Braid structures related to knot complements, handlebodies and 3–manifolds

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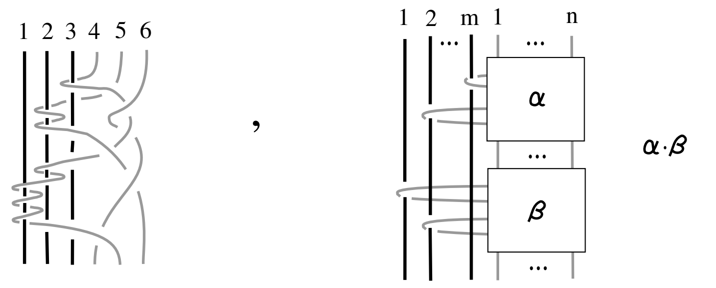
Abstract

We consider braids on $m+n$ strands, such that the first m strands are trivially fixed. We denote the set of all such braids by $B_{m,n}$. Via concatenation $B_{m,n}$ acquires a group structure. The objective of this paper is to find a presentation for $B_{m,n}$ using the structure of its corresponding pure braid subgroup, $P_{m,n}$, and the fact that it is a subgroup of the classical Artin group B_{m+n}. Then we give an irredundant presentation for $B_{m,n}$. The paper concludes by showing that these braid groups or appropriate cosets of them are related to knots in handlebodies, in knot complements and in c.c.o. 3–manifolds.

1 Introductory notions and motivations

Definition 1 *The set of all elements of the classical Artin group B_{m+n} for which, if we remove the last n strands we are left with the identity braid on m strands, shall be denoted by $B_{m,n}$ (see figure 1(a) below for an example in $B_{3,3}$). The elements of $B_{m,n}$ are special cases of 'mixed braids' (cf. section 6).*

Figure 1:



(a) (b)

Concatenation is a closed operation in $B_{m,n}$: the product $\alpha\cdot\beta$ of two elements $\alpha,\beta\in B_{m,n}$ is also an element of $B_{m,n}$ (see figure 1(b)). Thus $B_{m,n}\leq B_{m+n}$. Our purpose is to obtain a simple presentation for $B_{m,n}$.

The motivation for studying these braids comes from studying oriented knots and links in knot complements, in c.c.o. 3-manifolds and in handlebodies, since these spaces may be represented by a fixed braid or a fixed integer-framed braid in S^3. Then knots and links in these spaces may be represented by elements of the above braid groups $B_{m,n}$ or of appropriate cosets of these groups. More precisely, if M denotes the complement of the m-unlink or a connected sum of m lens spaces of type $L(p,1)$ or a handlebody of genus m, then knots and links in these spaces may be represented precisely by the mixed braids in $B_{m,n}$, for $n \in \mathbb{N}$. In the case $m=1$, $B_{1,n}$ is the Artin group of type $\mathcal{B}$ (cf. [4], [5], [6]). If M is generic, concatenation is no more a closed operation of mixed braids, but as we show in section 6, knots and links in M may be represented by mixed braids in $B_{m,n}$, for $n \in \mathbb{N}$, followed by a fixed part associated to M, i.e. by elements of a coset of $B_{m,n}$.

We recall now some facts about braids and pure braids. For more details and a complete study of the classical theory of braids the reader is referred to [1]. The pure braid group, P_n, corresponding to the classical Artin group on n strands, B_n, consists of all elements in B_n that induce the identity permutation in S_n, thus $P_n \lhd B_n$ and P_n is generated by the elements

$$\begin{aligned} a_{ij} &= {\sigma_i}^{-1}{\sigma_{i+1}}^{-1}\ldots{\sigma_{j-2}}^{-1}{\sigma_{j-1}}^{2}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i \\ &= \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}{\sigma_i}^{2}{\sigma_{i+1}}^{-1}\ldots{\sigma_{j-2}}^{-1}{\sigma_{j-1}}^{-1}, \quad 1 \leq i < j \leq n. \end{aligned}$$

The generators a_{ij} may be pictured geometrically as an elementary loop between the ith and jth strand (cf. figure 2).

The most important property of pure braids is that they have a canonical form, the so-called *'Artin's canonical form'*, which says that every element, A, of P_n may be written uniquely in the form:

$$A = U_1 U_2 \cdots U_{n-1}$$

where each U_i is a uniquely determined product of powers of the a_{ij} using only those with $i < j$. Geometrically, this means that any pure braid can be 'combed' i.e. can be written canonically as the pure braiding of the first strand with the rest, then keep the first strand fixed and uncrossed and have the pure braiding of the second strand and so on (cf. figure 3).

The main idea for finding a presentation for P_n is the following: The combing of a strand may be regarded as a loop in the complement space of the other strands, and as such is an element of a free group since the fundamental group of a punctured disc is free. Thus,

$$P_n = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1 = F_{n-1} \rtimes P_{n-1},$$

where each F_i is a free group on the generators $a_{1,i+1},\ldots,a_{i,i+1}$ (the elementary loops between the $(i+1)$st strand and all its previous ones), and where the action is induced by conjugation. It turns out that P_n has $\frac{n(n-1)}{2}$ generators and $\frac{1\cdot 2^2+2\cdot 3^2+\ldots+(n-2)(n-1)^2}{2}$ relations, the following:

$$
{a_{ij}}^{-1}a_{rs}a_{ij}=\begin{cases} a_{rs} & \text{if } i<j<r<s \quad \text{or } r<i<j<s, \\ a_{is}a_{js}{a_{is}}^{-1} & \text{if } i<j<s, \\ a_{is}a_{js}a_{is}{a_{js}}^{-1}{a_{is}}^{-1} & \text{if } i<j<s, \\ a_{is}a_{js}{a_{is}}^{-1}{a_{js}}^{-1}a_{rs}a_{js}a_{is}{a_{js}}^{-1}{a_{is}}^{-1} & \text{if } i<r<j<s. \end{cases}
$$

Based on these ideas, we introduce in section 2 the pure braid group $P_{m,n}$ and in section 3 we find a presentation for it. Then in section 4 we put together a presentation for $B_{m,n}$, which we simplify in section 5. In section 5 we also give a Dynkin-diagram related to $B_{m,n}$. Finally, in section 6 we explain that elements of $B_{m,n}$ represent oriented knots and links in certain spaces and that appropriate cosets of $B_{m,n}$ represent knots and links in the generic cases of knot complements and c.c.o. 3-manifolds.

The results here have been preliminary studied by the author in [4] and have been presented in various mathematical meetings since 1995. A. Sossinsky, independently, motivated by the same topological considerations, studies these groups in [8] and he conjectures the irredundant presentation for $B_{m,n}$. Moreover, V. Vershinin in [9] studies the groups $B_{m,n}$ in connection to handlebodies of genus g, taking a configuration-spaces approach. Back in 1993 Alastair Leeves had found a presentation for $B_{m,n}$, which was presented in [4], but a proof was never published. The author is thankfull to A. Leeves for inspiring discussions at the time. Also, her grateful thanks are due to Bernard Leclerc for his careful reading through this work and his very valuable comments.

This is the first paper in a sequel of three. The next one gives expressions for algebraic equivalence of braids reflecting knot isotopy in arbitrary knot complements and c.c.o. 3-manifolds. The case of handlebodies is joint work with Reinhard Häring-Oldenburg.

2 The pure braid group $P_{m,n}$

Definition 2 *The corresponding pure braid group $P_{m,n}$ of $B_{m,n}$ is defined as $P_{m,n}=B_{m,n}\cap P_{m+n}$, i.e. $P_{m,n}\leq P_{m+n}$ and it does not contain pure braiding among the first m strands.*

By its definition, $P_{m,n}$ is generated by the pure braid generators a_{ij} for $i\in\{1,\ldots,m+n-1\}$ and $j\in\{m+1,\ldots,m+n\}$ of P_{m+n} (see figure 2). Then, $B_{m,n}$

is clearly generated by the elementary mixed braids (drawn below) a_{ij} for $i \in \{1, \ldots, m+n-1\}$ and $j \in \{m+1, \ldots, m+n\}$ together with $\sigma_{m+1}, \ldots, \sigma_{m+n-1}$, the elementary crossings among the last n strands. Note that the inverses of the a_{ij}'s and σ_k's are represented by the same geometric pictures, but with the opposite crossings.

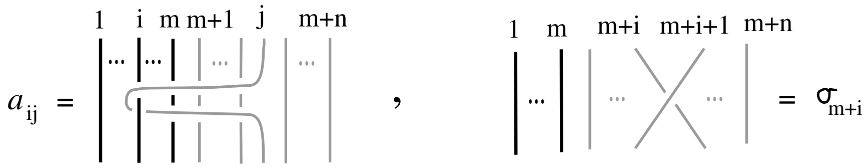


Figure 2

Also, by definition we have an exact sequence

$$1 \longrightarrow P_{m,n} \longrightarrow B_{m,n} \longrightarrow S_n \longrightarrow 1.$$

In particular, $P_{m,n} \lhd B_{m,n}$. More precisely, we have the following relations:

$$\sigma_k{}^{-1} a_{ij}{}^{\pm} \sigma_k = \begin{cases} a_{ij}{}^{\pm} & \text{if } \ k \leq i-2 \ \text{ or } \ i+1 \leq k \leq j-2 \ \text{ or } \ k \geq j+1, \\ a_{i-1,j}{}^{\pm} & \text{if } \ k = i-1, \\ a_{ij} a_{i+1,j}{}^{\pm} a_{ij}{}^{-1} & \text{if } \ k = i, \\ a_{i,j-1}{}^{\pm} & \text{if } \ k = j-1, \\ a_{ij} a_{i,j+1}{}^{\pm} a_{ij}{}^{-1} & \text{if } \ k = j. \end{cases} \qquad \square$$

We shall call these *mixed relations* and we shall denote them by M_1, M_2, M_3, M_4 and M_5 in the order they are written.

Thus $B_{m,n}$ is a group extension of $P_{m,n}$ by S_n. This will yield a presentation for $B_{m,n}$, conditionally to knowing a presentation for $P_{m,n}$.

3 A presentation for $P_{m,n}$

Theorem 1 *The pure braid group $P_{m,n}$ is generated by the elements a_{ij} for $i \in \{1, \ldots, m+n-1\}$, $j \in \{m+1, \ldots, m+n\}$ and $i < j$, which are subject to the relations:*

$$\begin{array}{llcll} (P_1) & a_{ij}{}^{-1} a_{rs} a_{ij} & = & a_{rs} & \textit{if} \ \ i<j<r<s \ \ \textit{or} \ \ r<i<j<s, \\ (P_2) & a_{ij}{}^{-1} a_{js} a_{ij} & = & a_{is} a_{js} a_{is}{}^{-1} & \textit{if} \ \ i<j<s, \\ (P_3) & a_{ij}{}^{-1} a_{is} a_{ij} & = & a_{is} a_{js} a_{is} a_{js}{}^{-1} a_{is}{}^{-1} & \textit{if} \ \ i<j<s, \\ (P_4) & a_{ij}{}^{-1} a_{rs} a_{ij} & = & a_{is} a_{js} a_{is}{}^{-1} a_{js}{}^{-1} a_{rs} a_{js} a_{is} a_{js}{}^{-1} a_{is}{}^{-1} & \textit{if} \ \ i<r<j<s. \end{array}$$

Relations P_1, P_2, P_3 and P_4 shall be called *pure braid relations*. Note that P_1 and P_4 involve the strands i, r, j, s, whilst P_2 and P_3 involve the strands i, j, s.

Proof By its definition and by the fact that the a_{ij}'s (for all indices) generate P_{m+n} follows that the above set of elements is indeed a set of generators for $P_{m,n}$. Since $P_{m,n} \leq P_{m+n}$ we can apply on its elements Artin's combing. As for P_n, the combing of a strand can be regarded as a loop in the complement space of the strands with smaller index (including the m fixed ones) and as such it is an element of a free group. Therefore we have:

$P_{m,1} = F_m = \langle a_{1,m+1}, a_{2,m+1}, \ldots, a_{m,m+1} \rangle$, the free group on m generators.

Further is: $P_{m,2} = F_{m+1} \rtimes P_{m,1} = \langle a_{1,m+2}, \ldots, a_{m,m+2}, a_{m+1,m+2} \rangle \rtimes P_{m,1}$, i.e. F_{m+1} is the free group on $m+1$ generators, and $P_{m,1}$ acts on F_{m+1} by conjugation, via the relations of the pure braid group P_{m+1} for appropriate indices. We proceed inductively to obtain:

$$\begin{aligned} P_{m,n} &= F_{m+n-1} \rtimes \cdots \rtimes F_{m+1} \rtimes F_m = F_{m+n-1} \rtimes P_{m,n-1} \\ &= \langle a_{1,m+n}, \ldots, a_{m,m+n}, \ldots, a_{m+n-1,m+n} \rangle \rtimes P_{m,n-1}, \end{aligned}$$

where F_{m+n-1} is the free group on $m+n-1$ generators, and where $P_{m,n-1}$ acts on F_{m+n-1} by conjugation, via the relations of the pure braid group P_{m+n} for appropriate indices, i.e. via the relations P_1, P_2, P_3 and P_4. $\square$

Some remarks are now due.

Remark 1 The groups $P_{m,n}$ and P_{m+n} have seemingly the same presentation. For $m \neq 1$ is, though, $P_{m,n} \neq P_{m+n}$. The difference lies in the restriction of the indices of the generators. In fact, $P_{m,n}$ has $\frac{n(n+2m-1)}{2}$ generators, which is the number of generators of P_{m+n} less the number of generators of P_m. Moreover, $P_{m,n}$ has $\frac{(m-1)\cdot m^2 + m\cdot(m+1)^2 + \cdots + (m+n-2)(m+n-1)^2}{2} - \frac{(m-1)\cdot m \cdot n \cdot (n+2m-1)}{4}$ relations, which is the number of relations of P_{m+n} less the number of relations of P_m. In the case $m = 1$ holds $P_{1,n} = P_{1+n}$, which follows immediately from the definition of $P_{m,n}$ or can be observed from its presentation for $m = 1$.

Remark 2 In [4] there is a discussion about the groups $B_{m,n}$ and a different line of proof is given for finding a presentation. There, by $P_{m,n}$ we denoted some smaller pure braid subgroups, for which it is rather complicated to find a presentation. But the case $m = 1$ was extensively treated, also in the sequel papers [5, 6]. In all these previous results $P_{1,n}$ denoted the free group $F_n = \langle a_{12}, a_{13}, \ldots, a_{1,n+1} \rangle$ and *not* the corresponding pure braid group of $B_{1,n}$. That's why we had then $B_{1,n} = F_n \rtimes B_n$. We hope that the readers familiar with those results will not be in confusion.

4 A presentation for $B_{m,n}$

In section 2 we showed that $1 \longrightarrow P_{m,n} \longrightarrow B_{m,n} \longrightarrow S_n \longrightarrow 1$ and in section 3 we found a presentation for $P_{m,n}$. Recall that S_n has the presentation:

$$\langle s_1, ..., s_{n-1} \mid s_i s_j = s_j s_i \text{ for } |i-j| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad {s_i}^2 = 1 \rangle.$$

We are now ready to put together a presentation for $B_{m,n}$. Namely, we can apply a result from the theory of group presentations (see [3], p.139), that gives a presentation for a group extension of two groups with known presentations. Indeed, the following is then a presentation for $B_{m,n}$.

$$\left\langle \begin{array}{l} a_{1,m+1}, \ldots, a_{1,m+n}, \ldots, a_{m,m+1}, \ldots, a_{m,m+n}, \\ a_{m+1,m+2}, \ldots, a_{m+1,m+n}, \ldots, a_{m+n-1,m+n}, \\ \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_{m+n-1} \end{array} \middle| \begin{array}{l} P_1, P_2, P_3, P_4, \\ M_1, M_2, M_3, M_4, M_5, \\ \Sigma_1, \Sigma_2, \Sigma_3. \end{array} \right\rangle,$$

where the relations Σ_1, Σ_2 and Σ_3 are satisfied by the $\sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_{m+n-1}$ and they are the following:

$$\begin{array}{lrcll} (\Sigma_1) & \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{if} \quad |i-j| > 1, \\ (\Sigma_2) & \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{if} \quad m+1 \leq i \leq m+n-2, \\ (\Sigma_3) & {\sigma_i}^2 & = & a_{i,i+1} & \text{if} \quad m+1 \leq i \leq m+n-2. \end{array}$$

Σ_1 and Σ_2 are the *'braid relations'*.

Notice now that relations Σ_3 for $i \in \{m+1, \ldots, m+n-1\}$ and $j \in \{m+2, \ldots, m+n\}$ do not involve any mixed braiding and so they may be taken as defining relations, namely:

$$\begin{array}{lcl} {a_{m+1,m+2}}^{\pm} & := & {\sigma_{m+1}}^{\pm 2}, \\ {a_{m+1,m+3}}^{\pm} & := & \sigma_{m+2} {\sigma_{m+1}}^{\pm 2} {\sigma_{m+2}}^{-1}, \ldots, \\ {a_{ij}}^{\pm} & := & \sigma_{j-1} \ldots \sigma_{i+1} {\sigma_i}^{\pm 2} {\sigma_{i+1}}^{-1} \ldots {\sigma_{j-1}}^{-1}, \\ & \vdots & \\ {a_{m+n-1,m+n}}^{\pm} & := & {\sigma_{m+n-1}}^{\pm 2}. \end{array}$$

Therefore, we want to omit eventually these a_{ij}'s from the list of generators of $B_{m,n}$ and subsequently to eliminate or simplify all relations involving these elements, applying Tietze transformations. Indeed, we examine one by one the relations and we have:

$\boldsymbol{P_1}$ **:** ${a_{ij}}^{-1} a_{rs} a_{ij} = a_{rs}$ for the case $r < i < j < s$. If all $r, i, j, s \in \{m+1, \ldots, m+n\}$, the relations follow from Σ_1 and Σ_2 and so we only keep the ones where $r \in \{1, \ldots, m\}$ and $i \in \{m+1, \ldots, m+n-1\}$ or $r, i \in \{1, \ldots, m\}$.

$\boldsymbol{P_2}$: ${a_{ij}}^{-1}a_{js}a_{ij} = a_{is}a_{js}{a_{is}}^{-1}$ for $i < j < s$. Since $j, s \in \{m+1, \ldots, m+n\}$ the only case to be kept is when $i \in \{1, \ldots, m\}$, as the relations for $i \in \{m+1, \ldots, m+n-1\}$ follow from the braid relations.

$\boldsymbol{P_3}$: ${a_{ij}}^{-1}a_{is}a_{ij} = a_{is}a_{js}a_{is}{a_{js}}^{-1}{a_{is}}^{-1}$ for $i < j < s$. As in the previous case, the only relations that do not follow from Σ_1 and Σ_2 are the ones where $i \in \{1, \ldots, m\}$.

$\boldsymbol{P_4}$: ${a_{ij}}^{-1}a_{rs}a_{ij} = a_{is}a_{js}{a_{is}}^{-1}{a_{js}}^{-1}a_{rs}a_{js}a_{is}{a_{js}}^{-1}{a_{is}}^{-1}$ for $i < r < j < s$. Since $j, s \in \{m+1, \ldots, m+n\}$ the only relations to be kept are those where either $i \in \{1, \ldots, m\}$ and $r \in \{m+1, \ldots, m+n-1\}$ or $i, r \in \{1, \ldots, m\}$.

$\boldsymbol{M_1}$: ${\sigma_k}^{-1}{a_{ij}}^{\pm}\sigma_k = {a_{ij}}^{\pm}$ for $k \leq i-2$ or $i+1 \leq k \leq j-2$ or $k \geq j+1$. Here also we have $j \in \{m+1, \ldots, m+n\}$ and $k \in \{m+1, \ldots, m+n-1\}$. Now, if $i \in \{m+1, \ldots, m+n-1\}$, all these relations follow from Σ_1 and Σ_2, whilst for $i \in \{1, \ldots, m\}$ it only makes sense to consider $k \leq j-2$ or $k \geq j+1$.

$\boldsymbol{M_2}$: ${\sigma_{i-1}}^{-1}{a_{ij}}^{\pm}\sigma_{i-1} = {a_{i-1,j}}^{\pm}$. Since $i-1 \in \{m+1, \ldots, m+n-1\}$ it must be $i > m+1$ and so all these relations follow from Σ_1 and Σ_2.

$\boldsymbol{M_3}$: ${\sigma_i}^{-1}{a_{ij}}^{\pm}\sigma_i = a_{ij}{a_{i+1,j}}^{\pm}{a_{ij}}^{-1}$. This is analogous to the above case, since $i \geq m+1$.

$\boldsymbol{M_4}$: ${\sigma_{j-1}}^{-1}{a_{ij}}^{\pm}\sigma_{j-1} = {a_{i,j-1}}^{\pm}$. Here also the only cases that do not follow from the braid relations are the ones with $i \in \{1, \ldots, m\}$.

$\boldsymbol{M_5}$: ${\sigma_j}^{-1}{a_{ij}}^{\pm}\sigma_j = a_{ij}{a_{i,j+1}}^{\pm}{a_{ij}}^{-1}$. As above, the only relations surviving are the ones where $i \in \{1, \ldots, m\}$.

Remark 3 The remaining relations of P_1 for $i \in \{m+1, \ldots, m+n-1\}$ follow from the simpler relations: $a_{ij}\sigma_k = \sigma_k a_{ij}$ for $k \leq j-2$ or $k \geq j+1$, which coincides with the remaining of M_1 above.

To summarize, we showed that the following is a presentation for $B_{m,n}$.

$$B_{m,n} = \left\langle \begin{array}{l} a_{1,m+1}, \ldots, a_{1,m+n}, \ldots, a_{m,m+1}, \ldots, a_{m,m+n}, \\ a_{m+1,m+2}, \ldots, a_{m+1,m+n}, \ldots, a_{m+n-1,m+n}, \\ \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_{m+n-1} \end{array} \left| \begin{array}{l} P_1', P_2', P_3', P_4', \\ M_1', M_2', M_3', \\ \Sigma_1, \Sigma_2 \end{array} \right. \right\rangle,$$

where we have:

$$
\begin{array}{llcll}
(P_1') & a_{ij}a_{rs} & = & a_{rs}a_{ij} & \text{for } \ r<i<j<s, \ 1 \leq i, r \leq m, \\
(P_2') & {a_{ij}}^{-1}a_{js}a_{ij} & = & a_{is}a_{js}{a_{is}}^{-1} & \text{for } \ i<j<s, \ 1 \leq i \leq m, \\
(P_3') & {a_{ij}}^{-1}a_{is}a_{ij} & = & a_{is}a_{js}a_{is}{a_{js}}^{-1}{a_{is}}^{-1} & \text{for } \ i<j<s, \ 1 \leq i \leq m, \\
(P_4') & {a_{ij}}^{-1}a_{rs}a_{ij} & = & a_{is}a_{js}{a_{is}}^{-1}{a_{js}}^{-1}a_{rs}a_{js}a_{is}{a_{js}}^{-1}{a_{is}}^{-1} & \text{for } \ i<r<j<s, \\
 & & & 1 \leq i \leq m, \ 1 \leq r \leq m+n-1, & \\
(M_1') & {\sigma_k}^{-1}{a_{ij}}^{\pm}\sigma_k & = & {a_{ij}}^{\pm} \quad \text{for } \ k \leq j-2 \ \text{ or } \ k \geq j+1 \ \text{ and } \ 1 \leq i \leq m, & \\
(M_2') & {a_{ij}}^{\pm} & = & \sigma_{j-1}{a_{i,j-1}}^{\pm}{\sigma_{j-1}}^{-1} \quad \text{for } \ 1 \leq i \leq m, & \\
(M_3') & {\sigma_j}^{-1}{a_{ij}}^{\pm}\sigma_j & = & a_{ij}{a_{i,j+1}}^{\pm}{a_{ij}}^{-1} \quad \text{for } \ 1 \leq i \leq m. &
\end{array}
$$

Having now done the first, 'obvious' clearing in the original presentation of $B_{m,n}$, we observe that many of the above relations are redundant or they simplify further, and that we may omit the a_{ij}'s with $i \geq m+1$.

Theorem 2 *The following is a presentation for $B_{m,n}$:*

$$
B_{m,n} = \left\langle \begin{array}{l} a_{1,m+1}, \ldots, a_{1,m+n}, \cdots, \\ a_{m,m+1}, \ldots, a_{m,m+n}, \\ \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_{m+n-1} \end{array} \ \middle| \ \begin{array}{l} \Sigma_1, \Sigma_2, (1), (2), (3), (4), \\ \textit{for all appropriate indices} \end{array} \right\rangle,
$$

where we have:

$$
\begin{array}{llcll}
(1) & {\sigma_k}^{-1}{a_{ij}}^{\pm}\sigma_k & = & {a_{ij}}^{\pm} & \textit{for } \ k \leq j-2 \ \textit{ or } \ k \geq j+1, \\
(2) & {a_{ij}}^{\pm} & = & \sigma_{j-1}{a_{i,j-1}}^{\pm}{\sigma_{j-1}}^{-1}, & \\
(3) & {\sigma_j}^{-1}{a_{ij}}^{\pm}\sigma_j & = & a_{ij}{a_{i,j+1}}^{\pm}{a_{ij}}^{-1} & \\
(4) & {a_{ij}}^{\pm}{a_{r,j+1}}^{\pm} & = & {a_{r,j+1}}^{\pm}{a_{ij}}^{\pm} & \textit{for } \ r<i.
\end{array}
$$

Proof Relations $(1), (2)$ and (3) are precisely M_1', M_2' and M_3', whilst relations (4) are a special case of relations P_1'. So we have to show that P_2', P_3', P_4' and P_5' as well as the rest cases of P_1' follow from $\Sigma_1, \Sigma_2, (1), (2), (3)$ and (4). Before continuing we note that, using (2), relation (3) is equivalent to

$$\sigma_j a_{ij} \sigma_j {a_{ij}}^{\pm} = {a_{ij}}^{\pm} \sigma_j a_{ij} \sigma_j$$

and relation (4) is equivalent to

$${a_{ij}}^{\pm}(\sigma_j {a_{rj}}^{\pm} {\sigma_j}^{-1}) = (\sigma_j {a_{rj}}^{\pm} {\sigma_j}^{-1}){a_{ij}}^{\pm}.$$

We shall also use these forms in the proof. We proceed now case by case. The underlining indicates the expressions involved in each step of the proof.

$$
\begin{aligned}
a_{ij}\underline{a_{rs}} &\overset{M_2'}{=} \underline{a_{ij}\sigma_{s-1}\dots\sigma_{j+1}a_{r,j+1}}{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1}\\
&\overset{M_1'}{=} \sigma_{s-1}\dots\sigma_{j+1}\underline{a_{ij}(\sigma_j a_{rj}{\sigma_j}^{-1})}{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1}\\
&\overset{(4)}{=} \sigma_{s-1}\dots\sigma_{j+1}(\sigma_j a_{rj}{\sigma_j}^{-1})a_{ij}\underline{{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1}}\\
&\overset{M_1'}{=} \underline{\sigma_{s-1}\dots\sigma_j a_{rj}{\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1}}a_{ij}\\
&\overset{M_2'}{=} a_{rs}a_{ij}.
\end{aligned}
$$

$$
\begin{aligned}
{a_{ij}}^{-1}\ a_{js}\ a_{ij} &\overset{M_2'}{=} {a_{ij}}^{-1}(\underline{\sigma_{s-1}\dots\sigma_{j+1}}{\sigma_j}^2\underline{{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1}})a_{ij}\\
&\overset{M_1'}{=} (\sigma_{s-1}\dots\sigma_{j+1})\underline{{a_{ij}}^{-1}{\sigma_j}^2}a_{ij}({\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\\
&\overset{M_3'}{=} (\sigma_{s-1}\dots\sigma_{j+1})\sigma_j a_{ij}{a_{i,j+1}}^{-1}\underline{{a_{ij}}^{-1}\sigma_j}a_{ij}({\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\\
&\overset{M_3'}{=} (\sigma_{s-1}\dots\sigma_{j+1})\sigma_j a_{ij}\underline{{a_{i,j+1}}^{-1}\sigma_j}a_{ij}{a_{i,j+1}}^{-1}{a_{ij}}^{-1}a_{ij}({\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\\
&\overset{M_2'}{=} (\sigma_{s-1}\dots\sigma_{j+1})\sigma_j a_{ij}{\sigma_j}^2{a_{ij}}^{-1}{\sigma_j}^{-1}({\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\\
&= \underline{(\sigma_{s-1}\dots\sigma_j)a_{ij}({\sigma_j}^{-1}\dots\sigma_{s-1}}^{-1}\underline{\sigma_{s-1}\dots\sigma_j)\sigma_j^{2}({\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1}}\\
&\quad\times\underline{\sigma_{s-1}\dots\sigma_j)a_{ij}^{-1}({\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1})}\\
&\overset{M_2'}{=} a_{is}a_{js}{a_{is}}^{-1}.
\end{aligned}
$$

For P_3' we have:

$$
\begin{aligned}
a_{is}\ \underline{a_{js}}\ a_{is}\ \underline{{a_{js}}^{-1}}\ {a_{is}}^{-1} &\overset{M_2'}{=} \underline{a_{is}}(\sigma_{s-1}\dots\sigma_{j+1}{\sigma_j}^2{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\underline{a_{is}}\\
&\quad\times(\sigma_{s-1}\dots\sigma_{j+1}{\sigma_j}^{-2}{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\underline{{a_{is}}^{-1}}\\
&\overset{M_2'}{=} (\sigma_{s-1}\dots\sigma_j a_{ij}{\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1})(\sigma_{s-1}\dots\sigma_{j+1}{\sigma_j}^2{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\\
&\quad\times(\sigma_{s-1}\dots\sigma_j a_{ij}{\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1})(\sigma_{s-1}\dots\sigma_{j+1}{\sigma_j}^{-2}{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1})\\
&\quad\times(\sigma_{s-1}\dots\sigma_j {a_{ij}}^{-1}{\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1})\\
&= (\sigma_{s-1}\dots\sigma_j)a_{ij}{\sigma_j}^2\underline{a_{ij}{\sigma_j}^{-2}{a_{ij}}^{-1}}({\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1}).
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
{a_{ij}}^{-1}\ \underline{a_{is}}\ a_{ij} &\overset{M_2'}{=} \underline{{a_{ij}}^{-1}\sigma_{s-1}\dots\sigma_{j+1}}\sigma_j a_{ij}{\sigma_j}^{-1}\underline{{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1}a_{ij}}\\
&\overset{M_1'}{=} \sigma_{s-1}\dots\sigma_{j+1}\underline{{a_{ij}}^{-1}\sigma_j a_{ij}{\sigma_j}^{-1}}a_{ij}{\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1}\\
&\overset{M_3'}{=} \sigma_{s-1}\dots\sigma_{j+1}\sigma_j a_{ij}\sigma_j(\sigma_j{\sigma_j}^{-1}){a_{ij}}^{-1}{\sigma_j}^{-2}a_{ij}(\sigma_j{\sigma_j}^{-1}){\sigma_{j+1}}^{-1}\dots{\sigma_{s-1}}^{-1}\\
&= (\sigma_{s-1}\dots\sigma_j)a_{ij}{\sigma_j}^2\underline{{\sigma_j}^{-1}{a_{ij}}^{-1}{\sigma_j}^{-2}a_{ij}\sigma_j}({\sigma_j}^{-1}\dots{\sigma_{s-1}}^{-1}).
\end{aligned}
$$

Therefore, it suffices to show that the underlined expressions in the last equation of either side are equal. Indeed we have:

$$\begin{aligned}
a_{ij}\quad \sigma_j^{-1}\quad & \underline{\sigma_j^{-1}a_{ij}^{-1}(\sigma_j^{-1}a_{ij}^{-1}}\sigma_j^{2}a_{ij}\sigma_j) \\
\stackrel{M_3'}{=}\quad & a_{ij}\underline{\sigma_j^{-1}a_{ij}^{-1}\sigma_j^{-1}a_{ij}^{-1}}\sigma_j^{-1}\sigma_j^{2}a_{ij}\sigma_j \\
\stackrel{M_3'}{=}\quad & a_{ij}a_{ij}^{-1}\sigma_j^{-1}a_{ij}^{-1}\sigma_j^{-1}\sigma_j a_{ij}\sigma_j \\
=\quad & 1.
\end{aligned}$$

Finally, for P_4' and $r \in \{1,\dots,m\}$ we have:

$$\begin{aligned}
a_{is}\quad a_{js}\quad & a_{is}^{-1}\ a_{js}^{-1}\ a_{rs}\ a_{js}\ a_{is}\ a_{js}^{-1}\ a_{is}^{-1} \\
\stackrel{M_2'}{=}\quad & (\sigma_{s-1}\dots\sigma_j a_{ij}\sigma_j^{-1}\dots\sigma_{s-1}^{-1})\cdot(\sigma_{s-1}\dots\sigma_{j+1}\sigma_j^{2}\sigma_{j+1}^{-1}\dots\sigma_{s-1}^{-1}) \\
& \times(\sigma_{s-1}\dots\sigma_j a_{ij}^{-1}\sigma_j^{-1}\dots\sigma_{s-1}^{-1})\cdot(\sigma_{s-1}\dots\sigma_{j+1}\sigma_j^{-2}\sigma_{j+1}^{-1}\dots\sigma_{s-1}^{-1}) \\
& \times(\sigma_{s-1}\dots\sigma_j a_{rj}\sigma_j^{-1}\dots\sigma_{s-1}^{-1})\cdot(\sigma_{s-1}\dots\sigma_{j+1}\sigma_j^{2}\sigma_{j+1}^{-1}\dots\sigma_{s-1}^{-1}) \\
& \times(\sigma_{s-1}\dots\sigma_j a_{ij}\sigma_j^{-1}\dots\sigma_{s-1}^{-1})\cdot(\sigma_{s-1}\dots\sigma_{j+1}\sigma_j^{-2}\sigma_{j+1}^{-1}\dots\sigma_{s-1}^{-1}) \\
& \times(\sigma_{s-1}\dots\sigma_j a_{ij}^{-1}\sigma_j^{-1}\dots\sigma_{s-1}^{-1}) \\
=\quad & \sigma_{s-1}\dots\sigma_{j+1}\underline{\sigma_j a_{ij}\sigma_j^{2}a_{ij}^{-1}\sigma_j^{-2}a_{rj}\sigma_j^{2}a_{ij}\sigma_j^{-2}a_{ij}^{-1}\sigma_j^{-1}}\sigma_{j+1}^{-1}\dots\sigma_{s-1}^{-1}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
a_{ij}^{-1}\quad \underline{a_{rs}}\quad a_{ij}\ \stackrel{M_2'}{=}\ & \underline{a_{ij}^{-1}\sigma_{s-1}\dots\sigma_{j+1}}\sigma_j a_{rj}\sigma_j^{-1}\underline{\sigma_{j+1}^{-1}\dots\sigma_{s-1}^{-1}a_{ij}} \\
\stackrel{M_1'}{=}\quad & \sigma_{s-1}\dots\sigma_{j+1}\underline{a_{ij}^{-1}\sigma_j a_{rj}\sigma_j^{-1}a_{ij}}\sigma_{j+1}^{-1}\dots\sigma_{s-1}^{-1}.
\end{aligned}$$

Again, it suffices to show that the underlined expressions in the last equation of either side are equal. Indeed we have:

$$\begin{aligned}
\sigma_j\quad a_{ij}\quad & \sigma_j^{2}\ a_{ij}^{-1}\ \sigma_j^{-2}\ a_{rj}\ \sigma_j^{2}\ a_{ij}\ \underline{\sigma_j^{-2}\ a_{ij}^{-1}\ \sigma_j^{-1}\ (a_{ij}^{-1}}\ \sigma_j\ a_{rj}^{-1}\ \sigma_j^{-1}\ a_{ij}) \\
\stackrel{M_3'}{=}\quad & \sigma_j a_{ij}\sigma_j^{2}a_{ij}^{-1}\sigma_j^{-2}a_{rj}\sigma_j^{2}a_{ij}a_{ij}^{-1}\sigma_j^{-1}a_{ij}^{-1}\sigma_j^{-2}\sigma_j a_{rj}^{-1}\sigma_j^{-1}a_{ij} \\
=\quad & \sigma_j a_{ij}\sigma_j^{2}a_{ij}^{-1}\sigma_j^{-2}a_{rj}\underline{\sigma_j a_{ij}^{-1}\sigma_j^{-1}a_{rj}^{-1}}\sigma_j^{-1}a_{ij} \\
\stackrel{(4)}{=}\quad & \sigma_j a_{ij}\sigma_j^{2}\underline{a_{ij}^{-1}\sigma_j^{-1}a_{ij}^{-1}\sigma_j^{-2}}a_{ij} \\
\stackrel{M_3'}{=}\quad & \sigma_j a_{ij}\sigma_j^{2}\sigma_j^{-2}a_{ij}^{-1}\sigma_j^{-1}a_{ij}^{-1}a_{ij} \\
=\quad & 1.
\end{aligned}$$

The case where $r \in \{m+1,\dots,m+n-1\}$ is completely analogous. $\square$

5 Irredundant presentation for $B_{m,n}$

Looking now at the last presentation for $B_{m,n}$ we observe that relations (2) may be seen as defining relations for $1 \leq i \leq m$ and $j \geq m+2$, namely:

$$\begin{array}{lcl} a^{\pm}_{i,m+2} & := & \sigma_{m+1} a^{\pm}_{i,m+1} \sigma^{-1}_{m+1}, \\ a^{\pm}_{i,m+3} & := & \sigma_{m+2}\sigma_{m+1} a^{\pm}_{i,m+1} \sigma^{-1}_{m+1}\sigma^{-1}_{m+2}, \\ & \vdots & \\ a^{\pm}_{i,m+n} & := & \sigma_{m+n-1} \ldots \sigma_{m+1} a^{\pm}_{i,m+1} \sigma^{-1}_{m+1} \ldots \sigma^{-1}_{m+n-1}. \end{array}$$

Therefore, we want to omit further these a_{ij}'s from the list of generators, and subsequently to eliminate or simplify all relations involving these elements. Indeed, we have:

Theorem 3 *The following is a presentation for $B_{m,n}$:*

$$B_{m,n} = \left\langle \begin{array}{l} a_{1,m+1}, a_{2,m+1}, \ldots, a_{m,m+1}, \\ \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_{m+n-1} \end{array} \middle| \begin{array}{l} \Sigma_1, \Sigma_2, (1'), (2'), (3'), \\ \textit{for all appropriate indices} \end{array} \right\rangle,$$

where:

$$\begin{array}{lrcl} (1') & {\sigma_k}^{-1} {a_{i,m+1}}^{\pm} \sigma_k & = & {a_{i,m+1}}^{\pm}, \quad k \geq m+2, \\ (2') & {a_{i,m+1}}^{\pm} \sigma_{m+1} a_{i,m+1} \sigma_{m+1} & = & \sigma_{m+1} a_{i,m+1} \sigma_{m+1} {a_{i,m+1}}^{\pm}, \\ (3') & {a_{i,m+1}}^{\pm} (\sigma_{m+1} {a_{r,m+1}}^{\pm} {\sigma_{m+1}}^{-1}) & = & (\sigma_{m+1} {a_{r,m+1}}^{\pm} {\sigma_{m+1}}^{-1}) {a_{i,m+1}}^{\pm}, \quad r < i. \end{array}$$

Proof Relations $(1')$ are a special case of relations (1), relations $(2')$ are a special case of relations (3) and relations $(3')$ are a special case of relations (4). Note that relations $(3')$ are equivalent to

$${a_{i,m+1}}^{\pm} {a_{r,m+2}}^{\pm} = {a_{r,m+2}}^{\pm} {a_{i,m+1}}^{\pm}, \quad r < i.$$

We show the sufficiency of the new relations by examining each case.

For relations (1) and for $m+1 \leq k \leq j-2$ we have:

$$\begin{array}{rcl} {\sigma_k}^{-1} \underline{{a_{ij}}^{\pm}} \sigma_k & \overset{(2)}{=} & \underline{{\sigma_k}^{-1} (\sigma_{j-1} \ldots \sigma_{k+2}} \sigma_{k+1} \ldots \sigma_{m+1} {a_{i,m+1}}^{\pm} \\ & & \times {\sigma_{m+1}}^{-1} \ldots {\sigma_{k+1}}^{-1} \underline{{\sigma_{k+2}}^{-1} \ldots {\sigma_{j-1}}^{-1}) \sigma_k} \\ & \overset{\Sigma_1}{=} & (\sigma_{j-1} \ldots \sigma_{k+2}) \underline{{\sigma_k}^{-1} \sigma_{k+1} \sigma_k} \ldots \sigma_{m+1} {a_{i,m+1}}^{\pm} \\ & & \times {\sigma_{m+1}}^{-1} \ldots \underline{{\sigma_k}^{-1} {\sigma_{k+1}}^{-1} \sigma_k} ({\sigma_{k+2}}^{-1} \ldots {\sigma_{j-1}}^{-1}) \\ & \overset{\Sigma_2}{=} & (\sigma_{j-1} \ldots \sigma_{k+2}) \sigma_{k+1} \sigma_k \underline{{\sigma_{k+1}}^{-1} \sigma_{k-1} \ldots \sigma_{m+1}} a_{i,m+1}{}^{\pm} \\ & & \times \underline{{\sigma_{m+1}}^{-1} \ldots {\sigma_{k-1}}^{-1} \sigma_{k+1}} {\sigma_k}^{-1} {\sigma_{k+1}}^{-1} ({\sigma_{k+2}}^{-1} \ldots {\sigma_{j-1}}^{-1}) \\ & \overset{\Sigma_1}{=} & (\sigma_{j-1} \ldots \sigma_{m+1}) \underline{{\sigma_{k+1}}^{-1} {a_{i,m+1}}^{\pm} \sigma_{k+1}} ({\sigma_{m+1}}^{-1} \ldots {\sigma_{j-1}}^{-1}) \\ & \overset{(1')}{=} & \sigma_{j-1} \ldots \sigma_{m+1} {a_{i,m+1}}^{\pm} {\sigma_{m+1}}^{-1} \ldots {\sigma_{j-1}}^{-1}, \ (k+1 \geq m+2) \\ & \overset{(2)}{=} & {a_{ij}}^{\pm}. \end{array}$$

Further, for relations (1) and for $k \geq j+1$ we have:

$$\begin{aligned}
{\sigma_k}^{-1}\underline{a_{ij}}^{\pm}\sigma_k &\stackrel{(2)}{=} \underline{{\sigma_k}^{-1}(\sigma_{j-1}\ldots\sigma_{m+1}}{a_{i,m+1}}^{\pm}\underline{{\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1})\sigma_k} \\
&\stackrel{\Sigma_1}{=} (\sigma_{j-1}\ldots\sigma_{m+1})\underline{{\sigma_k}^{-1}{a_{i,m+1}}^{\pm}\sigma_k}({\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1}) \\
&\stackrel{(1')}{=} \sigma_{j-1}\ldots\sigma_{m+1}{a_{i,m+1}}^{\pm}{\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1}) \\
&\stackrel{(2)}{=} {a_{ij}}^{\pm}.
\end{aligned}$$

In the last two equations we used that $k \geq m+2$. For relations (3) we have:

$$\begin{aligned}
\sigma_j\underline{a_{ij}}\sigma_j\underline{{a_{ij}}^{\pm}} &\stackrel{(2)}{=} \sigma_j(\sigma_{j-1}\ldots\sigma_{m+1}a_{i,m+1}\underline{{\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1})\sigma_j} \\
&\quad \times \underline{(\sigma_{j-1}\ldots\sigma_{m+1}}{a_{i,m+1}}^{\pm}{\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1}) \\
&\stackrel{\Sigma_2}{=} \sigma_j\ldots\sigma_{m+1}a_{i,m+1}\underline{\sigma_j}\ldots\underline{\sigma_{m+2}}\sigma_{m+1}\underline{{\sigma_{m+2}}^{-1}}\ldots\underline{{\sigma_j}^{-1}} \\
&\quad \times {a_{i,m+1}}^{\pm}{\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1} \\
&\stackrel{\Sigma_1,(1')}{=} \underline{\sigma_j(\sigma_{j-1}\sigma_j)\ldots(\sigma_{m+1}\sigma_{m+2})}a_{i,m+1}\sigma_{m+1}{a_{i,m+1}}^{\pm} \\
&\quad \times ({\sigma_{m+2}}^{-1}{\sigma_{m+1}}^{-1})\ldots({\sigma_j}^{-1}{\sigma_{j-1}}^{-1}) \\
&\stackrel{\Sigma_2}{=} (\sigma_{j-1}\sigma_j)\ldots(\sigma_{m+1}\sigma_{m+2})\underline{\sigma_{m+1}a_{i,m+1}\sigma_{m+1}{a_{i,m+1}}^{\pm}} \\
&\quad \times ({\sigma_{m+2}}^{-1}{\sigma_{m+1}}^{-1})\ldots({\sigma_j}^{-1}{\sigma_{j-1}}^{-1}) \\
&\stackrel{(2')}{=} (\sigma_{j-1}\sigma_j)\ldots(\sigma_{m+1}\sigma_{m+2}){a_{i,m+1}}^{\pm}\sigma_{m+1}a_{i,m+1} \\
&\quad \times \underline{\sigma_{m+1}({\sigma_{m+2}}^{-1}{\sigma_{m+1}}^{-1})\ldots({\sigma_j}^{-1}{\sigma_{j-1}}^{-1})} \\
&\stackrel{\Sigma_2}{=} (\sigma_{j-1}\underline{\sigma_j})\ldots(\sigma_{m+1}\underline{\sigma_{m+2}}){a_{i,m+1}}^{\pm}\sigma_{m+1}a_{i,m+1} \\
&\quad \times (\underline{{\sigma_{m+2}}^{-1}}{\sigma_{m+1}}^{-1})\ldots(\underline{{\sigma_j}^{-1}}{\sigma_{j-1}}^{-1})\sigma_j
\end{aligned}$$

$$\begin{aligned}
&\stackrel{\Sigma_1,(1')}{=} (\sigma_{j-1}\ldots\sigma_{m+1}){a_{i,m+1}}^{\pm}\underline{\sigma_j\ldots\sigma_{m+2}\sigma_{m+1}{\sigma_{m+2}}^{-1}\ldots{\sigma_j}^{-1}} \\
&\quad \times a_{i,m+1}({\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1})\sigma_j \\
&\stackrel{\Sigma_2}{=} \underline{(\sigma_{j-1}\ldots\sigma_{m+1}){a_{i,m+1}}^{\pm}{\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1}}\sigma_j \\
&\quad \times \underline{\sigma_{j-1}\ldots\sigma_{m+1}a_{i,m+1}({\sigma_{m+1}}^{-1}\ldots{\sigma_{j-1}}^{-1})}\sigma_j \\
&\stackrel{(2)}{=} {a_{ij}}^{\pm}\sigma_j a_{ij}\sigma_j.
\end{aligned}$$

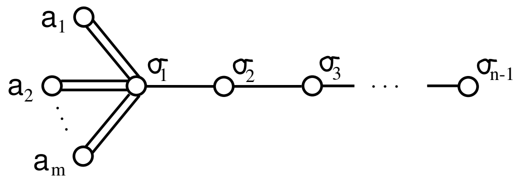
Finally, for relations (4) we have:

$$
\begin{aligned}
\underline{a_{ij}{}^{\pm}} \quad (\sigma_j \quad \underline{a_{rj}{}^{\pm}}\ \sigma_j{}^{-1}) &\overset{(2)}{=} (\sigma_{j-1}\dots\sigma_{m+1}a_{i,m+1}{}^{\pm}\underline{\sigma_{m+1}{}^{-1}\dots\sigma_{j-1}{}^{-1})\sigma_j} \\
&\quad \times \underline{(\sigma_{j-1}\dots\sigma_{m+1}}a_{r,m+1}{}^{\pm}\sigma_{m+1}{}^{-1}\dots\sigma_{j-1}{}^{-1})\sigma_j{}^{-1} \\
&\overset{\Sigma_2}{=} \sigma_{j-1}\dots\sigma_{m+1}a_{i,m+1}{}^{\pm}\underline{\sigma_j}\dots\underline{\sigma_{m+2}}\sigma_{m+1}\underline{\sigma_{m+2}{}^{-1}}\dots\underline{\sigma_j{}^{-1}}a_{r,m+1}{}^{\pm} \\
&\quad \times \sigma_{m+1}{}^{-1}\dots\sigma_j{}^{-1} \\
&\overset{\Sigma_1,(1')}{=} (\sigma_{j-1}\sigma_j)\dots(\sigma_{m+1}\sigma_{m+2})a_{i,m+1}{}^{\pm}\sigma_{m+1}a_{r,m+1}{}^{\pm} \\
&\quad \times \underline{(\sigma_{m+2}{}^{-1}\sigma_{m+1}{}^{-1})\dots(\sigma_j{}^{-1}\sigma_{j-1}{}^{-1})\sigma_j{}^{-1}} \\
&\overset{\Sigma_2}{=} (\sigma_{j-1}\sigma_j)\dots(\sigma_{m+1}\sigma_{m+2})\underline{a_{i,m+1}{}^{\pm}(\sigma_{m+1}a_{r,m+1}{}^{\pm}\sigma_{m+1}{}^{-1})} \\
&\quad \times (\sigma_{m+2}{}^{-1}\sigma_{m+1}{}^{-1})\dots(\sigma_j{}^{-1}\sigma_{j-1}{}^{-1}) \\
&\overset{(3')}{=} \underline{(\sigma_{j-1}\sigma_j)\dots(\sigma_{m+1}\sigma_{m+2})\sigma_{m+1}}a_{r,m+1}{}^{\pm}\sigma_{m+1}{}^{-1}a_{i,m+1}{}^{\pm} \\
&\quad \times (\sigma_{m+2}{}^{-1}\sigma_{m+1}{}^{-1})\dots(\sigma_j{}^{-1}\sigma_{j-1}{}^{-1}) \\
&\overset{\Sigma_2}{=} \sigma_j(\sigma_{j-1}\underline{\sigma_j})\dots(\sigma_{m+1}\underline{\sigma_{m+2}})a_{r,m+1}{}^{\pm}\sigma_{m+1}{}^{-1}a_{i,m+1}{}^{\pm} \\
&\quad \times (\underline{\sigma_{m+2}{}^{-1}}\sigma_{m+1}{}^{-1})\dots(\underline{\sigma_j{}^{-1}}\sigma_{j-1}{}^{-1}) \\
&\overset{\Sigma_1,(1')}{=} \sigma_j\dots\sigma_{m+1}a_{r,m+1}{}^{\pm}\underline{\sigma_j\dots\sigma_{m+2}\sigma_{m+1}{}^{-1}\sigma_{m+2}{}^{-1}\dots\sigma_j{}^{-1}} \\
&\quad \times a_{i,m+1}{}^{\pm}\sigma_{m+1}{}^{-1}\dots\sigma_{j-1}{}^{-1} \\
&\overset{\Sigma_2}{=} \sigma_j\underline{\sigma_{j-1}\dots\sigma_{m+1}a_{r,m+1}{}^{\pm}\sigma_{m+1}{}^{-1}\dots\sigma_{j-1}{}^{-1}}\sigma_j{}^{-1} \\
&\quad \times \underline{\sigma_{j-1}\dots\sigma_{m+1}a_{i,m+1}{}^{\pm}\sigma_{m+1}{}^{-1}\dots\sigma_{j-1}{}^{-1}} \\
&\overset{(2)}{=} \sigma_j(a_{rj}{}^{\pm}\sigma_j{}^{-1}a_{ij}{}^{\pm}). \qquad \square
\end{aligned}
$$

The system of generators in the last presentation of $B_{m,n}$ is irredundant, in the sense that no proper subset of it can generate $B_{m,n}$. In order now to simplify the notation we will relabel the generators $a_{1,m+1}, \dots, a_{m,m+1}, \sigma_{m+1}, \dots, \sigma_{m+n-1}$ to $a_1, \dots, a_m, \sigma_1, \dots, \sigma_{n-1}$ accordingly, to obtain the following, final presentation for $B_{m,n}$:

$$
B_{m,n} = \left\langle \begin{array}{l} a_1, \dots, a_m, \\ \sigma_1, \dots, \sigma_{n-1} \end{array} \left| \begin{array}{l} \sigma_k\sigma_j = \sigma_j\sigma_k, \quad |k-j| > 1 \\ \sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}, \quad 1 \leq k \leq n-1 \\ a_i\sigma_k = \sigma_k a_i, \quad k \geq 2,\ 1 \leq i \leq m, \\ a_i\sigma_1 a_i\sigma_1 = \sigma_1 a_i\sigma_1 a_i, \quad 1 \leq i \leq m \\ a_i(\sigma_1 a_r\sigma_1{}^{-1}) = (\sigma_1 a_r\sigma_1{}^{-1})a_i, \quad r < i. \end{array} \right. \right\rangle.
$$

Remark 4 It is worth mentioning that the above presentation of $B_{m,n}$ is very similar to that of the Artin braid group associated to the Dynkin diagram below.



In the diagram the single bonds mean relations of degree 3, the double bonds relations of degree 4, and if two generators are not connected by a bond they commute. The two presentations differ only in the last relation, which in the case of the Artin group (cf. [2]) is a mere commutation relation between a_i and a_r. Nevertheless, in the case $m = 1$, $B_{1,n}$ is the Artin group of type $\mathcal{B}$.

6 The cosets $C_{m,n}$

In this section the word *knot* will be used to mean *knots and links.*

Let now $S^3\backslash K$ be the complement of the oriented knot K in S^3. Obviously, $S^3\backslash K$ can be represented in S^3 by the knot K. By classical results of Lickorish and Wallace a closed, connected, orientable 3-manifold can be obtained (not uniquely) by surgery along an integer-framed knot in S^3, so it can be represented in S^3 by this knot. We shall denote by M either a knot complement or a c.c.o. 3-manifold. Then, by fixing M we may also fix a knot in S^3 representing M, and this knot may be assumed to be a closed braid, say $\widehat{B}$. It is shown in [7] that knots in these spaces can be represented by 'mixed braids', which contain the braid B as a fixed subbraid. More precisely, we have the following:

Definition 3 *A mixed braid is a special element of B_{m+n} consisting of two disjoint orbits of strands, such that the subbraid forming the one orbit consists of the first m strands and it is a fixed element of B_m.*

If now the manifold M is a handlebody of genus m, knots in M can clearly be represented by elements of $B_{m,n}$. Further, if M is the complement of the m-unlink or a connected sum of m lens spaces of type $L(p,1)$, then knots in M are represented by elements of $B_{m,n}$, for $n \in I\!N$. In the special case where M is the complement of the trivial knot or $L(p,1)$, knots in M are represented by elements of $B_{1,n}$, the Artin group of type $\mathcal{B}$ (cf. [4], [5], [6]). These are rather special cases of knot complements or c.c.o. 3-manifolds.

In the generic case the subbraid representing M will not be the identity braid (for an example see figure 3(a)). In the generic case the multiplication of two mixed braids in B_{m+n} related to M is not a closed operation, since concatenation does not preserve the structure of the manifold.

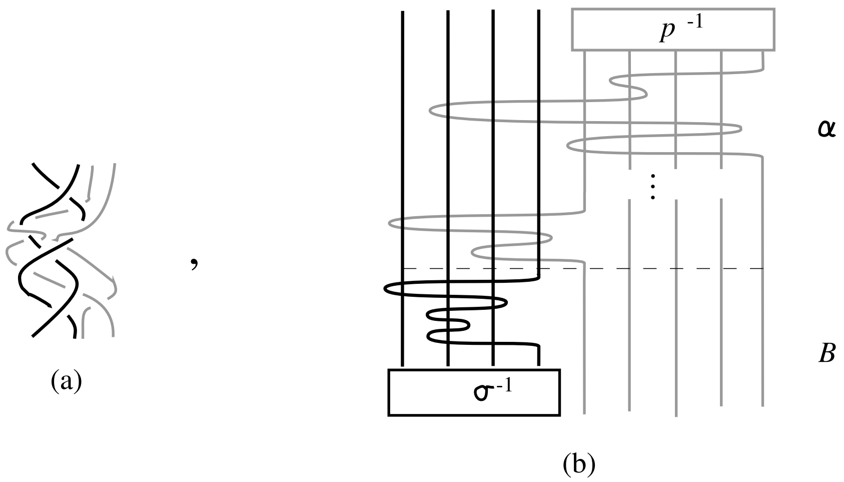


Figure 3

The following proposition shows that, nevertheless, we still have braid structures in M. Indeed, let $\bigcup_{n=1}^{\infty} C_{m,n}$ denote the disjoint union of the cosets of all mixed braids associated to a generic M. Then we have the folllowing:

Proposition 1 *For a fixed M, $C_{m,n}$ is a coset of $B_{m,n}$ in B_{m+n}.*

Proof Let $A \in C_{m,n}$. We shall show that A can be written as a product $\alpha \cdot B$, where $\alpha \in B_{m,n}$ and B is the fixed braid representing M in S^3. Indeed, we notice first that by symmetry, Artin's combing for pure braids can be also applied starting from the last strand of a pure braid. So, we multiply A from the top with a braid p on the last n strands and with a braid σ on the first m strands, such that $pA\sigma$ is a pure braid in B_{m+n}. Then we apply to it Artin's canonical form from the end. This will separate A into two parts, one being an element of $B_{m,n}$ and the other being the fixed braid B embedded in B_{m+n} (see figure 3(b)). $\square$

A final comment is now due.

Remark 5 The groups $B_{m,n}$ and their cosets $C_{m,n}$ will be used for yielding an algebraic version of Markov's theorem for isotopy of knots in knot complements and 3-manifolds. For the purpose of constructing knot invariants following the line of Jones-Ocneanu one can use the irredundant presentation of $B_{m,n}$ for considering appropriate quotient algebras that satisfy a quadratic skein relation for the σ_i's.

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