Knot theory related to generalized and cyclotomic Hecke algebras of type $\mathcal{B}$

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1 Introduction

After Jones's construction of the classical by now Jones polynomial for knots in S^3 using Ocneanu's Markov trace on the associated Hecke algebras of type $\mathcal{A}$, arised questions about similar constructions on other Hecke algebras as well as in other 3-manifolds.

In [12] is established that knot isotopy in a 3-manifold may be interpreted in terms of Markov braid equivalence and, also, that the braids related to the 3-manifold form algebraic structures. Moreover, the sets of braids related to the solid torus or to the lens spaces $L(p,1)$ form groups, which are in fact the Artin braid groups of type $\mathcal{B}$. As a consequence, in [12, 13] appeared the first construction of a Jones-type invariant using Hecke algebras of type $\mathcal{B}$, and this had a natural interpretation as an isotopy invariant for oriented knots in a solid torus. In a further 'horizontal' development and using a different technique we constructed in [8] all such solid torus knot invariants derived from the Hecke algebras of type $\mathcal{B}$. Furthermore, in [7] all Markov traces related to the Hecke algebras of type $\mathcal{D}$ were consequently constructed.

In this paper we consider all possible generalizations of the $\mathcal{B}$-type Hecke algebras, namely the cyclotomic and what we call 'generalized', and we construct Markov traces on each of them, so as to obtain *all possible different levels* of homfly-pt analogues in the solid torus related to the (Hecke) algebras of $\mathcal{B}$-type. Our strategy is based on the one in [13], which in turn followed [11]. So, in this sense, the construction in [12, 13] is incorporated here as the most basic level.

In more detail: It is well-understood from Jones's construction of the homfly-pt (2-variable Jones) polynomial, P_L, in [11], that $\mathcal{H}_n(q)$, the Iwahori-Hecke algebra of $\mathcal{A}_n$-type, is a quotient of the braid group algebra $\mathbb{Z}[q^{\pm 1}]B_n$

by factoring out the quadratic relations

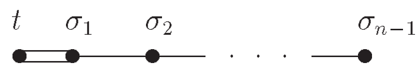
$$\sigma_i^2 = (q-1)\sigma_i + q$$

and that these relations reflect precisely the skein property of P_L:

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}\, P_{L_+} - \sqrt{q}\sqrt{\lambda}\, P_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})\, P_{L_0},$$

where L_+ is a regular projection of an oriented link containing a specified positive crossing, L_- the same projection with a negative crossing instead, and L_0 yet the same projection with no crossing.

We do now analogous considerations for the solid torus, which we denote by ST. Let us consider the following Dynkin diagram.

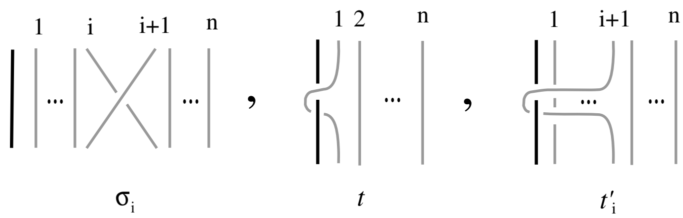
$(\mathcal{B}_n)$  $n \geq 1$

The symbols $t, \sigma_1, \ldots, \sigma_{n-1}$ labelling the nodes correspond to the generators of the Artin braid group of type $\mathcal{B}_n$, which we denote by $B_{1,n}$. $B_{1,n}$ is defined therefore by the relations

$$\begin{array}{rcll}
\sigma_1 t \sigma_1 t & = & t\sigma_1 t \sigma_1 & \\
t\sigma_i & = & \sigma_i t & \text{if} \quad i > 1 \\
\sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if} \quad |i-j| > 1 \\
\sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{if} \quad 1 \leq i \leq n-2
\end{array}$$

Relations of these types will be called *braid relations.*

$B_{1,n}$ may be seen as the subgroup of B_{n+1}, the classical braid group on $n+1$ strands, the elements of which keep the first strand fixed (this is the reason for having chosen the symbol $B_{1,n}$). This allows for a geometric interpretation of the elements of $B_{1,n}$ as mixed braids in S^3. Below we illustrate the generators σ_i, t and the element $t_i' = \sigma_i \ldots \sigma_1 t \sigma_1^{-1} \ldots \sigma_i^{-1}$ in $B_{1,n}$, which plays a crucial role in this work.



Note that the inverses of σ_i, t are represented by the same geometric pictures, but with the opposite crossings.

As shown in [12, 13], we can represent oriented knots and links inside ST by elements of the groups $B_{1,n}$, where the fixed strand represents the complementary solid torus in S^3, and the next n numbered strands represent the knot in ST. Also, that knot isotopy in ST can be translated in terms of equivalence classes in $\bigcup_{n=1}^{\infty} B_{1,n}$ (Markov theorem), the equivalence being generated by the following two moves.

(i) Conjugation: if $\alpha, \beta \in B_{1,n}$ then $\alpha \sim \beta^{-1}\alpha\beta$.
(ii) Markov moves: if $\alpha \in B_{1,n}$ then $\alpha \sim \alpha{\sigma_n}^{\pm 1} \in B_{1,n+1}$.

Consider now the classical Iwahori-Hecke algebra of type $\mathcal{B}_n$, $\mathcal{H}_n(q,Q)$, as a quotient of the group algebra $\mathbb{Z}[q^{\pm 1}, Q^{\pm 1}]B_{1,n}$ by factoring out the ideal generated by the relations $t^2 = (Q-1)t + Q$ and $g_i^2 = (q-1)g_i + q$ for all i, where we denote the image of σ_i in $\mathcal{H}_n(q,Q)$ by g_i. The idea in [12, 13, 8] was to construct invariants of knots in the solid torus by constructing trace functions τ on $\bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q)$ which support the Markov property:

$$\tau(hg_n) = z\tau(h),$$

for z an independent variable in $\mathbb{Z}[q^{\pm 1}, Q^{\pm 1}]$ and $h \in \mathcal{H}_n(q,Q)$. In other words, traces that respect the above braid equivalence on $\bigcup_{n=1}^{\infty} B_{1,n}$. The construction of such traces was only possible because we were able to find an appropriate inductive basis on $H_{n+1}(q,Q)$, every element of which involves the generator g_n or the element $t'_n := g_n \ldots g_1 t g_1^{-1} \ldots g_n^{-1}$ at most once (see picture above for the lifting of t'_i in $B_{1,n}$). In particular, the trace constructed in [12, 13] was well-defined inductively by the rules:

$$\begin{array}{lll} 1) & tr(ab) = tr(ba) & a, b \in \mathcal{H}_n(q,Q) \\ 2) & tr(1) = 1 & \text{for all } \mathcal{H}_n(q,Q) \\ 3) & tr(ag_n) = z\, tr(a) & a \in \mathcal{H}_n(q,Q) \\ 4) & tr(at'_n) = s\, tr(a) & a \in \mathcal{H}_n(q,Q) \end{array}$$

If we had not used the elements t'_n in the above constructions we would have not been able to define the trace with only four simple rules. The intrinsic reason for this is that $B_{1,n}$ splits as a semi-direct product of the classical braid group B_n and of its free subgroup $P_{1,n}$ generated precisely by the elements $t, t'_1, \ldots, t'_{n-1}$:

$$B_{1,n} = P_{1,n} \rtimes B_n.$$

The Jones-type invariants in ST constructed from the above traces on $\bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q)$ satisfy the skein rule related to the quadratic relations $g_i^2 = (q-1)g_i + q$ plus another one reflecting the quadratic relation $t^2 = (Q-1)t + Q$ (cf. [12, 13, 8] for an extensive treatment).

During the work of S.L. and J. Przytycki on the problem of computing the 3rd skein module of the lens spaces $L(p,1)$ following the above strategy, it turned out that the skein rule of the homfly-pt type invariants in [12, 13, 8] related to t was actually 'artificial', so far that knot invariants in ST were concerned, and that for analogous constructions in $L(p,1)$ it was needed to have constructed first the most generic 2-variable Jones analogue in ST, one that would not satisfy any skein relation involving t.

We drop then the quadratic relation of t, and we consider the quotient of the group algebra $\mathbb{Z}[q^{\pm 1}]B_{1,n}$ by factoring out only the relations

$$g_i^2 = (q-1)g_i + q$$

for all i. This is now a new infinite dimensional algebra, which we denote by $\mathcal{H}_n(q,\infty)$ and we shall call it *generalized Iwahori-Hecke algebra of type $\mathcal{B}$*. By g_i above we denote the image of σ_i in $\mathcal{H}_n(q,\infty)$, whilst the symbol ∞ was chosen to indicate that the generator t satisfies no order relation (since now any power t^k, for $k \in \mathbb{Z}$ may appear, like in $B_{1,n}$). For connections of these algebras with the affine Hecke algebras of type $\mathcal{A}$ see Remark 1.

But we would like now to go one step back and, instead of removing from $\mathcal{H}_n(q,Q)$ the quadratic relation for t, to require that t satisfies a relation given by a cyclotomic polynomial of degree d:

$$(t-u_1)(t-u_2)\cdots(t-u_d) = 0$$

Then we obtain a finite-dimensional algebra known as *cyclotomic Hecke algebra of type $\mathcal{B}$*, denoted here by $\mathcal{H}_n(q,d)$. The corresponding *cyclotomic Coxeter group of type $\mathcal{B}$*, which we denote by $W_{n,d}$, is obtained as a quotient of $B_{1,n}$ modulo the relations $g_i^2 = 1$ and $t^d = 1$. $\mathcal{H}_n(q,d)$ may be seen as a 'd-deformation' of $W_{n,d}$: In order to obtain the group algebra we have to substitute the parameters of the cyclotomic polynomial by the d th roots of unity (and not by 1 as in the classical case). These algebras have been introduced and studied independently by two groups of mathematicians in [1, 2, 4, 3]. It follows from the discussion above that the cyclotomic Hecke algebras are also related to the knot theory of the solid torus and, in fact, they make the bridge between $\mathcal{H}_n(q,Q)$ and $\mathcal{H}_n(q,\infty)$.

Like for the classical Hecke algebras of type $\mathcal{B}$, in order to construct linear Markov traces on $\bigcup_{n=1}^{\infty}\mathcal{H}_n(q,\infty)$ or on $\bigcup_{n=1}^{\infty}\mathcal{H}_n(q,d)$, we need to find appropriate inductive bases on both types of these algebras. The inductive bases are derived from known basic sets. This is the aim and the main result of Section 3. Note that, in the case of $\mathcal{H}_n(q,Q)$, we could easily yield such an inductive basis using the results in [6], whilst for $\mathcal{H}_n(q,d)$ we use the results in [2], [4]. For $\mathcal{H}_n(q,\infty)$ we study its structure in Section 2 and we construct a basis for it using the structure of the braid group $B_{1,n}$ and the known bases for $\mathcal{H}_n(q,d)$.

In Section 4 we construct Markov traces on $\bigcup_{n=1}^{\infty} \mathcal{H}_n(q,\infty)$ and on $\bigcup_{n=1}^{\infty} \mathcal{H}_n(q,d)$ using the inductive bases of Section 3. Finally in Section 5, we normalize the traces according to the Markov braid theorem in order to derive the corresponding knot invariants in ST, and we also give skein interpretations. The invariant related to $\mathcal{H}_n(q,\infty)$ is the most interesting one for us, and in this sense, this work may be seen as the required fundament for extending such constructions to knots in the lens spaces (see remarks at the end). In the special case of $\mathcal{H}_n(q,\infty)$ the derived knot invariant reproves the structure of the 3rd skein module of the solid torus (cf. [10, 16]). On the other hand, the knot invariants derived from $\mathcal{H}_n(q,d)$ are related to submodules of the 3rd skein module of ST. It may be worth noting that introducing and studying $\mathcal{H}_n(q,\infty)$ has been independent of the studies on the cyclotomic analogues.

Our method shows on one hand that the original strategy of [11] can carry through to so complicated structures. On the other hand it unifies the construction for all these different $\mathcal{B}$-type algebras and it highlights the algebraic background underlying these knot invariants in ST. The tedious calculations employed for constructing appropriate bases reflect the tedious arguments of a more combinatorial approach.

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2 Finding a basis for $\mathcal{H}_n(q,\infty)$

We start by introducing in more detail $\mathcal{H}_n(q,\infty)$, $\mathcal{H}_n(q,d)$ and their corresponding Coxeter-type groups $W_{n,\infty}$, $W_{n,d}$.

Definition 1 The generalized Iwahori-Hecke algebra of type $\mathcal{B}_n$ is defined as

$$\mathcal{H}_n(q,\infty) := \mathbb{Z}[q^{\pm 1}]\, B_{1,n} \,/ < {\sigma_i}^2 = (q-1)\,\sigma_i + q \;\text{ for all } i > .$$

The underlying generalized Coxeter group of type $\mathcal{B}_n$ is defined as

$$W_{n,\infty} := B_{1,n} \,/ < {\sigma_i}^2 = 1 \;\; \text{for all } i > .$$

It follows that if g_i denotes the image of σ_i in $\mathcal{H}_n(q,\infty)$, then $\mathcal{H}_n(q,\infty)$ is defined by the generators $t, g_1, g_2, \ldots, g_{n-1}$ and their relations:

$$\begin{array}{rcll} tg_1tg_1 & = & g_1tg_1t & \\ tg_i & = & g_it & \text{for} \quad i>1 \\ g_ig_{i+1}g_i & = & g_{i+1}g_ig_{i+1} & \text{for} \quad 1 \leq i \leq n-2 \\ g_ig_j & = & g_jg_i & \text{for} \quad |i-j|>1 \\ {g_i}^2 & = & (q-1)\,g_i + q & \text{for} \quad \text{all } i \end{array}$$

$\mathcal{H}_n(q,\infty)$ is an associative algebra with 1. Also, it is easily verified that, if S_n is the symmetric group, then

$$W_{n,\infty} = \mathbb{Z}^n \rtimes S_n \;\; (\text{ compare with the structure of} B_{1,n}).$$

Definition 2 Let $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, u_1^{\pm 1}, \ldots, u_d^{\pm 1}, \ldots]$, where $q, u_1, \ldots, u_d, \ldots$ are indeterminates. The cyclotomic Iwahori-Hecke algebra of type $\mathcal{B}_n$ and of degree d is defined as

$$\mathcal{H}_n(q,d) := \mathcal{R}\, B_{1,n} \,/ < {\sigma_i}^2 = (q-1)\,\sigma_i + q \;\; \text{all } i,\; (t-u_1)(t-u_2)\cdots(t-u_d) = 0 > .$$

The underlying cyclotomic Coxeter group of type $\mathcal{B}$ and of degree d is:

$$W_{n,d} := B_{1,n} \,/ < {\sigma_i}^2 = 1 \;\; \text{for all } i,\; t^d = 1,\; d \in \mathbb{N} > .$$

The relation $t^d = 1$ is derived by the cyclotomic polynomial by substituting the u_i's by the d'th roots of unity. Also, the Coxeter group of $\mathcal{B}_n$-type, in our notation $W_{n,2}$, is the quotient of $B_{1,n}$ over the relations $t^2 = {\sigma_i}^2 = 1$, for all i.

$\mathcal{H}_n(q,d)$ is an associative algebra with 1, and it is a free module over $\mathcal{R}$ of rank $d^n \cdot n!$, which is precisely the order of $W_{n,d}$ (cf. [2],[4]). If $d = 1$ and $u_1 = 1$, then $\mathcal{H}_n(q,1)$ is isomorphic to the Iwahori-Hecke algebra of type $\mathcal{A}$ (over $\mathbb{Z}[q^{\pm 1}]$). If $d = 2$, $u_1 = -1$ and $u_2 = Q$, we recover the familiar relation of $\mathcal{H}_n(q,Q)$, the Iwahori-Hecke algebra of type $\mathcal{B}$ (over $\mathbb{Z}[q^{\pm 1}, Q^{\pm 1}]$). In $\mathcal{H}_n(q,d)$ we have

$$\spadesuit \qquad t^d = a_{d-1}t^{d-1} + \cdots + a_0, \quad \text{where}$$

$a_{d-1} = u_1 + \cdots + u_d$, $a_{d-2} = -(u_1u_2 + \cdots + u_{d-1}u_d)$, $\ldots$, $a_0 = (-1)^d(u_1 \ldots u_d)$; from this we can derive easily a relation for t^{-1}.

$W_{n,d}$ may also be seen as the quotient $W_{n,\infty} / < t^d = 1 >, \ d \in I\!N$ of $W_{n,\infty}$, and it is easily verified that

$$W_{n,d} = \mathbb{Z}_d{}^n \rtimes S_n$$

Its order is $d^n \cdot n!$, whilst $W_{n,2} = \mathbb{Z}_2{}^n \rtimes S_n$ (compare with the structure of $B_{1,n}$).

Note 1 W.l.o.g. we extend the ground ring of $\mathcal{H}_n(q,\infty)$ to $\mathcal{R}$. Then $\mathcal{H}_n(q,d)$ may also be obtained from $\mathcal{H}_n(q,\infty)$ by factoring out the cyclotomic relation. In this sense $\mathcal{H}_n(q,d)$ is a 'bridge' between $\mathcal{H}_n(q,\infty)$ and $\mathcal{H}_n(q,Q)$, the classical Hecke algebra.

We shall now find a basis for $\mathcal{H}_n(q,\infty)$ as follows: We find first a canonical form for the braid group $B_{1,n}$, which yields a basis for $\mathbb{Z}[q^{\pm 1}]B_{1,n}$. The images of these basic elements in $\mathcal{H}_n(q,\infty)$ through the canonical map span $\mathcal{H}_n(q,\infty)$. In [2, 4] bases for $\mathcal{H}_n(q,d)$ have been constructed. We then treat the spanning set and using these bases we obtain a basis for $\mathcal{H}_n(q,\infty)$. This approach shows clearly the relation among the structures of $B_{1,n}$, $\mathcal{H}_n(q,\infty)$, $\mathcal{H}_n(q,d)$ and $W_{n,\infty}$, $W_{n,d}$.

In order to proceed we need to recall the notion of the pure braid group and Artin's canonical form for pure braids: The classical pure braid group, P_n, consists of all elements in B_n that induce the identity permutation in S_n; $P_n \lhd B_n$ and P_n is generated by the elements

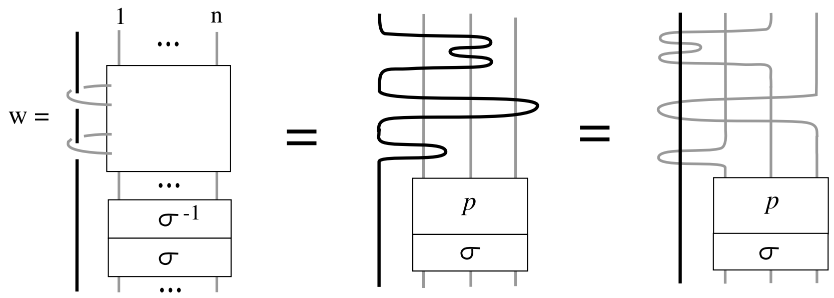
$$\begin{aligned} A_{rs} &= {\sigma_r}^{-1}{\sigma_{r+1}}^{-1}\ldots{\sigma_{s-2}}^{-1}{\sigma_{s-1}}^{2}\sigma_{s-2}\ldots\sigma_{r+1}\sigma_r \\ &= \sigma_{s-1}\sigma_{s-2}\ldots\sigma_{r+1}{\sigma_r}^{2}{\sigma_{r+1}}^{-1}\ldots{\sigma_{s-2}}^{-1}{\sigma_{s-1}}^{-1}, \quad 1 \leq r < s \leq n. \end{aligned}$$

Artin's canonical form says that every element, A, of P_n can be written uniquely in the form:

$$A = U_1 U_2 \cdots U_{n-1}$$

where each U_i is a uniquely determined product of powers of the A_{ij} using only those with $i < j$. Geometrically, this means that any pure braid can be 'combed' i.e. can be written canonically as: the pure braiding of the first string with the rest, then keep the first string fixed and uncrossed and have the pure braiding of the second string and so on (cf. [J.S. Birman, Braids, Links and Mapping Class Groups, *Ann. of Math. Stud.* 82, Princeton University Press, Princeton 1974] for a complete treatment).

We find now a canonical form for $B_{1,n}$. An element w of $B_{1,n}$ induces a permutation $\sigma \in S_n$ of the n numbered strands. We add at the bottom of the braid a standard braid in B_n corresponding to σ^{-1}, and then we add its inverse σ. Now, $w\sigma^{-1}$ is a pure braid on $n+1$ stands (including the first fixed one), and we apply to it Artin's canonical form. This separates the braiding of the fixed strand from the rest:



The above is in fact the proof of the decomposition of $B_{1,n}$ as a semidirect product:

Proposition 1 $B_{1,n} = P_{1,n} \rtimes B_n$.

From the uniqueness of Artin's canonical form, it follows that any $w \in B_{1,n}$ can be expressed uniquely as a product $v \cdot \sigma$ ('vector-permutation'), where v is an element of the free group $P_{1,n}$:

$$v = {t'_{i_1}}^{k_1} {t'_{i_2}}^{k_2} \ldots {t'_{i_r}}^{k_r}, \ k_1, \ldots, k_r \in \mathbb{Z}, \quad \text{where} \quad {t'_i}^k := \sigma_i \ldots \sigma_1 t^k \sigma_1^{-1} \ldots \sigma_i^{-1},$$

and $\sigma \in B_n$ is written in the induced by P_n canonical form. Thus the set $\{v \cdot \sigma\}$ forms a basis for the algebra $\mathbb{Z}[q^{\pm 1}]\, B_{1,n}$, and, therefore, it spans the quotient $\mathcal{H}_n(q,\infty)$. On the level of $\mathcal{H}_n(q,\infty)$ we can already improve this spanning set, since on this level σ is a word in $\mathcal{H}_n(q)$, the Iwahori-Hecke algebra of $\mathcal{A}_{n-1}$-type. So, σ can be written in terms of the standard basis of $\mathcal{H}_n(q)$ (cf. [11]):

$$\{(g_{i_1} g_{i_1 - 1} \ldots g_{i_1 - r_1})(g_{i_2} g_{i_2 - 1} \ldots g_{i_2 - r_2}) \ldots (g_{i_p} g_{i_p - 1} \ldots g_{i_p - r_p})\},$$
$$\text{for} \ \ 1 \le i_1 < \ldots < i_p \le n-1 \ \ \text{and} \ \ r_j \in \{0, 1, \ldots, i_j - 1\}.$$

Therefore we showed

Proposition 2 *The set*

$$\Sigma_1 = \{ {t'_{j_1}}^{k_1} {t'_{j_2}}^{k_2} \ldots {t'_{j_r}}^{k_r} \cdot \sigma \},$$

where $t'_0 := t$, ${t'_i}^k := g_i \ldots g_1 t^k g_1^{-1} \ldots g_i^{-1}$, $j_1, \ldots, j_r \in \{0, 1, \ldots, n-1\}$, $k_1, \ldots, k_r \in \mathbb{Z}$ *and* σ *a basic element of* $\mathcal{H}_n(q)$, *spans* $\mathcal{H}_n(q,\infty)$.

Notice that the indices of the 'vector' part are not ordered. Also, that the above canonical form for $B_{1,n}$ yields immediately the following canonical form $\{v \cdot \sigma\}$ for $W_{n,\infty}$:

$$\{v \cdot \sigma\} = \{{t_{j_1}}^{k_1} {t_{j_2}}^{k_2} \ldots {t_{j_r}}^{k_r} \cdot \sigma\},$$

where $t_0 := t$, ${t_i}^k := s_i \ldots s_1 t^k s_1 \ldots s_i$, for $0 \leq j_1 < \ldots < j_r \leq n-1$, $k_1, \ldots, k_r \in \mathbb{Z}$ and $\sigma \in S_n$ is an element of the canonical form of S_n (where s_i denotes the image of σ_i in $W_{n,\infty}$). Thus, this set also forms a basis for the group algebra $\mathbb{Z}[q^{\pm 1}]\, W_{n,\infty}$.

Notice here that the indices of the 'vector' part are ordered. This suggests that it may be possible to order the indices $j_1, \ldots, j_r$ of the words ${t'_{j_1}}^{k_1} {t'_{j_2}}^{k_2} \ldots {t'_{j_r}}^{k_r}$ in Σ_1, so as to be left with a canonical basis for $\mathcal{H}_n(q,\infty)$. To achieve this straight from Σ_1 is very difficult, because it is hard to get hold of an induction step, even though there are relations among the ${t'_i}^{k_i}$'s. Instead, we change the ${t'_i}^{k}$'s to the elements ${t_i}^k$, where $t_0 := t$, and $t_i := g_i \ldots g_1 t g_1 \ldots g_i$. These elements commute in $\mathcal{H}_n(q,\infty)$.

The following relations hold in $\mathcal{H}_n(q,\infty)$ and in $\mathcal{H}_n(q,d)$ and will be used repeatedly in the sequel.

Lemma 1 *For $\epsilon \in \{\pm 1\}$ the following hold:*

(i) ${g_i}^{\epsilon} = q^{\epsilon}\, {g_i}^{-\epsilon} + (q^{\epsilon} - 1)$,

${g_i}^{2\epsilon} = (q^{\epsilon} - 1)\, {g_i}^{\epsilon} + q^{\epsilon}$, *for* $q \neq 0$.

(ii) ${g_i}^{\epsilon}({g_k}^{\pm 1} g_{k-1}^{\pm 1} \ldots {g_j}^{\pm 1}) = ({g_k}^{\pm 1} g_{k-1}^{\pm 1} \ldots {g_j}^{\pm 1}) {g_{i+1}}^{\epsilon}$, *for* $k > i \geq j$,

${g_i}^{\epsilon}({g_j}^{\pm 1} g_{j+1}^{\pm 1} \ldots {g_k}^{\pm 1}) = ({g_j}^{\pm 1} g_{j+1}^{\pm 1} \ldots {g_k}^{\pm 1}) {g_{i-1}}^{\epsilon}$, *for* $k \geq i > j$,

where the sign of the ± 1 superscript is the same for all generators.

(iii) $g_i g_{i-1} \ldots g_{j+1} g_j g_{j+1} \ldots g_i = g_j g_{j+1} \ldots g_{i-1} g_i g_{i-1} \ldots g_{j+1} g_j$,

${g_i}^{-1} g_{i-1}^{-1} \ldots g_{j+1}^{-1} {g_j}^{\epsilon} g_{j+1} \ldots g_i = g_j g_{j+1} \ldots g_{i-1} {g_i}^{\epsilon} g_{i-1}^{-1} \ldots g_{j+1}^{-1} {g_j}^{-1}$.

(iv) ${g_i}^{\epsilon} \ldots {g_{n-1}}^{\epsilon} {g_n}^{\epsilon} {g_n}^{\epsilon} {g_{n-1}}^{\epsilon} \ldots {g_i}^{\epsilon} =$

$(q^{\epsilon} - 1) \sum_{r=0}^{n-i} q^{\epsilon r} \left({g_i}^{\epsilon} \ldots {g_{n-r-1}}^{\epsilon} {g_{n-r}}^{\epsilon} {g_{n-r-1}}^{\epsilon} \ldots {g_i}^{\epsilon}\right) + q^{\epsilon(n-i+1)} =$

$\sum_{r=0}^{n-i+1} (q^{\epsilon} - 1)^{\epsilon_r} q^{\epsilon r} \left({g_i}^{\epsilon} \ldots {g_{n-r-1}}^{\epsilon} {g_{n-r}}^{\epsilon} {g_{n-r-1}}^{\epsilon} \ldots {g_i}^{\epsilon}\right)$,

where $\epsilon_r = 1$ *if* $r \leq n-i$ *and* $\epsilon_{n-i+1} = 0$.

Similarly,

${g_i}^{\epsilon} \ldots {g_2}^{\epsilon} {g_1}^{\epsilon} {g_1}^{\epsilon} {g_2}^{\epsilon} \ldots {g_i}^{\epsilon} =$

$(q^\epsilon - 1)\sum_{r=0}^{i-1} q^{\epsilon r}\,({g_i}^\epsilon \ldots {g_{r+2}}^\epsilon {g_{r+1}}^\epsilon {g_{r+2}}^\epsilon \ldots {g_i}^\epsilon) + q^{\epsilon i} =$
$\sum_{r=0}^{i} (q^\epsilon - 1)^{\epsilon_r} q^{\epsilon r}\,({g_i}^\epsilon \ldots {g_{r+2}}^\epsilon {g_{r+1}}^\epsilon {g_{r+2}}^\epsilon \ldots {g_i}^\epsilon),$
where $\epsilon_r = 1$ *if* $r \leq i-1$ *and* $\epsilon_i = 0$.

(v) $t^\lambda g_1 t g_1 = g_1 t g_1 t^\lambda$ *for* $\lambda \in \mathbb{Z}$,
$g_i {t_k}^\epsilon = {t_k}^\epsilon g_i$ *for* $k > i,\ k < i-1$,
$g_i t_i = q\, t_{i-1} g_i + (q-1)\, t_i$,
$g_i t_{i-1} = q^{-1}\, t_i g_i + (q^{-1} - 1)\, t_i$,
$g_i {t_{i-1}}^{-1} = q\, {t_i}^{-1} g_i + (q-1)\, {t_{i-1}}^{-1}$,
$g_i {t_i}^{-1} = q^{-1}\, {t_{i-1}}^{-1} g_i + (q^{-1} - 1)\, {t_{i-1}}^{-1}$.

(vi) $g_i {t'_k}^\epsilon = {t'_k}^\epsilon g_i$ *for* $k > i,\ k < i-1$,
$g_i {t'_i}^\epsilon = {t'_{i-1}}^\epsilon g_i + (q-1)\, {t'_i}^\epsilon + (1-q)\, {t'_{i-1}}^\epsilon$,
$g_i {t'_{i-1}}^\epsilon = {t'_i}^\epsilon g_i$.

(vii) ${t_i}^k {t_j}^\lambda = {t_j}^\lambda {t_i}^k$ *for* $i \neq j$ *and* $k, \lambda \in \mathbb{Z}$.

(viii) ${t'_i}^k = g_i \ldots g_1 t^k {g_1}^{-1} \ldots {g_i}^{-1}$ *for* $k \in \mathbb{Z}$.
Therefore we have in $\mathcal{H}_n(q,d)$:
$(t'_i - u_1)(t'_i - u_2)\ldots(t'_i - u_d) = 0$, *which implies* ${t'_i}^d = a_{d-1}{t'_i}^{d-1} + \cdots + a_0$,
and where the a_i*'s are given in relation* (♠) *in Section 2.*

Proof. We point out first that in the rest of the paper and in order to facilitate the reader we underline in the proofs the expressions which are crucial for the next step. We also use the symbol '$\sum$' instead of the phrase 'linear combination of words of the type'.

Except for (iv), all relations are easy consequenses of the defining relations of $\mathcal{H}_n(q,\infty)$ respectively $\mathcal{H}_n(q,d)$. Relation (vii) can be also checked using braid diagrams. We prove (iv) by induction on the length $l = n-i+1$ of the word $g_n g_{n-1}\ldots g_i$. For $l=1$ we have ${g_n}^2 = (q-1)g_n + q^1$. Assume now (iv) holds up to $l = n-i$. Then for $l = n-i+1$ we have

$$g_i\underline{g_{i+1}\ldots g_n g_n \ldots g_{i+1}}g_i \overset{induction\ step}{=}$$

$$\underline{g_i}\,[(q-1)\sum_{r=0}^{n-(i+1)} q^r\,(g_{i+1}\ldots g_{n-r-1}g_{n-r}g_{n-r-1}\ldots g_{i+1}) + q^{n-i}]\,\underline{g_i} =$$

$$(q-1)\sum_{r=0}^{n-(i+1)} q^r\,(g_i\ldots g_{n-r-1}g_{n-r}g_{n-r-1}\ldots g_i) + q^{n-i}\underline{{g_i}^2} =$$

$$(q-1)\sum_{r=0}^{n-(i+1)} q^r\,(g_i\ldots g_{n-r-1}g_{n-r}g_{n-r-1}\ldots g_i) + (q-1)q^{n-i}g_i + q^{n-i+1} =$$

$(q-1)\sum_{r=0}^{n-i} q^r\,(g_i\ldots g_{n-r-1}g_{n-r}g_{n-r-1}\ldots g_i)+q^{n-i+1}$.
Furthermore note that in the Relations (v) and (vi) a t_i or a t'_i will *not* change to a ${t_i}^{-1}$ or a ${t'_i}^{-1}$ respectively and, therefore, these relations preserve the total sum of the exponents of the t_i's and the t'_i's in a word. Note also that for $j=i-1$ the relations (iii) boil down to the usual braid relation and its variations with inverses. $\square$

Theorem 1 *In $\mathcal{H}_n(q,\infty)$ the set*

$$\Sigma_2=\{{t_{i_1}}^{k_1}{t_{i_2}}^{k_2}\ldots {t_{i_r}}^{k_r}\cdot\sigma\}$$

for $0\le i_1<\ldots<i_r\le n-1$, $k_1,\ldots,k_r\in\mathbb{Z}$ and σ a basic element in $\mathcal{H}_n(q)$, forms a basis for $\mathcal{H}_n(q,\infty)$.

Notice that in Σ_2 the indices of the 'vector' part are ordered.

Proof. To show that Σ_2 spans $\mathcal{H}_n(q,\infty)$ it suffices, by Proposition 2, to show that an element of Σ_1 can be written as a linear combination of elements in Σ_2. Indeed, let

$$w={t'_{j_1}}^{k_1}{t'_{j_2}}^{k_2}\ldots{t'_{j_m}}^{k_m}\cdot\sigma\in\Sigma_1.$$

We do the proof by induction on

$$\rho=|k_1|+|k_2|+\cdots+|k_m|,$$

the absolute number of t's in w. For $\rho=1$ either $w=t'_i\cdot\sigma$ or $w={t'_i}^{-1}\cdot\sigma$:

$t'_i\cdot\sigma=g_i\ldots g_1tg_1{}^{-1}\ldots g_i{}^{-1}\cdot\sigma=$

$\underline{g_i\ldots g_1t(g_1\ldots g_i}g_i{}^{-1}\ldots g_1{}^{-1})g_1{}^{-1}\ldots g_i{}^{-1}\cdot\sigma=t_i\cdot\sigma_1$,

where $\sigma_1=g_i{}^{-1}\ldots g_1{}^{-1}g_1{}^{-1}\ldots g_i{}^{-1}\cdot\sigma\in\mathcal{H}_n(q)$, a linear combination of basic elements of $\mathcal{H}_n(q)$.

${t'_i}^{-1}\cdot\sigma=g_i\ldots g_1t^{-1}g_1{}^{-1}\ldots g_i{}^{-1}\cdot\sigma=$

$\underline{g_i\ldots g_1(g_1\ldots g_i}g_i{}^{-1}\ldots g_1{}^{-1})t^{-1}g_1{}^{-1}\ldots g_i{}^{-1}\cdot\sigma \stackrel{Lemma 1,(iv)}{=}$

$(q-1)\sum_{r=0}^{i-1} q^r\,(g_i\ldots g_{r+2}\underline{g_{r+1}g_{r+2}\ldots g_i)g_i{}^{-1}\ldots g_{r+1}{}^{-1}}g_r{}^{-1}\ldots g_1{}^{-1}t^{-1}g_1{}^{-1}\ldots$

$g_i{}^{-1}\cdot\sigma+q^i\,t_i^{-1}\cdot\sigma=$

$(q-1)\sum_{r=0}^{i-1} q^r\,(g_i\ldots g_{r+2})\underline{g_r{}^{-1}\ldots g_1{}^{-1}t^{-1}g_1{}^{-1}\ldots g_r{}^{-1}}\ldots g_i{}^{-1}\cdot\sigma+q^i\,t_i^{-1}\cdot\sigma=$

$(q-1)\sum_{r=0}^{i-1} q^r\,t_r{}^{-1}(g_i\ldots g_{r+2}g_{r+1}{}^{-1}\ldots g_i{}^{-1}\cdot\sigma)+q^i\,t_i^{-1}\cdot\sigma=$

$(q-1)\sum_{r=0}^{i-1} q^r\, {t_r}^{-1}\cdot\sigma_r + q^i\, t_i^{-1}\cdot\sigma$, where $\sigma_r = g_i \ldots g_{r+2}{g_{r+1}}^{-1}\ldots {g_i}^{-1}\cdot\sigma \in \mathcal{H}_n(q)$.

Suppose now the assumption holds for up to $\rho - 1$ t's in w. Then, the induction step holds in particular for all such words with $\sigma = 1$. So, for $|k_1| + |k_2| + \cdots + |k_m| = \rho$ we have:

$$
{t'_{j_1}}^{k_1}\ldots {t'_{j_m}}^{k_m}\cdot\sigma = \begin{cases} \underline{{t'_{j_1}}^{k_1}\ldots {t'_{j_m}}^{k_m-1}} t'_{j_m}\cdot\sigma, & \text{if } k_m > 0 \\ \underline{{t'_{j_1}}^{k_1}\ldots {t'_{j_m}}^{k_m+1}} {t'_{j_m}}^{-1}\cdot\sigma, & \text{if } k_m < 0 \end{cases}
$$

$$
\overset{by\, induction}{=} \begin{cases} \Sigma\, {t_{i_1}}^{\lambda_1}\ldots {t_{i_n}}^{\lambda_n}\cdot\sigma_1\cdot \underline{t'_{j_m}}\cdot\sigma, & \text{for some } \sigma_1\in\mathcal{H}_n(q), \\ \Sigma\, {t_{\mu_1}}^{\nu_1}\ldots {t_{\mu_n}}^{\nu_n}\cdot\sigma_2\cdot \underline{{t'_{j_m}}^{-1}}\cdot\sigma, & \text{for some } \sigma_2\in\mathcal{H}_n(q), \end{cases}
$$

$$
\begin{cases} 1\le i_1 < \ldots < i_n \le n-1, & |\lambda_1| + \cdots + |\lambda_n| = \rho - 1 \\ 1\le \mu_1 < \ldots < \mu_n \le n-1, & |\nu_1| + \cdots + |\nu_n| = \rho - 1 \end{cases}
$$

$$
= \begin{cases} \Sigma\, {t_{i_1}}^{\lambda_1}\ldots {t_{i_n}}^{\lambda_n}\cdot \underline{\sigma_1\cdot t_{j_m}}({g_{j_m}}^{-1}\ldots {g_1}^{-1}{g_1}^{-1}\ldots {g_{j_m}}^{-1})\cdot\sigma \\ \Sigma\, {t_{\mu_1}}^{\nu_1}\ldots {t_{\mu_n}}^{\nu_n}\cdot \underline{\sigma_2\cdot (g_{j_m}\ldots g_1 g_1\ldots g_{j_m}){t_{j_m}}^{-1}}\cdot\sigma \end{cases} .
$$

We apply Lemma 1,(v) on the underlying expressions in order to shift t_{j_m} and ${t_{j_m}}^{-1}$ to the left and we obtain sums of the words:

$$
\begin{cases} \Sigma\, {t_{i_1}}^{\lambda_1}\ldots {t_{i_n}}^{\lambda_n}\cdot t_{e_1}\cdot\sigma'_1, & \sigma'_1\in\mathcal{H}_n(q),\ e_1\in\{0,1,\ldots,n-1\} \\ \Sigma\, {t_{\mu_1}}^{\nu_1}\ldots {t_{\mu_n}}^{\nu_n}\cdot {t_{e_2}}^{-1}\cdot\sigma'_2, & \sigma'_2\in\mathcal{H}_n(q),\ e_2\in\{0,1,\ldots,n-1\} \end{cases}
$$

$$
\overset{Lemma 1,(vii)}{=} \begin{cases} \Sigma\, {t_{i_1}}^{\lambda_1}\ldots {t_{i_r}}^{\lambda_r}\cdot t_{e_1}\cdot {t_{i_{r+1}}}^{\lambda_{r+1}}\ldots {t_{i_n}}^{\lambda_n}\cdot\sigma_i\sigma', i_r < e_1 < i_{r+1}. \\ \Sigma\, {t_{\mu_1}}^{\nu_1}\ldots {t_{\mu_k}}^{\nu_k}\cdot {t_{e_2}}^{-1}\cdot {t_{\mu_{k+1}}}^{\nu_1}\ldots {t_{\mu_n}}^{\nu_n}\cdot\sigma'_i\sigma, \mu_k < e_2 < \mu_{k+1}. \end{cases}
$$

I.e. in either case we obtained a linear combination of elements of Σ_2.

We next show linear independency of the elements of Σ_2:
Let $\sum_{i=1}^m \lambda_i w_i = 0$ for $w_1, w_2, \ldots, w_m \in \Sigma_2$. We assume first that the exponents of the t_j's in the words w_i are all positive for all i, and we choose $d > k \in I\!N$, where k is the maximum of the exponents of the t_j's in $\sum_{i=1}^m \lambda_i w_i$. Then, the canonical epimorphism of $\mathcal{H}_n(q,\infty)$ onto $\mathcal{H}_n(q,d)$ applied on the equation $\sum_{i=1}^m \lambda_i w_i = 0$ in $\mathcal{H}_n(q,\infty)$ yields the equation $\sum_{i=1}^m \lambda_i w_i = 0$ in $\mathcal{H}_n(q,d)$. As shown in [2], Proposition 3.4 and Theorem 3.10, the elements of Σ_2 with $0 < k_1, \ldots, k_r \le d-1$ form a basis for $\mathcal{H}_n(q,d)$, $d \in I\!N$. (In [2] d is denoted by r, $\mathcal{H}_n(q,d)$ is denoted by $\mathcal{H}_{n,r}$ and σ is denoted by a_w.) This implies $\lambda_i = 0,\ i = 1, \ldots, m$.

Assume finally that some w_i's contain t_j's with negative exponents. The idea is to resolve the negative exponents and then refer to the previous case. One way is to proceed as above, and after we have projected $\sum_{i=1}^m \lambda_i w_i = 0$ on $\mathcal{H}_n(q,d)$, to resolve the t_j's with negative exponents using the algebra relations; finally, to conclude $\lambda_i = 0,\ i = 1,\ldots,m$, using induction and arguments from linear algebra. But we would rather give a more elegant argument, that was suggested by T. tom Dieck.

Namely, let P be the product of all t_j^k, $k \in \mathbb{N}$ for all j,k such that t_j^{-k} is in some w_i. Since P is an invertible element of $\mathcal{H}_n(q,\infty)$, we have $\sum_{i=1}^m \lambda_i w_i = 0 \Leftrightarrow P\cdot\sum_{i=1}^m \lambda_i w_i = 0$. The last equation is eqivalent to $\sum_{i=1}^m \lambda_i P w_i = 0$, where the elements Pw_i are pairwise different and the exponents of the t_j's contained in each Pw_i are positive for all i. We then refer to the previous case, and the proof of Theorem 1 is now concluded.

Thus Σ_2 is a basis of $\mathcal{H}_n(q,\infty)$, and therefore $\mathcal{H}_n(q,\infty)$ is a free module. $\square$

Remark 1 In [5], (8.23) tom Dieck establishes an isomorphism between $\mathcal{H}_n(q,\infty)$ and the twisted tensor product of the Hecke algebra of the Coxeter group of the affine type $\tilde{\mathcal{A}}_{n-1}$. One can also use the extended affine Hecke algebra of type $\tilde{\mathcal{A}}_{n-1}$ and study quotient maps onto $\mathcal{H}_n(q,d)$ as defined in [1], Section 2.1. The same map also works for $\mathcal{H}_n(q,\infty)$ and it is in fact an isomorphism.

3 Inductive bases for $\mathcal{H}_n(q,\infty)$ and $\mathcal{H}_n(q,d)$

The basis of $\mathcal{H}_n(q,\infty)$ constructed in the previous section as well as the corresponding one for $\mathcal{H}_n(q,d)$ yields an inductive basis for $\mathcal{H}_n(q,\infty)$ respectively $\mathcal{H}_n(q,d)$, which gives rise to another two inductive bases, the last one being the appropriate for constructing Markov traces on these algebras. Here we give these three inductive bases and we conclude this section by giving another basic set for $\mathcal{H}_n(q,\infty)$ respectively $\mathcal{H}_n(q,d)$, which is analogous to the set Σ_2, but using t_i''s instead of t_i's.

From now on we shall denote by $\mathcal{H}_n$ both $\mathcal{H}_n(q,\infty)$ and $\mathcal{H}_n(q,d)$ and by W_n both $W_{n,\infty}$ and $W_{n,d}$. Also, whenever we refer to $k \in \mathbb{Z}$ respectively $k \in \mathbb{Z}_d$ we shall assume $k \neq 0$. We now find the first inductive basis for $\mathcal{H}_{n+1}$. This on the group level is an inductive canonical form, and it provides a set of right coset representatives of W_n into W_{n+1}, which is completely analogous to [6], p. 456 for $\mathcal{B}$-type Coxeter groups.

Lemma 2 *For $k \in \mathbb{Z}$ the following hold in $\mathcal{H}_{n+1}(q,\infty)$ respectively $\mathcal{H}_{n+1}(q,d)$:*

(i) $t_n{}^k g_n \;=\; (q-1)\,\sum_{j=0}^{k-1}\, q^j\, t_{n-1}{}^j t_n{}^{k-j} + q^k\, g_n t_{n-1}{}^k, \quad$ *if* $k \in \mathbb{N}$ *and*

$t_n{}^k g_n \;=\; (1-q)\,\sum_{j=0}^{k-1}\, q^j\, t_{n-1}{}^j t_n{}^{k-j} + q^k\, g_n t_{n-1}{}^k, \quad$ *if* $k \in \mathbb{Z} - \mathbb{N}$.

(ii) $t_n{}^k g_n g_{n-1} \ldots g_i \;=$

$$\begin{aligned}
&(q-1)\,\sum_{j=0}^{k-1}\, q^j\,(t_{n-1}{}^j g_{n-1} g_{n-2} \ldots g_i) t_n{}^{k-j} +\\
&(q-1) q^k\,\sum_{j=0}^{k-1}\, q^j\,(t_{n-2}{}^j g_{n-2} g_{n-3} \ldots g_i) g_n t_{n-1}{}^{k-j} +\\
&(q-1) q^{2k}\,\sum_{j=0}^{k-1}\, q^j\,(t_{n-3}{}^j g_{n-3} \ldots g_i) g_n g_{n-1} t_{n-2}{}^{k-j}\\
&+ \cdots +\\
&(q-1) q^{(n-i)k}\,\sum_{j=0}^{k-1}\, q^j\,(t_{i-1}{}^j) g_n g_{n-1} \ldots g_{i+1} t_i{}^{k-j}\\
&+ q^{(n-i+1)k}\, g_n g_{n-1} \ldots g_i t_{i-1}{}^k, \quad \text{if } k \in \mathbb{N},
\end{aligned}$$

whilst for $k \in \mathbb{Z} - \mathbb{N}$ *we have an analogous formula, only* $(q-1)$ *is replaced by* $(1-q)$, $q^k = q^{-|k|}$ *and* $|k-j| + |j| = |k|$.

Proof. We prove (i) for the case $k > 0$ by induction on k. (For $k < 0$ completely analogous.) For $k = 1$ we have $t_n g_n = (q-1) t_n + q^1 g_n t_{n-1}$. Suppose the assumption holds for $k-1$. Then for k we have:

$$\begin{aligned}
&t_n{}^k g_n = t_n \underline{t_n{}^{k-1} g_n} \overset{by\, induction}{=}\\
&\underline{t_n}\,[(q-1)\,\sum_{j=0}^{k-2}\, q^j\, t_{n-1}{}^j t_n{}^{k-1-j} + q^{k-1}\, g_n t_{n-1}{}^{k-1}] \overset{Lemma\, 1,(vii)}{=}\\
&(q-1)\,\sum_{j=0}^{k-2}\, q^j\, t_{n-1}{}^j t_n{}^{k-j} + q^{k-1}\, \underline{t_n g_n} t_{n-1}{}^{k-1} \overset{Lemma\, 1,(v)}{=}\\
&(q-1)\,\sum_{j=0}^{k-2}\, q^j\, t_{n-1}{}^j t_n{}^{k-j} + q^{k-1}(q-1)\, \underline{t_n t_{n-1}{}^{k-1}} + q^k\, g_n t_{n-1}{}^k \overset{Lemma\, 1,(vii)}{=}\\
&(q-1)\,\sum_{j=0}^{k-1}\, q^j\, t_{n-1}{}^j t_n{}^{k-j} + q^k\, g_n t_{n-1}{}^k.
\end{aligned}$$

We prove (ii) for the case $k > 0$ by decreasing induction on i. (For $k < 0$ completely analogous.) For $i = n$ we have (i). Assume it holds for $i+1 < n$ ($\Leftrightarrow i \leq n-2 \Leftrightarrow n-i \geq 2$). Then for i we have:

$$\begin{aligned}
&\underline{t_n{}^k g_n \ldots g_{i+1}} g_i \overset{by\, induction}{=}\\
&[(q-1)\,\sum_{j=0}^{k-1}\, q^j\,(t_{n-1}{}^j g_{n-1} g_{n-2} \ldots g_{i+1}) \underline{t_n{}^{k-j}] g_i} + \cdots +\\
&[(q-1) q^{(n-(i+1))k}\,\sum_{j=0}^{k-1}\, q^j\,(t_i{}^j) g_n g_{n-1} \ldots g_{i+2} \underline{t_{i+1}{}^{k-j}] g_i} +\\
&[q^{(n-i)k}\, g_n g_{n-1} \ldots g_{i+1} \underline{t_i{}^k] g_i} \overset{\text{Lemma1,(v)\&Lemma2,(i)}}{=}\\
&(q-1)\,\sum_{j=0}^{k-1}\, q^j\,(t_{n-1}{}^j g_{n-1} \ldots g_{i+1} g_i) t_n{}^{k-j} + \cdots +
\end{aligned}$$

$$(q-1)q^{(n-(i+1))k}\sum_{j=0}^{k-1} q^j\,(t_i{}^j g_i)g_n g_{n-1}\dots g_{i+2}t_{i+1}{}^{k-j}+$$
$$q^{(n-i)k}(q-1)\sum_{j=0}^{k-1} q^j\, g_n g_{n-1}\dots g_{i+1}\underline{t_{i-1}{}^j}t_i{}^{k-j}+$$
$$q^{(n-i)k}q^k\, g_n g_{n-1}\dots g_i t_{i-1}{}^k =$$
$$(q-1)\sum_{j=0}^{k-1} q^j\,(t_{n-1}{}^j g_{n-1}\dots g_{i+1}g_i)t_n{}^{k-j}+\dots+$$
$$(q-1)q^{(n-(i+1))k}\sum_{j=0}^{k-1} q^j\,(t_i{}^j g_i)g_n g_{n-1}\dots g_{i+2}t_{i+1}{}^{k-j}+$$
$$q^{(n-i)k}(q-1)\sum_{j=0}^{k-1} q^j\,(t_{i-1}{}^j)g_n g_{n-1}\dots g_{i+1}t_i{}^{k-j}+$$
$$q^{(n-i)k}q^k\, g_n g_{n-1}\dots g_i t_{i-1}{}^k.$$

□

Theorem 2 *Every element of $\mathcal{H}_{n+1}(q,\infty)$ respectively $\mathcal{H}_n(q,d)$ is a unique linear combination of words, each of one of the following types:*

1) w_{n-1}

2) $w_{n-1}g_n g_{n-1}\dots g_i$

3) $w_{n-1}g_n g_{n-1}\dots g_i t_{i-1}{}^k,\ k\in\mathbb{Z}$ *respectively* $k\in\mathbb{Z}_d$

4) $w_{n-1}t_n{}^k,\ k\in\mathbb{Z}$ *respectively* $k\in\mathbb{Z}_d$

where w_{n-1} is some word in $\mathcal{H}_n(q,\infty)$ respectively $\mathcal{H}_n(q,d)$. Thus, the above words furnish an inductive basis for $\mathcal{H}_{n+1}(q,\infty)$ respectively $\mathcal{H}_n(q,d)$.

Proof. By Theorem 1 it suffices to show that every element $v\cdot\sigma_n\in\Sigma_2$, where v is a product of t_i's and $\sigma_n\in\mathcal{H}_{n+1}(q)$, can be expressed uniquely in terms of 1), 2), 3) and 4). We prove this by induction on n: For $n=0$ there are no g_i's in the word, so $v\cdot\sigma_0=t^k\cdot 1$, a word of type 1). Suppose the assertion holds for all basic words in Σ_2 with indices up to $n-1$, and let $w\in\Sigma_2$ such that w contains elements of index n. We examine the different cases:

- $w=t_{i_1}{}^{k_1}t_{i_2}{}^{k_2}\dots t_{i_r}{}^{k_r}\underline{t_n{}^k\cdot\sigma_{n-1}}$, $1\le i_1<\dots<i_r<n$ and $\sigma_{n-1}\in\mathcal{H}_n(q)$.

Then, by Lemma 1,(v), $w=\underline{t_{i_1}{}^{k_1}\dots t_{i_r}{}^{k_r}\cdot\sigma_{n-1}}\cdot t_n{}^k=w_{n-1}t_n{}^k$, a word of type 4).

- $w=t_{i_1}{}^{k_1}t_{i_2}{}^{k_2}\dots t_{i_r}{}^{k_r}\cdot\sigma_n$, where $i_r<n$ and $\sigma_n=\sigma_{n-1}\cdot(g_n g_{n-1}\dots g_i)\in\mathcal{H}_{n+1}(q)$. Then $w=\underline{t_{i_1}{}^{k_1}t_{i_2}{}^{k_2}\dots t_{i_r}{}^{k_r}\cdot\sigma_{n-1}}\cdot(g_n g_{n-1}\dots g_i)=w_{n-1}g_n g_{n-1}\dots g_i$, a word of type 2).

- Finally, let $w=t_{i_1}{}^{k_1}t_{i_2}{}^{k_2}\dots t_{i_r}{}^{k_r}t_n{}^k\cdot\sigma_n$, where $\sigma_n=\sigma_{n-1}\cdot(g_n g_{n-1}\dots g_i)\in$

$\mathcal{H}_{n+1}(q)$. Then $w = t_{i_1}{}^{k_1} \ldots t_{i_r}{}^{k_r} \underline{t_n{}^k \cdot \sigma_{n-1}} \cdot g_n g_{n-1} \ldots g_i \overset{Lemma\,1,(v)}{=}$

$t_{i_1}{}^{k_1} \ldots t_{i_r}{}^{k_r} \cdot \sigma_{n-1} \cdot \underline{t_n{}^k g_n g_{n-1} \ldots g_i} \overset{Lemma\,2,(ii)}{=}$

$w_{n-1} t_n{}^{k-j} + \Sigma w_{n-1} g_n g_{n-1} \ldots g_s t_{s-1}{}^{k-j}$, for $j = 0, \ldots, k-1$.

I.e. w is a sum of words of type 4) and type 3). The uniqueness of these expressions follows from Lemma 1 and Lemma 2. □

Theorem 2 rephrased weaker says that the elements of the inductive basis contain *either* g_n *or* $t_n{}^k$ *at most once.* But, as explained in the beginning, our aim is to find an inductive basis for $\mathcal{H}_{n+1}$ using the elements $t'_i = g_i g_{i-1} \ldots g_1 t g_1{}^{-1} \ldots g_{i-1}{}^{-1} g_i{}^{-1}$, as these are the right ones for constructing Markov traces on $\bigcup_{n=1}^{\infty} \mathcal{H}_n$. We go from the t_i's to the t'_i's via the 'intermediate' elements

$$T_i^k := g_i g_{i-1} \ldots g_1 t^k g_1 \ldots g_{i-1} g_i, \;\; k \in \mathbb{Z}.$$

Theorem 3 *Every element of $\mathcal{H}_{n+1}(q, \infty)$ respectively $\mathcal{H}_n(q, d)$ is a unique linear combination of words, each of one of the following types:*

1′) w_{n-1}

2′) $w_{n-1} g_n g_{n-1} \ldots g_i$

3′) $w_{n-1} g_n g_{n-1} \ldots g_i T_{i-1}^k, \;\; k \in \mathbb{Z}$ *respectively* $k \in \mathbb{Z}_d$

4′) $w_{n-1} T_n^k, \;\; k \in \mathbb{Z}$ *respectively* $k \in \mathbb{Z}_d$

where w_{n-1} *is some word in* $\mathcal{H}_n(q, \infty)$ *respectively* $\mathcal{H}_n(q, d)$.

Proof. It suffices to show that elements of the inductive basis given in Theorem 2 can be expressed uniquely as sums of the above words. For this we need the following three lemmas.

Lemma 3 *For $k \in \mathbb{N}$ respectively $k \in \mathbb{Z}_{d-1}$ and $\epsilon \in \{\pm 1\}$ the following hold in $\mathcal{H}_{n+1}(q, \infty)$ respectively $\mathcal{H}_{n+1}(q, d)$:*

$t_n{}^{\epsilon(k+1)} \;=\; \sum_{r_1, \ldots, r_k = 0}^{n} (q^\epsilon - 1)^{\epsilon_{r_1} + \cdots + \epsilon_{r_k}} q^{\epsilon(r_1 + \cdots + r_k)}.$

$g_n{}^\epsilon g_{n-1}{}^\epsilon \ldots g_1{}^\epsilon t^\epsilon (g_1{}^\epsilon \ldots g_{n-r_1}{}^\epsilon \ldots g_1{}^\epsilon) t^\epsilon \ldots t^\epsilon (g_1{}^\epsilon \ldots g_{n-r_k}{}^\epsilon \ldots g_1{}^\epsilon) t^\epsilon g_1{}^\epsilon \ldots g_{n-1}{}^\epsilon g_n{}^\epsilon,$

where $\epsilon_{r_i} = 1$ *if* $r_i = 0, \ldots, n-1, \;\; \epsilon_n = 0$ *and* $g_0{}^\epsilon := 1$.

Proof. We show the case $\epsilon = +1$ by induction on k. The proof for $\epsilon = -1$ is completely analogous. For $k = 1$ we have:

$$t_n{}^2 = g_n g_{n-1} \dots g_1 t \underline{g_1 \dots g_{n-1} g_n g_n g_{n-1} \dots g_1} t g_1 \dots g_{n-1} g_n \overset{Lemma\, 1,(iv)}{=}$$
$$\sum_{r=0}^{n} (q-1)^{\epsilon_r} q^r \, g_n g_{n-1} \dots g_1 t (g_1 \dots g_{n-r} \dots g_1) t g_1 \dots g_{n-1} g_n.$$

Assume that the statement holds for any $k \in I\!N$. Then for $k+1$ we have:

$$t_n{}^{k+1} = t_n{}^k t_n \overset{by\, induction}{=} \sum_{r_1,\dots,r_{k-1}=0}^{n} (q-1)^{\epsilon_{r_1}+\dots+\epsilon_{r_{k-1}}} q^{r_1+\dots+r_{k-1}} \, g_n \dots g_1 \cdot$$
$$t(g_1 \dots g_{n-r_1} \dots g_1) t \dots t (g_1 \dots g_{n-r_{k-1}} \dots g_1) t \underline{g_1 \dots g_n (g_n \dots g_1} t g_1 \dots g_n)$$
$$\overset{Lemma\, 1,(iv)}{=} \sum_{r_1,\dots,r_k=0}^{n} (q-1)^{\epsilon_{r_1}+\dots+\epsilon_{r_k}} q^{r_1+\dots+r_k} \, g_n \dots g_1 \cdot$$
$$t(g_1 \dots g_{n-r_1} \dots g_1) t \dots t (g_1 \dots g_{n-r_k} \dots g_1) t g_1 \dots g_n. \qquad \square$$

Lemma 4 *For $k \in I\!N$ and $\epsilon \in \{\pm 1\}$ the following hold in $\mathcal{H}_n(q,\infty)$ respectively $\mathcal{H}_n(q,d)$:*

$$\begin{array}{llll} (i) & t^{\epsilon} g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} & = & g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} t^{\epsilon} + (q^{\epsilon}-1) t^{\epsilon} g_1{}^{\epsilon} t^{\epsilon k} + (1-q^{\epsilon}) t^{\epsilon k} g_1{}^{\epsilon} t^{\epsilon} \quad \textit{and} \\ (ii) & t^{-\epsilon} g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} & = & g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} t^{-\epsilon} + (q^{\epsilon}-1) t^{\epsilon(k-1)} g_1{}^{\epsilon} + (1-q^{\epsilon}) g_1{}^{\epsilon} t^{\epsilon(k-1)}. \end{array}$$

Proof. We only prove (i) for the case $\epsilon = +1$, by induction on k. All other statements are proved similarly. For $k=1$ we have $tg_1tg_1 = g_1tg_1t$. Assume the assertion is correct for k. Then for $k+1$ we have:

$$t g_1 t^{k+1} g_1 = t g_1 t^k g_1 \underline{g_1^{-1}} t g_1 \overset{Lemma\, 1,(i)}{=}$$
$$q^{-1} \, \underline{t g_1 t^k g_1} g_1 t g_1 + (q^{-1}-1) \, \underline{t g_1 t^k g_1} t g_1 \overset{induction\, step}{=}$$
$$q^{-1} \, g_1 t^k \underline{g_1 t g_1} t g_1 + q^{-1}(q-1) \, \underline{t g_1 t^k g_1} t g_1 + q^{-1}(1-q) \, t^k \underline{g_1 t g_1} t g_1 +$$
$$(q^{-1}-1) \, g_1 t^k g_1 t^2 g_1 + (q^{-1}-1)(q-1) \, t g_1 t^{k+1} g_1 +$$
$$(q^{-1}-1)(1-q) \, t^k g_1 t^2 g_1 \overset{rels.,\, induction\, step}{=} q^{-1} \, g_1 t^{k+1} g_1 t \underline{g_1^2} +$$
$$(1-q^{-1}) \, g_1 t^k g_1 t^2 g_1 + (1-q^{-1})(q-1) \, t g_1 t^{k+1} g_1 + (1-q^{-1})(1-q) \, t^k g_1 t^2 g_1 +$$
$$(q^{-1}-1) \, t^{k+1} g_1 t \underline{g_1^2} + (q^{-1}-1) \, g_1 t^k g_1 t^2 g_1 + (q^{-1}-1)(q-1) \, t g_1 t^{k+1} g_1 +$$
$$(q^{-1}-1)(1-q) \, t^k g_1 t^2 g_1 \overset{Lemma\, 1,(i)}{=}$$
$$q^{-1}(q-1) \, g_1 \underline{t^{k+1} g_1 t g_1} + g_1 t^{k+1} g_1 t + (q^{-1}-1)(q-1) \, t^{k+1} g_1 t g_1 +$$
$$(q^{-1}-1) q \, t^{k+1} g_1 t \overset{Lemma\, 1,(v)}{=}$$
$$(1-q^{-1}) \, \underline{g_1^2} t g_1 t^{k+1} + g_1 t^{k+1} g_1 t + (q^{-1}-1)(q-1) \, t^{k+1} g_1 t g_1 + (1-q) \, t^{k+1} g_1 t =$$
$$(1-q^{-1})(q-1) \, g_1 t g_1 t^{k+1} + (1-q^{-1}) q \, t g_1 t^{k+1} + g_1 t^{k+1} g_1 t +$$

$$(q^{-1}-1)(q-1)\,t^{k+1}g_1tg_1+(1-q)\,t^{k+1}g_1t \overset{Lemma\,1,(v)}{=}$$
$$g_1t^{k+1}g_1t+(q-1)\,tg_1t^{k+1}+(1-q)\,t^{k+1}g_1t. \qquad \square$$

Lemma 5 (Fundamental Lemma (F.L.)) *For $i,k\in I\!N$ and for $\epsilon\in\{\pm 1\}$ the following hold in $\mathcal{H}_n(q,\infty)$ respectively $\mathcal{H}_n(q,d)$:*

(i) $t^{\epsilon i}{g_1}^{\epsilon}t^{\epsilon k}{g_1}^{\epsilon}={g_1}^{\epsilon}t^{\epsilon k}{g_1}^{\epsilon}t^{\epsilon i}+$
$$(q^{\epsilon}-1)\,[t^{\epsilon}{g_1}^{\epsilon}t^{\epsilon(k+i-1)}+t^{2\epsilon}{g_1}^{\epsilon}t^{\epsilon(k+i-2)}+\cdots+t^{\epsilon i}{g_1}^{\epsilon}t^{\epsilon k}]+$$
$$(1-q^{\epsilon})\,[t^{\epsilon k}{g_1}^{\epsilon}t^{\epsilon i}+t^{\epsilon(k+1)}{g_1}^{\epsilon}t^{\epsilon(i-1)}+\cdots+t^{\epsilon(k+i-1)}{g_1}^{\epsilon}t^{\epsilon}]$$ *and*

(ii) $t^{-\epsilon i}{g_1}^{\epsilon}t^{\epsilon k}{g_1}^{\epsilon}={g_1}^{\epsilon}t^{\epsilon k}{g_1}^{\epsilon}t^{-\epsilon i}+$
$$(q^{\epsilon}-1)\,[t^{\epsilon(k-1)}{g_1}^{\epsilon}t^{-\epsilon(i-1)}+t^{\epsilon(k-2)}{g_1}^{\epsilon}t^{-\epsilon(i-2)}+\cdots+t^{\epsilon(k-i)}{g_1}^{\epsilon}]+$$
$$(1-q^{\epsilon})\,[t^{-\epsilon(i-1)}{g_1}^{\epsilon}t^{\epsilon(k-1)}+t^{-\epsilon(i-2)}{g_1}^{\epsilon}t^{\epsilon(k-2)}+\cdots+{g_1}^{\epsilon}t^{\epsilon(k-i)}].$$

Proof. We prove (i) for the case $\epsilon=+1$, by induction on i. The proof for $\epsilon=-1$ is completely analogous. For $i=1$ the assertion is true by Lemma 4,(i). Assume it holds for i. Then for $i+1$ we have:

$$t^{i+1}g_1t^kg_1=t\underline{t^ig_1t^kg_1}\overset{induction\ step}{=}\underline{tg_1t^kg_1}t^i+$$
$$(q-1)\,[t^2g_1t^{k+i-1}+t^3g_1t^{k+i-2}+\cdots+t^{i+1}g_1t^k]+$$
$$(1-q)\,[t^{k+1}g_1t^i+t^{k+2}g_1t^{i-1}+\cdots+t^{k+i}g_1t]\overset{Lemma\,4,(i)}{=}$$
$$g_1t^kg_1t^{i+1}+(q-1)\,tg_1t^{k+i}+(1-q)\,t^kg_1t^{i+1}+$$
$$(q-1)\,[t^2g_1t^{k+i-1}+t^3g_1t^{k+i-2}+\cdots+t^{i+1}g_1t^k]+$$
$$(1-q)\,[t^{k+1}g_1t^i+t^{k+2}g_1t^{i-1}+\cdots+t^{k+i}g_1t].$$
$\square$

We go back now to the proof of Theorem 3. By Lemma 3, a typical summand of ${t_n}^{\epsilon(k+1)}\in\mathcal{H}_{n+1}$ is:

$${g_n}^{\epsilon}\ldots{g_1}^{\epsilon}t^{\epsilon\lambda_1}({g_1}^{\epsilon}\ldots{g_{n-l_1}}^{\epsilon}\ldots{g_1}^{\epsilon})t^{\epsilon\lambda_2}\ldots t^{\epsilon\lambda_N}({g_1}^{\epsilon}\ldots{g_{n-l_N}}^{\epsilon}\ldots{g_1}^{\epsilon})t^{\epsilon\lambda_{N+1}}{g_1}^{\epsilon}\ldots{g_n}^{\epsilon},$$

where $\lambda_1,\lambda_2,\ldots,\lambda_{N+1}\in I\!N$ such that $\lambda_1+\cdots+\lambda_{N+1}=k+1$ and $l_i<n$ for $i=1,\ldots,N$ (since the cases $l_i=n$ are incorporated in $t^{\epsilon\lambda_i}$). In order to prove the theorem we want to show that such a word can be expressed in terms of words of the form $1')$, $2')$, $3')$ and $4')$. This is a very slow process as we shall readily see. In order to obtain an inductive argument on the number $N+1$ of the intermediate powers of t, we show first the following, seemingly more

general result, where an unsymmetric expression appears also in the word. It is 'seemingly more general' because this unsymmetry of the word appears anyhow in a later stage of the calculations.

Proposition 3 *Let $k \in I\!N$ respectively $k \in Z\!\!Z_{d-1}$, $\epsilon \in \{\pm 1\}$, $l, m, l_2, \ldots, l_N \leq n$ and let $\lambda_1, \lambda_2, \ldots, \lambda_{N+1} \in I\!N$ such that $\lambda_1 + \cdots + \lambda_{N+1} = k+1$. Then it holds in $\mathcal{H}_{n+1}(q, \infty)$ respectively $\mathcal{H}_{n+1}(q, d)$ that words of the form:*

$$w_{n-1} g_n{}^\epsilon \ldots g_1{}^\epsilon t^{\epsilon\lambda_1} (g_1{}^\epsilon \ldots g_l{}^\epsilon)(g_1{}^\epsilon \ldots g_m{}^\epsilon \ldots g_1{}^\epsilon) t^{\epsilon\lambda_2} (g_1{}^\epsilon \ldots g_{l_2}{}^\epsilon \ldots g_1{}^\epsilon) t^{\epsilon\lambda_3} \ldots$$

$$t^{\epsilon\lambda_{N+1}} g_1{}^\epsilon \ldots g_n{}^\epsilon$$

where only between the first two powers of t appears the unsymmetric expression $(g_1{}^\epsilon \ldots g_l{}^\epsilon)(g_1{}^\epsilon \ldots g_m{}^\epsilon \ldots g_1{}^\epsilon)$, can be expressed as sums of words of the form 1′), 2′), 3′) and 4′). Note that if $l = 0$ we obtain the generic summand of $t_n{}^{\epsilon(k+1)}$.

Proof. We prove the statement for $\epsilon = +1$ by induction on the number $N+1$ of intermediate powers of t. The proof for $\epsilon = -1$ is completely analogous. For $N = 0$ we have $w_{n-1} g_n \ldots g_1 t^{\lambda_1} g_1 \ldots g_n$, where $\lambda_1 = k+1$ i.e. $w_{n-1} T_n^{k+1}$. Suppose the assertion holds for N. Then for $N+1$ we have:

$$A = w_{n-1} g_n \ldots g_1 t^\lambda (g_1 \ldots g_l)(g_1 \ldots g_m \ldots g_1) t^\mu (g_1 \ldots g_{l_2} \ldots g_1) t^{\lambda_3} \ldots t^{\lambda_{N+1}} g_1 \ldots g_n$$

$$= w_{n-1} g_n \ldots g_1 t^\lambda \underline{(g_1 \ldots g_l)} (g_m \ldots g_1 \ldots g_m) t^\mu (g_{l_2} \ldots g_1 \ldots g_{l_2}) t^{\lambda_3} \ldots t^{\lambda_{N+1}} g_1 \ldots g_n.$$

Here we also use the symbol '$\sum$' to mean 'linear combination of words of the type', the symbol 'w_{n-1}' for *not* always the same word in $\mathcal{H}_n$, and, in order to shorten the words, we substitute the expression $g_{l_2} \ldots g_1 \ldots g_{l_2} t^{\lambda_3} \ldots t^{\lambda_{N+1}} g_1 \ldots g_n$ by S.

We proceed by examining the cases $l < m$, $l > m$ and $l = m$.

- For $l < m$ we have:

$$A = w_{n-1} g_n \ldots g_1 t^\lambda \underline{g_m \ldots g_2} g_1 \ldots g_m (g_1 \ldots g_l) t^\mu \cdot S =$$

$$w_{n-1}(g_{m-1} \ldots g_1) g_n \ldots g_1 t^\lambda g_1 \ldots g_m \underline{g_1} g_2 \ldots g_l t^\mu \cdot S \overset{m \geq 1}{=}$$

$$w_{n-1}(g_{m-1} \ldots g_1 g_1) g_n \ldots g_1 t^\lambda g_1 \ldots g_m g_2 \ldots g_l \underline{t^\mu} \cdot S =$$

$$\underline{w_{n-1}(g_{m-1} \ldots g_1^2)} g_n \ldots \underline{g_1 t^\lambda g_1 t^\mu} g_2 \ldots g_m g_2 \ldots g_l \cdot S \overset{F.L.}{=}$$

$$w_{n-1} g_n \ldots g_2 \underline{t^\mu} g_1 t^\lambda g_1 g_2 \ldots g_m g_2 \ldots g_l \cdot S +$$

$$\sum_{i+j=\lambda+\mu} w_{n-1} g_n \ldots g_2 \underline{t^i} g_1 \underline{t^j} g_2 \ldots g_m g_2 \ldots g_l \cdot S =$$

$$w_{n-1} t^\mu g_n \ldots g_1 t^\lambda g_1 \ldots g_m \underline{g_2 \ldots g_l} \cdot S +$$

$$\sum_{i+j=\lambda+\mu} w_{n-1}t^i g_n\dots g_1\underline{g_2\dots g_m g_2\dots g_l t^j}\cdot S \overset{Lemma\ 1,(ii),\,l<m}{=}$$

$$\underline{(w_{n-1}t^\mu g_2\dots g_l)}g_n\dots g_1 t^\lambda g_1\dots g_m\cdot S+$$

$$\sum_{i+j=\lambda+\mu}\underline{(w_{n-1}t^i g_1\dots g_{m-1}g_1\dots g_{l-1})}g_n\dots g_1 t^j\cdot S =$$

$$(w_{n-1}g_n\dots g_1t^\lambda g_1\dots g_m\cdot S+\sum_{i+j=\lambda+\mu} w_{n-1}g_n\dots g_1t^j\cdot S$$

and the number of intermediate powers of t has reduced to N in all summands of ${t_n}^{k+1}$.

- For $l > m$ we have:

$$A = w_{n-1}g_n\dots g_1t^\lambda(g_1\dots g_l)\underline{g_m\dots g_1\dots g_m}t^\mu\cdot S \overset{m<l,\,Lemma\ 1,(ii)}{=}$$

$$\underline{(w_{n-1}g_m\dots g_1\dots g_m)}g_n\dots g_1t^\lambda g_1\dots g_l\underline{t^\mu}\cdot S =$$

$$w_{n-1}g_n\dots\underline{g_1t^\lambda g_1t^\mu}g_2\dots g_l\cdot S \overset{F.L.}{=} w_{n-1}g_n\dots g_2\underline{t^\mu}g_1t^\lambda g_1\dots g_l\cdot S+$$

$$\sum_{i+j=\lambda+\mu} w_{n-1}g_n\dots g_2\underline{t^i}g_1\underline{t^j}g_2\dots g_l\cdot S =$$

$$\underline{w_{n-1}t^\mu}g_n\dots g_1t^\lambda g_1\dots g_l\cdot S+\sum_{i+j=\lambda+\mu} w_{n-1}t^i g_n\dots g_1\underline{(g_2\dots g_l)}t^j\cdot S \overset{Lemma\ 1,(ii)}{=}$$

$$w_{n-1}g_n\dots g_1t^\lambda g_1\dots g_l\cdot S+\sum_{i+j=\lambda+\mu}\underline{(w_{n-1}t^i g_1\dots g_{l-1})}g_n\dots g_1t^j\cdot S =$$

$$w_{n-1}g_n\dots g_1t^\lambda g_1\dots g_l+\sum_{i+j=\lambda+\mu} w_{n-1}g_n\dots g_1t^j$$

and the number of intermediate powers of t has reduced to N in all summands of ${t_n}^{k+1}$.

- Finally if $l = m$ we have:

$$A = w_{n-1}g_n\dots g_1t^\lambda\underline{(g_1\dots g_m)g_m\dots g_1}\dots g_mt^\mu\cdot S \overset{Lemma 1,\,(iv)}{=}$$

$$w_{n-1}g_n\dots g_1t^\lambda\underline{g_2\dots g_m}t^\mu\cdot S+$$

$$\sum_{r=0}^{m-1} w_{n-1}g_n\dots g_1t^\lambda(\underline{g_{m-r}\dots g_2}g_1\dots g_{m-r})g_2\dots g_m\underline{t^\mu}\cdot S \overset{Lemma 1,\,(ii)}{=}$$

$$\underline{(w_{n-1}g_1\dots g_{m-1})}g_n\dots g_1t^{\lambda+\mu}\cdot S+$$

$$\sum_{r=0}^{m-1}\underline{(w_{n-1}g_{m-r-1}\dots g_1)}g_n\dots\underline{g_1t^\lambda g_1t^\mu}(g_2\dots g_{m-r})g_2\dots g_m\cdot S \overset{F.L.}{=}$$

$$w_{n-1}g_n\dots g_1t^{\lambda+\mu}\cdot S+\sum_{r=0}^{m-1} w_{n-1}g_n\dots g_2\underline{t^\mu}g_1t^\lambda(g_1\dots g_{m-r})g_2\dots g_m\cdot S+$$

$$\sum_{i+j=\lambda+\mu}\sum_{r=0}^{m-1} w_{n-1}g_n\dots g_2\underline{t^i}g_1\underline{t^j}(g_2\dots g_{m-r})g_2\dots g_m\cdot S =$$

$$w_{n-1}g_n\dots g_1t^{\lambda+\mu}\cdot S+\sum_{r=0}^{m-1} w_{n-1}t^\mu g_n\dots g_1t^\lambda(g_1\dots g_{m-r})\underline{g_2\dots g_{m-r-1}}\dots g_m\cdot$$

$$S+\sum_{i+j=\lambda+\mu}\sum_{r=0}^{m-1} w_{n-1}t^i g_n\dots g_1\underline{(g_2\dots g_{m-r})g_2\dots g_mt^j}\cdot S \overset{Lemma 1,\,(ii)}{=}$$

$$w_{n-1}g_n\dots g_1t^{\lambda+\mu}\cdot S+$$
$$\sum_{r=0}^{m-1}\underline{(w_{n-1}t^{\mu}g_2\dots g_{m-r-1})}g_n\dots g_1t^{\lambda}(g_1\dots g_{m-r-1}\underline{g_{m-r}^2}g_{m-r+1}\dots g_m)\cdot S+$$
$$\sum_{i+j=\lambda+\mu}\sum_{r=0}^{m-1}\underline{(w_{n-1}t^{i}g_1\dots g_{m-r-1}g_1\dots g_{m-1})}g_n\dots g_1t^{j}\cdot S=$$
$$w_{n-1}g_n\dots g_1t^{\lambda+\mu}\cdot S+\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{\lambda}(g_1\dots g_{m-r-1}\underline{g_{m-r+1}\dots g_m})\cdot S+$$
$$\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{\lambda}(g_1\dots g_m)\cdot S+\sum_{i+j=\lambda+\mu}\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{j}\cdot S=$$
$$w_{n-1}g_n\dots g_1t^{\lambda+\mu}\cdot S+\sum_{r=0}^{m-1}\underline{w_{n-1}g_{m-r+1}\dots g_m}g_n\dots g_1t^{\lambda}(g_1\dots g_{m-r-1})\cdot S+$$
$$\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{\lambda}(g_1\dots g_m)\cdot S+\sum_{i+j=\lambda+\mu}\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{j}\cdot S=$$
$$w_{n-1}g_n\dots g_1t^{\lambda+\mu}\cdot S+\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{\lambda}(g_1\dots g_{m-r-1})\cdot S+$$
$$\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{\lambda}(g_1\dots g_m)\cdot S+\sum_{i+j=\lambda+\mu}\sum_{r=0}^{m-1}w_{n-1}g_n\dots g_1t^{j}\cdot S$$

and the number of the intermediate powers of t has reduced to N in all summands of ${t_n}^{k+1}$.

$\square$

We can now conclude the proof of Theorem 3, since for the different possibilities of a word $w\in\mathcal{H}_{n+1}$ we have:

Case 1. If $w=w_{n-1}$ or $w=w_{n-1}g_n\dots g_i$ for $i=0,\dots,n$ there is nothing to show.

Case 2. If $w=w_{n-1}t_n^k,\ k\in\mathbb{Z}$ respectively $\mathbb{Z}_d$, then by Proposition 3, w is a unique linear combination of words of type $1')$, $2')$, $3')$ and $4')$.

Case 3. Finally, if $w=w_{n-1}g_n\dots g_{i+1}t_i^k,\ k\in\mathbb{Z}$ respectively $\mathbb{Z}_d$, by Proposition 3, t_i^k is written in terms of words $w_{i-1}, w_{i-1}g_i\dots g_r$ for $r\le i$, $w_{i-1}g_i\dots g_{r+1}T_r^k$ and $w_{i-1}T_i^k$. Therefore w can be written uniquely in terms of the words

$w_{n-1}g_n\dots g_{i+1}w_{i-1}g_i\dots g_r$ for $r=0,\dots,i$,

$w_{n-1}g_n\dots g_{i+1}w_{i-1}g_i\dots g_{r+1}T_r^k$ and

$w_{n-1}g_n\dots g_{i+1}w_{i-1}T_i^k$.

w_{i-1} commutes with $g_n\dots g_{i+1}$, unless $i=0$, where the word is already arranged in a trivial manner. So the above words reduce to the types $w_{n-1}g_n\dots g_r$ or $w_{n-1}g_n\dots g_{j+1}T_j^k$.

$\square$

Theorem 3 rephrased weaker says that the elements of the inductive basis contain *either* g_n *or* T_n^k *at most once*. We can now pass easily to the inductive basis that we need for constructing Markov traces on $\bigcup_{n=1}^{\infty} \mathcal{H}_n$. Indeed we have the following:

Theorem 4 *Every element of $\mathcal{H}_{n+1}(q,\infty)$ respectively $\mathcal{H}_{n+1}(q,d)$ can be written uniquely as a linear combination of words, each of one of the following types:*

1″) w_{n-1}

2″) $w_{n-1}g_n g_{n-1}\dots g_i$

3″) $w_{n-1}g_n g_{n-1}\dots g_{i+1}{t'_i}^k, k \in \mathbb{Z}$ *respectively* $\mathbb{Z}_d$

4″) $w_{n-1}{t'_n}^k, k \in \mathbb{Z}$ *respectively* $\mathbb{Z}_d$

where w_{n-1} is some word in $\mathcal{H}_n(q,\infty)$ respectively $\mathcal{H}_n(q,d)$.

Proof. By Theorem 3 it suffices to show that expressions of the forms 3′) and 4′) can be written (uniquely) in terms of 1″), 2″), 3″) and 4″). Indeed, for $k \in \mathbb{Z}$, let

$$w = w_{n-1}g_n g_{n-1}\dots g_{i+1}T_i^k = w_{n-1}g_n g_{n-1}\dots g_{i+1}g_i\dots g_1 t^k \underline{g_1\dots g_i}.$$

We apply the relation $g_r = q\cdot g_r^{-1} + (q-1)\cdot 1$ to all letters of the word $g_1\dots g_i$ to get:

$$w = w_{n-1}g_n\dots g_{i+1}g_i\dots g_1 t^k g_1^{-1}\dots g_i^{-1} + \sum w_{n-1}g_n\dots g_1 t^k g_{j_1}^{-1}\dots g_{j_k}^{-1},$$

where in the words $g_{j_1}^{-1}\dots g_{j_k}^{-1}$ there are possible gaps of indices. Let the closest to t^k gap occur at the index ρ; then

$$\begin{aligned} w &= w_{n-1}g_n\dots g_{i+1}{t'_i}^k + \sum w_{n-1}g_n\dots g_1 t^k g_1^{-1}\dots g_{\rho-1}^{-1}\underline{g_{\rho+1}^{-1}\dots g_{j_k}^{-1}} = \\ &\quad w_{n-1}g_n\dots g_{i+1}{t'_i}^k + \sum \underline{(w_{n-1}g_\rho^{-1}\dots g_{j_k-1}^{-1})}g_n\dots g_1 t^k g_1^{-1}\dots g_{\rho-1}^{-1} = \\ &\quad w_{n-1}g_n\dots g_{i+1}{t'_i}^k + \sum w_{n-1}g_n\dots g_\rho {t'_{\rho-1}}^k. \end{aligned}$$

Hence w is a sum of words of type 3″. In the case where $w = w_{n-1}T_n^k,\ k \in \mathbb{Z}$, we apply the same reasoning as above.

$\square$

Theorem 4 rephrased weaker says that the elements of the inductive basis contain *either* g_n *or* ${t'_n}^k$ *at most once*. Notice also that if we were working on the level of the Iwahori-Hecke algebra $\mathcal{H}_n(q,Q)$, we would omit Theorem 3.

Remark 2 All three inductive bases of $\mathcal{H}_{n+1}(q,\infty)$ respectively $\mathcal{H}_{n+1}(q,d)$ given in Theorems 2, 3 and 4 induce the same complete set of right coset representatives, S_{n+1}, of $W_{n,\infty}$ respectively $W_{n,d}$ in $W_{n+1,\infty}$ respectively $W_{n+1,d}$, namely:

$$S_{n+1} := \{s_n s_{n-1} \dots s_i \,|\, i = 1, \dots, n\} \bigcup$$
$$\{s_n s_{n-1} \dots s_1 t^k s_1 \dots s_i \,|\, i = 1, \dots, n-1, \ k \in \mathbb{Z} \text{ respectively } k \in \mathbb{Z}_d, k \neq 0\} \bigcup$$
$$\{{t_n}^k \,|\, k \in \mathbb{Z} \text{ respectively } k \in \mathbb{Z}_d\}.$$

We now give the final result of this section, namely, a basic set of $\mathcal{H}_{n+1}$ which is a proper subset of Σ_1.

Theorem 5 *The set*

$$\Sigma = \{{t'_{i_1}}^{k_1} {t'_{i_2}}^{k_2} \dots {t'_{i_r}}^{k_r} \cdot \sigma\}$$

for $1 \leq i_1 < i_2 < \ldots < i_r \leq n$, $k_1, \dots, k_r \in \mathbb{Z}$ *respectively* $\mathbb{Z}_d$ *and* $\sigma \in \mathcal{H}_{n+1}(q)$ *forms a basis in* $\mathcal{H}_{n+1}(q, \infty)$ *respectively* $\mathcal{H}_{n+1}(q, d)$.

Proof. By Theorem 4 it suffices to show that words in the inductive basis 1″), 2″), 3″) and 4″) can be written in terms of elements of Σ. Indeed, by induction on n we have: if $n = 0$ the only non-empty words are powers of t, which are of type 4″) and which are elements of Σ trivially. Assume the result holds for $n - 1$. Then for n we have:

Case 1. If $w = w_{n-1}$ there is nothing to show (by induction).

Case 2. If $w = w_{n-1} g_n \dots g_i$, then, by induction $w_{n-1} = {t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} \cdot \sigma$, a word of Σ restricted on $\mathcal{H}_n$. Thus $w = {t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} \cdot \sigma \cdot g_n \dots g_i \in \Sigma$, since $\sigma \cdot g_n \dots g_i$ is an element of the canonical basis of $\mathcal{H}_{n+1}(q)$.

Case 3. If $w = w_{n-1} g_n \dots g_{i+1} {t'_i}^k$, then, by induction step $w_{n-1} = {t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} \cdot \sigma$, a word of Σ restricted on $\mathcal{H}_n$, so

$$w = {t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} \cdot \sigma \cdot \underline{g_n \dots g_{i+1} {t'_i}^k} \overset{Lemma 1, (vi)}{=}$$

$${t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} \cdot \underline{\sigma \cdot {t'_n}^k} g_n \dots g_{i+1} \overset{Lemma 1, (vi)}{=}$$

$${t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} {t'_n}^k \cdot \sigma \cdot g_n \dots g_i.$$

Now $\sigma \cdot g_n \dots g_i$ is a basic element of $\mathcal{H}_{n+1}(q)$, thus $w \in \Sigma$.

Case 4. Finally, if $w = w_{n-1} {t'_n}^k$, by induction step we have $w_{n-1} = {t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} \cdot \sigma$, a word of Σ restricted on $\mathcal{H}_n$. Then

$$w = {t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} \cdot \underline{\sigma \cdot {t'_n}^k} \overset{Lemma 1, (vi)}{=} {t'_{i_1}}^{k_1} \dots {t'_{i_r}}^{k_r} {t'_n}^k \cdot \sigma \in \Sigma.$$

$\square$

4 Construction of Markov traces

The aim of this section is to construct Markov linear traces on the generalized and on each level of the cyclotomic Iwahori-Hecke algebras of $\mathcal{B}$-type. As these algebras are quotients of the braid groups, the constructed traces will actually attach to each braid a Laurent polynomial. The traces as well as the strategy of their construction are based on and include as special case the one constructed on the classical $\mathcal{B}$-type Hecke algebras in [12], [13] (Theorem 5), which in turn was based on Ocneanu's trace on Hecke algebras of $\mathcal{A}$-type, cf. [11] (Theorem 5.1). In the next section we combine these results with the Markov braid equivalence for knots in a solid torus, so as to obtain analogues of the homfly-pt polynomial for the solid torus.

Let $\mathcal{R} = \mathbb{Z}[q^{\pm 1}, u_1^{\pm 1}, \ldots, u_d^{\pm 1}, \ldots]$ and let $\mathcal{H}_n$ denote either $\mathcal{H}_n(q,\infty)$ or $\mathcal{H}_n(q,d)$. Note that the natural inclusion of the group $B_{1,n}$ into $B_{1,n+1}$ (geometrically, by adding one more strand at the end of the braid) induces a natural inclusion of $\mathcal{H}_n$ into $\mathcal{H}_{n+1}$. Therefore it makes sense to consider $\mathcal{B} := \bigcup_{n=1}^{\infty} B_{1,n}$ and $\mathcal{H} := \bigcup_{n=1}^{\infty} \mathcal{H}_n$. Then we have the following result:

Theorem 6 *Given z, s_k, specified elements in $\mathcal{R}$ with $k \in \mathbb{Z}$ respectively $\mathbb{Z}_d$ and $k \neq 0$, there exists a unique linear trace function*

$$tr: \ \mathcal{H} := \bigcup_{n=1}^{\infty} \mathcal{H}_n \longrightarrow \mathcal{R}(z, s_k), \ \ k \in \mathbb{Z} \textit{ respectively } \mathbb{Z}_d$$

determined by the rules:

$$\begin{array}{lll} 1) & tr(ab) = tr(ba) & a, b \in \mathcal{H}_n \\ 2) & tr(1) = 1 & \textit{for all } \mathcal{H}_n \\ 3) & tr(ag_n) = z\, tr(a) & a \in \mathcal{H}_n \\ 4) & tr(a{t'_n}^k) = s_k\, tr(a) & a \in \mathcal{H}_n, \ \ k \in \mathbb{Z} \textit{ respectively } \mathbb{Z}_d \end{array}$$

Proof. The idea of the proof of Theorem 6 is to construct tr on $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ inductively using Theorem 4 and the two last rules of the statement above. For this we need the following lemma. In order to avoid confusion with the indices we introduce here the symbol 'Z' to mean '$\mathbb{Z}$' or '$\mathbb{Z}_d$' respectively.

Lemma 6 *The map*

$$\begin{array}{lrcl} c_n: \ (\mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n) \ \bigoplus_{k \in Z} \mathcal{H}_n & & \longrightarrow & \mathcal{H}_{n+1} \\ \textit{given by} & c_n(a \otimes b \oplus_k e_k) & := & a g_n b + \sum_{k \in Z} e_k {t'_n}^k \end{array}$$

is an isomorphism of $(\mathcal{H}_n, \mathcal{H}_n)$-bimodules.

Proof. It follows from Theorem 4 that the set $\mathcal{L}_n$ below provides a basis of $\mathcal{H}_n$ as a free $\mathcal{H}_{n-1}$-module (compare with Remark 2 for W_{n+1}):

$$\mathcal{L}_n := \{g_{n-1}g_{n-2}\ldots g_i \,|\, i=1,\ldots,n-1\} \bigcup \{{t'_{n-1}}^k \,|\, k \in Z\} \bigcup$$
$$\{g_{n-1}g_{n-2}\ldots g_1 t^k g_1{}^1 \ldots g_i{}^1 \,|\, i=1,\ldots,n-2,\ k\in Z,\ k\neq 0\}.$$

Then we have: $\mathcal{H}_n \;=\; \bigoplus_{b\in\mathcal{L}_n} \mathcal{H}_{n-1}\cdot b,$

and using the universal property of tensor product we obtain:

$$\begin{aligned}
\mathcal{H}_n \bigotimes_{\mathcal{H}_{n-1}} \mathcal{H}_n &= \mathcal{H}_n \bigotimes_{\mathcal{H}_{n-1}} (\bigoplus_{b\in\mathcal{L}_n} \mathcal{H}_{n-1}\cdot b) \\
&= \bigoplus_{b\in\mathcal{L}_n} (\mathcal{H}_n \bigotimes_{\mathcal{H}_{n-1}} \mathcal{H}_{n-1}\cdot b) \\
&= \bigoplus_{b\in\mathcal{L}_n} \mathcal{H}_n \cdot b.
\end{aligned}$$

Therefore:

$$\mathcal{H}_n \bigotimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \bigoplus_{k\in Z} \mathcal{H}_n \;=\; \bigoplus_{b\in\mathcal{L}_n} \mathcal{H}_n \cdot b \bigoplus_{k\in Z} \mathcal{H}_n.$$

Applying now the same reasoning as above, the set $\mathcal{L}_{n+1}$ below provides a basis of $\mathcal{H}_{n+1}$ as a free $\mathcal{H}_n$-module:

$$\mathcal{L}_{n+1} := \{g_n g_{n-1}\ldots g_i \,|\, i=1,\ldots,n\} \bigcup \{{t'_n}^k \,|\, k\in Z\} \bigcup$$
$$\{g_n g_{n-1}\ldots g_1 t^k {g_1}^{-1}\ldots {g_i}^{-1} \,|\, i=1,\ldots,n-1,\ k\in Z,\ k\neq 0\}.$$

The latter isomorphism then proves that c_n is indeed an isomorphism of $(\mathcal{H}_n,\mathcal{H}_n)$-bimodules, since it corresponds bijectively basic elements to elements of the set $\mathcal{L}_{n+1}$.

□

We can now define inductively a trace, tr, on $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ as follows: assume tr is defined on $\mathcal{H}_n$ and let $x\in\mathcal{H}_{n+1}$ be an arbitrary element. By Lemma 6 there exist $a, b, e_k \in \mathcal{H}_n,\ k\in Z$, such that

$$x := c_n(a\otimes b \oplus_k e_k).$$

Define now:

$$tr(x) \;:=\; z\cdot tr(ab) + tr(e_0) + \sum_{k\in Z} s_k \cdot tr(e_k).$$

Then tr is well-defined. Furthermore, it satisfies the rules 2), 3) and 4) of the statement of Theorem 6. Rule 3) reflects the Markov property (recall the discussion in Introduction), and therefore, if the function tr is a trace then it is in particular a Markov trace. In fact one can check easily using induction and linearity, that tr satisfies the following seemingly stronger condition:

$$(3') \quad tr(ag_n b) = z\, tr(ab), \quad \text{for any } a, b \in \mathcal{H}_n.$$

In order to prove the existence of tr, it remain to prove the conjugation property, i.e. that tr is indeed a trace. We show this by examining case by case the different possibilities. Before continuing with the proof, we note that having proved the existence, the uniqueness of tr follows immediately, since for any $x \in \mathcal{H}_{n+1}$, $tr(x)$ can be clearly computed inductively using rules 1), 2), 3), 4) and linearity.

We now proceed with checking that $tr(ax) = tr(xa)$ for all $a, x \in \mathcal{H}$. Since tr is defined inductively the assumption holds for all $a, x \in \mathcal{H}_n$, and we shall show that $tr(ax) = tr(xa)$ for $a, x \in \mathcal{H}_{n+1}$. For this it suffices to consider $a \in \mathcal{H}_{n+1}$ arbitrary and x one of the generators of $\mathcal{H}_{n+1}$. I.e. it suffices to show:

$$\begin{array}{rcll} tr(ag_i) & = & tr(g_i a) & a \in \mathcal{H}_{n+1},\ i = 1, \ldots, n \\ tr(at) & = & tr(ta) & a \in \mathcal{H}_{n+1}. \end{array}$$

By Theorem 4, a is of one of the following types:

i) $a = w_{n-1}$

ii) $a = w_{n-1} g_n g_{n-1} \ldots g_i$

iii) $a = w_{n-1} g_n g_{n-1} \ldots g_{i+1} {t'_i}^k, k \in \mathbb{Z}$ respectively $\mathbb{Z}_d$

iv) $a = w_{n-1} {t'_n}^k, k \in \mathbb{Z}$ respectively $\mathbb{Z}_d$, where w_{n-1} is some word in $\mathcal{H}_n$.

If $a = w_{n-1}$ and $x = t$ or $x = g_i$ for $i = 1, \ldots, n-1$ the assumption holds from the induction step, whilst for $x = g_n$ it follows from $(3')$ that $tr(w_{n-1} g_n) = z\, tr(a) = tr(g_n w_{n-1})$.

If a is of type ii) or of type iii) and $x = t$ or $x = g_i$ for $i = 1, \ldots, n-1$ we apply the same reasoning as above using rule $(3')$. So we have to check still the cases where $a = w_{n-1} g_n g_{n-1} \ldots g_i$ or $a = w_{n-1} g_n g_{n-1} \ldots g_{i+1} {t'_i}^k$ and $x = g_n$, i.e.

$$\begin{array}{rcl} tr(w_{n-1} g_n \ldots g_i g_n) & = & tr(g_n w_{n-1} g_n \ldots g_i) \\ tr(w_{n-1} g_n \ldots g_{i+1} {t'_i}^k g_n) & = & tr(g_n w_{n-1} g_n \ldots g_{i+1} {t'_i}^k) \end{array} \qquad (*)$$

If a is of type iv) and $x = t$ or $x = g_i$ for $i = 1, \ldots, n-1$ we have to check:

$$\begin{array}{rcl} tr(w_{n-1} {t'_n}^k t) & = & tr(t w_{n-1} {t'_n}^k) \\ tr(w_{n-1} {t'_n}^k g_i) & = & tr(g_i w_{n-1} {t'_n}^k) \end{array} \qquad (**)$$

Finally, if a is of type iv) and $x = g_n$ we have to check:

$$tr(w_{n-1}{t'_n}^k g_n) \;=\; tr(g_n w_{n-1}{t'_n}^k) \qquad (***)$$

Before checking $(*), (**)$ and $(***)$ we need the following:

Lemma 7 *The function tr satisfies the following stronger version of rule 4):*

$$(4') \quad tr(x{t'_n}^k y) = s_k\, tr(xy),$$

for any $x, y \in \mathcal{H}_n$, $k \in \mathbb{Z}$ respectively $\mathbb{Z}_d$.

Proof. It suffices to prove $(4')$ for the case that y is of the form $y = y_1 t^\lambda y_2$, where $y_1$is a product of the g_i's for $i = 1, \ldots, n-1$, $\lambda \in \mathbb{Z}$ respectively $\mathbb{Z}_d$ and y_2 an arbitrary word in $\mathcal{H}_n$. Indeed we have:

$$tr(x{t'_n}^k y) = tr(x\underline{{t'_n}^k y_1} t^\lambda y_2) \stackrel{Lemma 1,(vi)}{=} tr(xy_1{t'_n}^k t^\lambda y_2)$$

$$= tr(xy_1 g_n \ldots g_1 t^k {g_1}^{-1} \underline{{g_2}^{-1} \ldots {g_n}^{-1} t^\lambda} y_2) \stackrel{Lemma 1,(vi)}{=}$$

$$= tr(xy_1 g_n \ldots \underline{g_1 t^k {g_1}^{-1} t^\lambda} {g_2}^{-1} \ldots {g_n}^{-1}) = A$$

The latter underlined expression says that we have to consider four possibilities depending on k, λ being positive or negative. We show here the case where both k, λ are positive. The rest are proved completely analogously. For k, λ positive, Lemma 5,(i) says:

$$\begin{aligned} g_1 t^k {g_1}^{-1} t^\lambda &= t^\lambda g_1 t^k {g_1}^{-1} + (q^{-1} - 1)\,[t^{\lambda-1} g_1 t^{k+1} + \cdots + g_1 t^{k+\lambda}] \\ &+ (1 - q^{-1})\,[t^k g_1 t^\lambda + \cdots + t^{k+\lambda-1} g_1 t]. \end{aligned}$$

We substitute then in A to obtain:

$$\begin{aligned} A &= tr(xy_1 \underline{g_n \ldots g_2 t^\lambda} g_1 t^k {g_1}^{-1} \ldots {g_n}^{-1} y_2) \\ &\quad + (q^{-1} - 1)\,[tr(xy_1 \underline{g_n \ldots g_2 t^{\lambda-1}} g_1 \underline{t^{k+1} {g_2}^{-1} \ldots {g_n}^{-1}} y_2) + \cdots \\ &\quad + tr(xy_1 g_n \ldots g_1 \underline{t^{k+\lambda} {g_2}^{-1} \ldots {g_n}^{-1}} y_2)] \\ &\quad + (1 - q^{-1})\,[tr(xy_1 \underline{g_n \ldots g_2 t^k} g_1 \underline{t^\lambda {g_2}^{-1} \ldots {g_n}^{-1}} y_2) + \cdots \\ &\quad + tr(xy_1 \underline{g_n \ldots g_2 t^{k+\lambda-1}} g_1 \underline{t {g_2}^{-1} \ldots {g_n}^{-1}} y_2)] \stackrel{Lemma\,1,(vi)}{=} \\ &= tr(xy_1 t^\lambda {t'_n}^k y_2) \\ &\quad + (q^{-1} - 1)\,[tr(xy_1 t^{\lambda-1} \underline{g_n \ldots g_1 {g_2}^{-1} \ldots {g_n}^{-1}} t^{k+1} y_2) + \cdots \\ &\quad + tr(xy_1 \underline{g_n \ldots g_1 {g_2}^{-1} \ldots {g_n}^{-1}} t^{k+\lambda} y_2)] \end{aligned}$$

$$+(1-q^{-1})\,[tr(xy_1t^k\underline{g_n\ldots g_1{g_2}^{-1}\ldots {g_n}^{-1}}t^{\lambda}y_2)+\cdots$$
$$+tr(xy_1t^{k+\lambda-1}\underline{g_n\ldots g_1{g_2}^{-1}\ldots {g_n}^{-1}}ty_2)]\overset{Lemma\,1,(iii)}{=}$$
$$=tr(xy_1t^{\lambda}{t'_n}^ky_2)$$
$$+(q^{-1}-1)\,[tr(xy_1t^{\lambda-1}{g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1t^{k+1}y_2)+\cdots$$
$$+tr(xy_1{g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1t^{k+\lambda}y_2)]$$
$$+(1-q^{-1})\,[tr(xy_1t^k{g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1t^{\lambda}y_2)+\cdots$$
$$+tr(xy_1t^{k+\lambda-1}{g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1ty_2)]\overset{(3')}{=}$$
$$=tr(xy_1t^{\lambda}{t'_n}^ky_2)+(q^{-1}-1)z\,[tr(xy_1t^{\lambda+k}y_2)+(1-q^{-1})z\,[tr(xy_1t^{k+\lambda}y_2)$$
$$=tr(xy_1t^{\lambda}{t'_n}^ky_2).$$

□

The relations $(**)$ follow now immediately from Lemma 7, since:

$tr(w_{n-1}\underline{{t'_n}^k}g_i)\overset{(4')}{=}s_k\,tr(w_{n-1}g_i)\overset{induction\,step}{=}s_k\,tr(g_iw_{n-1})=tr(g_iw_{n-1}{t'_n}^k)$,

for all $i<n$, and similarly for $x=t$.

We next show $(*)$ for $a=w_{n-1}g_n\ldots g_i$. The case $a=w_{n-1}g_n\ldots g_{i+1}{t'_i}^k$ is shown similarly. On the one hand we have:

$$tr(w_{n-1}g_ng_{n-1}\underline{\ldots g_ig_n})=tr(w_{n-1}\underline{g_ng_{n-1}g_n}g_{n-2}\ldots g_i)$$
$$=tr(w_{n-1}g_{n-1}\underline{g_n}g_{n-1}g_{n-2}\ldots g_i)\overset{(3')}{=}z\,tr(w_{n-1}\underline{{g_{n-1}}^2}g_{n-2}\ldots g_i)$$
$$=(q-1)z\,tr(w_{n-1}g_{n-1}\ldots g_i)+qz\,tr(w_{n-1}g_{n-2}\ldots g_i).$$

On the other hand in order to calculate $tr(g_nw_{n-1}g_n\ldots g_i)$ we examine the different possibilities for w_{n-1}:

– If $w_{n-1}\in\mathcal{H}_{n-1}$, then $tr(\underline{g_nw_{n-1}}g_n\ldots g_i)=tr(w_{n-1}\underline{{g_n}^2}g_{n-1}\ldots g_i)$

$$=(q-1)z\,tr(w_{n-1}g_{n-1}\ldots g_i)+qz\,tr(w_{n-1}g_{n-2}\ldots g_i).$$

– If $w_{n-1}=bg_{n-1}c$, where $b,\,c\in\mathcal{H}_{n-1}$, then $tr(\underline{g_nb}g_{n-1}\underline{cg_n}g_{n-1}\ldots g_i)$

$$=tr(bg_{n-1}\underline{g_n}g_{n-1}cg_{n-1}\ldots g_i)\overset{(3')}{=}z\,tr(b\underline{{g_{n-1}}^2}cg_{n-1}\ldots g_i)$$
$$=(q-1)z\,tr(bg_{n-1}cg_{n-1}\ldots g_i)+qz\,tr(bcg_{n-1}\ldots g_i)$$
$$=(q-1)z\,tr(bg_{n-1}cg_{n-1}\ldots g_i)+qz^2\,tr(bcg_{n-2}\ldots g_i)$$
$$=(q-1)z\,tr(bg_{n-1}cg_{n-1}\ldots g_i)+qz\,tr(bg_{n-1}cg_{n-2}\ldots g_i)$$

$$= (q-1)z\,tr(w_{n-1}g_{n-1}\dots g_i) + qz\,tr(w_{n-1}g_{n-2}\dots g_i).$$

– Finally, if $w_{n-1} = b{t'_{n-1}}^{k}$, where $b, \in \mathcal{H}_{n-1}$, then

$$tr(\underline{g_n b}{t'_{n-1}}^{k} g_n \dots g_i) = tr(b g_n {t'_{n-1}}^{k} \underline{g_n} \dots g_i)$$

$$= q\,tr(b\underline{{t'_n}^{k} g_{n-1}} \dots g_i) + (q-1)\,tr(b\underline{g_n}{t'_{n-1}}^{k} g_{n-1} \dots g_i) \overset{(4'),(3')}{=}$$

$$= qz\,tr(b{t'_{n-1}}^{k} g_{n-2} \dots g_i) + (q-1)z\,tr(b{t'_{n-1}}^{k} g_{n-1} \dots g_i)$$

$$= qz\,tr(w_{n-1}g_{n-2}\dots g_i) + (q-1)z\,tr(w_{n-1}g_{n-1}\dots g_i).$$

Note 2 The relations $(*)$ imply that $tr(xg_nyg_n) = tr(g_nxg_ny)$ for any $x,\ y \in \mathcal{H}_n$.

It remains now to show $(***)$. On the one hand we have:

$$tr(w_{n-1}\underline{{t'_n}^{k} g_n}) \overset{Lemma\,1,(vi)}{=} tr(w_{n-1}\underline{g_n}{t'_{n-1}}^{k}) \overset{(3')}{=} z\,tr(w_{n-1}{t'_{n-1}}^{k}).$$

On the other hand in order to calculate $tr(g_nw_{n-1}{t'_n}^{k})$ we examine the different possibilities for w_{n-1}:

– If $w_{n-1} \in \mathcal{H}_{n-1}$, then $tr(\underline{g_nw_{n-1}}{t'_n}^{k}) = tr(w_{n-1}\underline{g_n}^2{t'_{n-1}}^{k}g_n{}^{-1})$

$$= (q-1)\,tr(w_{n-1}\underline{{t'_n}^{k}}) + q\,tr(w_{n-1}{t'_{n-1}}^{k}\underline{g_n{}^{-1}}) = (q-1)\,tr(w_{n-1}{t'_{n-1}}^{k})$$

$$+z\,tr(w_{n-1}{t'_{n-1}}^{k}) + (1-q)\,tr(w_{n-1}{t'_{n-1}}^{k}) = z\,tr(w_{n-1}{t'_{n-1}}^{k}).$$

– If $w_{n-1} = ag_{n-1}b$, where $a,\ b \in \mathcal{H}_{n-1}$, then

$$tr(\underline{g_na}g_{n-1}b\underline{{t'_n}^{k}}) = tr(a\underline{g_ng_{n-1}g_n}b{t'_{n-1}}^{k}\,\underline{g_n{}^{-1}}$$

$$= q^{-1}\,tr(\underline{ag_{n-1}}g_n\underline{g_{n-1}b{t'_{n-1}}^{k}}g_n) + (q^{-1}-1)\,tr(ag_{n-1}\underline{g_n}g_{n-1}b{t'_{n-1}}^{k}) =$$

(applying Note 2 for $x = ag_{n-1}$ and $y = g_{n-1}b{t'_{n-1}}^{k}$)

$$= q^{-1}\,tr(\underline{g_na}g_{n-1}g_ng_{n-1}b{t'_{n-1}}^{k}) + (q^{-1}-1)z\,tr(ag_{n-1}{}^2b{t'_{n-1}}^{k})$$

$$= q^{-1}\,tr(ag_{n-1}\underline{g_n}g_{n-1}{}^2b{t'_{n-1}}^{k}) + (q^{-1}-1)z\,tr(ag_{n-1}{}^2b{t'_{n-1}}^{k})$$

$$= q^{-1}z(q^2-q+1)\,tr(\underline{ag_{n-1}b}{t'_{n-1}}^{k}) + q^{-1}zq(q-1)\,tr(ab{t'_{n-1}}^{k})$$

$$+(q^{-1}-1)z(q-1)\,tr(a\underline{g_{n-1}}b{t'_{n-1}}^{k}) + (q^{-1}-1)zq\,tr(ab{t'_{n-1}}^{k}) = z\,tr(w_{n-1}{t'_{n-1}}^{k}).$$

Before proving the last case we need to deform the expression ${t'_{n-1}}^{l}{t'_n}^{k}$. Indeed we have:

$${t'_{n-1}}^{l}{t'_n}^{k} = g_{n-1}\dots g_1t^l\underline{g_1{}^{-1}\dots g_{n-1}{}^{-1}g_n\dots g_1}t^kg_1{}^{-1}\dots g_n{}^{-1}$$

$$
\begin{aligned}
&= g_{n-1}\dots g_1 t^l \underline{g_n\dots g_2} g_1 \underline{g_2{}^{-1}\dots g_n{}^{-1}} t^k g_1{}^{-1}\dots g_n{}^{-1} \\
&= (g_{n-1}g_n)\dots(g_1g_2)t^l g_1 t^k (g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})\underline{g_n{}^{-1}} \\
&= (g_{n-1}g_n)\dots(g_1g_2)t^l g_1 t^k g_1{}^{-1}(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1}) \\
&= q^{-1}\,(g_{n-1}g_n)\dots(g_1g_2)\underline{t^l g_1 t^k g_1}(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1}) \\
&+(q^{-1}-1)\,(g_{n-1}g_n)\dots(g_1g_2)t^l g_1 t^k g_1(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1}) \overset{Lemma\ 5,(i)}{=} \\
&= q^{-1}\,(g_{n-1}g_n)\dots(g_1g_2)\underline{g_1}t^k g_1 t^l(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1}) \\
&+(1-q^{-1})\,[(g_{n-1}g_n)\dots(g_1g_2)t g_1 t^{k+l-1}(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})+\cdots \\
&+(g_{n-1}g_n)\dots(g_1g_2)t^l g_1 t^k(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})] \\
&+(q^{-1}-1)\,[(g_{n-1}g_n)\dots(g_1g_2)t^k g_1 t^l(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})+\cdots \\
&+(g_{n-1}g_n)\dots(g_1g_2)t^{k+l-1} g_1 t(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1})] \\
&+(q^{-1}-1)\,(g_{n-1}g_n)\dots(g_1g_2)t^l g_1 t^k(g_2{}^{-1}g_1{}^{-1})\dots(g_n{}^{-1}g_{n-1}{}^{-1}) \\
&= q^{-1}\,g_n g_{n-1}\dots g_1 t^k \underline{g_n\dots g_1 g_2{}^{-1}\dots g_n{}^{-1}} t^l g_1{}^{-1}\dots g_{n-1}{}^{-1} \\
&+(1-q^{-1})\,[g_{n-1}\dots g_1 t \underline{g_n\dots g_1 g_2{}^{-1}\dots g_n{}^{-1}} t^{k+l-1} g_1{}^{-1}\dots g_{n-1}{}^{-1}+\cdots \\
&+g_{n-1}\dots g_1 t^l \underline{g_n\dots g_1 g_2{}^{-1}\dots g_n{}^{-1}} t^k g_1{}^{-1}\dots g_{n-1}{}^{-1}] \\
&+(q^{-1}-1)\,[g_{n-1}\dots g_1 t^k \underline{g_n\dots g_1 g_2{}^{-1}\dots g_n{}^{-1}} t^l g_1{}^{-1}\dots g_{n-1}{}^{-1}+\cdots \\
&+g_{n-1}\dots g_1 t^{k+l-1} \underline{g_n\dots g_1 g_2{}^{-1}\dots g_n{}^{-1}} t g_1{}^{-1}\dots g_{n-1}{}^{-1}] \\
&+(q^{-1}-1)\,g_{n-1}\dots g_1 t^l \underline{g_n\dots g_1 g_2{}^{-1}\dots g_n{}^{-1}} t^k g_1{}^{-1}\dots g_{n-1}{}^{-1} \\
&= q^{-1}\,g_n\dots g_1 t^k g_1{}^{-1}\dots g_{n-1}{}^{-1} l \underline{g_n}\dots g_1 t^l g_1{}^{-1}\dots g_{n-1}{}^{-1} \\
&+(1-q^{-1})\,[g_{n-1}\dots g_1 t g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n\dots g_1 t^{k+l-1} g_1{}^{-1}\dots g_{n-1}{}^{-1}+\cdots \\
&+g_{n-1}\dots g_1 t^{l-1} g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n\dots g_1 t^{k+1} g_1{}^{-1}\dots g_{n-1}{}^{-1}] \\
&+(q^{-1}-1)\,[g_{n-1}\dots g_1 t^k g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n\dots g_1 t^l g_1{}^{-1}\dots g_{n-1}{}^{-1}+\cdots \\
&+g_{n-1}\dots g_1 t^{k+l-1} g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n\dots g_1 t g_1{}^{-1}\dots g_{n-1}{}^{-1}] \\
&= g_n\dots g_1 t^k g_1{}^{-1}\dots g_n^{-1} g_{n-1}\dots g_1 t^l g_1{}^{-1}\dots g_{n-1}{}^{-1} \\
&+(1-q^{-1})\,g_n\dots g_1 t^{k+l} g_1{}^{-1}\dots g_{n-1}{}^{-1} \\
&+(1-q^{-1})\,[g_{n-1}\dots g_1 t g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n\dots g_1 t^{k+l-1} g_1{}^{-1}\dots g_{n-1}{}^{-1}+\cdots \\
&+g_{n-1}\dots g_1 t^{l-1} g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n\dots g_1 t^{k+1} g_1{}^{-1}\dots g_{n-1}{}^{-1}] \\
&+(q^{-1}-1)\,[g_{n-1}\dots g_1 t^k g_1{}^{-1}\dots g_{n-1}{}^{-1} g_n\dots g_1 t^l g_1{}^{-1}\dots g_{n-1}{}^{-1}+\cdots
\end{aligned}
$$

$+g_{n-1}\ldots g_1 t^{k+l-1} {g_1}^{-1}\ldots {g_{n-1}}^{-1} g_n \ldots g_1 t {g_1}^{-1}\ldots {g_{n-1}}^{-1}]$.

Notice that with applying the other cases of Lemma 5 we obtain analogous results.

– If, finally, $w_{n-1} = b{t'_{n-1}}^l$, where $b \in \mathcal{H}_{n-1}$, we have: $tr(\underline{g_n} b\underline{{t'_{n-1}}^l {t'_n}^k})$

$$
\begin{aligned}
&= tr(b\underline{{g_n}^2} g_{n-1}\ldots g_1 t^k {g_1}^{-1}\ldots g_n^{-1} g_{n-1}\ldots g_1 t^l {g_1}^{-1}\ldots {g_{n-1}}^{-1})\\
&+(1-q^{-1})\, tr(b\underline{{g_n}^2} g_{n-1}\ldots g_1 t^{k+l} {g_1}^{-1}\ldots {g_{n-1}}^{-1})\\
&+(1-q^{-1})\,[tr(bg_n g_{n-1}\ldots g_1 t {g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1 t^{k+l-1}{g_1}^{-1}\ldots {g_{n-1}}^{-1})\\
&+\cdots + tr(bg_n g_{n-1}\ldots g_1 t^{l-1} {g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1 t^{k+1}{g_1}^{-1}\ldots {g_{n-1}}^{-1})]\\
&+(q^{-1}-1)\,[tr(bg_n g_{n-1}\ldots g_1 t^k {g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1 t^{l}{g_1}^{-1}\ldots {g_{n-1}}^{-1}) + \cdots\\
&+tr(g_{n-1}\ldots g_1 t^{k+l-1} {g_1}^{-1}\ldots {g_{n-1}}^{-1}\underline{g_n}\ldots g_1 t {g_1}^{-1}\ldots {g_{n-1}}^{-1})]\\
&= (q-1)\, tr(b{t'_n}^k {t'_{n-1}}^l) + q\, tr(b{t'_{n-1}}^k \underline{{g_n}^{-1}{t'_{n-1}}^l}) + (1-q^{-1})[(q-1)z+q]\, tr(b{t'_{n-1}}^{k+l})\\
&+(1-q^{-1})\,[q\, tr(bt'_n {t'_{n-1}}^{k+l-1}) + (q-1)z\, tr(b{t'_{n-1}}^{k+l}) + \cdots\\
&+q\, tr(b{t'_n}^{l-1} {t'_{n-1}}^{k+1}) + (q-1)z\, tr(b{t'_{n-1}}^{k+l})]\\
&+(q^{-1}-1)\,[q\, tr(b{t'_n}^k {t'_{n-1}}^{l}) + (q-1)z\, tr(b{t'_{n-1}}^{k+l}) + \cdots\\
&+q\, tr(b{t'_n}^{k+l-1} t'_{n-1}) + (q-1)z\, tr(b{t'_{n-1}}^{k+l})]\\
&= (q-1)s_k s_l\, tr(b) + q[q^{-1}z + (q^{-1}-1)]\, tr(b{t'_{n-1}}^{k+l}) +\\
&[(q+q^{-1}-2)z + (q-1)]\, tr(b{t'_{n-1}}^{k+l}) + (q^{-1}-1)(q-1)z\, tr({t'_{n-1}}^{k+l}) tr(b)\\
&+(q-1)s_1\, tr({t'_{n-1}}^{k+l-1}) tr(b) + \cdots + (q-1)s_{l-1}\, tr({t'_{n-1}}^{k+1}) tr(b)\\
&+(1-q)\, tr({t'_n}^k) s_l\, tr(b) + \cdots + (1-q)\, tr({t'_n}^{k+l-1}) s_1\, tr(b).
\end{aligned}
$$

And since $tr({t'_n}^i) = tr({t'_{n-1}}^i)$ in all algebras $\mathcal{H}_n$, we conclude that

$$tr(g_n b{t'_{n-1}}^l {t'_n}^k) = z\, tr(b{t'_{n-1}}^{k+l}) = z\, tr(b{t'_{n-1}}^l {t'_{n-1}}^k) = z\, tr(w_{n-1}{t'_{n-1}}^k).$$

The proof of Theorem 6 is now concluded. □

As already mentioned in the Introduction, we can define tr with so few rules, because the elements $t^k, \ldots, {t'_i}^k$ in rule 4) are all conjugate, and this reflects the fact that $B_{1,n}$ splits as a semi-direct product of the classical braid group B_n and of its free subgroup $P_{1,n}$ generated precisely by the elements $t, t'_1, \ldots, t'_{n-1}$: $B_{1,n} = P_{1,n} \rtimes B_n$.

Note that if $k \in \mathbb{Z}_2$ we are in the case of the classical Iwahori-Hecke algebras of type $\mathcal{B}$, and from the above construction we recover the trace given in [12, 13]. Moreover, if a word $x \in \mathcal{H}_n$ does not contain any t's (that is, if x is an element of the Iwahori-Hecke algebra of type $\mathcal{A}_n$), then $tr(x)$ can be computed using only rules 1), 2), and 3) of Theorem 6, and in this case tr agrees with Ocneanu's trace (cf. [11]).

Remark 3 A word seen as an element of different $\mathcal{B}$-type Hecke algebras will aquire in principle different values for the different traces. This difference consists in substituting – if necessary – the parameters s_i according to the defining relation (♠) of $\mathcal{H}_n(q,d)$: $t^d = a_{d-1}t^{d-1} + \cdots + a_0$. So, in $\mathcal{H}_n(q,d)$ we have: $tr({t'_n}^k) = s_k$ for $k \in \mathbb{Z}_d$ and $tr({t'_n}^d) = a_{d-1}s_{d-1} + \cdots + a_0$.

For example in $\mathcal{H}_n(q,\infty)$ and in $\mathcal{H}_n(q,d)$ for $d > 5$ we have $tr(t^5) = s_5$.

In $\mathcal{H}_n(q,5)$ is $tr(t^5) = a_4s_4 + \cdots + a_0$, whilst in $\mathcal{H}_n(q,3)$ is

$tr(t^5) = ({a_2}^3 + 2a_1a_2 + a_0)s_2 + ({a_1}^2 + a_1{a_2}^2 + a_0a_2)s_1 + (a_0a_1 + a_0{a_2}^2)$.

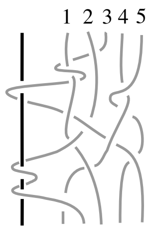
In order to calculate the trace of a word in $\mathcal{H}_n$ we bring it to the canonical form of Theorem 5 applying at the same time the rules of the trace. As an example we calculate below $tr(g_2g_1t^3g_1^{-1}g_3g_2g_3)$. We have:

$$
\begin{aligned}
tr(g_2g_1t^3g_1^{-1}\underline{g_3g_2g_3}) &= tr(g_2g_1t^3g_1^{-1}g_2\underline{g_3}g_2) = z\,tr(g_2g_1t^3g_1^{-1}\underline{{g_2}^2}) \\
&= z(q-1)\,tr(g_2g_1t^3g_1^{-1}\underline{g_2}) + zq\,tr(g_2g_1t^3g_1^{-1}) \\
&= z(q-1)q\,tr({t'_2}^3) + z(q-1)^2\,tr(\underline{g_2}g_1t^3g_1^{-1}) + zq\,tr(\underline{g_2}g_1t^3g_1^{-1}) \\
&= q(q-1)z\,tr({t'_2}^3) + z^2(q^2-q+1)\,tr({t'_1}^3).
\end{aligned}
$$

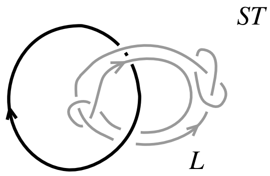
5 Invariants of knots in the solid torus

The aim of this section is to construct all analogues of the 2-variable Jones polynomial homfly-pt) for oriented knots in the solid torus derived from the cyclotomic and generalized Hecke algebras of type $\mathcal{B}$, using their Markov equivalence and the Markov traces constructed in Theorem 6. All knots/links will be assumed to be oriented, and we shall say 'knots' for both knots and links.

As mentioned in the Introduction the elements of the braid groups $B_{1,n}$, which we call 'mixed braids', are represented geometrically by braids in $n+1$ strands in S^3, which keep the first strand fixed. The closure of a mixed braid represents a knot inside the oriented solid torus, ST, where the fixed strand represents the complementary solid torus in S^3, and the next n numbered strands represent the knot in ST. Below we illustrate a mixed braid in $B_{1,5}$ and a knot in ST.



,



Moreover, it has been well-understood that all knots in ST may be represented by mixed braids, and their isotopy in ST is reflected by equivalence classes of braids in $\bigcup_{n=1}^{\infty} B_{1,n}$ through the following:

Theorem 7 (cf. [13], Theorem 3.)
Let L_1, L_2 be two oriented links in ST and β_1, β_2 be mixed braids in $\bigcup_{n=1}^{\infty} B_{1,n}$ corresponding to L_1, L_2. Then L_1 is isotopic to L_2 in ST if and only if β_1 is equivalent to β_2 in $\bigcup_{n=1}^{\infty} B_{1,n}$ under equivalence generated by the braid relations together with the following two moves:

(i) Conjugation: If $\alpha, \beta \in B_{1,n}$ then $\alpha \sim \beta^{-1}\alpha\beta$.

(ii) Markov moves: If $\alpha \in B_{1,n}$ then $\alpha \sim \alpha{\sigma_n}^{\pm 1} \in B_{1,n+1}$.

Let now π denote the canonical quotient map $B_{1,n} \longrightarrow \mathcal{H}_n$ given in Definition 1, and consider the trace constructed in Theorem 6 for a specified algebra $\mathcal{H}_n$. Then a braid in $B_{1,n}$ can be mapped through $tr \circ \pi$ to an expression in the variables $q, u_1^{\pm 1}, \ldots, u_d^{\pm 1}, \ldots, z, (s_k),\ k \in \mathbb{Z}$ respectively $\mathbb{Z}_d$. Let also $\widehat{\alpha}$ denote the knot obtained by closing the mixed braid α. Theorem 7 combined with Theorem 6 say that in order to obtain a knot invariant $\mathcal{X}$ in ST from any specified trace of Theorem 6 we have to normalize first g_i to $\sqrt{\lambda}g_i$ so that

$$tr(a(\sqrt{\lambda}g_n)) = tr(a((\sqrt{\lambda}g_n)^{-1})) \quad \text{for} \quad a \in \mathcal{H}_n.$$

This normalization has been done in [13], (5.1), where Jones's normalization of Ocneanu's trace (cf. [11]) was followed, and it yields

$$\lambda := \frac{z+1-q}{qz}, \quad z := \frac{1-q}{q\lambda - 1}.$$

Then we have to normalize tr so that

$$\mathcal{X}(\widehat{\alpha}) = \mathcal{X}(\widehat{\alpha\sigma_n}) = \mathcal{X}(\widehat{\alpha\sigma_n^{-1}}).$$

Let finally A be the field of rational functions over $\mathbb{Q}$ in indeterminates $\sqrt{\lambda}$, $\sqrt{q}$, $a_{d-1}, \ldots, a_0, \ldots, (s_k)$, $k \in \mathbb{Z}$ respectively $\mathbb{Z}_d$. (The reason for having square root of q becomes clear in the recursive formula † below.) Then the normalizations result the following

Definition 3 (cf. [13], Definition 1.) For α, tr, π as above let

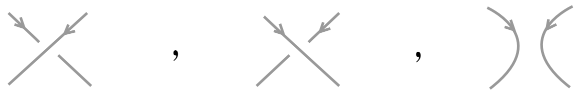
$$\mathcal{X}_{\widehat{\alpha}} = \mathcal{X}_{\widehat{\alpha}}(q, a_{d-1}, \ldots, a_0, \sqrt{\lambda}, s_1, s_2, \ldots) := \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} (\sqrt{\lambda})^e \, tr(\pi(\alpha)),$$

where e is the exponent sum of the σ_i's that appear in α. (Note that the t_i''s do not affect the estimation of e, so they can be ignored.) Then $\mathcal{X}_{\hat{\alpha}}$ depends only on the isotopy class of the mixed knot $\widehat{\alpha}$, which represents an oriented knot in ST. (For example, in $\mathcal{H}_n(q,d)$ and for $k \in \mathbb{Z}_d$ we have: $\alpha = t^k$, then $\mathcal{X}_{\hat{\alpha}} = s_k$.)

Note that if a knot in ST can be enclosed in a 3-ball then it may be seen as a knot in S^3 and there exists a mixed braid representative, α, which does not contain t_i''s. Then $\mathcal{X}_{\hat{\alpha}}$ has the same value as the 2-variable Jones polynomial (homfly-pt) as given in [11], Definition 6.1. On the lower level of $\mathcal{H}_n(q,Q)$ $\mathcal{X}$ yields the invariants constructed in [13], Section 5 and [8], Section 5.

Remark 4 Note furthermore that one could also define $\mathcal{H}_n(q,d)$ as a quotient of $B_{1,n}$ by sending the generator t of $B_{1,n}$ to t^{-1} of $\mathcal{H}_n(q,d)$. Then the traces and the knot invariants in ST constructed above exhaust the whole range of such constructions related to *all possible* Hecke and Hecke-related algebras of type $\mathcal{B}$.

On recursive formulae: We shall now show how to interpret the above in terms of knot diagrams, and how to calculate alternatively the above knot invariants in ST by applying recursive *skein relations* and *initial conditions* on the mixed link diagrams. Let L_+, L_-, L_0 be oriented mixed link diagrams that are identical, except in one crossing, where they are as depicted below:

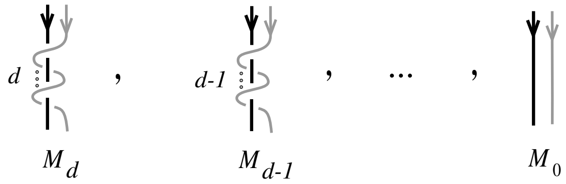


L_+ , L_- , L_0

With analogous reasoning as in [13], (5.2) (cf. also [11]) the defining quadratic relation of $\mathcal{H}_n$ induces the invariant $\mathcal{X}$ to satisfy the following recursive linear formula, which is the well-known skein rule used for the evaluation of the homfly-pt polynomial.

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}\,\mathcal{X}_{L_+} - \sqrt{q}\sqrt{\lambda}\,\mathcal{X}_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})\,\mathcal{X}_{L_0} \qquad \dagger$$

In the case of $\mathcal{H}_n(q,\infty)$ there is no other skein relation that $\mathcal{X}$ satisfies.
In the case of $\mathcal{H}_n(q,d)$, let $M_d, M_{d-1}, \ldots, M_0$ be oriented mixed link diagrams that are identical, except in the regions depicted below:



Using conjugation we may assume that $M_d = \widehat{\alpha\, {t'_i}^d}, M_{d-1} = \alpha\, \widehat{{t'_i}^{d-1}}, \ldots, M_0 = \widehat{\alpha}$ for some $\alpha \in B_{1,n}$. And so by Lemma 1, (viii) we obtain:

$$tr(\pi(\alpha\, {t'_i}^d)) = a_{d-1}\, tr(\pi(\alpha\, {t'_i}^{d-1})) + \cdots + a_0\, tr(\pi(\alpha)),$$

If we multiply now the above equation by

$$\Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} (\sqrt{\lambda})^e$$

we obtain the following skein relation for $\mathcal{X}$:

$$\mathcal{X}_{\widehat{\alpha {t'_i}^d}} = a_{d-1}\, \mathcal{X}_{\widehat{\alpha {t'_i}^{d-1}}} + \cdots + a_0\, \mathcal{X}_{M_0} \qquad \ddagger$$

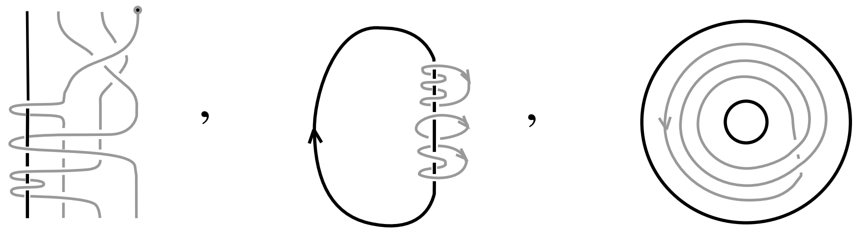
(compare with Remark 3). We next find the initial conditions that are also needed for evaluating $\mathcal{X}$ for any knot diagram in ST using the skein relations $\dagger$ and $\ddagger$. Clearly

$$\mathcal{X}_{unknot} = 1$$

should be one of them. Recall now the canonical basis of $\mathcal{H}_{n+1}$ given in Theorem 5. With appropriate changes of crossings (using the quadratic relations for the g_i's) this basis yields a canonical enumeration of descending diagrams related to $B_{1,n+1}$. Let now α be such a descending diagram. Applying tr on α means geometrically that we close the braid α and we apply the Markov moves. Using Rule (4) we extract and re-insert $tr({t'_i}^k)$ so as to obtain:

$$\mathcal{X}_{\widehat{\alpha}} = \mathcal{X}_{{t'_{i_1}}^{k_1}\widehat{{t'_{i_2}}^{k_2}}\ldots {t'_{i_r}}^{k_r}}.$$

This provides the second set of initial conditions, namely the values of $\mathcal{X}$ at all links consisting of stucks of loops of different twists with same orientation around the 'axis' solid torus. If $\mathcal{X}$ is derived by the cyclotomic Hecke algebra $\mathcal{H}_n(q,d)$ the number of twists of each loop cannot exceed $d-1$. In the case of $\mathcal{H}_n(q,\infty)$ the number of twists is arbitrary. We illustrate below an example of a descending diagram with the starting point at the top of the last strand, the basic link $t^4 t_1 {t_2}^{-1}$ and the projection of t^3 on a punctured disc.



We conclude with some remarks.

Remarks (i) On the level of $\mathcal{H}_n(q,\infty), \mathcal{X}$ is defined by all initial conditions (with unrestricted number of twists) and only by the first skein rule. Therefore the set of mixed links of the form $\widehat{{t'_{i_1}}^{k_1} {t'_{i_2}}^{k_2} \ldots {t'_{i_r}}^{k_r}}$, for $k_1, \ldots, k_r \in \mathbb{Z}$ forms the basis of the 3rd skein module of the solid torus. Thus the result of J.Hoste and M.Kidwell in [10], and of V.Turaev in [16] is recovered with this method. If $\mathcal{X}$ is derived by the cyclotomic Hecke algebras $\mathcal{H}_n(q,d)$ the set of mixed links of the form $\widehat{{t'_{i_1}}^{k_1} {t'_{i_2}}^{k_2} \ldots {t'_{i_r}}^{k_r}}$, for $k_1, \ldots, k_r \in \mathbb{Z}_d$ forms the basis of the corresponding submodule of the 3rd skein module of ST. In [15] the algebra $\mathcal{H}_n(q,\infty)$ has been studied independently and the corresponding ST-invariant has been constructed using similar methods.

(ii) If on the level of $\mathcal{H}_n(q,\infty)$ we use the skein rule

$$\frac{1}{t}\,\mathcal{Y}_{L_+} - t\,\mathcal{Y}_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})\,\mathcal{Y}_{L_0}$$

instead of †, and the initial conditions $\mathcal{Y}_{unknot} = 1$ and $\mathcal{Y}_{\widehat{t}} = s$ we obtain an analogue of the Jones polynomial for oriented knots in the *oriented* ST. If ST is *unoriented* we have to allow an extra isotopy move for knots in ST, namely to flip over the diagram around the x-axis, where the knot diagram is projected on a punctured disc. The invariant $\mathcal{Y}$ is preserved under the flipping over move, so $\mathcal{Y}$ is the analogue of the Jones polynomial in the *orientable* ST. For details and for the Kauffman bracket approach of this invariant see [9].

(iii) The invariant $\mathcal{X}$ related to $\mathcal{H}_n(q,\infty)$ is the appropriate one for extending the results to the lens spaces $L(p,1)$. The combinatorial setup is similar to the one for ST, only the Markov braid equivalence includes one more move, which reflects the surgery description of $L(p,1)$. So, in order to construct a homfly-pt analogue for knots in $L(p,1)$ or, equivalently, in order to compute for $L(p,1)$ the 3rd skein module and its quotients we have to normalize the ST-invariants further so that

$$\mathcal{X}_{\widehat{\alpha}} = \mathcal{X}_{sl(\widehat{\alpha})}, \text{ for } \alpha \in B_{1,n},$$

for all possible slidings of α. This is the subject of [S. Lambropoulou, J. Przytycki, Hecke algebra approach to the skein module of lens spaces, in preparation].

(iv) Analogous combinatorial setup, Markov braid equivalence and braid structures in arbitrary c.c.o. 3-manifolds and knot complements has already been done in [14],[S. Lambropoulou, Braid structures in 3-manifolds, to appear in JKTR]. Therefore it is possible in principle to extend such algebraic constructions to other 3-manifolds, by means of constructing appropriate quotient algebras and Markov traces on them, followed by appropriate normalizing, in order to derive knot invariants.

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