RELIABILITY BASED STRUCTURAL OPTIMIZATION

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Abstract. In this paper a robust and efficient methodology is presented for treating large-scale reliability-based, structural optimization problems. The optimization part is performed with evolution strategies, while the reliability analysis is carried out with the Monte Carlo simulation (MCS) method incorporating the importance sampling technique for the reduction of the sample size. The elasto-plastic analysis phase, required by the MCS, is replaced by a neural network predictor in order to compute the necessary data for the MCS procedure. The use of neural networks is motivated by the approximate concepts inherent in reliability analysis and the time consuming repeated analyses required by MCS. A training algorithm is implemented for training the NN utilizing available information generated from selected elasto-plastic analyses.

1 INTRODUCTION

Reliability analysis methods have been developed significantly over the last decades and have stimulated the interest for the probabilistic optimum design of structures (Schueller1). Despite the theoretical advancements in the field of reliability analysis serious computational obstacles arise when treating realistic problems. In particular, the reliability-based optimization (RBO) of large-scale structural systems is an extremely computationally intensive task, as shown by Tsompanakis and Papadrakakis2. Despite the improvement on the efficiency of the computational aspects of the reliability analysis techniques, they still require disproportionate computational effort for treating practical reliability problems. This is the reason why very few successful numerical investigations are known in the field of RBO and are mainly restricted to 2-D frames and trusses.

In the present study the reliability-based sizing optimization of large-scale multi-storey 3-D frames is investigated. The objective function is the weight of the structure while the constraints are both deterministic (stress and displacement limitations) and probabilistic (the overall probability of failure of the structure). Randomness of loads, material properties, and member geometry are taken into consideration in reliability analysis using Monte Carlo simulation. The probability of failure of the frame structures is determined via a limit elasto-plastic analysis.

The optimization part is solved using evolution strategies (ES), which in most cases are more robust and present a better global behaviour than mathematical programming methods (Papadrakakis et al.3). The limit elasto-plastic analyses required during the MCS are replaced by NN predictions. The use of NN is motivated by the approximate concepts inherent in reliability analysis and the time consuming repeated analyses required for MCS. An NN is trained first utilizing available information generated from selected conventional elasto-plastic analyses. The limit state analysis data is processed to obtain input and output pairs, which are used to produce a trained NN. The trained NN is then used to predict the critical load factor due to different sets of basic random variables. It appears that the use of a properly selected and trained NN can eliminate any limitation on the sample size used for MCS and on the dimensionality of the problem, due to the drastic reduction of the computing time required for the repeated limit elasto-plastic analyses.
2 STRUCTURAL RELIABILITY ANALYSIS

The reliability of a structure or its probability of failure is an important factor in the design procedure since it quantifies the probability under which a structure will fulfill its design requirements. Structural reliability analysis is a tool that assists the design engineer to take into account all possible uncertainties during the design, construction phases and lifetime of a structure in order to calculate its probability of failure $p_f$. A time invariant reliability analysis produces the following relationship

$$p_f = p[R < S] = \int_{-\infty}^{\infty} F_R(t) f_S(t) dt = 1 - \int_{-\infty}^{\infty} F_S(t) f_R(t) dt$$

in which $R$ denotes the structure bearing capacity and $S$ the external loads. The randomness of $R$ and $S$ can be described by known probability density functions $f_R(t)$ and $f_S(t)$, with $F_R(t)=p[R < t]$, $F_S(t)=p[S < t]$ being the cumulative probability density functions of $R$ and $S$, respectively.

Most often a limit state function is defined as $G(R, S)=S-R$ and the probability of structural failure is given by

$$p_f = p[G(R, S) \geq 0] = \int_{G \geq 0} f_R(R) f_S(S) dR dS$$

It is practically impossible to evaluate $R$ analytically for complex and/or large-scale structures. In such cases the integral of Eq. (2) can be calculated only approximately using either simulation methods, such as the Monte Carlo simulation method, or by using approximation methods. First and second order approximation methods (FORM and SORM) lead to formulations that require prior knowledge of the means and variances of the random variables and the definition of a differentiable failure function. On the other hand, MCS methods require that the probability density functions of all random variables must be known prior to the reliability analysis. For small-scale problems FORM and SORM implementations have been proved very efficient, but when the number of random variables increases and the problems become more complex MCS based methods have been proven more reliable.

2.1 The Monte Carlo simulation method

In reliability analysis the MCS method is often employed when the analytical solution is not attainable and the failure domain cannot be expressed or approximated by an analytical form. This is mainly the case in problems of complex nature with a large number of basic variables where all other reliability analysis methods are not applicable. Although the mathematical formulation of the MCS is relatively simple and the method has the capability of handling practically every possible case regardless of its complexity, the computational effort involved in conventional MCS is excessive. For this reason a lot of sampling techniques, also called variance reduction techniques, have been developed in order to improve the computational efficiency of the method by reducing the statistical error that is inherent in MCS methods and keeping the sample size to the minimum possible. Expressing the limit state function as $G(x)$, where $x=(x_1, x_2, ..., x_M)$ is the vector of the random variables, Eq. (2) can be written as

$$p_f = \int_{G(x) \geq 0} f(x) dx$$

where $f(x)$ denotes the joint probability of failure for all random variables. Since MCS is based on the theory of large numbers ($N_{\infty}$) an unbiased estimator of the probability of failure is given by

$$p_f = \frac{1}{N_\infty} \sum_{j=1}^{N_\infty} I(x_j)$$

in which $I(x_j)$ is an indicator for successful and unsuccessful simulations defined as

$$I(x_j) = \begin{cases} 1 & \text{if } G(x_j) \geq 0 \\ 0 & \text{if } G(x_j) < 0 \end{cases}$$

In order to estimate $p_f$ an adequate number of $N$ independent random samples is produced using a specific, usually uniform, probability density function of the vector $x$. The value of the failure function is computed for each random sample $x_j$ and the Monte Carlo estimation of $p_f$ is given in terms of sample mean by
where \( N_{\text{H}} \) is the number of unsuccessful simulations.

2.2 Importance Sampling

Various reduction techniques have been proposed in order to improve the efficiency and the accuracy of the MCS method. Importance Sampling (IS) is generally recognized as the most efficient reduction technique\(^4\)\(^6\). The key-idea of this technique is to obtain a non-negative sampling density located in the neighbourhood of the most probable failure point. The selection of an appropriate important sampling density function \( g_*(x) \) is of critical importance for both the efficiency and the accuracy of the MCS. A successful choice of \( g_*(x) \) yields reliable results and reduces significantly the number of simulations, while a misleading choice produces inaccurate results.

3 BASIC PRINCIPLES OF ARTIFICIAL NEURAL NETWORKS THEORY

The aim of the present study is to train a neural network to provide computationally inexpensive estimates of analysis outputs required during the MCS process. A trained neural network presents some distinct advantages over the numerical computing paradigm. It provides a rapid mapping of a given input into the desired output quantities, thereby enhancing the efficiency of the structural analysis process. This major advantage of a trained NN over the conventional procedure, under the provision that the predicted results fall within acceptable tolerances, leads to results that can be produced in a few clock cycles, representing orders of magnitude less computational effort than the conventional computational process. The learning algorithm, which was employed for the training, is the well-known Back Propagation (BP) algorithm\(^7\).

In the present implementation the objective is to investigate the ability of the NN to predict the collapse load for different values of the basic random variables. The calculation of the collapse load is used by the MCS method for reliability analysis. The results of the reliability analyses are used to verify the feasibility or not of the design with respect to the probabilistic constraint functions. This is achieved with a proper training of the NN. The NN training comprises the following tasks: (i) select the proper training set, (ii) find a suitable network architecture and (iii) determine the appropriate values of characteristic parameters such as the learning rate and momentum term.

The learning rate coefficient and the momentum term are two user defined BP parameters that effect the learning procedure of NN. The training is sensitive to the choice of these parameters. The learning rate coefficient, employed during the adjustment of weights, is used to speed-up or slow-down the learning process. A bigger learning coefficient increases the weight changes, hence large steps are taken toward the global minimum of error level, while smaller learning coefficients increase the number of steps taken to reach the desired error level. If an error curve shows a downward trend but with poor convergence rate the learning rate coefficient is likely to be too high. Although these learning rate coefficients are usually taken to be constant for the whole net, local learning rate coefficients for each individual layer or unit may be applied as well.

The basic NN configuration employed in this study is selected to have one hidden layer. An important factor governing the success of the learning procedure of NN architecture is the selection of the training set. A sufficient number of input data properly distributed in the design space together with the output data resulting from complete structural analyses are needed for the BP algorithm in order to provide satisfactory results. Overloading the network with unnecessary similar information results to over training without increasing the accuracy of the predictions. A few tens of limit elasto-plastic analyses have been found sufficient for the example considered to produce a satisfactory training of the NN.

In this work a fully connected network is used. The number of conventional step-by-step limit analysis calculations performed in order to built up the proper data for the training set is in the range of thirty\(^8\). This selection is based on the requirement that the full range of possible results should be represented in the training procedure. For the application of the NN simulation and for the selection of the suitable training pairs, the sample space for each random variable is divided into equally spaced distances. The central points within the intervals are used as inputs for the limit state analyses.

4 RELIABILITY-BASED STRUCTURAL OPTIMIZATION

During the last ten years various methodologies have evolved which deal with the reliability-based optimum design of structures. These attempts are restricted to relatively moderate size truss and frame structural problems using FORM and SORM reliability analysis methods\(^9\). In the present study the reliability-based sizing optimization of large-scale multi-storey 3-D frames is investigated.

In sizing optimization problems the aim is to minimize the weight of the structure under certain deterministic
behavioral constraints usually on stresses and displacements. In reliability-based optimal design additional probabilistic constraints are imposed in order to take into account various random parameters and to ensure that the probability of failure of the structure is within acceptable limits. The probabilistic constraints enforce the condition that the probability of a local or the system failure is smaller than a certain value (i.e. $10^{-3}$). In this work the overall probability of failure of the structure, as a result of a limit elasto-plastic analysis, is taken as the global reliability constraint.

The probabilistic design variables are chosen to be the cross-sectional dimensions of the structural members and the material properties ($E$, $\sigma_y$). Due to engineering practice demands the members are divided into groups having the same design variables. This linking of elements results in a trade-off between the use of more material and the need of symmetry and uniformity of structures due to practical considerations. Furthermore, it has to be taken into account that due to manufacturing limitations the design variables are not continuous but discrete since cross-sections belong to a certain set.

A discrete RBO problem can be formulated in the following form

$$
\min \ F(s) \\
\text{subject to} \ g_j(s) \leq 0 \ \ j=1,...,m \\
s_i \in \mathbb{R}^k, \ \ i=1,...,n \\
p_r \leq p_s
$$

(7)

$F(s)$ is the objective function (i.e. the structural weight), $s$ is the vector of geometric design variables, which can take values only from the given discrete set $\mathbb{R}^k$, $g_j(s)$ are the deterministic constraints and $p_r$ is the probability of failure of the design. Most frequently the deterministic constraints of the structure are the member stresses and nodal displacements or inter-storey drifts. For rigid frames with I-shape cross sections, the stress constraints, under allowable stress design requirements specified by Eurocode3, are expressed by the non-dimensional ratio $q$ of the following formulas

$$
q = \frac{f_a}{F_a} + \frac{f_y}{F_y} + \frac{f_z}{F_z} \leq 1.0 \quad \text{if} \quad \frac{f_a}{F_a} \leq 0.15
$$

(8)

and

$$
q = \frac{f_a}{F_a} + \frac{C_m t_b y}{(1-f_a/F_a)F_b} + \frac{C_m t_b z}{(1-f_a/F_a)F_b} \leq 1.0 \quad \text{if} \quad \frac{f_a}{F_a} > 0.15
$$

(9)

where $f_a$ is the computed compressive axial stress, $f_y^y, f_z^z$ are the computed bending stresses for y and z axis, respectively. $F_a$ is the allowable compressive axial stress, $F_y$ is the allowable bending stresses, $F_z$ is the Euler stress divided by a safety factor, $C_m$ is a coefficient depending upon element’s curvature caused by the applied moments, $F_y=0.60\times\sigma_y$, is the allowable axial stress, $F_y=0.66\times\sigma_y$ is the allowable bending stress and $\sigma_y$ is the yield stress. The allowable inter-storey drift is limited to 1.5% of the height of each storey.

The proposed reliability-based sizing optimization methodology proceeds with the following steps:

1. At the outset of the optimization procedure the member geometry, the boundaries and the loads of the structure under investigation have to be defined.
2. The mean values of the design variables are properly selected and the constraints are also defined in order to formulate the optimization problem as in Eq. (7).
3. The optimization phase is carried out with ES where feasible designs are produced at each cycle (generation). The feasibility of the designs is checked for each design vector with respect to deterministic and probabilistic constraints of the problem.
4. The satisfaction of the deterministic constraints is monitored through a finite elements analysis of the structure.
5. The satisfaction of the probabilistic constraints is realized with the reliability analysis of the structure and the MCS technique in order to evaluate the probability of failure of the structure.
6. If the convergence criteria for the optimisation algorithm are satisfied then the optimum solution has been found and the process is terminated, else the whole process is repeated from step 3 with the new set of design variables.

### 4.1 Reliability-based structural optimization using MCS, ES and NN

In reliability analysis of elasto-plastic structures using MCS the computed critical load factors are compared to the corresponding external loading leading to the computation of the probability of structural failure. The probabilistic constraints enforce the condition that the probability of a local failure of the system or the global...
system failure is smaller than a certain value (i.e. $10^{-2}$-$10^{-3}$). In this work the overall probability of failure of the structure, as a result of limit elasto-plastic analyses, is taken as the global reliability constraint. The probabilistic design variables are chosen to be the cross-sectional dimensions of the structural members and the material properties ($E$, $\sigma_y$).

At each ES cycle (generation) a number of MCS are carried out. In order to replace the time consuming limit elasto-plastic analyses needed by MCS for each design, a training procedure is performed based on the data collected from M conventional limit elasto-plastic analyses. After the selection of the suitable NN architecture the training procedure is performed with $M=30$ data sets, in order to obtain the I/O pairs needed for the NN training. After the training phase is concluded the trained NN replaces the conventional limit elasto-plastic analyses, for the current design.

The Algorithm

1. Selection step : selection of $s_i$ (i = 1, 2, ..., $\mu$) parent vectors of the design variables.
2. Deterministic constraint check : all parent vectors become feasible.
3. Monte Carlo Simulation step : for each parent vectors
   3a. Selection of the NN training set
   3b. NN training
   3c. Perform Monte Carlo Simulations
4. Probabilistic constraint check : all parent vectors become feasible.
5. Offspring generation : generate $s_j$ (j=1,2,...,$\lambda$) offspring vectors of the design variables.
6. Deterministic constraint check : all parent vectors become feasible.
7. Monte Carlo Simulation step : for each offspring vectors
   7a. Selection of the NN training set
   7b. NN training
   7c. Perform Monte Carlo Simulations
8. Probabilistic constraint check : if satisfied continue, else change $s_j$ and go to step 6.
9. Selection step : selection of the next generation parents according to ($\mu+\lambda$) or ($\mu$,$\lambda$) selection schemes.
10. Convergence check : If satisfied stop, else go to step 5.

5 TEST EXAMPLE

A realistic test example has been investigated in the present study in order to illustrate the efficiency of the proposed methodology for reliability-based sizing optimization problems. The cross section of each member of the space frame is assumed to be a I-shape and for each member two design variables are allocated. The objective function of the problem is the weight of the structure. The deterministic constraints are imposed on the inter-storey drifts and for each group of structural members on the maximum non-dimensional ratio $q$ of Eqs. (8) and (9) which combines axial forces and bending moments. The values of allowable axial and bending stresses are $F_a=150$ MPa and $F_b=165$ MPa, respectively, whereas the allowable inter-storey drift is limited to 1.5% of the height of each storey.

The probabilistic constraint is imposed on the probability of structural collapse due to successive formation of plastic nodes and is set to $p_a=0.001$. The probability of failure caused by uncertainties related to material properties, member geometry and loads of the structures is estimated using MCS with the Importance Sampling technique. External loads, yield stresses, elastic moduli and the dimensions of the cross-sections of the structural members are considered to be random variables. The loads follow a log-normal probability density function, while random variables associated with material properties and cross-section dimensions follow a normal probability density function. The required importance sampling function $g(x)$ for the loads is assumed to follow a normal distribution. The mean value of $g(x)$ corresponds to the failure load when all other random values are kept fixed to their mean values.

A Six-storey space frame

This example consists of 63 elements with 180 degrees of freedom as shown in Figure 2. The length of the beams and the columns of the frame is $L_1=7.32$ m and $L_2=3.66$ m, respectively. The structure is loaded with a 19.16 kPa gravity load on all floor levels and a lateral load of 110 kN applied at each node in the front elevation along the z direction. The members of the structure are divided into five groups, as shown in Figure 1, each one having two design variables. The deterministic constraints are eleven, two for the stresses of each element group and one for the inter-storey drift. The type of probability density functions, mean values, and variances of the random parameters are presented in Table 1. For each geometric variable (i.e. the cross-sectional dimensions $b$, $h$) the mean value is taken as the current value of the corresponding design variable $s_i$. 
Figure 1: Description of the six-storey frame

<table>
<thead>
<tr>
<th>Random variable</th>
<th>Probability density function (pdf)</th>
<th>Mean value</th>
<th>Standard deviation (σ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>N</td>
<td>200</td>
<td>20</td>
</tr>
<tr>
<td>σ_y</td>
<td>N</td>
<td>25.0</td>
<td>2.5</td>
</tr>
<tr>
<td>b,h</td>
<td>N</td>
<td>s_i</td>
<td>0.1s_i</td>
</tr>
<tr>
<td>Loads</td>
<td>Log-N</td>
<td>640</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 1: Characteristics of the random variables for the six-storey frame

<table>
<thead>
<tr>
<th>Optimization procedure</th>
<th>ES cycles</th>
<th>p_γ</th>
<th>Optimum weight (tn)</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DBO</td>
<td>41</td>
<td>0.166</td>
<td>67.5</td>
<td>177</td>
</tr>
<tr>
<td>RBO</td>
<td>79</td>
<td>0.001</td>
<td>77.8</td>
<td>54,126</td>
</tr>
<tr>
<td>RBO-NN</td>
<td>81</td>
<td>0.001</td>
<td>77.9</td>
<td>9,471</td>
</tr>
</tbody>
</table>

Table 2: Performance of the methods for the six-storey frame

For this test case the (μ+λ)-ES approach is used with μ=λ=5, while a sample size of 500 simulations is taken for the MCS combined with the Important Sampling technique. As it can be observed from Table 2 the optimum weight achieved by the RBO is 15% more than the deterministic one (DBO). On the other hand, the probability of failure for the deterministic optimum is inapplicable since it exceeds the accepted value 10^{-3}. The proposed RBO-NN combination manages to achieve the optimum weight in one sixth of the CPU time required by the conventional RBO procedure.

6 CONCLUSIONS

The solution of realistic RBO problems in structural mechanics is an extremely computationally intensive task. In the test example considered the conventional RBO procedure was found over forty times more expensive than the corresponding deterministic optimization procedure. The aim of the proposed RBO procedure is to increase the safety margins of the optimized structures under various model uncertainties, while at the same time minimizing the weight of the structure as well as the additional computational cost. This goal was achieved using NN predictions to perform the structural analyses involved in MCS.

7 REFERENCES

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