Steady State Probabilistic Response of a Half Oscillator Under Colored, Gaussian or non-Gaussian Excitation

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ABSTRACT

In the present work the probabilistic characteristics of the long-time (when dynamic statistical equilibrium has been reached), steady-state response of a half oscillator, subject to a colored, asymptotically stationary, Gaussian or non Gaussian (cubic Gaussian) excitation, are derived by means of the Response-Excitation (RE) theory, first introduced by Athanassoulis & Sapsis (2006). RE theory permits us to derive an evolution equation for the joint Response-Excitation Probability Density Function (REPDF) of any dynamical system with polynomial nonlinearities under arbitrary stochastic excitation. This evolution equation is derived by projecting an exact, infinite dimensional, Functional Differential Equation, for the joint response-excitation characteristic functional, to finite dimensions. Application of this theory to the ship roll motion has been presented by Athanassoulis *et al* (2009). The joint REPDF evolution equation is a peculiar equation, involving two times (one for the excitation and one for the response), and two sets of probability arguments (one for the excitation variables, and another one for the response variables), and partial derivatives only with respect to the response time and the response probability arguments. In general it is not uniquely solvable and, thus, it needs to be completed by an appropriate closure scheme. In this paper we present a closure technique via localized linear problems, and a numerical solution to the steady-state case, providing the long-time, statistical equilibrium PDFs. The method of numerical solution is based on a representation of the sought-for REPDF by means of kernel density functions, and a Galerkin-type numerical scheme. The obtained PDFs are compared with results from Monte Carlo simulation for the same problem. The method can be extended to treat full nonlinear oscillators, with polynomial nonlinearities, subjected to Gaussian or non Gaussian excitation.

Keywords: *Probabilistic characterization of responses, non markovian responses, non gaussian excitation, stochastic modeling of non-linear systems, half oscillator.*

1. INTRODUCTION

Wave loads on ships are generally modelled as colored (smoothly-correlated) stochastic processes (in contrast to delta-correlated processes, which are commonly used in stochastic dynamic analysis; see e.g., Naess (2000), Iourtchenko (2003), Pirrotta (2007)). Thus, ship responses are also smoothly-correlated stochastic processes, lacking to obey the Markovian property, which essentially simplifies the analysis of stochastic systems. The determination of probability density functions (PDFs) associated with ship responses is straightforward as far as the assumption of linearity is (approximately) valid, and the excitation can be simplified as Gaussian. When strong nonlinearities are present, and the excitation cannot be considered neither Gaussian nor delta correlated, as in the case of roll motion, the classical theory of diffusion processes and the **Fokker-Planck-Kolmogorov** (FPK) **equation** is not applicable (see e.g. Neves *et al*, 2011, for a comprehensive review on issues involving dynamic behaviour and probabilistic nature of extreme events in a seaway). In such cases, the proba-

bilistic characterization of the responses is a difficult problem, calling for specific modelling techniques and advanced mathematical tools.

In cases where the correlation time of the excitation is small, in comparison with the relaxation time of the dynamical system, it is possible to reformulate the problem as an averaged Ito SDE and advance to the corresponding averaged FPK. This technique, known as **the stochastic averaging method**, was first introduced by Stratonovitch (1963) and made rigorous by Khasminkii (1966). Variants of this methodology have been applied to the ship rolling problem by Roberts (1982), Roberts and Vasta (2000), where also an extensive survey of the previous works is presented, Kreuzer and Sichermann (2007).

An approach which can resolve the non-Markovian character of the response, keeping a close connection with the standard treatment of stochastic differential equations, is the **filtering approach**. This method is implemented by making the colored excitation to be the output of an appropriate linear filter, coupled with the original dynamic system. In this way, an augmented stochastic system is obtained, which can be treated by means of a FPK equation involving the state-space variables of the original system and the filter (Spanos, 1986, Muscolino, 1995, Pugachev and Sinitsyn, 2001). The method has been applied to the ship roll motion by Francescutto and Naito (2004) and others. It is rather general and effective as far as the excitation is Gaussian and the appropriate filter is of low order. Hybrid techniques, combining some of the above methods have also been developed (Di Paola & Floris, 2008).

Another generic approach, applicable to quite general dynamical systems under colored stochastic excitation, introduced in 2006 (Athanassoulis & Sapsis, 2006), further developed in Sapsis & Athanassoulis (2008) (hereafter referred to as [1]), and applied to ship rolling motion by Athanassoulis *et al* (2009) (hereafter referred to as [2]). This approach is based on the use of the joint, response-excitation characteristic functional, which is an infinite-dimensional mathematical object modelling the complete probabilistic structure of the involved stochastic processes, without any simplifying assumptions. See Hopf (1952), who introduced the characteristic functional for the study of turbulent flows, and its extensions to other stochastic dynamical problems presented by Beran (1968). A Hopf-type functional differential equation (FDE) governing the evolution of the joint, response-excitation characteristic functional for quite general stochastic differential equations is easy to obtain. Since, however, the numerical solution of infinite-dimensional FDEs is not feasible (at least, for the time being), the goal of this approach is to come back to probabilistic equations for finite-dimensional PDFs, by appropriate projections of the FDE. For example, if we assume that the ship roll motion obeys the following dynamical equation (see, e.g., Belenky & Sevastianov, 2003).

$$(I+A)\ddot{x}(t)+b_1\dot{x}(t)+b_3\dot{x}^3(t)+ \\ +K_1x(t)+K_3x^3(t) = Y(t) \qquad (1)$$

where $Y(t)$ is the external stochastic excitation, the Response-Excitation (RE) theory developed in [1,2] leads to the following evolution equation for the joint Response–Excitation, Probability Density Function (REPDF) $f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)$ ($x_1(t)=x(t)$, $x_2(t)=\dot{x}_1(t)$; see [2] for a detailed description of all coefficients appearing in equ. (1)):

$$\frac{\partial}{\partial t} f_{x_1(t)x_2(t)y(s)}(\alpha_1\alpha_2,\beta)\Big|_{s\to t} + \\ + \mathsf{L}_{\alpha_1\alpha_2} f_{x_1(t)x_2(t)y(t)}(\alpha_1\alpha_2,\beta) + \\ + \beta\frac{\partial f_{x_1(t)x_2(t)y(t)}(\alpha_1\alpha_2,\beta)}{\partial\alpha_2} = 0 \qquad (2a)$$

supplemented by the marginal compatibility condition

$$f_{y(s)}(\beta)=\iint_{\mathbb{R}^2} f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)d\alpha_1 d\alpha_2 = \\ = \textit{known density function}, \quad \forall s\ge t_0 \qquad (2b)$$

and the initial condition.

$$f_{x_1(t_0)x_2(t_0)}(\alpha_1,\alpha_2) = \textit{known density function} \qquad (2c)$$

$\mathsf{L}_{\alpha_1\alpha_2}(\bullet)$, appearing in (2a), is a linear differential operator given by

$$\mathsf{L}_{\alpha_1\alpha_2}(\bullet) = \left[-\frac{1}{I+A}\left(b_1 + 3\,b_3\,\alpha_2^2 - \alpha_2\frac{\partial}{\partial\alpha_1} + \right.\right.$$
$$+ K_1\,\alpha_1\frac{\partial}{\partial\alpha_2} + K_3\,\alpha_1^3\frac{\partial}{\partial\alpha_2} + b_1\,\alpha_2\frac{\partial}{\partial\alpha_2} +$$
$$\left.\left. + b_3\,\alpha_2^3\frac{\partial}{\partial\alpha_2} + b_1\alpha_2\frac{\partial}{\partial\alpha_2} + b_3\alpha_2^3\frac{\partial}{\partial\alpha_2}\right)\right](\bullet) \qquad (3)$$

Equ. (2a) is a peculiar equation, involving two times and three probability arguments (one for the excitation, one for the motion and one for the velocity), and partial derivatives only with respect to one (response) time and two (motion-velocity) probability arguments. This peculiarity gives rise to fundamental questions regarding both the well-posedness of the problem (2a,b,c) and the methods of its numerical solution. Recently, our approach re-examined by Venturi *et al* (2012), using a different (but essentially equivalent) method. They confirmed the validity of equation derived in [1] (and, thus, indirectly, of equation (2a), derived in [2]), and answered in negative the question regarding the well-posedness of problem (2), by presenting a simple example in which our equation is valid but it does not ensures uniqueness. Accordingly, it becomes clear that a kind of completion of problem (2) is necessary. The type of completion proposed by Venturi *et al* (2012) results in a much more complicated equation, including the entire history of the response process in a functional integral form, which cannot be considered as an attractive alternative.

The crucial problem with the equ. (2a) is that the joint-REPDF $f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)$ is differentiated only with respect to the response time t. However, since the function $f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)$ is unknown, the partial-time (called hereafter *half-time*) derivative $\partial f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)/\partial t$ cannot be properly evaluated, because there is no way to separate the effect of the response time t from the effect of the excitation time s, without knowing the specific form of $f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)$. This fact calls for a kind of *a priori* approximation of the term $\lim_{s\to t}\partial f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)/\partial t$, before any attempt to formulate a numerical scheme for solving.

In the present paper we develop an *a priori* closure technique, for the long-time, steady-state REPDF evolution equation, by formulating and using localized linear problems. Up to now, this, improved, response-excitation theory has been developed only for first-order dynamical systems, that is, half-oscillators, under Gaussian excitation. (Athanassoulis *et al*, 2012). Accordingly, we shall present herewith the method of closure, the scheme of the numerical solution, and numerical results for a half-oscillator of the form:

$$\dot{x}(t;\theta) = H\big(x(t;\theta)\big) + \Psi\big(y(t;\theta)\big) \qquad (4a)$$

$$x(0;\theta) = x_0(\theta) \qquad (4b)$$

where θ is the stochastic argument (the sample-point indicator), $H(\bullet)$ and $\Psi(\bullet)$ are polynomial functions, and $y(t;\theta)$ is a given, smoothly correlated, asymptotically stationary, Gaussian stochastic process (regular colored noise). Accordingly, the polynomial excitation $\Psi\big(y(t;\theta)\big)$ can model (strongly) non-Gaussian processes. We shall focus on the long-time, steady-state response of equ. (4), assuming that the Gaussian process $y(t;\theta)$, shaping the excitation, becomes quickly stationary, as $t\to\infty$. Then, the evolution equation for the long time steady state REPDF $f_{x(t)y(s)}(\alpha,\beta)$, which is valid in the limit $s\to t$ (for s and t in the long-time regime) and for every point $(\alpha,\beta)\in\mathbb{R}^2$ of the RE phase space, reads as follows:

$$\frac{\partial}{\partial t} f_{x(t)y(s)}(\alpha,\beta)\Big|_{s\to t} + \qquad (5a)$$
$$+ \frac{\partial}{\partial\alpha}\Big[\big(H(\alpha)+\Psi(\beta)\big)\cdot f_{x(t)y(s)}(\alpha,\beta)\Big|_{s\to t}\Big] = 0$$

supplemented by the marginal-compatibility constraint,

$$\int_{\alpha\in\mathbb{R}} f_{x(t)y(s)}(\alpha,\beta)\,d\alpha = \\ = f_{y(s)}(\beta) = \text{ a known pdf} \tag{5b}$$

ensuring that $f_{x(t)y(s)}(\alpha,\beta)$ complies with the given marginal, as well as by the conditions:

$$f_{x(t)y(s)}(\alpha,\beta)\geq 0, \tag{5c}$$

$$\int_{\beta\in\mathbb{R}}\int_{\alpha\in\mathbb{R}} f_{x(t)y(s)}(\alpha,\beta)\,d\beta\,d\alpha = 1. \tag{5d}$$

No initial condition needs to be assumed, since in the long-time statistical equilibrium system's response is no longer dependent on the initial conditions.

2. LOCAL DESCRIPTION OF THE RESPONSE-EXCITATION CORRELATION STRUCTURE

In order to implement our *a priori* closure scheme for equ. (5a), we define and solve localized linear problems, providing us with information concerning the local RE-correlation structure. Focusing on a (any) specific point y_0 of the excitation state space, we find the corresponding (deterministic) long-time equilibrium point x_0 in the response state space, by solving the equation $H(x_0)=-\Psi(y_0)$. Introducing a localized, around the point (x_0,y_0), linear approximation of the right-hand side of equ.(4a), we get:

$$\dot{x}(t;\theta) \approx H'(x_0)\cdot\big(x(t;\theta)-x_0\big) + \\ + \Psi'(y_0)\cdot\big(y(t;\theta)-y_0\big) \tag{6}$$

and formulate the following ***localized version*** of equ. (4a):

$$\dot{x}_{\ell oc}(t;\theta) = H'(x_0)\cdot\big(x_{\ell oc}(t;\theta)-x_0\big) + \\ + \Psi'(y_0)\cdot\big(y_{\ell oc}(t;\theta)-y_0\big) \tag{7}$$

To ensure stability we always assume that $H'(x_0)<0$. The localized excitation $y_{\ell oc}(t;\theta)$ is considered Gaussian (as $y(t;\theta)$), with mean value $m_{y_{\ell oc}}=y_0$ and an appropriate autocovariance function $C_{y_{\ell oc}y_{\ell oc}}(t;s)$. Since we are interested in the long-time regime, we choose a scaled version of the long-time limit of the global autocovariance function $C_{yy}^{(\infty)}(t-s)$, that is $C_{y_{\ell oc}y_{\ell oc}}(t-s) = \sigma^2_{y_{\ell oc}}\cdot C_{yy}^{(\infty)}(t-s)/\sigma_y^2$. However, the long-time correlation matrix of the local linear problem is scaled uniformly by $\sigma^2_{y_{\ell oc}}/\sigma_y^2$, and we can assume $\sigma^2_{y_{\ell oc}}=\sigma_y^2$ and introduce the scaling later on.

The localized, linear, stochastic differential equation (7) is readily solved analytically. The long-time covariances $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(w)$ and $C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(w)$ are given by the formulae:

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(w) = \lim_{\substack{t\to\infty \\ w=\text{const.}}} C_{x_{\ell oc}y_{\ell oc}}(t,t-w) = \\ = \Psi'(y_0)\cdot\int_{-w}^{\infty} e^{H'(x_0)\cdot(u+w)}\cdot C_{y_{\ell oc}y_{\ell oc}}(u)\,du, \tag{8}$$

and

$$C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(w) = \frac{\big(\Psi'(y_0)\big)^2}{2\big(-H'(x_0)\big)}\times \\ \times\int_{v=-\infty}^{v=+\infty} C_{y_{\ell oc}y_{\ell oc}}(v)\cdot e^{\,H'(x_0)\cdot|v-w|}\,dv \tag{9}$$

We shall now specialize the results (8) and (9) for the covariance function

$$C_{y_{\ell oc}y_{\ell oc}}(t-s) = C_{yy}^{(\infty)}(t-s) = \\ = \sigma_y^2\cdot\exp\big(-a(t-s)^2\big) \tag{10}$$

The integrals appearing in the right-hand side of equs. (8) and (9) can be calculated explicitly. The results read as follows:

$$C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s) = \frac{\sqrt{\pi}}{2\sqrt{a}}\cdot\sigma_y^2\cdot\Psi'(y_0)\times$$
$$\times\exp\left(\frac{(H'(x_0))^2}{4\cdot a}+H'(x_0)\cdot(t-s)\right)\times$$
$$\times\left\{\mathrm{erf}\left(\sqrt{a}\cdot(t-s)+\frac{H'(x_0)}{2\cdot\sqrt{a}}\right)+1\right\} \quad (11)$$

$$C^{(\infty)}_{x_{\ell oc}x_{\ell oc}}(t-s) = \frac{\Psi'(y_0)}{-H'(x_0)}\cdot\frac{1}{2}\times$$
$$\times\left\{C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)+C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(s-t)\right\} \quad (12)$$

Using equ. (11) we are able to determine the partial-time derivative of $C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)$ with respect to the excitation time t. After some algebraic manipulations, the final result reads as follows:

$$\frac{\partial C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)}{\partial t} = \quad (13)$$
$$= H'(x_0)\cdot C^{(\infty)}_{x_{\ell oc}y_{\ell oc}}(t-s)+\Psi'(y_0)\cdot C^{(\infty)}_{y_{\ell oc}y_{\ell oc}}(t-s)$$

The derived auxiliary conditions (11)-(13), provide the local correlation structure between $x(t;\theta)$ and $y(s;\theta)$, as $s\to t$, in the vicinity of the RE phase space point $(\alpha,\beta)=(x_0,y_0)$. This information will be exploited in the next section, in order to implement the long time limit $\lim_{s\to t}\partial f_{x(t)y(s)}(\alpha,\beta)/\partial t$.

3. KERNEL DENSITY REPRESENTATION FOR THE JOINT RESPONSE-EXCITATION AND MARGINAL PDFS

The target of the numerical solution to equ. (5a), supplemented by all appropriate auxiliary conditions (already discussed), is to find the time-independent (statistical equilibrium) joint REPDF $f_{xy}(\alpha,\beta)=f_{x(t)y(t)}(\alpha,\beta)$. However, in order to cope with the appearance of the unusual, response-time (half-time) derivative in equ. (5a), a representation of the lag-time dependent joint REPDF $f_{x(t)y(s)}(\alpha,\beta)$ capable of embedding the acquired local conditions shall be introduced. On the basis of the above and previous (successful) experience in representing PDFs by superposition of kernel density functions (Athanassoulis and Belibassakis 2002, Athanassoulis and Gavriliadis 2002), the following Kernel Density Representation (KDR) is adopted:

$$f_{x(t)y(s)}(\alpha,\beta) = \sum_{i,j} p_{ij}\cdot K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\right) \quad (14)$$

where $(\alpha_i,\beta_j),(i,j)\in\mathbb{Z}\times\mathbb{Z}$, is a grid of points in the state space $\mathbb{R}\times\mathbb{R}$, each (α_i,β_j) serving as the center of the ***Gaussian kernel density function*** $K(\alpha,\beta;\cdot,\cdot,\bullet)$, while

$$\Sigma_{\alpha i,\beta j}(t-s) = \begin{pmatrix} C_{\alpha_i\alpha_i}(0) & C_{\alpha_i\beta_j}(t-s) \\ C_{\alpha_i\beta_j}(t-s) & C_{\beta_j\beta_j}(0)\end{pmatrix} \quad (15)$$

is the covariance matrix of $K(\alpha,\beta;\cdot,\cdot,\bullet)$. To ensure that (14) is a legitimate PDF, the following constraints need to be imposed on the unknown coefficients $p_{ij},(i,j)\in\mathbb{Z}\times\mathbb{Z}$:

$$0\le p_{ij} \quad \text{and} \quad \sum_{i,j}p_{ij}=1. \quad (16a,b)$$

In principle, $f_{x(t)y(t)}(\alpha,\beta)$ is supported on the whole plane $\mathbb{R}\times\mathbb{R}$. For computational reasons, we focus on its form in its essential support D_{ess}, conventionally defined as the subset of $\mathbb{R}\times\mathbb{R}$ where $f_{xy}(\alpha,\beta)>\varepsilon\approx 10^{-3}\cdot\max\{f_{xy}(\alpha,\beta)\}$. This choice, restricts the approximation in a compact subdomain $D_{\alpha\beta}=[\alpha_{\min},\alpha_{\max}]\times[\beta_{\min},\beta_{\max}]$, of $\mathbb{R}\times\mathbb{R}$, such that $D_{ess}\subseteq D_{\alpha\beta}$ (tail questions are not considered herewith), and the indices (i,j) run over the finite set $\mathcal{N}(I)\times\mathcal{N}(J)$, where $\mathcal{N}(I)=\{1,2,\ldots,I\}$ and $\mathcal{N}(J)$ is similarly defined. Since D_{ess} is not known a priori some preliminary information is necessary in order to choose the computational domain $D_{\alpha\beta}$. This information is provided by the essential support of the known excitation $f_y(\beta)$, in conjunction

with an estimate of the total response variance, approximated by solving a global moment problem using Gaussian closure.

Introducing the KDR in equs. (5a) (the constitutive conditions, equs. (5c,d) are automatically satisfied thanks to the defining properties of the KDR), we obtain the following reformulation of problem (5a,c,d):

$$\sum_{i,j} p_{ij}\left[\frac{\partial}{\partial t}K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\right)\Big|_{s\to t} + \right.$$
$$+\frac{\partial}{\partial\alpha}\Big\{\big(H(\alpha)+\Psi(\beta)\big)\times$$
$$\left.\times K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(0)\right)\Big|_{s\to t}\Big\}\right] = 0 \quad (17a)$$

under the marginal compatibility constraint

$$\sum_{i,j} p_{ij} K_{\beta_{j(i)}}\left(\beta;\beta_{j(i)},\sigma_{\beta j\,(i)}\right) - f_y(\beta) = 0 \quad (17b)$$

where $K_{\beta_{j(i)}}\left(\beta;\beta_{j(i)},\sigma_{\beta j\,(i)}\right) =$

$$= \int_{\alpha\in\mathbb{R}} K\left(\alpha,\beta;\alpha_i,\beta_j,\Sigma_{\alpha i,\beta j}(t-s)\right)d\alpha$$

Note that, the half-time derivative $\partial K\left(\cdots,\Sigma_{\alpha i,\beta j}(t-s)\right)/\partial t$, appearing in equ. (17a), is now reduced to the corresponding derivative of the covariance matrix $\Sigma_{\alpha i,\beta j}(t-s)$, which is a priori estimated by means of the results of the previous section.

4. GALERKIN TYPE DISCRETIZATION OF THE PROBLEM

On the basis of the KDR, equ. (14), the determination of the sought-for joint REPDF has been reduced to the determination of the coefficients p_{ij}, $(i,j)\in\mathcal{N}(I)\times\mathcal{N}(J)$, from the system of equs. (17a,b). This problem can be solved using a Galerkin type, weighted-residual method (Kantorovich and Krylov, 1964, Zeidler, 1990) to find a discrete system of equations, approximately equivalent to equs. (17a,b). Using Gaussian Galerkin Kernels $\Lambda_{\kappa,\lambda}(\alpha,\beta)$ and $\tilde{\Lambda}_{\tilde{\lambda}}(\beta)$ (see Athanassoulis *et al* 2012) we obtain the following linear system for p_{ij} coefficients:

$$\sum_{i,j} p_{ij}\cdot G_{ij,\kappa\lambda} = 0\,,\quad \forall(\kappa,\lambda)\in\mathcal{N}(K)\times\mathcal{N}(L)\,,$$
$$\sum_{i,j} p_{ij}\cdot\tilde{G}_{j(i),\tilde{\lambda}} = g_{\tilde{\Lambda}_{\tilde{\lambda}}}(f_y),\quad \forall\tilde{\lambda}\in\mathcal{N}(\tilde{L}) \quad (18a,b)$$

where $G_{ij,\kappa\lambda}$, $\tilde{G}_{j(i),\tilde{\lambda}}$ are integrals of Gaussian functions (calculated analytically) and

$$g_{\tilde{\Lambda}_{\tilde{\lambda}}}(f_y) = \int_{\mathbb{R}} f_y(\beta)\cdot\tilde{\Lambda}_{\tilde{\lambda}}(\beta)\,d\beta\,.$$

On the basis of the above discussion, the problem of calculating the expansion coefficients p_{ij} of the joint-REPDF takes the following form:

> Problem P : Find p_{ij}, $(i,j)\in\mathcal{N}(I)\times\mathcal{N}(J)$, satisfying the homogeneous equation (18a), under the marginal compatibility constraint (18b) and the constitutive constraints (16a,b).

5. RESULTS - SOLUTION OF THE HALF-OSCILLATOR PROBLEM

In this section numerical results will be presented for the special case of equ. (4a), describing a cubic half-oscillator under cubic excitation:

$$\dot{x}(t;\theta) = \left(\mu_1+\mu_2\cdot x(t;\theta)^2\right)\cdot x(t;\theta) + \left(\kappa_1+\kappa_2\cdot y(t;\theta)^2\right)\cdot y(t;\theta) \quad (19)$$

where $y(t;\theta)$ is a Gaussian stochastic process, with long-time correlation function given by equ. (10). Two cases have been considered: the case of a linear half-oscillator under non Gaussian (cubic) excitation, with parameter values $\mu_1=-1$, $\mu_2=0$, $\kappa_1=0$, $\kappa_2=1$, (hereby called **Case 1)** and the case of a non-linear half-oscillator under Gaussian excitation, with parameter values $\mu_1=\mu_2=-1$, $\kappa_1=1$, $\kappa_2=0$ (hereby called **Case 2**). The correlation time of the Gaussian excitation process $y(t;\theta)$

is given, by $\tau_{\text{corr}}^{(y)} = \int_0^\infty C_{yy}(\tau)d\tau / C_{yy}(0) = = 0.5\sqrt{\pi / a}$, while the relaxation time of the system is estimated by the linear relaxation time $\tau_{\text{relax}}^{(\text{lin})} = 1$ (which is an overestimation regarding Case 2). Two methods have been used for the derivation of the joint REPDF $f_{xy}(\alpha,\beta)$ and the marginal PDF $f_x(\alpha)$. The numerical solution of the constraint optimization problem P (hereby referred to as **RE solution**), and a conventional Monte Carlo scheme (hereby referred to as **MC simulation**). Systematic comparisons of the results obtained by the two methods are also presented.

In the context of the RE theory, Problem P is solved numerically in three steps. In the **first step,** representation kernels and Galerkin kernels are identified. Their centers α_i, β_j, $i \in \mathcal{N}(I)$, $j \in \mathcal{N}(J)$, are placed on a regularly spaced grid over the computational domain $D_{\alpha\beta}$. The kernel variances $C_{\alpha_i\alpha_i}, C_{\beta_j\beta_j}$ are adjusted to $D_{\alpha\beta}$ and the resolution of the grid, aiming at a certain degree of overlapping between contiguous kernels. The kernel covariances $C_{\alpha_i\beta_j}$, are defined by means of the formula $C_{\alpha_i\beta_j} = \rho_{loc}^{(\infty)}(0)\cdot\sqrt{C_{\alpha_i\alpha_i}}\cdot\sqrt{C_{\beta_j\beta_j}}$, where the local correlation coefficient $\rho_{loc}^{(\infty)}(0)$ is calculated from the localized linear problem. The long-time limit of the half-time derivative $\partial C_{\alpha_i\beta_j} = \lim_{s\to t}\partial C_{\alpha_i\beta_j}(t-s)/\partial t$, necessary in order to fully specify the coefficients $G_{ij,\kappa\lambda}$, is estimated from the localized approximation, equ. (13). The Galerkin kernels have been selected to be Gaussian kernels, identical with the representation ones. With these choices all coefficients of equs. (18) are fully specified and thus we can proceed to the **second step**, namely, the numerical solution of problem P. This is performed using LSQLIN, the constrained least squares MATLAB® function. The solution yields p_{ij}, from which a first estimate of the joint-REPDF is obtained. In the **third step** the solution obtained in the second step is exploited in order to estimate the essential support D_{ess}, to redistribute the kernels, and redefine the kernel parameters. Within usually one or two iterations of step 3, the essential support converges, and the final solution $f_{xy}(\alpha,\beta)$ is extracted.

The Monte Carlo simulation is obtained by generating 4000 samples of the excitation process $y(t;\theta)$, using the 1-D random-phase model. Equ. (19) (with zero initial condition) is solved using ODE45, a MATLAB® implementtation of the Dormant-Prince method (Dormand and Prince 1980), based on an explicit Runge-Kutta (4,5) formula. The MC PDF estimations are computed using the kernel density estimation via diffusion, introduced by Botev *et al* (2010) and coded in MATLAB® functions by the same author.

Results are presented for two different values of parameter a, namely $a =$ 3 and 7, corresponding to excitation correlation time: $\tau_{\text{corr}}^{(y)} =$ 0.51 and 0.33 respectively. Since the nonlinearity of equ. (19) contributes to damping terms, the relaxation time of Case 2 is smaller than that of Case 1, for which $\tau_{\text{relax}}^{(\text{lin})} = 1$. Thus, the values $a =$ 3 and 7 correspond to $\lambda = \tau_{\text{corr}}^{(y)} / \tau_{\text{relax}}^{(\text{lin})} =$ 0.51, 0.33. In all cases $\sigma_y^2 = 1$.

In Figures 1 and 5, the joint REPDF is shown, as calculated by the MC simulation for Case 1 and 2 respectively. The corresponding calculations using RE theory are illustrated in Figures 2 and 6. The latter Figures also depict the marginals obtained by the two methods. The absolute difference between the MC and RE method is shown in Figure 3 for Case 1, and Figure 7 for case 2. This difference is, in general, less than 5% in both Cases, except for the high probability areas in the strongly colored examples ($a =$ 3), where it locally reaches a maximum of 20%. The latter, local, high mismatch should be associated with the local steepness of the corresponding PDFs. In general, the PDFs shapes, as obtained by the

two methods, are very similar, as can be seen from Figs. 1, 2 and 5, 6. In addition, the response marginal PDFs calculated by the two methods, compare very satisfactorily in both Cases, regardless of the color strength

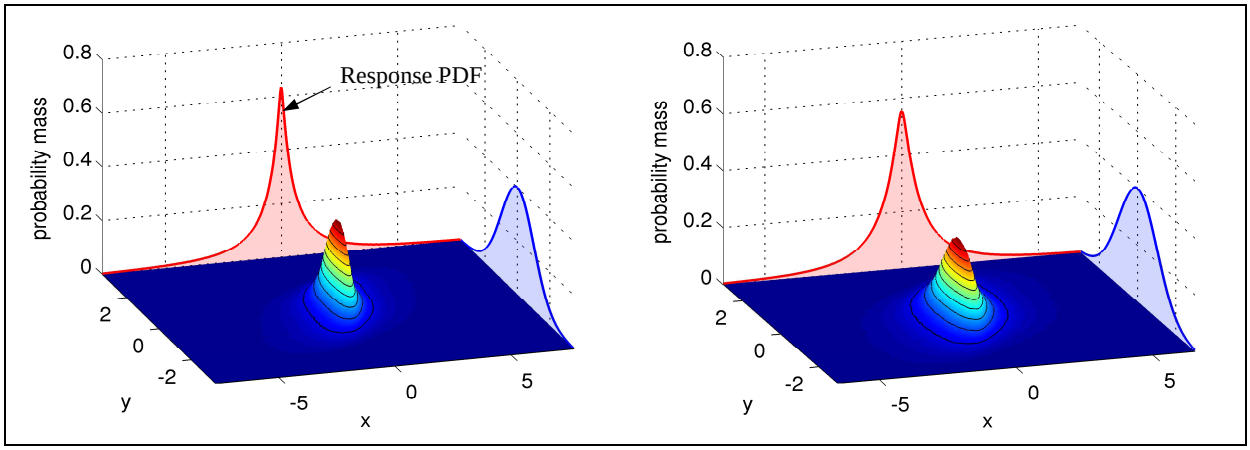

Figure 1: (Color online). REPDFs as calculated via MC solution for Case 1 and $a = 3, 7$. The projections depict the marginal PDFs.

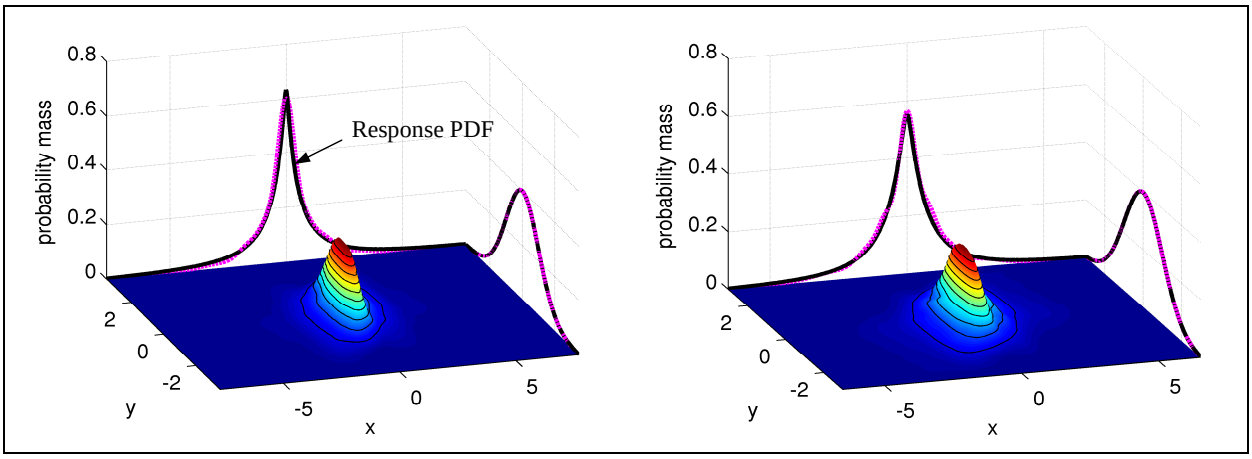

Figure 2: (Color online). REPDFs as calculated using the RE solution for Case 1 and $a = 3, 7$. The marginal projections depict both MC (solid lines) and RE solutions (dashed lines).

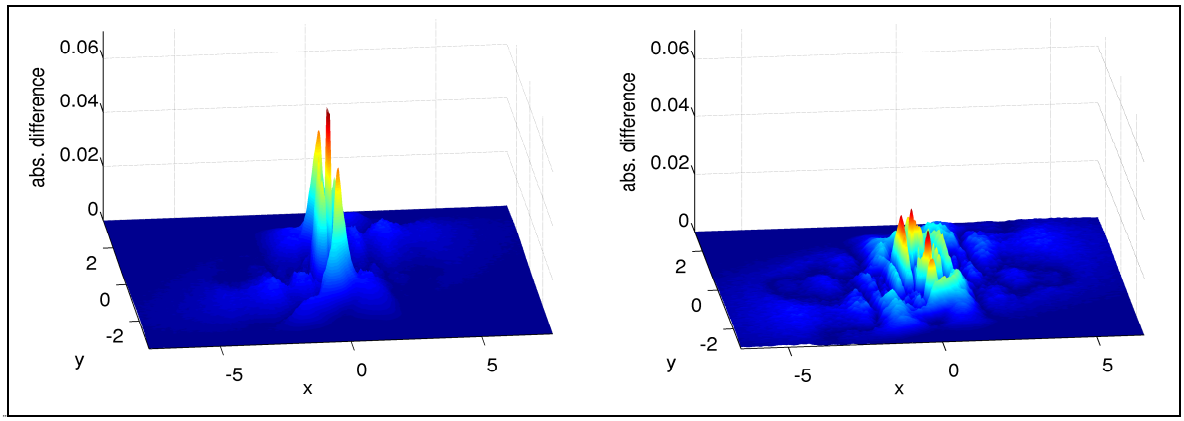

Figure 3: (Color online). The absolute difference between RE and MC solutions: Case 1, $a = 3, 7$.

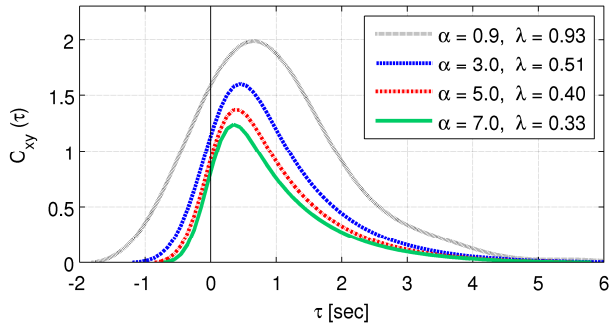

Figure 4: (Color online). The RE covariance function for Case 1.

It is interesting to notice the strong deviations of the calculated PDFs from the "equivalent" 2D Gaussian distributions, for both examined cases. The same also holds for the response densities, as intuitively expected. Apart from the examples shown here, the RE solution is also applicable for higher values of ratio λ, as shown in recent work (Athanassoulis *et al*, 2012), where the joint REPDF becomes bi-modal, although the examined system is mono-stable (regarding the bi-modality of bi-stable systems see also Grigolini *et al*, 1988, Jung & Risken, 1985).

In Figs. 4 and 8, the response-excitation covariance $C_{xy}(\tau)$ for Cases 1 and 2 is plotted, as obtained by MC simulation. $\tau < 0$ corresponds to future lag values (excitation in advance of response). In contrast to cases of delta-correlated excitation, there is a correlation between the current response value and the future excitation. It is interesting to point out that, after a rescaling, $C_{xy}(\tau)$ has a very similar shape for both examined cases, when a values are equal.

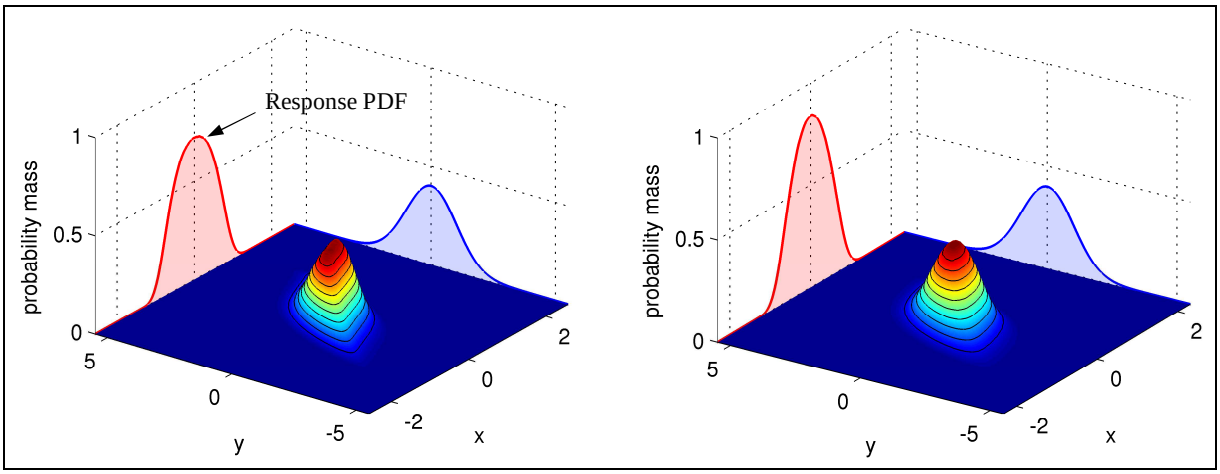

Figure 5: (Color online). REPDFs as calculated via MC solution for Case 2 and $a = 3, 7$. The projections depict the marginal PDFs.

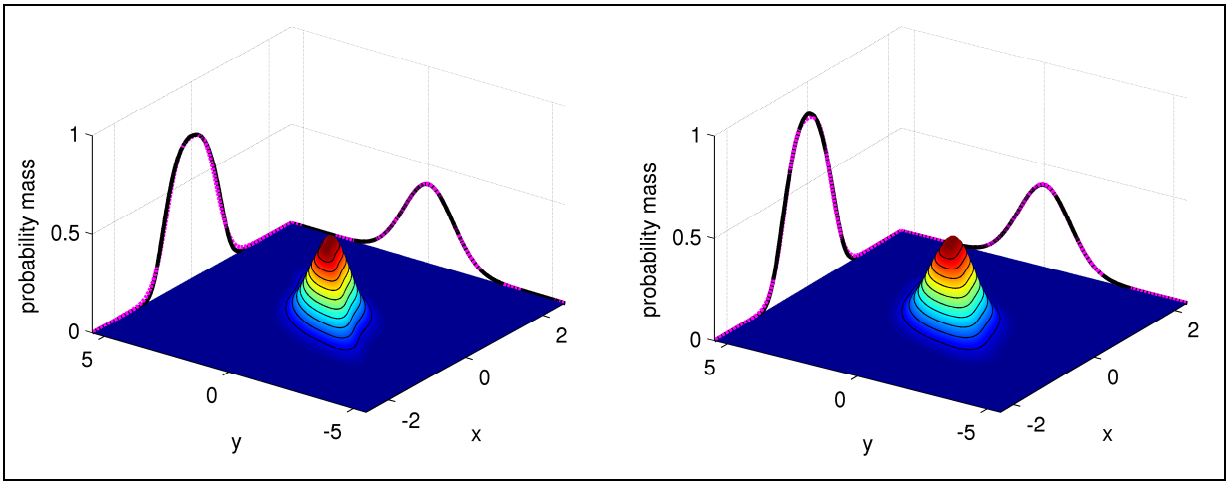

Figure 6: (Color online). REPDFs as calculated using the RE solution for Case 2 and $a = 3, 7$. The marginal projections depict both MC (solid lines) and RE solutions (dashed lines).

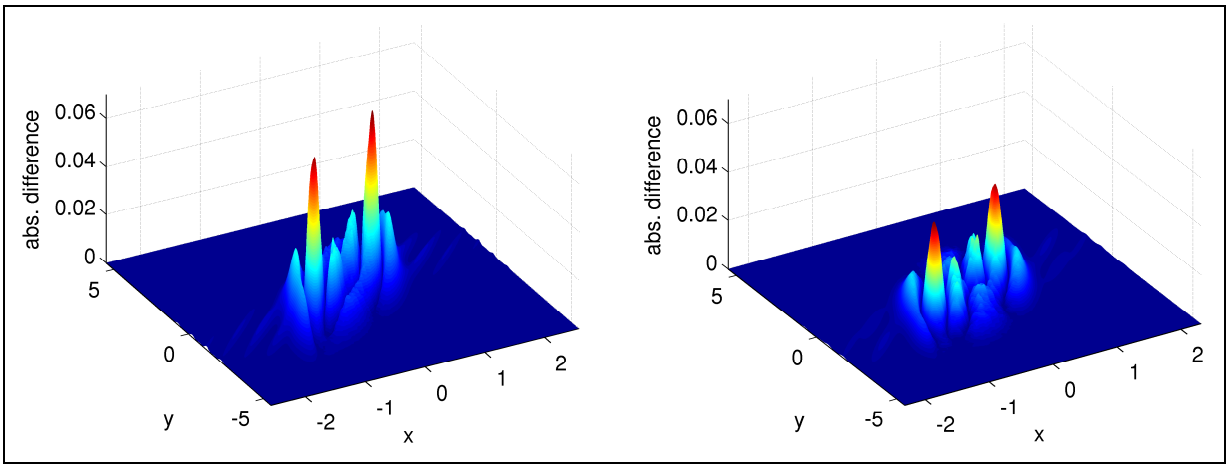

Figure 7: (Color online). The absolute difference between RE and MC solutions: Case 2, $a = 3, 7$.

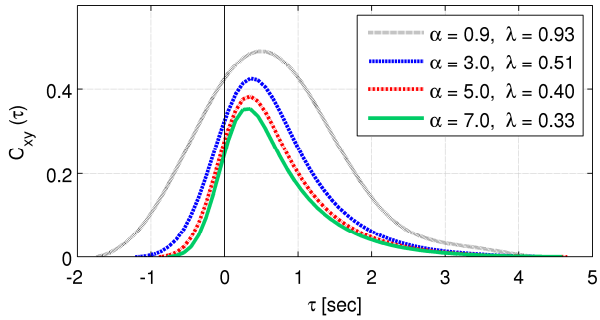

Figure 8: (Color online). The RE covariance function for Case 2.

6. CONCLUSIONS

In this paper we present, for the first time, an *a priori* closure scheme, and a method for numerical solution of the joint REPDF evolution equation, introduced in [1] and applied to ship rolling motion in [2]. The method is used to solve numerically two examples, a linear half-oscillator excited by a non-Gaussian random process, and a non-linear half-oscillator excited by Gaussian noise (see also Athanassoulis *et al*, 2012). The numerical results obtained have been confirmed via MC simulation. It is clear that the present method can be

extended straightforwardly to the general case of equ. (19) (nonlinear half-oscillator under non Gaussian excitation), covering both monostable and bistable cases. The extension of the presented methodology to full (2D) oscillators (e.g., to the 2nd-order ship roll motion equ. (1)), is currently under investigation.

7. ACKNOWLEDMENTS

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