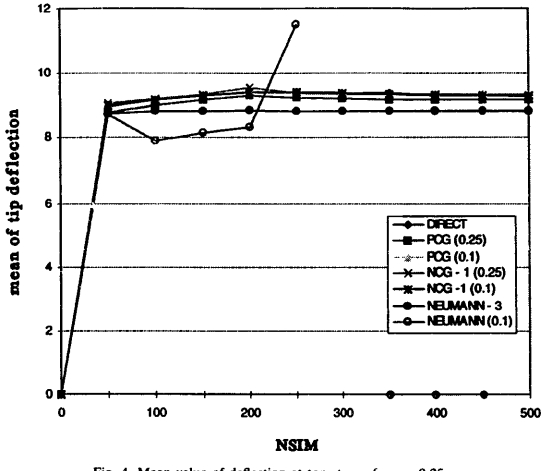

Fig. 4. Mean value of deflection at top storey for $\sigma_f = 0.25$.

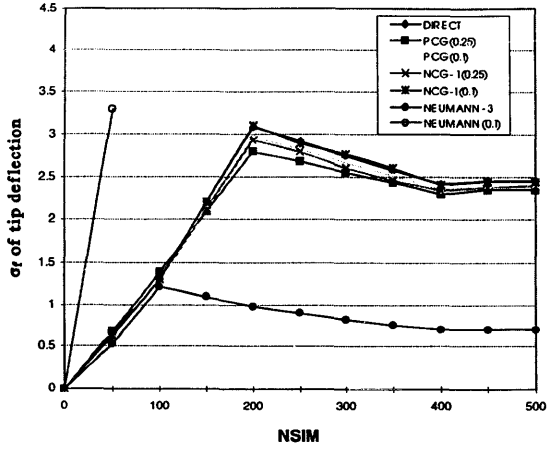

Fig. 5. Standard deviation of deflection at top storey for $\sigma_f \approx 0.25$.

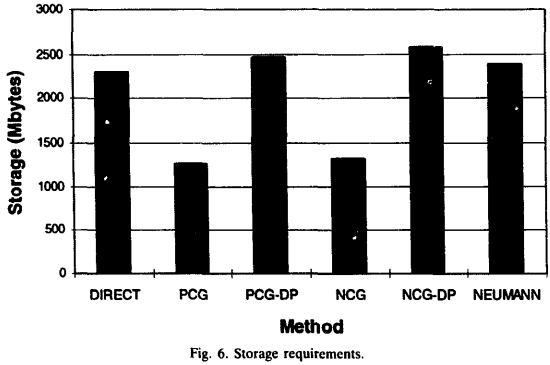

Fig. 6. Storage requirements.

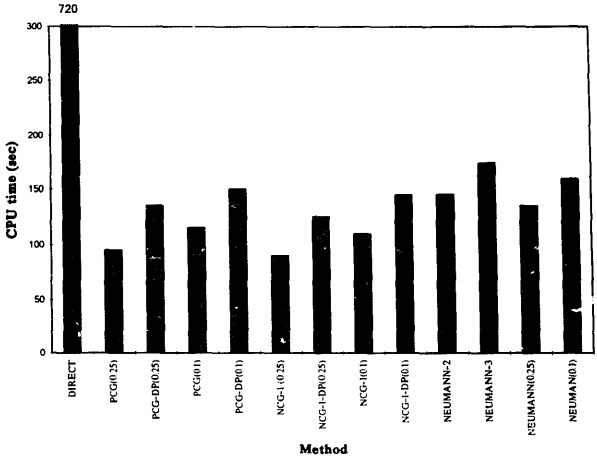

Fig. 7. Total CPU time for $\sigma_f = 0.15$ after 100 simulations.

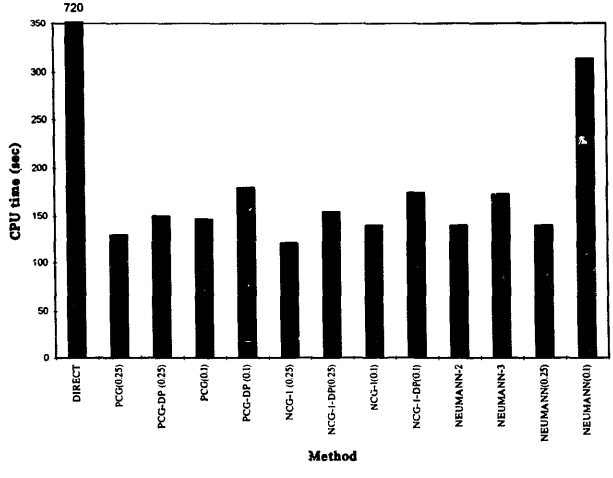

Fig. 8. Total CPU time for $\sigma_f = 0.25$ after 100 simulations.

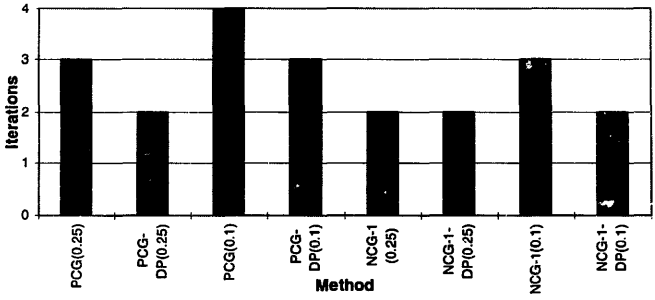

Fig. 9. Number of conjugate gradient iterations for $\omega_f = 0.25$ in a typical MC simulation.

5. Conclusions

This paper presents a methodology for accurately and efficiently estimating the response variability of stochastic finite element systems with application to space frames. Accurate solutions are obtained using the preconditioned conjugate gradient method and the Neumann conjugate gradient method in the context of Monte Carlo simulation, while significant reduction in computing time and storage requirements is achieved compared to the conventional direct solution and the truncated Neumann expansion method without correction. This improvement is even more pronounced for large variations of the modulus of elasticity where the results obtained from the Neumann expansion method without correction are often erroneous.

For the case of a 0.15 variation of the modulus of elasticity, the results obtained from the first order NCG method (NCG-1) and the termination criterion $\varepsilon = 0.1$ coincide with those of the direct MCS, while all other methods are in good agreement within an error of 7% and 12% for $\varepsilon = 0.1$ and $\varepsilon = 0.25$, respectively. The second order Neumann expansion method (NEUMANN-2), produces an error of 25% and represents the most computationally efficient case in the context of the Neumann expansion method. The results given by the Neumann expansion method with termination criterion $\varepsilon_2 = 0.1$ are in good agreement with those of the direct MCS. A reduction of 40% and 30% in CPU time is achieved, using the NCG-1 method compared to the Neumann expansion method, for $\varepsilon = \varepsilon_2 = 0.1$ and $\varepsilon = \varepsilon_2 = 0.25$, respectively. NCG and PCG methods are more computationally efficient even if compared to the most efficient second order Neumann expansion (NEUMANN-2).

For the case of a 0.25 variation of the modulus of elasticity, the Neumann expansion method can not achieve satisfactory results, producing an error of 70%, while the Neumann expansion method with $\varepsilon_2 = 0.1$ diverges as shown in Figs. 4 and 5. Results obtained by the NCG-1 method with $\varepsilon = 0.1$ coincide with those obtained by the direct MCS. All other versions of PCG and NCG methods produce errors of less than 5%. For this case, the NCG-1 method requires less computing time compared to the PCG method $\varepsilon = 0.1$, while a reduction of 85% in computing time is achieved compared to the direct MCS. For the case of $\varepsilon = 0.25$ NCG and PCG methods require the same computing time. Additionally, Fig. 9 shows the number of iterations of the methods for a typical simulation, while the average number of iterations for NCG-1 and PCG is 2.45 and 4.70, respectively.

Finally, a reduction of about 50% in storage requirements is achieved using PCG or NCG methods compared to the direct MCS and the Neumann expansion method, as well as to PCG and NCG methods with double precision arithmetic. Double precision arithmetic versions require 20% more CPU time compared to the corresponding mixed precision ones.

It may therefore be concluded that PCG and NCG methods are superior compared to the conventional direct solution and the truncated Neumann expansion method without correction for the stochastic finite element analysis using the Monte Carlo simulation approach, while NCG proved to be

superior compared to the PCG method. In all cases studied, a considerable improvement in both accuracy and computational efficiency is achieved regardless of the amount of the variation of the stochastic parameters.

Acknowledgment

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Appendix A

A.1. Stochastic element stiffness matrix of a 3-D beam element

Using the standard displacement-based finite element analysis, the matrix containing the shape functions for the 3-D beam element is given by

$$N^{(e)} = \begin{bmatrix} N_{11}^{(e)} & 0 & 0 & 0 & 0 & 0 & N_{17}^{(e)} & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{22}^{(e)} & 0 & 0 & 0 & N_{26}^{(e)} & 0 & N_{28}^{(e)} & 0 & 0 & 0 & N_{2,12}^{(e)} \\ 0 & 0 & N_{33}^{(e)} & 0 & N_{35}^{(e)} & 0 & 0 & 0 & N_{39}^{(e)} & 0 & N_{3,11}^{(e)} & 0 \\ 0 & 0 & 0 & N_{44}^{(e)} & 0 & 0 & 0 & 0 & 0 & N_{4,10}^{(e)} & 0 & 0 \end{bmatrix}$$

where

$$N_{11}^{(e)} = 1 - \frac{x}{L^{(e)}}, \qquad N_{17}^{(e)} = \frac{x}{L^{(e)}}$$

$$N_{22}^{(e)} = 1 - 3\left(\frac{x}{L^{(e)}}\right)^2 + 2\left(\frac{x}{L^{(e)}}\right)^3, \qquad N_{26}^{(e)} = x\left(1 - \frac{x}{L^{(e)}}\right)^2$$

$$N_{28}^{(e)} = 3\left(\frac{x}{L^{(e)}}\right)^2 - 2\left(\frac{x}{L^{(e)}}\right)^3, \qquad N_{2,12}^{(e)} = x\left[-\frac{x}{L^{(e)}} + \left(\frac{x}{L^{(e)}}\right)^2\right]$$

$$N_{33}^{(e)} = 1 - 3\left(\frac{x}{L^{(e)}}\right)^2 + 2\left(\frac{x}{L^{(e)}}\right)^3, \qquad N_{35}^{(e)} = -x\left(1 - \frac{x}{L^{(e)}}\right)^2$$

$$N_{39}^{(e)} = 3\left(\frac{x}{L^{(e)}}\right)^2 - 2\left(\frac{x}{L^{(e)}}\right)^3, \qquad N_{3,11}^{(e)} = -x\left[-\frac{x}{L^{(e)}} + \left(\frac{x}{L^{(e)}}\right)^2\right]$$

$$N_{44}^{(e)} = 1 - \frac{x}{L^{(e)}}, \qquad N_{4,10}^{(e)} = \frac{x}{L^{(e)}}$$

The matrix containing the derivatives of the shape functions is given by

$$B^{(e)} = \begin{bmatrix} B_{11}^{(e)} & 0 & 0 & 0 & 0 & 0 & B_{17}^{(e)} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{22}^{(e)} & 0 & 0 & 0 & B_{26}^{(e)} & 0 & B_{28}^{(e)} & 0 & 0 & 0 & B_{2,12}^{(e)} \\ 0 & 0 & B_{33}^{(e)} & 0 & B_{35}^{(e)} & 0 & 0 & 0 & B_{39}^{(e)} & 0 & B_{3,11}^{(e)} & 0 \\ 0 & 0 & 0 & B_{44}^{(e)} & 0 & 0 & 0 & 0 & 0 & B_{4,10}^{(e)} & 0 & 0 \end{bmatrix}$$

where

$$B_{11}^{(e)} = -\frac{1}{L^{(e)}}, \qquad B_{17}^{(e)} = \frac{1}{L^{(e)}}$$

$$B_{22}^{(e)} = -\frac{6}{L^{(e)2}} + \frac{12x}{L^{(e)3}}, \qquad B_{26}^{(e)} = -\frac{4}{L^{(e)}} + \frac{6x}{L^{(e)2}}$$

$$B_{28}^{(e)} = \frac{6}{L^{(e)2}} - \frac{12x}{L^{(e)3}}, \qquad B_{2,12}^{(e)} = -\frac{2}{L^{(e)}} + \frac{6x}{L^{(e)2}}$$

$$B_{33}^{(e)} = -\frac{6}{L^{(e)2}} - \frac{12x}{L^{(e)3}}, \qquad B_{35}^{(e)} = \frac{4}{L^{(e)}} - \frac{6x}{L^{(e)2}}$$

$$B_{39}^{(e)} = \frac{6}{L^{(e)2}} - \frac{12x}{L^{(e)3}}, \qquad B_{3,11}^{(e)} = \frac{2}{L^{(e)}} - \frac{6x}{L^{(e)2}}$$

$$B_{44}^{(e)} = B_{11}^{(e)}, \qquad B_{4,10}^{(e)} - B_{17}^{(e)}$$

The deterministic matrices $K_0^{(e)}$, $\Delta K_0^{(e)}$, $\Delta K_1^{(e)}$, $\Delta K_2^{(e)}$ involved in Eq. (5) are defined as follows:

$$K_0^{(e)} = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 & 0 \\ & 12I_3 & 0 & 0 & 0 & 6I_3/L & 0 & -12I_3 & 0 & 0 & 0 & 6I_3/L \\ & & 12I_2 & 0 & -6I_2/L & 0 & 0 & 0 & -12I_2 & 0 & -6I_2/L & 0 \\ & & & G & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 \\ & & & & 4I_2/L^2 & 0 & 0 & 0 & 6I_2/L & 0 & 2I_2/L^2 & 0 \\ & & & & & 4I_3/L^2 & 0 & -6I_3/L & 0 & 0 & 0 & 2I_3/L^2 \\ & & & & & & A & 0 & 0 & 0 & 0 & 0 \\ & & & \text{Symm.} & & & & 12I_3 & 0 & 0 & 0 & -6I_3/L \\ & & & & & & & & 12I_2 & 0 & 6I_2/L & 0 \\ & & & & & & & & & G & 0 & 0 \\ & & & & & & & & & & 4I_2/L^2 & 0 \\ & & & & & & & & & & & 4I_3/L^2 \end{bmatrix}$$

where $A = A^{(e)}E_0^{(e)}/L^{(e)}$, $I_2 = E_0^{(e)}I_2^{(e)}/L^{(e)3}$, $I_3 = E_0^{(e)}I_3^{(e)}/L^{(e)3}$, $G = E_0^{(e)}J^{(e)}/(2(1+\nu)L^{(e)})$ and $L = L^{(e)}$

$$\Delta K_0^{(e)} = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 & 0 \\ & 36I_3L^2 & 0 & 0 & 0 & 24I_3L^3 & 0 & -36I_3L^2 & 0 & 0 & 0 & 12I_3L^3 \\ & & 36I_2L^2 & 0 & -24I_2L^3 & 0 & 0 & 0 & -36I_2L^2 & 0 & -12I_2L^3 & 0 \\ & & & G & 0 & 0 & 0 & 0 & 0 & -G & 0 & 0 \\ & & & & 16I_2L^4 & 0 & 0 & 0 & 24I_2L^3 & 0 & 8I_2L^4 & 0 \\ & & & & & 16I_3L^4 & 0 & -24I_3L^3 & 0 & 0 & 0 & 8I_3L^4 \\ & & & & & & A & 0 & 0 & 0 & 0 & 0 \\ & & & \text{Symm.} & & & & 36I_3L^2 & 0 & 0 & 0 & -12I_3L^3 \\ & & & & & & & & 36I_2L^2 & 0 & 6I_2/L & 0 \\ & & & & & & & & & G & 0 & 0 \\ & & & & & & & & & & 4I_2L^4 & 0 \\ & & & & & & & & & & & 4I_3L^4 \end{bmatrix}$$

where $A = A^{(e)}E_0^{(e)}/L^{(e)2}$, $I_2 = E_0^{(e)}I_2^{(e)}/L(e)^6$, $I_3 = E_0^{(e)}I_3^{(e)}/L^{(e)6}$, $G = E_0^{(e)}J^{(e)}/(2(1+\nu)L^{(e)2})$ and $L = L^{(e)}$

$$\Delta K_1^{(e)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 144I_1L & 0 & 0 & 0 & -84I_1L^2 & 0 & 144I_1L & 0 & 0 & 0 & -60I_3L^2 \\ & & 144I_2L & 0 & 84I_2L^2 & 0 & 0 & 0 & 144I_2L & 0 & 60I_2L^2 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & -48I_2L^3 & 0 & 0 & 0 & -84I_2L^2 & 0 & -36I_2L^3 & 0 \\ & & & & & -48I_1L^3 & 0 & 84I_1L^2 & 0 & 0 & 0 & -36I_3L^3 \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & -144I_1L & 0 & 0 & 0 & 60I_3L^2 \\ & & & \text{Symm.} & & & & & -144I_2L & 0 & -60I_2L^2 & 0 \\ & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & & -24I_2L^3 & 0 \\ & & & & & & & & & & & -24I_3L^3 \end{bmatrix}$$

where $I_2 = E_0^{(e)} I_2^{(e)} / L^{(e)^6}$, $I_3 = E_0^{(e)} I_3^{(e)} / L^{(e)^6}$ and $L = L^{(e)}$

$$\Delta K_2^{(e)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 144I_3 & 0 & 0 & 0 & 72I_3L & 0 & -144I_3 & 0 & 0 & 0 & 72I_3L \\ & & 144I_2 & 0 & -72I_2L & 0 & 0 & 0 & -144I_2 & 0 & -72I_2L & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 36I_2L^2 & 0 & 0 & 0 & 72I_2L & 0 & 36I_2L^2 & 0 \\ & & & & & 36I_3L^2 & 0 & -72I_3L & 0 & 0 & 0 & 36I_3L^2 \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & \text{Symm.} & & & & 144I_3 & 0 & 0 & 0 & -72I_3L \\ & & & & & & & & 144I_2 & 0 & 72I_2L & 0 \\ & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & & 36I_2L^2 & 0 \\ & & & & & & & & & & & 36I_3L^2 \end{bmatrix}$$

where $I_2 = E_0^{(e)} I_2^{(e)} / L^{(e)^6}$, $I_3 = E_0^{(e)} I_3^{(e)} / L^{(e)^6}$ and $L = L^{(e)}$.

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