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Non-linear Science and the Laws of Nature

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ABSTRACT: *Simple examples such as the Bernouilli shift and the anharmonic lattice are studied. It is shown that instability as well as the thermodynamic limit lead to a new formulation of laws of nature in terms of probabilities (instead of trajectories or wave functions).* © 1997 The Franklin Institute. Published by Elsevier Science Ltd

I. Introduction

In recent years, a radical change of perspective has been witnessed in science following the realization that large classes of systems may exhibit abrupt transitions, a multiplicity of states, coherent structures or a seemingly erratic motion characterized by unpredictability often referred to as *deterministic chaos*. Classical science emphasized stability and equilibrium; now we see instabilities, fluctuations and evolutionary trends in a variety of areas ranging from atomic and molecular physics through fluid mechanics, chemistry and biology to large-scale systems of relevance in environmental and economic sciences. Concepts such as 'dissipative structures' and 'self-organisation' have become quite popular. Distance from equilibrium, and therefore the arrow of time, plays an essential role in these processes, somewhat like temperature in equilibrium physics. When we lower the temperature we have, in succession, various states of matter. In non-equilibrium physics and chemistry, when we change the distance from equilibrium the observed behavior is even more varied. How can these findings be interpreted from the point of view of the basic laws of physics? These questions are at the heart of our present description of nature.

The 19th century has left us with a conflicting heritage. On one side, there are the 'laws of nature' such as Newton's law which relates acceleration to force. This law is time reversible and deterministic. If we know the initial condition of a dynamical system, we can predict its state at an arbitrary time, be it in the future or in the past. There is no distinction between past and future. These characteristics remain true in relativity and quantum mechanics, as the Einstein or Schrödinger equations are also reversible and deterministic. On the other hand, the famous 'second law' of thermodynamics, associated with the increase of entropy, expresses the arrow of time.

The nature and reality of the flow of time has fascinated artists, philosophers and scientists over the years. The fact that the fundamental equations of dynamics are time reversible whereas a thermodynamic description is time oriented may be called the time paradox. It is interesting that the time paradox was only identified in the second half of the 19th century. It was then that the Viennese physicist Ludwig Boltzmann tried to emulate what Charles Darwin had achieved in biology and to formulate an evolutionary approach to physics. At that time, the laws of Newtonian physics had long been accepted as expressing the ideal of objective knowledge. As Newton's laws imply the equivalence between past and future, any attempt to confer a fundamental meaning onto the arrow of time was resisted as a threat to the ideal of objective knowledge. Newton's laws were considered to be final in their domain of application, somewhat as quantum mechanics is considered today by many physicists. How could unidirectional time associated with entropy be understood without destroying these amazing achievements of the human mind?

Here we come to the relation between non-linear science and the time paradox. It has been realized in recent years that irreversibility may become a source of order. This is already clear in classical experiments such as thermal diffusion. We heat one wall of a box containing two components and cool the other. The system evolves to a steady state in which one component is enriched in the hot part and the other in the cold part. We have an ordering process that would be impossible at equilibrium.

Close to equilibrium, entropy is maximum or free energy minimum. Fluctuations are harmless as they are followed by responses which bring the system back to equilibrium. The situation changes drastically, however, when we go far from equilibrium. Then there is no longer any minimum principle and fluctuations can grow. If the equations of evolution are non-linear we observe, in general, bifurcations which lead to new spatio-temporal structures. The 'dissipative structures' achieved in this way are therefore the consequences of non-linearity. In this sense we may even consider life, with its essential non-equilibrium properties, as the manifestation of non-linearity. We shall not go into details concerning these macroscopic aspects which are treated in many publications (**1, 2**). Let us only note that these results indicate that irreversibility (associated with the flow of time) has a fundamental constructive role, be it in chemistry or biology. For this reason, irreversible processes cannot correspond to approximations that we would introduce into the basic laws. On the contrary, what we need is an extension of both classical and quantum physics which includes irreversibility. Our main subject here will be the relation between non-linearity and time symmetry breaking on the fundamental dynamical level. For a general presentation see Refs. (**9, 10**).

II. Chaotic Maps

The basic idea is simple. We know, since the pioneering work of Gibbs and Einstein at the beginning of this century, that we can describe dynamics from two points of view. On the one hand we have the individual description in terms of trajectories in classical dynamics or of wave functions in quantum theory. On the other hand we have the description in terms of ensembles associated with a probability distribution (called the density matrix in quantum mechanics). An ensemble represents a superposition of trajectories or wave functions. For Gibbs and Einstein, the ensemble point of view was merely a convenient computational tool when exact initial conditions were not available. In their view, probabilities express ignorance or lack of information. Moreover, it has always been admitted that the consideration of individual trajectories and probability distributions were equivalent problems. We can start with individual trajectories and then derive the evolution of probability functions or vise-versa.

Is this always so? For stable systems where we do not expect any irreversibility, this is indeed true. Gibbs and Einstein were right. The individual point of view (in terms of trajectories) and the statistical point of view (in terms of probabilities) are then, indeed, equivalent, but for unstable dynamical systems this is no longer so. The equivalence between the individual level and the statistical level is then broken. We obtain new solutions for the probability distribution which are 'irreducible' as they do not apply to single trajectories. In this new formulation irreversibility is included because the symmetry between past and future is broken. Laws of nature acquire a new meaning. They no longer express certitudes but possibilities, as is appropriate for the evolutionary world that we observe (**3**).

The simplest way to illustrate how irreversibility emerges from unstable dynamics is to consider chaotic maps, which are discrete-time dynamical process. The simplest chaotic map is known as the Bernoulli map (**4, 5**). We have a variable x defined on the interval from 0 to 1. This interval is the 'phase space' of the system. The map is given by the rule that the value of x at some given time step is twice the value at the previous time step. In order to stay in the interval from 0 to 1 though, if the new value exceeds 1 only the fractional part is kept. The rule for the map is thus concisely written as $x_{n+1} = 2x_n \pmod 1$, where n represents time, which takes integer values.

This very simple system has the remarkable property that even though successive values of x are completely determined, they also have quite random properties. If x is written in binary notation then successive values are obtained simply by removing the first digit in the expansion and shifting over the remaining digits. This means that after m time steps, information about the initial value to an accuracy of 2^{-m} is now amplified to give whether the value of x is between 0 and 1/2 or 1/2 and 1. This amplification in any initial uncertainty of the value of x makes following trajectories for more than a few time steps a practical impossibility.

A generic initial value of x would be an irrational number with an infinite non-repeating expansion. This would lead to a trajectory that wanders forever throughout the phase space, but rational numbers, with repeating or terminating binary expansions, thus leading to periodic or fixed-point trajectories are densely distributed among irrational numbers. This means that qualitatively different behavior, in the sense of trajectory dynamics, arises from initial conditions that are infinitesimally close. This kind of complicated microstructure of phase space, typical for chaotic systems, is in contrast to systems with regular dynamics where initial conditions throughout large regions of phase spaces lead to similar behaviors.

These facts suggest that a much more natural way to consider the time evolution in chaotic systems is in terms of ensembles of trajectories defined by probability distributions. The probability distribution evolves through the application of an operator known as the Frobenius–Perron operator. In contrast to the erratic behavior of trajectories, the evolution of a 'smooth' probability distribution is regular and approaches an equilibrium state. By a smooth distribution we mean one that does not just represent a trajectory, which would be a distribution localized at a single point. Then we would return to the problems with trajectories. A point distribution is written in terms of a

'Dirac delta function'. This object is not a normal function but a so-called generalized function. Its value is zero except at a single point where its value is infinity. It has a well-defined meaning only when integrated with a normal function where it acts to sift out the value of the normal function at the point where the delta function is non-zero.

How to understand the difference between the individual behavior (the trajectories) and the statistical behavior associated with distribution functions? Here we come to one of the main chapters of modern mathematics associated with operators. These methods are essential in orthodox quantum theory. It is remarkable that they also play a fundamental role in overcoming the time paradox and permit the extension of the laws of nature to include irreversible processes.

Since the formulation of quantum mechanics in the 1920s, operator calculus has become a basic tool for physicists. How indeed to understand the appearance of discrete energy levels in atomic physics? In classical mechanics, the basic quantity determining the dynamics of a system is the Hamiltonian (that is the energy expressed in terms of coordinators and momenta). It is a continuous function. The basic step to derive quantization is to associate with each physical quantity an *operator*. An operator is nothing more than a prescription of how to act on a given function. It may involve multiplication, differentiation or any other mathematical operation. In general, an operator O acting on a function $f(x)$ transforms it into a different function. (For example, if O is the derivative operator $\mathrm{d}/\mathrm{d}x$, then $Ox^2 = 2x$.)

However, there are functions which remain invariant when we apply O; they are only multiplied by a number. These special functions are called *eigenfunctions* of the operator. (In our example above, e^{ikx} is an eigenfunction for all values of k.) The number which multiplies the eigenfunction is called the *eigenvalue* (in our example, it is ik). The basic idea of quantum mechanics is to associate the numerical values of observables to the eigenvalues of the corresponding operator. It is a daring step as it entails a conceptual difference between a physical quantity (represented by an operator) and the numerical values this quantity can take (the eigenvalues of this operator). The ensemble of the eigenvalues forms the spectrum and a fundamental theorem is that we can express an operator in terms of its eigenfunctions and eigenvalues. This is the 'spectral decomposition' of the operator, but there is an additional element which is important for us.

The spectral decomposition of an operator depends, not only on how the operator acts on a function but on the type of functions the operator is considered to act upon. In quantum mechanical problems, the operators are considered to act on 'nice' normalizable functions that are members of a collection of functions known as a Hilbert space. Time evolution operators, even in classical mechanics, have traditionally been analyzed in Hilbert space. A class of operators known as Hermitian operators plays a special role. These operators have only real eigenvalues in Hilbert space. The time evolution is then expressed as $e^{i\omega t}$, which is a purely oscillating function because ω is a real number. In order to have an explicit approach to equilibrium expressed by decay modes as $e^{-\gamma t}$ it is necessary to go outside the Hilbert space where Hermitian operators may have complex eigenvalues.

The main point is that by extending the functional space, we can include in the spectral decomposition of the operator new states. This is an essential point to understanding the appearance of new solutions in the statistical description. Time evolution operators for unstable dynamical systems, when extended in this way, include irreversibility, which was 'hidden' as long as we remained in the Hilbert space.

As we have seen, the statistical description of chaotic involves the Perron–Frobenius operator. Only recently has the Frobenius–Perron operator come under serious investigation. Already by the late 1970s, David Ruelle of the IHES in France, and Mark Pollicott, among others, noticed that the eigenvalues of the Frobenius–Perron operator which characterize the approach to equilibrium could be determined. However, a difficulty in obtaining the spectral decomposition (i.e. both the eigenvalues and eigenfunctions) is that for chaotic one-dimensional maps the Frobenius–Perron operator does not have a spectral decomposition in Hilbert space. For this reason, Ioannis Antoniou of the International Solvay Institutes in Brussels, suggested that the decomposition be performed in a more general function space (the 'rigged' Hilbert space we introduced above).

In the last few years, several members of our group have obtained the spectral decomposition for the Bernoulli map and other chaotic systems (**4, 5**). An essential result of this work is that the eigenfunctions corresponding to the spectral decomposition with decay modes are indeed generalized functions. As noted, in the sense of a probability distribution, a trajectory is also a generalized function — a delta function. Since a generalized function cannot be multiplied by another one, this spectral decomposition cannot be applied to trajectories. (We have already noted that a delta function has to be associated with a continuous function, a product of delta functions has no sense.) The new solution on the statistical level includes the approach to equilibrium, but cannot be applied to single trajectories. The equivalence between the individual description and the statistical description is broken.

The Bernoulli map has from the start a broken time symmetry. The trajectory description is not invertible, unlike real mechanical systems for which it is. However, by adding an extra dimension to the phase space we may extend the Bernoulli map to obtain an invertible system known as the baker map. The dynamics associated with the new dimension is contracting so the map preserves the area of an initial element of phase space. These two features of invertibility and preservation of phase space area are the essential features of dynamical systems in the world around us.

Contrary to the Bernoulli map, there is a representation of the Frobenius–Perron operator of the baker map in Hilbert space but no irreversibility is apparent there. The generalized spectral decomposition for the baker map has also been recently constructed. In this decomposition, irreversibility emerges from the unstable dynamics by selecting a class of distribution functions which approach equilibrium in our future. There is also a class oriented to the past. Which class to choose? Experience shows that all objects in our universe share the same direction of time. We all age together. Therefore it is natural to retain the class of distributions which approach equilibrium in our future. Irreversibility already appears at the most basic level in this system. The generalized spectral decompositions for other model systems with more realistic features, such as diffusion, have also been constructed.

III. Probabilistic Extension of Classical Mechanics

We come now to systems governed by classical mechanics. Here the individual description in terms of trajectories also has, in general, to be replaced by a statistical description in terms of distribution functions. We understand that this is a strong statement. Many people look at present for an extension of quantum mechanics, but that even classical mechanics has to be extended is certainly unexpected. Let us indicate in which systems we expect irreversibility to emerge and to which our extension will apply.

We come first to the problem of integrability closely connected to the problem of non-linearity. These problems are at the center of Henri Poincaré at the end of the 19th century. The Hamiltonian generally contains two terms corresponding, respectively, to the kinetic and the potential energy. Poincaré asked the question (which we simplify somewhat): is it possible to eliminate the potential energy by an appropriate choice of variables? Then the system would become isomorphic to independent particles and the solutions of the equation of motion would be immediate. Poincaré has shown that this is, in general, impossible and fortunately so. If the answer had been in the affirmative there would be no possibility of coherence, no organization and no life. The importance of Poincaré resonances is well-recognized today. It led to the KAM theory, so called in honor of its founders Kolmogorov, Arnold and Moser.

Poincaré moreover identified the reason for non-integrability: the existence of resonances between the various degrees of freedom. For each degree of freedom there is an associated frequency ω. Consider then a system characterized by two degrees of freedom. The corresponding frequencies are ω_1 and ω_2; whenever $(n_1\omega_1+n_2\omega_2) = 0$, with n_1 and n_2 non-vanishing integers, we have resonance. These resonances lead to the problem of small denominators by showing up in perturbation calculations as $1/(n_1\omega_1+n_2\omega_2)$. The resonances give rise to random trajectories. In this sense, Poincaré resonances and non-integrability are also associated with chaos. Resonances are responsible for fundamental phenomena such as emission or absorption of light, decay of unstable particles and scattering of particles, to name a few.

Besides non-integrability there is a second condition for irreversibility. In typical macroscopic situations where we observe irreversible processes, molecules collide continuously with each other. We have 'persistent' interactions. This is in contrast to 'transitory' interactions as considered, e.g. in ordinary scattering experiments (described by the so-called '*S-matrix*' theory) in which we have free asymptotic 'in' and 'out' states. To describe persistent interactions we have to introduce 'delocalized' distribution functions spread out in space.

In this paper we shall consider in detail the problems of anharmonic lattices (**6**). We shall also limit ourselves to classical dynamics. We consider the thermodynamics limit for which the number of particles $N\to\infty$. Moreover, we require that the distinction between intensive variables and extensive variables be maintained in this limit. For example, the displacement of a single particle has to remain finite as well as the density of energy H/N.

We first consider the case of a harmonic lattice and consider the limit $N\to\infty$. The dynamic description can then be performed on the level of trajectories or in terms of distributions ρ associated with a Hilbert space (Section IV). We then consider the case of anharmonic lattices (Section V) and show that the thermodynamic limit destroys the Hilbert space structure. Moreover, due to Poincaré resonances, new diffusive terms appear which destroy trajectories. The trajectory 'collapses' to use the terminology of quantum mechanics.

IV. Harmonic Lattices

We first summarize briefly the situation for harmonic lattices. For simplicity, we consider one-dimensional lattices. We assume that N atoms with mass m are equally spaced with a distant a in the equilibrium position, and the equilibrium potential energy U_0. For the excited lattice, the potential energy U is the quadratic form

$$U - U_0 = \frac{1}{2}\sum_{nn'} A_{\mathrm{nn'}} u_{\mathrm{n}} u_{\mathrm{n'}}, \tag{1}$$

where u_n is the displacement of the nth atom from its equilibrium position. We impose cyclic boundary conditions $u_{n+N} = u_n$.

We then introduce normal coordinates q_k

$$u_n = \sum_k q_k \mathrm{e}^{ikna} \tag{2}$$

where (with integer j)

$$k = \frac{2\pi}{Na} j \tag{3}$$

and angle a_k and action variable J_k related to q_k though $\omega_{-k} = \omega_k$

$$q_k = \frac{1}{\sqrt{2Nm}}\left\{\left(\frac{J_k}{\omega_k}\right)^{1/2} \mathrm{e}^{\mathrm{i}\alpha_k} + \left(\frac{\mathrm{J}_{-k}}{\omega_{-k}}\right)^{1/2} \mathrm{e}^{\mathrm{i}\alpha_{-k}}\right\} \tag{4}$$

$$q_k = \frac{1}{\sqrt{2Nm}}\left\{\left(J_k\omega_k\right)^{1/2} \mathrm{e}^{\mathrm{i}\alpha_k} - \left(\mathrm{J}_{-k}\omega_{-k}\right)^{1/2} \mathrm{e}^{\mathrm{i}\alpha_{-k}}\right\}.$$

Neglecting anharmonic terms, we obtain the expected form for the Hamiltonian

$$H_0 = \sum_k \omega_k \mathbf{J}_k. \tag{5}$$

The equations of motion are obviously

$$\dot{J}_k = 0 \tag{6}$$

$$\dot{\alpha}_k = \omega_k.$$

We next consider the statistical description in terms of distribution functions $\rho(J,\alpha,t)$ which satisfy the Liouville equation

$$\mathrm{i}\frac{\partial \rho}{\partial t} = L_0\rho, L_0 \equiv -\mathrm{i}\omega\frac{\partial}{\partial \alpha}. \tag{7}$$

The eigenfunctions $\varphi(\{n\})$ and eigenvalues $l_{\{n\}}$ are $\varphi(\{n\}) = c\exp\left[i\sum_k n_k\alpha_k\right]$ and $l_{\{n\}} = \sum_k n_k\omega_k$ where c is a normalization constant.

Now let us consider more closely the limit $N \to \infty$. Using Eqs (2) and (4) we obtain terms of the form

$$u_n = \sqrt{\frac{2}{Nm}}\sum_k e^{ikna}\sqrt{\frac{J_k}{\omega_k}}e^{i\alpha_k}. \tag{9}$$

In the limit $N \to \infty$

$$\frac{2\pi}{Na}\sum_k \to \int dk \tag{10}$$

$$\text{The condition } u_n \to \text{finite for } N \to \infty \tag{11}$$

imposes that

$$\sum_k e^{i\alpha_k} \sim \sqrt{N} \text{ for } N \to \infty. \tag{12}$$

The angle variables must therefore behave as 'stochastic variables' to which we can apply the law of large numbers. Not all initial conditions are compatible with Eq (11). If Eq (12) is not satisfied, we have to leave the model of a harmonic solid. Note that this condition means that the sequence $\alpha_{k1}, \alpha_{k2}, \ldots$ with $k_j = (2\pi j/Na)$ is 'incompressible' and therefore has a larger probability to realize the situation in Eq (12). They correspond to stochastic sequences among the real number sequences for $0 \leq \alpha_{kj} \leq 2\pi$.

We now turn to the statistical description of Eq (7). As could be expected, there is complete equivalence of this description to the individual description. Indeed let us impose a Hilbert space structure for the statistical description. We expand $\rho(J,\alpha)$ in a Fourier series. With obvious notations

$$\rho(J,\alpha) = \sum_{\{n\}} \rho_{\{n\}}(J)\exp\left[i\sum n_k\alpha_k\right]. \tag{13}$$

The Hilbert norm is therefore

$$\langle \rho | \rho \rangle = \int dJ \sum_{\{n\}} |\rho_{\{n\}}(J)|^2. \tag{14}$$

This norm is preserved in time. To obtain a finite Hilbert norm for $N \to \infty$, well-defined conditions have to be satisfied. Indeed the norm of Eq (14) contains such terms as (with $n_k = \ldots, -1_k, 0, 1_k, 2_k, \ldots$)

$$|\rho_0|^2 + \sum_k |\rho_{1k}|^2 + \sum_{kk'} |\rho_{1k1k'}|^2 + \sum_{kk'k''} |\rho_{1k1k'1k''}|^2 + \ldots \tag{15}$$

which have to converge for $N \to \infty$.† This implies

$$\rho_0 \sim 0(1), \rho_{1k} \sim \frac{1}{\sqrt{N}}, \rho_{1k1k'1k''} \sim \frac{1}{N^{3/2}}. \tag{16}$$

The Hilbert space structure is equivalent to the trajectory description including the randomness condition Eq (12). Indeed using Eq (9)Eq (13)Eq (16) we have

†If $\sum_{kk'k''}$ is over $k+k'+k''=0$ (conservation of "momentum") the condition is $\rho_{1k,1k',1k''} \sim \frac{1}{N}$.

$$\langle u_n \rangle \approx \frac{1}{\sqrt{N}} \sum_k \int \mathrm{d}J \sqrt{J_k \rho_{-1k}} \sim 0(1). \tag{17}$$

We may calculate in the same way other averages such as $\langle u_n u_{n'} \rangle$, or $\langle u_n u_{n'} u_{n''} \rangle$. Note that using Eqs (9) and (16)

$$\langle u_{n1} u_{n2} u_{n3} \rangle \approx \frac{1}{N^{3/2}} \sum_{kk'k''} \int \mathrm{d}J \rho_{1k1k'1k''} \sim \frac{1}{N^{3/2}} N^3 \frac{1}{N^{3/2}} \sim 0(1). \tag{18}$$

V. Anharmonic Lattices

We now come to anharmonic lattices. The potential energy is now:

$$U - U_0 = \frac{1}{2} \sum_{nn'} A_{nn'} u_n u_{n'} + \frac{1}{6} \sum_{nn'n''} B_{nn'n''} u_n u_{n'} u_{n''}. \tag{19}$$

The Hamiltonian H becomes

$$H = H_0 + \lambda V \tag{20}$$

with, after a few calculations,

$$V = \sum_{kk'k''} \left(\frac{J_k J_{k'} J_{k''}}{\omega_k \omega_{k'} \omega_{k''}} \right)^{1/2} \left[V_{kk'k''} \mathrm{e}^{\mathrm{i}(\alpha_k + \alpha_{k'} + \alpha_{k''})} + 3 V_{kk'-k''} \mathrm{e}^{\mathrm{i}(\alpha_k + \alpha_{k'} - \alpha_{k''})} + \text{c.c.} \right] \tag{21}$$

where the summation $\Sigma_{kk'k''}$ is over $k + k' + k'' = 0$ or over a vector of the reciprocal lattice and we have introduced the parameter λ for the coupling constant. Note that, as can be easily verified,

$$V_{kk'k''} \approx \frac{1}{\sqrt{N}}. \tag{22}$$

We shall show that both the trajectory description and the Hilbert space structure are incompatible with the thermodynamic limit $N \to \infty$. In thermodynamic equilibrium (equipartition theorem)

$$\langle V \rangle \approx N. \tag{23}$$

Because the summations over the wave vectors are restricted on the reciprocal lattice, we now have $\rho_{1k1k'1k''} \sim 1/\sqrt{N}$ in the Hilbert space. Hence, in contrast to Eq (23) we obtain, at most,

$$\langle V \rangle \sim \sum_{kk'k''} \int \mathrm{d}J \, V_{kk'k''} \rho_{1k1k'1k''} \sim \frac{1}{\sqrt{N}} N^2 \frac{1}{N} \sim \sqrt{N}. \tag{24}$$

This shows already that thermodynamics equilibrium of Eq (23) lies outside the Hilbert space. To obtain Eq (23) we need stronger 'correlations', such as

$$\rho_{1k1k'1k''} \sim 1/\sqrt{N} \tag{25}$$

but then the Hilbert space norm diverges.

This is a strong indication that the approach to equilibrium requires one to give up the Hilbert space description, as is also the case for interacting particles. There is, however, an interesting difference. In the case of interacting particles, the Hilbert space norm vanishes in the limit $N \to \infty$, while here it diverges.

Let us now describe the time evolution of anharmonic lattices in the Liouville formulation. For interacting systems we have

$$L = L_0 + \lambda L_V. \tag{26}$$

We use the matrix notation

$$\langle \{n\} | L_V | \{n'\} \rangle = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N \times \exp\left[-\mathrm{i} \sum_k n_k \alpha_k \right] L_V \exp\left[\mathrm{i} \sum_k n_k{}' \alpha_k \right]. \tag{27}$$

We obtain directly for the only non-vanishing matrix elements

$$\langle n_k n_{k'} n_{k''} | L_V | n_k \pm 1, n_{k''} \pm 1, n_{k''} \pm 1 \rangle$$
$$= V_{\mp k, \mp k', \mp k''} \left[\frac{n_k}{2J_k} + \frac{n_{k'}}{2J_{k'}} + \frac{n_{k''}}{2J_{k''}} \pm \frac{\partial}{\partial J_k} \pm \frac{\partial}{\partial J_{k'}} \pm \frac{\partial}{\partial J_{k''}} \right] \left(J_k J_{k'} J_{k''} \right)^{1/2}. \tag{28}$$

Note that this is still an operator acting on the actions J_k.

Starting from Eq (26) we can now introduce the dynamics of 'correlations'. The contribution ρ_0 is called the 'vacuum of correlations' and plays a especially important role. As a result of Eq (25) we may have 'destruction (of correlation) processes' such as represented graphically in Fig. 1.

Now using Eqs (22) and (28) we see immediately that Fig. 2 leads precisely to Eq (25). Correlations are amplified by anharmonic effects and bring us outside the Hilbert space. The trajectory description is also destroyed. Indeed, because of Poincaré resonances, we now have 'diffusive processes' such as represented in Fig. 3. Each vertex contains derivative operators $\partial/\partial J$, and Fig. 3 leads therefore to a diffusive process containing second-order operators $\partial^2/\partial J^2$ characteristic of diffusive process.

Let us mention that these diffusive processes to order λ^2 represent Fokker–Planck

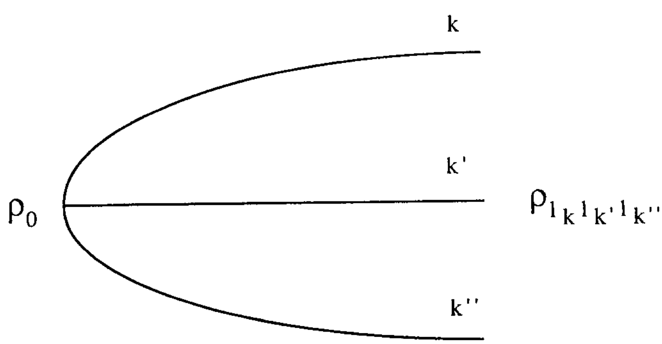

Fig. 1. Destruction process (see text).

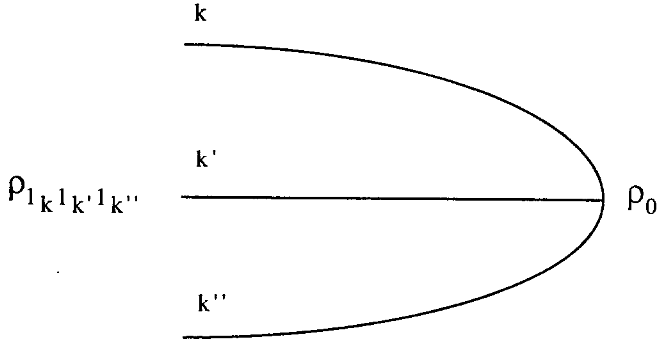

Fig. 2. Creation process (see text).

type contributions which break time symmetry. The operator appearing in Fig. 3 can be easily obtained explicitly. It is

$$\lim_{N\to\infty}\sum_{kk'k''}\pi\delta\left(\omega_k+\omega_{k'}-\omega_{k''}\right)\frac{|V_{kk'-k''}|^2}{\omega_k\omega_{k'}\omega_{k''}}\times\left(\frac{\partial}{\partial J_k}+\frac{\partial}{\partial J_{k'}}-\frac{\partial}{\partial J_{k''}}\right)J_kJ_{k'}J_{k''}\left(\frac{\partial}{\partial J_k}+\frac{\partial}{\partial J_{k'}}-\frac{\partial}{\partial J_{k''}}\right). \tag{29}$$

The action variable now becomes a stochastic variable. As a result, even if we start with well-defined action variables $\delta(J_k-J_k^0)$ trajectories are destroyed by diffusion. Eq (29) leads, for times of the order of the relaxation time, ($\sim\lambda^{-2}$ for weak interactions) to

$$\langle J_k^2\rangle-\langle J_k\rangle^2\sim t. \tag{30}$$

These are 'non-Newtonian contributions'. In this sense, the trajectory collapses.

There are, of course, many comments which could be made, but this would bring us outside the range of this article. Let us simply emphasize the role of the δ-function for the frequency in Eq (29) which comes from Poincaré resonances. For Poincaré integrable systems, there would be no 'collapse' of the trajectory and the Hilbert space structure

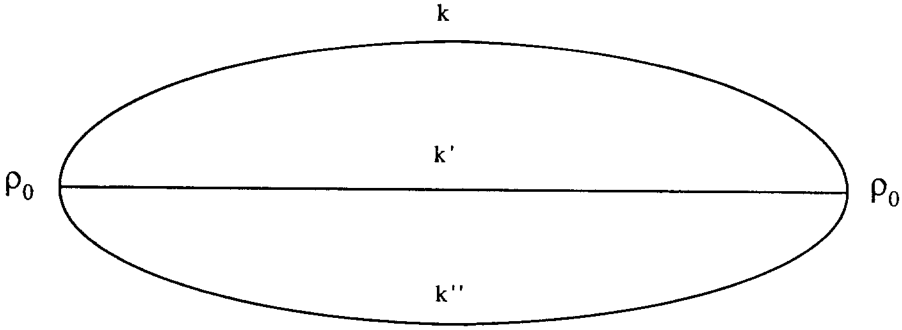

Fig. 3. Diffusive processes ('collision').

would be preserved since we could then introduce cyclic action-angle variables. The problem would then be similar to that of harmonic oscillators.

VI. Quantum Mechanics

The results obtained in classical mechanics can be extended to quantum mechanics. The role of Newton's equation is now played by the Schrödinger equation which governs the time evolution of the wave function. Again, for non-integrable systems and delocalized distribution functions we have to go outside the Hilbert space. We then obtain new spectral decompositions of the quantum Liouville operator which lead to complex eigenvalues and which are irreducible to wave functions. We may associate the new solutions with 'quantum chaos'. This result is at the core of the solution of the quantum paradox.

In spite of the immense success of quantum mechanics, discussions about its foundations continue. It is generally admitted that the wave function determines 'potentialities'. We need therefore an additional mechanism to go from potentialities to the 'actualities' we measure. This introduces the 'collapse' of the wave function and leads to irreversibility, but this means a dual structure at the basis of quantum theory. When to use Schrödinger equation; when to introduce the collapse? This leads to the quantum paradox. Many proposals to elucidate the conceptual foundations of quantum theory can be found in the literature, but as is the case for the time paradox they are mostly based on approximations we as observers would introduce in the basic quantum laws. Also, none of these proposals leads to new predictions which could be tested.

Quantum theory started with the observations that spectroscopic frequencies are differences of two energy levels but this is not true for the imaginary part of the eigenvalues, which correspond to irreversible processes. Quantum relaxations times, as observed or calculated, are not differences between two levels. This already shows that irreversible processes cannot be described in terms of wave functions in Hilbert space but only in terms of distribution functions.

This approach leads to a unified and testable formulation (**7**, **10**). The basic description is now on the statistical level. The collapse corresponds to situations where the initial state is outside the Hilbert space (like for plane waves) and where Poincaré resonances lead to diffusive behavior. In our theory the observer no longer plays any special role. Our theory permits us to describe the approach to equilibrium of quantum systems and eliminates the quantum paradox. We obtain a dynamical description of the measurement process as the measuring device is a thermodynamic system and the thermodynamic limit leads to a broken time symmetry. It is the common arrow of time which is the necessary condition for our communication with the physical world as it is the condition of our communication with our fellow humans.

Is it too ambitious to speculate that this work could have reconciled Niels Bohr and Albert Einstein? Bohr insisted that in order to communicate with the microscopic world, we need an apparatus described in classical terms (**8**). This is the famous Copenhagen interpretation, but how to imagine a classical apparatus in a quantum world? We have seen that this is not necessary; what is essential is the broken time symmetry. On the other hand, Einstein was opposed to the extraordinary role quantum mechanics attributed to the observer. The observer would be responsible for the appearance of irreversibility — but likely irreversible processes would exist in nature be there human beings or not. In our approach, irreversibility has a purely dynamical origin similar both in classical and quantum mechanics.

Statistical physics and thermodynamics have played an immense historical role in the evolution of physics in the 20th century. The divergence in the classical description of the specific heat of black-body radiation led to quantum theory. We are today in a somewhat similar situation. Non-linearity and instability force us to adopt a different point of view concerning the formulation of the laws of nature. They now express possibilities instead of certainties. There is no longer any contradiction between the dynamical and the thermodynamical descriptions of nature; far from being a measure of our ignorance, entropy expresses a fundamental property of the physical world, the existence of a broken time symmetry leading to a distinction between past and future which is a universal property of both the nature we observe as well as a prerequisite for the existence of life and consciousness.

At the end of the 19th century, when the debate about Boltzmann's work raged, Henry Poincaré expressed his belief that the evolutionary picture associated with entropy only has a meaning in a non-deterministic world. Today we have no longer to turn away in horror from this conclusion; it is the necessary outcome of the thermodynamic limit, be it in classical or quantum mechanics. For such systems, probability is no longer associated with ignorance and certitude with reason. We have to find the narrow path between the alienating deterministic picture in which there is no place for creativity and innovation and a purely random world in which there would also be no place for human endeavor.

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