LOGARITHMIC SINGULARITIES

Chairman: C. N. Yang
The Lambda Transition in Liquid Helium*

W. M. Fairbank
Stanford University, Stanford, Calif.

and

C. F. Kellers
Wells College, Aurora, N.Y.

This talk is based on work done at Duke University several years ago by C. F. Kellers, M. J. Buckingham and myself [1]. Much of what I will say has been previously reported [1]. Still, when Dr. Green asked me to give this talk I consented because I believed a review of the lambda transition in liquid helium is particularly pertinent to the present conference on critical phenomena. Not only were the Ehrenfest relationships for a second order transition enunciated first for the lambda transition in liquid helium, but liquid helium also served, through the experiments which I will describe, to first call attention to the logarithmic character of the lambda transition which is so much a part of the discussion of this conference.

The lambda transition in liquid helium is, of course, of particular interest because of the unique momentum ordering that takes place below the lambda point which makes it perhaps similar but certainly different from all other transitions. In addition, however, of all the cooperative transitions, it is the one which offers the best chance for experimental observations very close to the transition. Because liquid helium does not have any crystal structure, because it is a pure liquid changing at the lambda point only in its momentum ordering, it is apparently possible with sufficiently good thermal equilibrium to make measurements as close to the transition as is experimentally feasible. Very close to the transition, because liquid helium does not have any crystal structure, because it is a pure liquid changing at the lambda point only in its momentum ordering, it is apparently possible with sufficiently good thermal equilibrium to make measurements as close to the transition as is experimentally feasible.

Onsager and Feynman have suggested that liquid helium as a superfluid is required to be irrotational. If the helium rotates, it rotates according to the Onsager-Feynman theory, with quantized vortex motion [2]. Helium rotating slowly enough is predicted to rotate in one single quantized state of rotation with each helium atom having angular momentum \( h \) regardless of the size of the bucket in which it is contained [2, 3]. The theoretical suggestion is then the following: If a bucket of helium no matter how large is cooled through the lambda transition into the superfluid state while rotating slowly enough, each superfluid helium atom will attain the same angular momentum. For the slowest stage the superfluid will stop rotating and the bucket, if suspended freely, will rotate faster [4]. This experiment has never been performed. Some physicists, including a graduate student George Hess at Stanford, are trying to perform this experiment [5]. This then is the type of order transition which is predicted for liquid helium, long-range order in momentum space of a very special kind.

Blatt, Butler, and Schafroth have taken another approach to the problem [6, 7, 8, 9]. They questioned whether a real liquid, as contrasted with an ideal Bose gas can exhibit arbitrarily long-range order or correlation in momentum space. They suggest that there may be a finite correlation length of approximately \( 10^{-5} \) cm over which helium atoms are correlated in momentum space, but beyond which there is no correlation at all. In particular, they developed a theory of a Bose liquid in which helium atoms are correlated over a distance of about \( 10^{-4} \) cm, the correlated atoms representing the superfluid. By calculating the amount of this correlated phase, \( \rho \), they calculate the ratio of uncorrelated to correlated phase (normal to superfluid, \( \rho / \rho_s \)) as a function of temperature, the correlation length being an adjustable parameter. With a cor-
Thus, no matter how large the bucket, liquid helium was cooled below the lambda point and the rotational speed must decrease as \( l/r_z \) and is down by \( 10^{-4} \) deg which is a direct result of a finite correlation length. This rounding could not have been observed in the then existing data. It also follows from their theory that \( r_z = 0 \) is not an equilibrium property of superfluid helium II.

Feynmann [10] took issue with the premise that the correlation length in liquid helium could be cutoff at some finite distance, and suggested that the lambda point in liquid helium is not only sharp at one-thousandth of a degree from the lambda point but it is sharp at a hundred-thousandth of a degree. He believed that even though the correlation in liquid helium II gets weaker and weaker as \( \lambda \) approaches the lambda point, the specific heat is rounded at the lambda point for a temperature interval of approximately \( 10^{-4} \) deg which is a quasi-transition. Instead of being a second-order transition with a discontinuity in the specific heat at the lambda point, the specific heat is rounded at \( \lambda \) but it is sharp at a hundred-thousandth of a degree. This was only told to me by Feynmann and therefore is intuition.

The specific heat, \( C_v \), is plotted as ordinant against \( T - T_\lambda \), the temperature difference from the lambda point. It can be seen that as the specific heat is displayed in a vacuum and a mechanical heat switch provided for contact with the bath when required.

FIGURE 1. This characteristic feature of the specific heat transition is that it is a quasi-transition. Instead of being a second-order transition with a discontinuity in the specific heat at the lambda point, the specific heat is rounded at the lambda point for a temperature interval of approximately \( 10^{-4} \) deg which is a direct result of a finite correlation length. This rounding could not have been observed in the then existing data. It also follows from their theory that \( r_z = 0 \) is not an equilibrium property of superfluid helium II.

Feynmann [10] took issue with the premise that the correlation length in liquid helium could be cutoff at some finite distance, and suggested that the lambda point in liquid helium is not only sharp at one-thousandth of a degree from the lambda point but it is sharp at a hundred-thousandth of a degree. He believed that even though the correlation in liquid helium II gets weaker and weaker as \( \lambda \) approaches the lambda point, the specific heat is rounded at the lambda point for a temperature interval of approximately \( 10^{-4} \) deg which is a quasi-transition. Instead of being a second-order transition with a discontinuity in the specific heat at the lambda point, the specific heat is rounded at \( \lambda \) but it is sharp at a hundred-thousandth of a degree. This was only told to me by Feynmann and therefore is intuition.

The specific heat, \( C_v \), is plotted as ordinant against \( T - T_\lambda \), the temperature difference from the lambda point. It can be seen that as the specific heat is displayed in a vacuum and a mechanical heat switch provided for contact with the bath when required.

Measurement of carbon dioxide with a minimal detectable change representing \( 2 \times 10^{-7} \) deg. It was possible to make measurements both while increasing and decreasing the temperature. Figure 4 shows the data. These of course are not new data just taken, but the reason I was asked to present them is that perhaps they are very similar to data which are now being obtained on other transitions. The specific heat, \( C_v \), is plotted as ordinant against \( T - T_\lambda \).

In order to show the nature of the transition very near the lambda point, the data are shown on successively expanded temperature scales. To aid in a visualization of the very large amount of expansion of each successive curve, a small vertical line has been drawn just above the origin. The width of the line indicating the fraction of the curve which is shown expanded in the curve directly to the right. The ratio of the expansion between the first and last curve is about \( 5 \times 10^4 \). Thus if the projected slide of the first diagram is 10 ft, it would have to be expanded to 100 miles to obtain a 10 ft projection of the third figure.

It can be seen that as the specific heat is displayed on a more and more expanded scale, it maintains the same geometric shape. There is certainly no indication of a rounded quasi-transition as predicted by the theory of Blatt, Butler, and Schrauth.

Figure 5 shows the data in a form which you are all now familiar. \( C_v \) has been plotted as ordinant against \( T - T_\lambda \), degrees Kelvin on a logarithmic scale. It is seen in this kind of a plot that near the lambda point on each side there is a factor of \( 10^4 \) in \( T - T_\lambda \) over which the data fall in two
parallel straight lines which are branches of the expression

\[ C_s = 4.55 - 3.00 \log_{10} |T - T_\lambda| - 5.20\Delta \]  

(1)

where \( \Delta = 0 \) for \( T < T_\lambda \) and \( \Delta = 1 \) for \( T > T_\lambda \). One reason that this was very interesting and exciting at the time, is that Onsager had published the exact solution of the two-dimensional Ising model \([11]\). The two-dimensional Ising model gives a logarithmic transition but only for an exact solution.

Thus we see from the experimental data that the lambda point, instead of being a rounded quasi-transition as suggested by the theory of Blatt, Butler, and Schafroth assuming a finite correlation length, is in fact, sharp to at least 2 orders of magnitude closer in temperature to the lambda point than predicted by their theory. The simple logarithmic behavior of the data shown in figure 4 which extrapolates to a logarithmic singularity of the lambda point has stimulated more detailed experimental and theoretical investigation of the lambda point, as witnessed so completely by this conference. Although the two-dimensional Ising model has been solved exactly by Onsager giving such a logarithmic singularity, it is obvious from this conference that an exact solution to the three-dimensional Ising model will not be easily found.

In figure 5 the curve of the form

\[ \frac{A}{|T - T_\lambda|^2} + B|T - T_\lambda| + C \]

which has been fitted to the data for \( T = T_\lambda \) between \( 2 \times 10^{-3} \) and \( 2 \times 10^{-4} \) as shown. This is the fundamental form suggested by the Padé-approximation of the three-dimensional Ising model. It is seen that although it fits the data rather well for \( |T - T_\lambda| > 2 \times 10^{-1} \)K, it departs very dramatically for temperatures closer to the lambda point. Unless data are obtained close enough to the lambda point one could not differentiate between this expression and the logarithmic singularity.

Much of this conference will be taken up with the exact nature of other kinds of lambda transitions. However, I wish to compare quickly two results by Robinson and Friedberg [12] and Skalyo and Friedberg [13] on the lambda transition in hydrated nickel and cobalt chloride with specific heat data on liquid helium.

Figure 7 shows the specific heat data of nickel chloride to within 0.05°K of the lambda point from above, and within 0.2°K from below the transition temperature. Figure 8 shows the same data with \( C_v \) plotted against \( \log_{10}|T - T_\lambda| \). If one defines a reduced temperature by \( t = (T - T_\lambda)/T_\lambda \) and compares the curve for nickel chloride with the helium curve, one sees for the same value of \( t \) both curves show nonparallel straight lines. Figure 9 shows the data for cobalt chloride taken to smaller values of \( t \). These data show two parallel straight lines and then a flattening out of the data.
A comparison with helium indicates agreement as to the two parallel straight lines. The flattening out at values of $t$ closer to the lambda point would be expected in either an experiment or theoretical model where long-range correlation is cutoff. These two sets of data are then seen to be consistent with the lambda transition for helium and indicates the possibility that other lambda points have the same form as the helium transition.

The suggestion that the specific heat of helium might become infinite at the lambda transition was first made by Tisza [14]. The first experimental evidence concerning the logarithmic nature of the lambda transition was obtained by Atkins and Edwards [15]. They suggested that a logarithmic term could be used to derive the result of their measurements of the thermal expansion coefficient below the lambda point. Figure 10 shows the data on the expansion coefficient of Atkins and Edwards, Chase and Maxwell, and Kerr and Taylor. It is seen that the data can be represented by two parallel straight lines from the plot of the expansion coefficient versus log($T - T_\lambda$). This curve is reproduced from the paper by Kerr and Taylor [16]. Figure 11 shows the relative molar volume of helium in the vicinity of the lambda point as given by Kerr and Taylor. It is seen that the lambda point comes where the slope is infinite rather than at the point where the molar volume is minimum.

Buckingham [17] has derived rigorously the thermodynamic consequences of lambda transitions characterized by the absence of a latent heat, but at which the specific heat at constant pressure becomes infinite. Pippard [17] had previously considered such a transition and worked out thermodynamic relationships based on the assumption that the entropy surface is cylindrical near the lambda point. The thermodynamic relationships worked out by Buckingham and Pippard can be used to compare the behavior of various thermodynamic properties in the neighborhood of the lambda line [16]. In reference 16 the specific heat and coefficient of expansion are compared using the Buckingham relationship. The slope of the lambda line ($\Delta S/\Delta T$) is taken from the data of Lounasmaa and Kojo [18] and Lounasmaa and Kaunisto [19].

The data can be plotted in a parametric plot which should give asymptotically a single straight line. Two parallel straight lines are obtained by Kerr and Taylor using the data shown in figures 4 and 10. This shows that the fundamental form of the data is correct but the details need to be refined. It is difficult to obtain exactly the same experimental conditions in different experiments very close to the lambda point. It seems worthwhile to make simultaneous specific heat and expansion coefficient measurements to within $10^{-6}$ K of the lambda transition.

The behavior of the velocity of sound in the vicinity of the lambda point can also be predicted using specific heat and other thermodynamic data [17]. Here again there is approximate but not exact agreement. Since there will be a separate talk on this subject I will not present further data here.

In summary, the specific heat expansion coefficient and velocity of sound all give evidence to the logarithmic nature of the lambda transition in helium although refined details of comparison still need to be worked out. I would like to end by again mentioning that liquid helium presents the unique opportunity to obtain data close to the lambda point. The question naturally arises, is the transition in liquid helium...
The Specific Heat of He\textsubscript{3} and He\textsubscript{4} in the Neighborhood of Their Critical Points*  

M. R. Moldover and W. A. Little

Stanford University, Stanford, Calif.

**Introduction**

Recently Bagatskii, Voronel', and Gusak [1] showed that the specific heat at constant volume of argon exhibited what appears to be a logarithmic singularity at the critical temperature (T\textsubscript{c}) for measurements taken at a density near the critical density. This singular behavior is in sharp contrast to the predictions of the traditional viscous flow theory of this phase transition by Landau and Lifshitz [2]. However, the behavior is precisely that to be expected for the so-called "lattice gas" model for the liquid-gas transition. Lee and Yang [3] have shown that the partition function of a classical gas of particles moving on a discrete lattice with a repulsive force preventing double occupation of any site, and a nearest neighbor attraction can be mapped precisely onto the partition function of an Ising model of a spin system in an external magnetic field. The specific heat for this Ising model in zero field exhibits a logarithmic singularity at the Curie point. The specific heat for the corresponding lattice gas on the critical isochore exhibits a logarithmic singularity at the critical point. The measurements on argon then indicate that for a real gas the specific heat behaves in a similar manner to that of a lattice gas. We have investigated this point further by studying the specific heat at constant volume (C\textsubscript{V}) of both He\textsubscript{3} and He\textsubscript{4} at densities close to the critical density. We have done this for two main reasons. Firstly, to see whether the behavior observed for argon is also observed for helium, for which quantum effects should be important, and secondly, to investigate the detailed nature of the pressure-density plane, not only on the critical density, but also in its immediate neighborhood. Yang and Yang [4] have conjectured that the quantum effects would reduce the magnitude of the singular contribution to the specific heat in helium. Our results confirm this view.

**Experimental Procedure**

There are several practical advantages to using helium rather than another noble gas for C\textsubscript{V} measurements near the critical point. The low heat capacity of metals at liquid helium temperature permits one to use a massive calometer of large surface to volume ratio. Thus the path for heat transfer through the helium may be kept short. In addition, more nearly adiabatic conditions and high resolution thermometry are most easily attained at liquid helium temperatures. Our calorimeter was built of two OFHC copper parts. The helium was contained in the lower part in 50 slots. Each slot was 0.01 cm wide to facilitate good thermal contact between the calormeter and the helium. The slots were made only 0.3 cm deep in an effort to minimize possible gravitational effects. In contrast a thin stainless steel sheet 10 cm high and 4 cm in diameter containing a magnetic stirrer was used for the work on argon [1, 5]. This construction was necessary to obtain a low ratio of heat capacity of the calormeter to its contents while maintaining constant volume at the high critical pressure of argon.](\textsuperscript{[1]}) This construction was necessary to obtain a low ratio of heat capacity of the calormeter to its contents while maintaining constant volume at the high critical pressure of argon.\textsuperscript{[2]}

The lower part of our calormeter was wound with a carbon resistor cemented to its surface. The helium was admitted to the calormeter by means of nylon threads in an evacuated chamber. Thermal contact to the bath was made with a mechanical heat switch. A stainless steel filling capillary 5 in. in length and 0.005 in. in I.D. led from the calormeter to a needle valve. The dead volume was about 95 percent of the total volume of the calormeter. The helium was admitted to the calormeter via a Toeppler pump which was used to measure the volume of gas to an accuracy of about 0.2 percent.

One of the precautions taken was the measurement of the strain heat input to the calormeter before and after each data point. The approach to temperature equilibrium of the calormeter was observed after each heating interval. As T\textsubscript{c} was