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The Magnetic Fusion Energy Formulary

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Preface

This formulary is the product of two graduate students who became frustrated with referring to a dozen textbooks while studying for the MIT qualifying exams.

The guiding principle behind this work was to create a comprehensive reference for students and scientists working in the field of magnetic confinement fusion. We view the MFE Formulary as a complement to, rather than a competitor of, the NRL Plasma Formulary; it contains a far greater breadth and depth of mathematics and physics that is specific to magnetic fusion at the expense of a wealth of pure plasma physics.

The formulary consists of three broad sections. The first section (Chapters 1–2) covers the mathematics, fundamental units, and physical constants relevent to magnetic fusion. The second section (Chapters 3–9) covers the basic physics of thermonuclear fusion plasmas, beginning with electrodynamics as a foundation and developing single particle physics, plasma parameters, plasma models, plasma transport, plasma waves, and nuclear physics. The third and final section (Chapters 10–13) covers the physics of toroidally confined core and edge plasmas, as well as the fundamentals of magnetic fusion energy in deuterium-tritium tokamaks. Chapter 13 contains a large table of parameters for major tokamaks of the world.

With very few exceptions, everything found in the formulary has been taken from an original source, such as peer-reviewed literature, evaluated nuclear data tables, or the pantheon of "standard" mathematics and physics textbooks commonly used in magnetic fusion energy. The user will find that most items have a reference and, in the case of a textbook reference, a page number such that he/she may consult the original source with ease. In addition to providing transparency, this unique feature transforms the formulary, a useful collection of information, into a gateway to a deeper understanding of the critical equations, derivations, and physics for magnetic fusion energy.

References are given immediately following the cited item in superscript form as "a:b", where "a" is the number of the reference and "b" is the page number if the reference is a textbook. Full bibliographic entries for all references may be found at the end of the formulary. These references are naturally not the only ones that exist for these concepts, but they are what the authors used to generate this work.

As this is the first edition of the MFE formulary, it is by no means complete or error-free. We welcome suggestions for additional material, comments on the layout and usability, and particularly corrections to the pesky errors and typos we have tried so hard to eliminate. Please contact us at `mfe_formulary@mit.edu`.

Ultimately, we hope that this work is useful to all those trying to make magnetic fusion energy a reality.

Z & Y

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Disclaimer

Despite vigorous proofreading, no guarantee is provided by the authors as to the acuracy of the material in the MFE Formulary. The authors shall have no liability for direct, indirect, or other undesirable consequences of any character that may result from the use of the material in this work. This may include, but is not limited to, your analysis code not working or your tokamak not igniting. The reader is encouraged to consult the original references in all cases and is advised that the use of the materials in this work is at his or her own risk.

Chapter 1

Mathematics

$\mathbf{A}$,$\mathbf{B}$, ..., are vector functions
$\overleftrightarrow{\mathbf{T}}$ is a tensor
ψ and ξ are scalar functions
σ and τ refer to surfaces and volumes, respectively
$d\boldsymbol{\sigma}$ is a differential surface element pointing away from the volume
$d\tau$ is a differential volume element
$d\boldsymbol{r}$ is a differential line element

1.1 Vector Identities

1.1.1 Identities Involving Only Vectors [12:4]

(a) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$

(b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

(c) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$

(d) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}(\mathbf{A} \times \mathbf{B} \cdot \mathbf{D}) - \mathbf{D}(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})$

1.1.2 Identities Involving ∇ [12:4] [10]

(a) $\nabla \cdot (\psi \mathbf{A}) = \psi(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla \psi)$

(b) $\nabla \times (\psi \mathbf{A}) = \psi(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla \psi)$

(c) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(d) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

(e) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(f) $\nabla \cdot (\mathbf{A}\mathbf{B}) = (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B}$

(g) $\nabla \cdot (\psi \overleftrightarrow{\mathbf{T}}) = \nabla \psi \cdot \overleftrightarrow{\mathbf{T}} + \psi \nabla \cdot \overleftrightarrow{\mathbf{T}}$

(h) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(i) $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$

(j) $\nabla(\psi\xi) = \nabla(\phi\xi) = \psi\nabla\xi + \xi\nabla\psi$

(k) $\nabla \cdot (\nabla\psi \times \nabla\xi) = 0$

(l) $\nabla \cdot \nabla\psi = \nabla^2\psi$

(m) $\nabla \times \nabla\psi = 0$

1.1.3 Identities Involving $\int$ [12:5]

(a) $$\int_{volume} \nabla\psi \, d\tau = \int_{surface} \psi \, d\boldsymbol{\sigma}$$

(b) $$\int_{volume} \nabla \times \mathbf{A} \, d\tau = \oint_{surface} d\boldsymbol{\sigma} \times \mathbf{A}$$

(c) $$\int_{surface} d\boldsymbol{\sigma} \cdot \nabla \times \mathbf{A} = \oint_{boundary} d\boldsymbol{r} \cdot \mathbf{A}$$

(d) $$\oint_{boundary} d\mathbf{r} \times \mathbf{A} = \int_{surface} (d\boldsymbol{\sigma} \times \nabla) \times \mathbf{A}$$

1.2 Curvilinear Coordinate Systems

1.2.1 Cylindrical Coordinates (r, θ, z) [12:6–7 10]

Differential volume: $d\tau = r\, dr\, d\theta\, dz$

Relation to cartesian coordinates:

$$\begin{aligned} x &= r\cos\theta & \qquad \hat{\boldsymbol{x}} &= \cos\phi\,\hat{\boldsymbol{r}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ y &= r\sin\theta & \hat{\boldsymbol{y}} &= \sin\phi\,\hat{\boldsymbol{r}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ z &= z & \hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}} \end{aligned}$$

Unit vector differentials

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \hat{\boldsymbol{\theta}}\frac{d\theta}{dt} \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\hat{\boldsymbol{r}}\frac{d\theta}{dt}$$

Gradient

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

Curl

$$\begin{aligned}\nabla \times \mathbf{A} = &\left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\boldsymbol{r}} \\ &+ \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} \\ &+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{z}}\end{aligned}$$

Laplacian

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial \theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

Vector-dot-grad

$$\begin{aligned}\left(\mathbf{A}\cdot\nabla\right)\mathbf{B} = &\left(A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r}\right)\hat{\boldsymbol{r}} \\ &+ \left(A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + A_z\frac{\partial B_\theta}{\partial z} + \frac{A_\theta B_r}{r}\right)\hat{\boldsymbol{\theta}} \\ &+ \left(A_r\frac{\partial B_z}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta} + A_z\frac{\partial B_z}{\partial z}\right)\hat{\boldsymbol{z}}\end{aligned}$$

1.2.2 Spherical Coordinates (r, θ, ϕ) [12:8–9] [10]

Differential volume: $d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Relation to cartesian coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi & \hat{\boldsymbol{x}} &= \sin\theta\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ y &= r\sin\theta\sin\phi & \hat{\boldsymbol{y}} &= \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ z &= r\cos\theta & \hat{\boldsymbol{z}} &= \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{aligned}$$

Gradient

$$\nabla\psi = \frac{\partial \psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Curl

$$\nabla \times \mathbf{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}}$$
$$+ \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\right)\hat{\boldsymbol{\theta}}$$
$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$$

Laplacian

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Vector-dot-grad

$$(\mathbf{A}\cdot\nabla)\,\mathbf{B} = \left(A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r}\right)\hat{\boldsymbol{r}}$$
$$+ \left(A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi} + \frac{A_\theta B_r}{r} - \frac{\cot\theta A_\phi B_\phi}{r}\right)\hat{\boldsymbol{\theta}}$$
$$+ \left(A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} + \frac{A_\phi B_r}{r} + \frac{\cot\theta A_\phi B_\theta}{r}\right)\hat{\boldsymbol{\phi}}$$

1.3 Integral Relations of Vector Calculus

In this section, let $\mathbf{A} \equiv \mathbf{A}(\mathrm{x}_i,\mathrm{x}_j,\mathrm{x}_k)$ be a vector function that defines a vector field.

1.3.1 The Fundamental Theorem of Calculus

If $f(x)$ is a single-valued function on the interval [a,b] [14: 88]

$$\int_a^b f(x)dx = F(b) - F(a)$$

1.3.2 Gauss's (or the Divergence) Theorem

If τ is a volume enclosed by a surface σ, where $\boldsymbol{d\sigma} = \hat{\boldsymbol{n}}d\sigma$ and $\hat{\boldsymbol{n}}$ is a unit vector pointing away from τ [10: 31]

$$\int_{volume} (\nabla\cdot\mathbf{A})\; d\tau \;=\; \oint_{surface} \mathbf{A}\cdot\mathbf{d}\boldsymbol{\sigma}$$

1.3.3 Stoke's (or the Curl) Theorem

If σ is an open surface defined by a boundary contour at the surface edge [10: 34]

$$\int_{surface} (\nabla\times\mathbf{A})\cdot\mathbf{d}\boldsymbol{\sigma} \;= \oint_{contour} \mathbf{A}\cdot d\mathbf{r}$$

1.4 Legendre Polynomials

Legendre's equation [14: 337]

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \qquad -1 \le x \le 1 \text{ and } l = 0, 1, 2, \cdots$$

Legendre Polynomials [14: 289]

Order	Corresponding polynomial
$l = 0$	$P_0(x) = 1$
$l = 1$	$P_1(x) = x$
$l = 2$	$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$l = 3$	$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$l = 4$	$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
$l = 5$	$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$
$\cdots$	$\cdots$

Rodrigues' formula [14: 286]

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

Orthonormality [14: 286]

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = \int_0^{\pi} P_l(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \frac{2}{2l+1}\delta_{lm}$$

where δ_{lm} is the Kronecker delta: $l = m, \delta_{lm} = 1; l \neq m, \delta_{lm} = 0.$

1.5 Bessel Functions

1.5.1 Bessel's Equation

The most general form of Bessel's equation is [14: 269]

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\lambda^2 - \frac{p^2}{x^2}\right)y = 0$$

which has the general solution [14: 270]

$$y = AJ_p(\lambda x) + BY_p(\lambda x)$$

where J_p are Bessel functions of the first kind and Y_p are Bessel functions of the second kind (also known as Neumann Functions N_p), both of order p. Bessel functions of the first kind have no closed form representation;

however, they can be used to define Bessel functions of the second kind: [14: 270]

$$Y_p(x) = \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}$$

1.5.2 Bessel Function Relations

The following relationships are also valid for $Y_p(x)$ by replacing $J_p(x)$ with $Y_p(x)$ [14: 278–279]

(a) $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$

(b) $\frac{d}{dx}[J_0(x)] = -J_1(x)$

(c) $\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$

(d) $\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$

(e) $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x)$

(f) $J_{p-1}(x) - J_{p+1}(x) = 2\frac{d}{dx}J_p(x)$

(g) $\frac{d}{dx}J_p(x) = -\frac{p}{x}J_p(x) + J_{p-1}(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$

1.5.3 Asymptotic forms of Bessel Functions

For $x \to \infty$ [14: 273]

$$J_p(x) \approx \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{1}{2}p\pi - \frac{1}{4}\pi\right)\right]$$

$$Y_p(x) \approx \sqrt{\frac{2}{\pi x}}\left[\sin\left(x - \frac{1}{2}p\pi - \frac{1}{4}\pi\right)\right]$$

For $p \to \infty$ [14: 273]

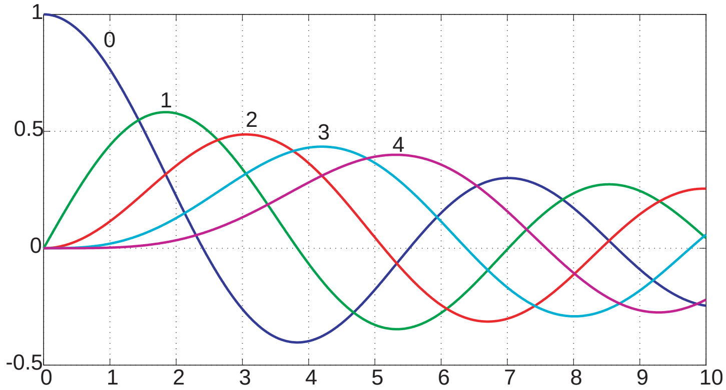
$$J_p(x) \approx \frac{1}{\sqrt{2\pi p}}\left(\frac{ex}{2p}\right)^p \qquad Y_p(x) \approx -\sqrt{\frac{2}{\pi p}}\left(\frac{ex}{2p}\right)^{-p}$$

1.6 Modified Bessel Functions

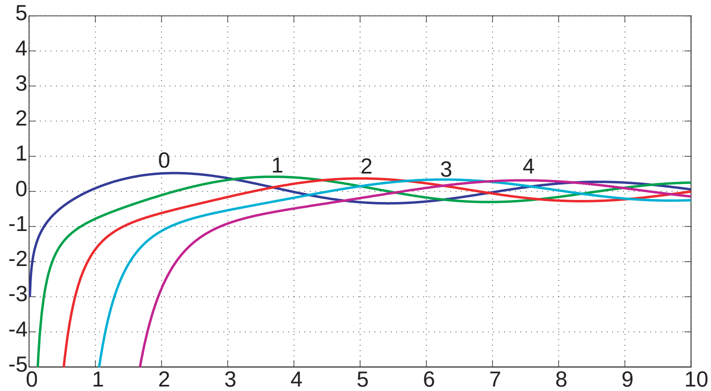
1.6.1 Bessel's Modified Equation

The most general form of Bessel's modified equation is [14: 274]

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(\lambda^2 + \frac{p^2}{x^2}\right)y = 0$$



Plots of $J_p(x)$



Plots of $Y_p(x)$

which has the general solution [14: 275]

$$y = AI_p(\lambda x) + BK_p(\lambda x)$$

where I_p are Modified Bessel functions of the first kind and K_p are modified Bessel functions of the second kind. Modified Bessel functions of the first kind have no closed form representation; however, they can be used to define Bessel functions of the second kind: [14: 275]

$$K_p(x) = \frac{\pi}{2}\frac{I_{-p}(x) - I_p(x)}{\sin(p\pi)}$$

1.6.2 Modified Bessel Functions Relations

Relations involving $I_p(x)$ [14:280]

(a) $xI_{p-1}(x) - xI_{p+1}(x) = 2pI_p(x)$

(b) $I_{p-1}(x) - I_{p+1}(x) = 2\frac{d}{dx}I_p(x)$

(c) $x\frac{d}{dx}\left[I_p(x)\right] + pI_p(x) = xI_{p-1}(x)$

(d) $x\frac{d}{dx}\left[I_p(x)\right] - pI_p(x) = xI_{p+1}(x)$

(e) $\frac{d}{dx}\left[I_0(x)\right] = I_1(x)$

(f) $I_2(x) = -\frac{2}{x}I_1(x) + I_0(x)$

Relations involving $K_p(x)$ [14:280]

(g) $xK_{p-1}(x) - xK_{p+1}(x) = -2pK_p(x)$

(h) $K_{p-1}(x) + K_{p+1}(x) = -2\frac{d}{dx}K_p(x)$

(i) $x\frac{d}{dx}\left[K_p(x)\right] + pK_p(x) = -xK_{p-1}(x)$

(j) $x\frac{d}{dx}\left[K_p(x)\right] - pK_p(x) = -xK_{p+1}(x)$

(k) $\frac{d}{dx}\left[K_0(x)\right] = -K_1(x)$

(l) $K_2(x) = \frac{2}{x}K_1(x) + K_0(x)$

1.6.3 Asymptotic Forms of Modified Bessel Functions

For $x \to \infty$ [14:278]

$$I_p(x) \approx \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4p^2-1}{8x}\right)$$
$$K_p(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\left(1 + \frac{4p^2-1}{8x}\right)$$

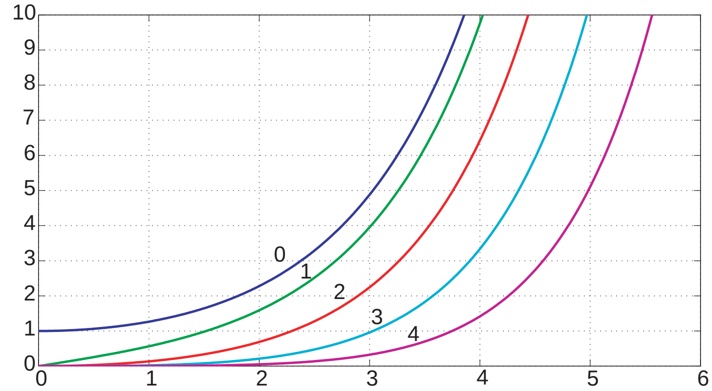
1.7 Partial Differential Equations

1.7.1 Basis Functions for Laplace's Equation

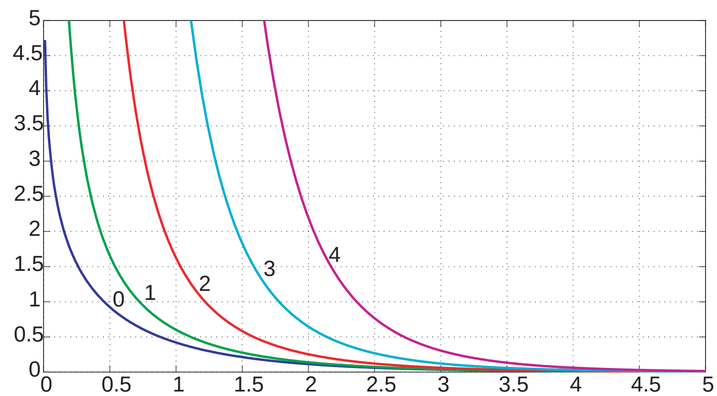
Basis functions are the most general solutions to $\nabla^2\psi = 0$.[18]

*If m is an integer, $J_{-m} \to Y_m$. If k is imaginary, $J_m(kr) \to I_m(|k|\,r)$ and $Y_m(kr) \to K_m(|k|\,r)$

†$Y_{lm}(\theta,\phi)$ is the spherical harmonic function



Plots of $I_p(x)$



Plots of $K_p(x)$

Geometry	Basis Function ψ
2D Cartesian	$\psi = (A \sin kx + B \cos kx)\left(Ce^{ky} + De^{-ky}\right)$
2D Cylindrical	$\psi = A_0 + B_0 \ln r + (A \sin n\theta + B \cos n\theta) \times (Cr^n + Dr^{-n})$
3D Cartesian	$\psi = Ae^{ik_x x}e^{ik_y y}e^{k_z z}$ with $k_x^2 + k_y^2 - k_z^2 = 0$
3D Cylindrical	$\psi = Ae^{\pm im\theta}e^{\pm kz}J_{\pm m}(kr)$ *
2D Spherical	$\psi = \left(Ar^l + Br^{-(l+1)}\right) P_l(\cos\theta)$
3D Spherical	$\psi = \sum_{m=-l}^{l} \left(A_{lm}r^l + B_{lm}r^{-(l+1)}\right) Y_{lm}(\theta,\phi)^\dagger$

1.8 Gaussian Integrals

Definite integral relations of Gaussian integrals [14:255]

$$\text{(a)}\quad \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}$$

$$\text{(b)}\quad \int_{-\infty}^\infty e^{-ax^2} dx = \left(\frac{\pi}{a}\right)^{1/2}$$

$$\text{(c)}\quad \int_{-\infty}^\infty e^{-ax^2} e^{-2bx} dx = \left(\frac{\pi}{a}\right)^{1/2} e^{\frac{b^2}{a}} \qquad \text{for a>0}$$

$$\text{(d)}\quad \int_{-\infty}^\infty x e^{-a(x-b)x^2} dx = b\left(\frac{\pi}{a}\right)^{1/2}$$

$$\text{(e)}\quad \int_{-\infty}^\infty x^2 e^{-ax^2} dx = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}$$

$$\text{(f)}\quad \int_0^\infty x^n e^{-ax^2} dx = \begin{cases} \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)/a^{(n+1)/2} & \text{a>0} \\ \frac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\frac{\pi}{a}} & \text{n=2k,a>0} \\ \frac{k!}{2a^{k+1}} & \text{n=2k+1,a>0} \end{cases}$$

Definite integrals of common Gaussian relations [21:65]

n	$\int_0^\infty x^n e^{-ax^2}\, dx$	$\int_{-\infty}^\infty x^n e^{-ax^2}\, dx$	$\int_0^\infty x^{\frac{1}{2}(n-1)} e^{-ax}\, dx$
0	$\frac{\pi^{1/2}}{2a^{1/2}}$	$\frac{\pi^{1/2}}{a^{1/2}}$	$\frac{\pi^{1/2}}{a^{1/2}}$
1	$\frac{1}{2a}$	0	$\frac{1}{a}$
2	$\frac{\pi^{1/2}}{4a^{3/2}}$	$\frac{\pi^{1/2}}{2a^{3/2}}$	$\frac{\pi^{1/2}}{2a^{3/2}}$
3	$\frac{1}{2a^2}$	0	$\frac{1}{a^2}$
4	$\frac{3\pi^{1/2}}{8a^{5/2}}$	$\frac{3\pi^{1/2}}{4a^{5/2}}$	$\frac{3\pi^{1/2}}{4a^{5/2}}$
5	$\frac{1}{a^3}$	0	$\frac{2}{a^3}$
6	$\frac{15\pi^{1/2}}{16a^{7/2}}$	$\frac{15\pi^{1/2}}{8a^{7/2}}$	$\frac{15\pi^{1/2}}{8a^{7/2}}$

1.9 Error functions

The error function [14:242]

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

Taylor expansion of the error function [14: 242]

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots \right)$$

The complimentary error function [14: 242]

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt = 1 - \operatorname{erf}(x)$$

Taylor expansion of the complimentary error function [1: 469]

$$\operatorname{erfc}(x) \approx \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{3}{(2x^2)^2} - \frac{15}{(2x^2)^3} + \dots \right)$$

Chapter 2

Fundamental Constants and SI Units

Numerical values for all constants taken from the 2006 CODATA *Internationally Recommended Values of the Fundamental Physical Constants*.

Universal Constants

Constant	Symbol	Value	Unit
Avagadro's Constant	N_A	6.022 141 79(30)$\times 10^{23}$	mol^{-1}
Boltzmann's Constant	k_B	1.380 650 4(24)$\times 10^{-23}$	J/K
		8.617 343 1(15)$\times 10^{-5}$	eV/K
Elementary Charge	e	1.602 176 487(40)$\times 10^{-19}$	C
Impedance of Vacuum $\sqrt{\mu_0/\epsilon_0}$	Z_0	376.730 313 461	Ω
Permittivity of Vacuum	ϵ_0	8.854 187 817$\times 10^{-12}$	F/m
Permeability of Vacuum	μ_0	$4\pi \times 10^{-7}$	N/A^2
Planck's Constant	h	6.626 068 96(33)$\times 10^{-34}$	J·s
		4.135 667 33(10)$\times 10^{-15}$	eV·s
H-bar ($h/2\pi$)	$\hbar$	1.054 571 628(53)$\times 10^{-34}$	J·s
		6.582 118 99(16)$\times 10^{-16}$	eV·s
Speed of Light (vacuum)	c	2.997 924 58$\times 10^{8}$	m/s

Atomic and Nuclear Constants

Constant	Symbol	Value	Unit
Electron Rest Mass	m_e	$9.109\ 382\ 15(45)\times10^{-31}$	kg
		$5.485\ 799\ 0943(23)\times10^{-4}$	u
		$0.510\ 998\ 910(13)$	MeV/c^2
Proton Rest Mass	m_p	$1.672\ 621\ 637(83)\times10^{-27}$	kg
		$1.007\ 276\ 466\ 77(10)$	u
		$938.272\ 013(23)$	MeV/c^2
Neutron Rest Mass	m_n	$1.674\ 927\ 211(84)\times10^{-27}$	kg
		$1.008\ 664\ 915\ 97(43)$	u
		$939.565\ 346(23)$	MeV/c^2
Deuteron (^{2}H) Rest Mass	m_d	$3.343\ 583\ 20(17)\times10^{-27}$	kg
		$2.013\ 553\ 212\ 724(78)$	u
		$1875.612\ 793(47)$	MeV/c^2
Triton (^{3}H) Rest Mass	m_t	$5.007\ 355\ 88(25)\times10^{-27}$	kg
		$3.015\ 500\ 7134(25)$	u
		$2808.9209\ 06(70)$	MeV/c^2
Helion (^{3}He) Rest Mass	m_h	$5.006\ 411\ 92(25)\times10^{-27}$	kg
		$3.014\ 932\ 2473(26)$	u
		$2808.391\ 383(70)$	MeV/c^2
Alpha (^{4}He) Rest Mass	m_α	$6.644\ 656\ 20(33)\times10^{-27}$	kg
		$4.001\ 506\ 179\ 127(62)$	u
		$3727.379\ 109(93)$	MeV/c^2
Proton to Electron Mass Ratio	m_p/m_e	$1836.152\ 672\ 47(80) \approx 6\pi^5$	
Bohr Radius ($a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2$)	a_0	$0.529\ 177\ 208\ 59(36)\times10^{-10}$	m
Classical Electron Radius ($r_e = e^2/4\pi\epsilon_0 m_e c^2$)	r_e	$2.817\ 940\ 289\ 4(58)\times10^{-15}$	m
Inverse Fine Structure Constant	$1/\alpha$	$137.035\ 999\ 679(94)$	
Rydberg Constant	R_∞	$1.097\ 373\ 156\ 852\ 7(73)\times10^{7}$	m^{-1}
	$R_\infty hc$	$13.605\ 691\ 93(34)$	eV
Stefan-Boltzmann Constant	σ	$5.670\ 400(40)\times10^{-8}$	W/m^2/K^4
Thomson Cross Section ($\sigma_{\mathrm{Th}} = (8\pi/3)\, r_e^2$)	σ_{Th}	$0.665\ 245\ 855\ 8(27)\times10^{-28}$	m^2
		$0.665\ 245\ 855\ 8(27)$	barns

The System of International Units

Quantity	Symbol	SI Unit	Dimensions
Activity	$\mathcal{A}$	Becquerel	s^{-1}
Capacitance	C	farad (F)	$\frac{s^2 \cdot C^2}{kg \cdot m^2}$
Charge	q	*coulomb (C)	C
Conductance		siemens (S)	$\frac{s \cdot C^2}{kg \cdot m^2}$
Conductivity	σ	siemens/meter (S/m)	$\frac{s \cdot C^2}{kg \cdot m^3}$
Current	I	ampére (A)	$\frac{C}{s}$
Displacement	$\mathbf{D}$	coulomb/m^2	$\frac{C}{m^2}$
Electric Field	$\mathbf{E}$	volt/meter	$\frac{kg \cdot m}{s^2 \cdot C}$
Electromotance	ε	volt (V)	$\frac{kg \cdot m^2}{s^2 \cdot C}$
Energy	W	joule (J)	$\frac{kg \cdot m^2}{s^2}$
Force	$\mathbf{F}$	newton (N)	$\frac{kg \cdot m}{s^2}$
Frequency	ν	hertz (Hz)	s^{-1}
Impedance	Z	ohm (Ω)	$\frac{kg \cdot m^2}{s \cdot C^2}$
Inductance	L	henry (H)	$\frac{kg \cdot m^2}{C^2}$
Length	l	*meter (m)	m
Magnetic Flux	Φ	weber (Wb)	$\frac{kg \cdot m^2}{s \cdot C}$
Magnetic Flux Density	$\mathbf{B}$	tesla (T)	$\frac{kg}{s \cdot C}$
Magnetic Moment	μ	ampere-m^2	$\frac{m^2 \cdot C}{s}$
Magnetization	$\mathbf{M}$	ampere-turn/m	$\frac{C}{s \cdot m}$
Permeability	μ	henry/meter	$\frac{kg \cdot m}{C^2}$
Permittivity	ϵ	farad/meter	$\frac{s^2 \cdot C^2}{kg \cdot m^3}$
Polarization	$\mathbf{P}$	coulomb/m^2	$\frac{C}{m^2}$
Electric Potential	V	volt (V)	$\frac{kg \cdot m^2}{s^2 \cdot C}$
Power	P	watt (W)	$\frac{kg \cdot m^2}{s^3}$
Pressure	p	pascal (Pa)	$\frac{kg}{m \cdot s^2}$
Resistance	R	ohm (Ω)	$\frac{kg \cdot m^2}{s \cdot C^2}$
Resistivity	η	ohm-meter	$\frac{kg \cdot m^3}{s \cdot C^2}$
Temperature	T	*kelvin (K)	K
Thermal Conductivity	κ	watt/meter/kelvin	$\frac{kg \cdot m}{s^3}$
Time	s	*second	s
Velocity	$\mathbf{v}$	meter/second	$\frac{m}{s}$

* denotes a fundamental SI base unit

Energy Conversion Factors

Energy	$\leftrightarrow$	Temperature	1 eV = 1.602189×10^{-19} J
			1 eV = 1.1604505×10^{4} K
Energy	$\leftrightarrow$	Mass	1 u = 931.501 MeV/c^2 = 1.660566×10^{-27} kg
Energy	$\leftrightarrow$	Wavelength	hc = 1239.8419 MeV·fm = 1239.8419 eV·nm
			$\hbar$c = 197.329 MeV·fm = 197.329 eV·nm
			$e^2/4\pi\epsilon_0$ = 1.439976 MeV·fm

Chapter 3

Electricity and Magnetism

In this chapter, all units are SI.

e is the elementary electric charge
q is the total particle charge
Z is the particle atomic (proton) number
n is the particle density
U is energy
$\mathbf{r}$ is the particle position
$\mathbf{v}$ is the particle velocity
ρ is volumetric charge density
σ is surface charge density
$\mathbf{J}$ is volumetric current density
$\mathbf{K}$ is surface current density
τ and σ are the volume and surface, respectively
$d\tau$, $d\boldsymbol{\sigma}$, and $d\mathbf{l}$ are the volume, surface, and line elements, respectively
b and f subscripts refer to bound and free charges

3.1 Electromagnetic in Vacuum

3.1.1 Fundamental Equations

Maxwell's equations [10: 326]

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Electrostatic scalar potential relations [10: 87]

$$\mathbf{E} = -\nabla V \qquad V = -\int \mathbf{E} \cdot d\mathbf{l}$$
$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{r}\, d\tau$$

Electrostatic vector potential relations [10:240]

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r} d\tau$$

Electromagnetic energy stored in the fields [10:348]

$$U_{\text{em}} = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau$$

Coulomb Force [10:59]

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|^2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Lorentz force law [10:204]

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Biot-Savart law [10:215]

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2} dl$$

3.1.2 Boundary Conditions

For given surface $\mathcal{S}$, + and - refer to above and below $\mathcal{S}$, respectively. $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\mathcal{S}$.

Electrostatic boundary conditions on **E** [10:179]

$$E_+^\perp - E_-^\perp = \sigma/\epsilon_0$$

$$E_+^{||} - E_-^{||} = 0$$

Magnetostatic boundary conditions on **B** [10:241]

$$B_+^\perp - B_-^\perp = 0$$

$$B_+^{||} - B_-^{||} = \mu_0 K$$

$$\mathbf{B}_+ - \mathbf{B}_- = \mu_0 \left(\mathbf{K} \times \hat{\mathbf{n}}\right)$$

3.2 Electromagnetics in Matter

3.2.1 Fundamental Equations

Maxwell's equations in matter [10: 330]

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

The polarization in linear media (χ_e is the polarizability) [10: 179]

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad \text{[electric dipole moments per m}^{-3}\text{]}$$

The magnetization in linear media (χ_m is the magnetization) [10: 274]

$$\mathbf{M} = \chi_m \mathbf{H} \quad \text{[electric dipole moments per m}^{-3}\text{]}$$

The displacement field [10: 175,180]

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ &= \epsilon \mathbf{E} \quad \text{(linear media only where } \epsilon \equiv \epsilon_0(1+\chi_e)\text{)} \end{aligned}$$

The H-field (Magnetic field) [10: 269,275]

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \\ &= \frac{1}{\mu}\mathbf{B} \quad \text{(linear media only where } \mu \equiv \mu_0(1+\chi_m)\text{)} \end{aligned}$$

Associated bound charges (σ_b, ρ_b) and currents ($\mathbf{K}_b$, $\mathbf{J}_b$) [10: 167,168,267,268]

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b = -\nabla \cdot \mathbf{P}$$
$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

3.2.2 Boundary Conditions

For given surface $\mathcal{S}$, + and - refer to above and below $\mathcal{S}$, respectively. $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\mathcal{S}$. [10: 178,273]

$$D_+^\perp - D_-^\perp = \sigma_f \qquad \mathbf{D}_+^{||} - \mathbf{D}_-^{||} = \mathbf{P}_+^{||} - \mathbf{P}_-^{||}$$
$$H_+^\perp - H_-^\perp = -\left(M_+^\perp - M_-^\perp\right) \qquad \mathbf{H}_+^{||} - \mathbf{H}_-^{||} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

3.3 Dipoles

In this section, **p** and **m** are electric and magnetic dipoles, respectively. **N** is the torque and **F** is the force generated by the dipole.

Definition [10: 149,244]	Fields [10: 153,155,246]	Potentials [10: 166,244]
$\mathbf{p} \equiv \int \mathbf{r}\rho(\mathbf{r})d\tau$	$\mathbf{E}_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]$ $\mathbf{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$	$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$
$\mathbf{m} \equiv I\int d\boldsymbol{\sigma}$	$\mathbf{B}_{dip} = \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]$ $\mathbf{B}_{dip}(r,\theta) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$	$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$

Electric [10: 164,165]	Magnetic [10: 257,258,281]
$\mathbf{F} = (\mathbf{p}\cdot\nabla)\,\mathbf{E}$	$\mathbf{F} = \nabla\,(\mathbf{m}\cdot\mathbf{B})$
$\mathbf{N} = \mathbf{p}\times\mathbf{E}$	$\mathbf{N} = \mathbf{m}\times\mathbf{B}$
$\mathrm{U} = -\mathbf{p}\cdot\mathbf{E}$	$\mathrm{U} = -\mathbf{m}\cdot\mathbf{B}$

3.4 Circuit Electrodynamics

Microscopic Ohm's law [10: 285]

$$\mathbf{J} = \sigma_c \mathbf{E}$$

where σ_c is the conductivity. Resistivity, ρ_r, is defined as $\rho_r = 1/\sigma_c$.

Macroscopic Ohm's law [10: 287]

$$V = IR$$

where V is the voltage, I is the current, and R is the resistance.

The voltage due to a changing magnetic field (Faraday's Law) [10: 295,296] ,

$$V = -\frac{d\Phi}{dt} \qquad \Phi = \int\limits_{surface} \mathbf{B}\cdot d\mathbf{A}$$

Capacitance is written as C, and inductance is written as L.[18]

$$Q = CV \qquad \Phi = LI$$

$$I = -C\frac{dV}{dt} \qquad V = -L\frac{dI}{dt}$$

Energy stored in capacitance and inductance [10: 106,317]

$$U = \frac{1}{2}LI^2 + \frac{1}{2}CV^2$$

3.5 Conservation Laws

Conservation of charge [10:214]

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

Poynting vector [10:347]

$$\mathbf{S} \equiv \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$

Poynting's theorem (integral form) [10:347]

$$\frac{dU}{dt} = -\frac{d}{dt} \int\limits_{\text{volume}} \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau - \frac{1}{\mu_0} \oint\limits_{\text{surface}} (\mathbf{E} \times \mathbf{B}) \cdot d\boldsymbol{\sigma}$$

Poynting's theorem (differential form) [10:348]

$$\frac{\partial}{\partial t} (U_{\text{mechanical}} + U_{\text{em}}) = -\nabla \cdot \mathbf{S}$$

Maxwell's stress tensor [10:352]

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

Electromagnetic force density on collection of charges [10:352]

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t}$$

Total electromagnetic force on collection of charges [10:353]

$$\mathbf{F} = \oint\limits_{\text{surface}} \overleftrightarrow{\mathbf{T}} \cdot d\boldsymbol{\sigma} - \epsilon_0 \mu_0 \frac{d}{dt} \int\limits_{\text{volume}} \mathbf{S} \, d\tau$$

Momentum density in electromagnetic fields [10:355]

$$p_{em} = \mu_0 \epsilon_0 \mathbf{S}$$

Conservation of momentum in electromagnetic fields [10:356]

$$\frac{\partial}{\partial t} (p_{mech} + p_{em}) = \nabla \cdot \overleftrightarrow{\mathbf{T}}$$

3.6 Electromagnetic Waves

In this section,

λ is the wavelength
$k = 2\pi/\lambda$ is the wave number
ν is the frequency

$\omega = 2\pi\nu$ is the angular frequency
$T = 1/\nu$ is the period
$\mathbf{k} = k\hat{\mathbf{k}}$ is the wave number vector
$\hat{\mathbf{n}}$ is the polarization vector in the direction of electric field
$\tilde{\mathbf{X}}$ is a complex vector

The wave equation in three dimensions [10:376]

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

is satisfied by two transformations of Maxwell's equations in vacuum [10:376]

$$\nabla^2 \mathbf{E} = \mu_0\epsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{B} = \mu_0\epsilon_0\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

These have sinusoidal solutions, but it is more convenient to work with imaginary exponentials and take the real parts [10:379]

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}}$$

$$\tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}}\times\hat{\mathbf{n}}) = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}}$$

EM Wave Relations [10:381–382]

Parameter	Symbol	Equation
Averaged energy per unit volume	$\langle u \rangle$	$\frac{1}{2}\epsilon_0 E_0^2$
Averaged energy flux density	$\langle \mathbf{S} \rangle$	$\frac{1}{2}c\epsilon_0 E_0^2\hat{\mathbf{k}}$
Averaged momentum density	$\langle P \rangle$	$\frac{1}{2c}\epsilon_0 E_0^2\hat{\mathbf{k}}$
Intensity	I	$\langle S \rangle$
Radiation pressure	P	$\frac{I}{c}$

3.6.1 EM Waves in Matter

In this section, θ is measured from the normal to the surface

Assuming no free charge or current in a linear media, the EM wave equations become [10:383]

$$\nabla\cdot\mathbf{E} = 0 \qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}$$

Speed of light in a material [10:383]

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$$

Index of refraction [10:383]

$$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\epsilon_r}$$

Intensity [10:383]

$$I = \frac{1}{2}\epsilon v E_0^2$$

Boundary conditions at a material surface [10:384]

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \mathbf{E}_1^{||} = \mathbf{E}_2^{||}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\mathbf{B}_1^{||} = \frac{1}{\mu_2}\mathbf{B}_2^{||}$$

Reflection and transmission coefficients [10:386]

$$R \equiv \frac{I_{ref}}{I_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \qquad T \equiv \frac{I_{trans}}{I_{inc}} = \frac{4n_1n_2}{(n_1+n_2)^2} \qquad R + T = 1$$

Snell's Laws for oblique incidence on material surface [10:388]

$$k_{inc}\sin\theta_{inc} = k_{ref}\sin\theta_{ref} = k_{trans}\sin\theta_{trans}$$

$$\theta_{inc} = \theta_{ref}$$

$$\frac{\sin\theta_{trans}}{\sin\theta_{inc}} = \frac{n_1}{n_2}$$

Fresnel Equations [13:305–306]

Polarization to incident plane	E_{trans}/E_{inc}	E_{ref}/E_{inc}
Perpendicular	$\frac{2n_1\cos\theta_{inc}}{n_1\cos\theta_{inc}+(\mu_1/\mu_2)\sqrt{n_2^2-n_1^2\sin^2\theta_{inc}}}$	$\frac{n_1\cos\theta_{inc}-(\mu_1/\mu_2)\sqrt{n_2^2-n_1^2\sin^2\theta_{inc}}}{n_1\cos\theta_{inc}+(\mu_1/\mu_2)\sqrt{n_2^2-n_1^2\sin^2\theta_{inc}}}$
Parallel	$\frac{2n_1n_2\cos\theta_{inc}}{(\mu_1/\mu_2)n_2^2\cos\theta_{inc}+n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{inc}}}$	$\frac{(\mu_1/\mu_2)n_2^2\cos\theta_{inc}-n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{inc}}}{(\mu_1/\mu_2)n_2^2\cos\theta_{inc}+n_1\sqrt{n_2^2-n_1^2\sin^2\theta_{inc}}}$

Brewster's angle (no reflection of perpendicular incidence wave) [10:390]

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2 - \beta^2}$$

where $\beta = \mu_1 n_2/\mu_2 n_1$

If the wave is in a conductor, it will experience damping due to the presence of free charges, subject to $\mathbf{J}_f = \sigma \mathbf{E}$. Solving Maxwell's equations gives [10: 394]

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

Decomposing gives real and imaginary parts of the wave vector $\tilde{k} = k + i\kappa$

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \qquad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

Knowing the imaginary part of the wave number, allows to know the damping of the wave which is characterized by a skin depth or the e-folding length $d \equiv 1/\kappa$. [10: 394]

3.7 Electrodynamics

We are allowed to choose $\nabla \cdot \mathbf{A}$; two common gauges used are the Lorentz and Coulomb gauge. [10: 421−422]

Gauge	$\nabla \cdot \mathbf{A}$	V Equation	$\mathbf{A}$ Equation
Lorentz	$-\mu_0\epsilon_0\frac{\partial V}{\partial t}$	$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$	$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}$
Coulomb	0	$\nabla^2 V = -\frac{\rho}{\epsilon_0}$	$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \mu_0\epsilon_0\nabla\left(\frac{\partial V}{\partial t}\right)$

For any scalar function λ, any potential formulation is valid if [10: 420]

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad V' = V - \frac{\partial\lambda}{\partial t}$$

3.7.1 Fields of Moving Charges

In this section, $||A||$ evaluates A at the retarded time[18]

Definition of retarded time [10: 423]

$$t_{ret} \equiv t - \frac{|\mathbf{r}(t) - \mathbf{r}(t_{ret})|}{c}$$

The Lienard-Wiechert potentials [10: 432−433]

$$V = \frac{q}{4\pi\epsilon_0}\left|\left|\frac{1}{r\kappa}\right|\right| \qquad \mathbf{A} = \frac{\mu_0 q}{4\pi}\left|\left|\frac{\mathbf{v}}{r\kappa}\right|\right|$$

Electric field of a moving point charge [10:438]

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left|\left|(\hat{\mathbf{r}} - \mathbf{v}/c)\left(1 - v^2/c^2\right)\frac{1}{\kappa^3 r^2} - \hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \mathbf{v}/c) \times \mathbf{a}/c)\frac{1}{\kappa^3 r c}\right|\right|$$

Magnetic field of a moving point charge [10:438]

$$\mathbf{B} = ||\hat{\mathbf{r}}|| \times \mathbf{E}/c$$

where $\kappa = 1 - \hat{\boldsymbol{r}} \cdot \frac{\mathbf{v}}{c}$.

3.7.2 Radiation by Charges

In this section, $\mathbf{u} \equiv c\hat{\mathbf{r}} - \mathbf{v}$, $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$ is the relativistic gamma factor, and $\mathbf{a}$ is the acceleration.

Poynting vector associated with a moving charge [10:460]

$$\mathbf{S} = \frac{1}{\mu_0 c}\left[E^2\hat{\mathbf{r}} - (\hat{\mathbf{r}} \cdot \mathbf{E})\,\mathbf{E}\right]$$

Non-relativistic power radiated by a moving charge [10:462]

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

Relativistic power radiated by a moving charge per solid angle [10:463]

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\frac{\left|\hat{\mathbf{r}} \times (\mathbf{u} \times \mathbf{a})\right|^2}{(\hat{\mathbf{r}} \cdot \mathbf{u})^5}$$

Relativistic power radiated by a moving charge[10:463]

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\mathbf{v} \times \mathbf{a}}{c}\right|^2\right)$$

Relativistic force of a moving charge [10:467]

$$\mathbf{F}_{rad} = \frac{\mu_0 q^2}{6\pi c}\frac{d\mathbf{a}}{dt}$$

Chapter 4

Single Particle Physics

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

e is the elementary electric charge
q is the total particle charge
Z is the particle atomic (proton) number
m is the particle mass
$\mathbf{r}$ is the particle position
$\mathbf{v}$ is the particle velocity
U is energy
T is temperature; $T_{\mathrm{keV}} = T$ in units of kiloelectron-volts
$\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields
$\hat{\mathbf{b}}$ is a unit vector in the direction of $\mathbf{B}$
$||$ and $\perp$ indicate parallel and perpendicular to $\hat{\mathbf{b}}$

4.1 Single Particle Motion in E and B Fields

4.1.1 General Formulation

Single particle trajectories result from solving Newton's second law for a particle with charge q and mass m in electric and magnetic fields: [8: 141]

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \qquad \frac{d\mathbf{r}}{dt} = \mathbf{v}$$

If $\mathbf{E}$ and $\mathbf{B}$ are independent of time, the particle's kinetic and potential energy is conserved [8: 142]

$$\frac{1}{2}mv^2 + qV = \text{constant}$$

where V is the scalar potential ($\mathbf{E} = -\nabla V$).

4.1.2 Gyro Motion Solutions for $\mathbf{B} = \mathrm{B}_0\hat{z}$; $\mathbf{E} = 0$

Particle initially has $\mathbf{r} = (x_0, y_0, z_0), \mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}$, and arbitrary phase, ϕ.

Parallel to the field: [8:143]

$$z(t) = z_0 + v_\parallel t$$

Perpendicular to the field: [8:144]

$$x(t) = x_g + \rho_L \sin(\Omega_c t - \phi) \qquad x_g \equiv x_0 + \rho_L \sin\phi$$
$$y(t) = y_g + \rho_L \cos(\Omega_c t - \phi) \qquad y_g \equiv y_0 - \rho_L \cos\phi$$

The guiding center position is (x_g, y_g); the larmor (or gyro) radius is $\rho_L = v_\perp/\Omega_c = mv_\perp/qB$; the larmor (or gyro) frequency is Ω_c

4.1.3 Single Particle Drifts

In this section, $\mathbf{R}_c$ is the particle's radius of curvature in a magnetic field and is defined as $\mathbf{b} \cdot \nabla\mathbf{b} = -\mathbf{R}_c/R_c^2$

E×B drift [8:149]

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

∇B drift [8:153]

$$\mathbf{v}_{\nabla B} = \frac{mv_\perp^2}{2qB}\frac{\mathbf{B} \times \nabla B}{B^2}$$

Curvature drift [8:159]

$$\mathbf{v}_\kappa = \frac{mv_\parallel^2}{qB}\frac{\boldsymbol{R_c} \times \mathbf{B}}{R_c^2 B}$$

Polarization drift [8:162]

$$\mathbf{v}_p = \frac{m}{qB}\hat{\boldsymbol{b}} \times \frac{d\mathbf{v}_E}{dt}$$

Vacuum field only $\rightarrow \nabla \times \mathbf{B} = 0$ [8:160]

$$\mathbf{v}_{\nabla B} + \mathbf{v}_\kappa = \frac{m}{qB}(v_\parallel^2 + \frac{v_\perp^2}{2})\frac{\boldsymbol{R_c} \times \mathbf{B}}{R_c^2 B}$$

Particle drift velocity for a general force $\mathbf{F}$ [8:153]

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2}$$

4.1.4 Magnetic Moment And Mirroring

In this section, i refers to the initial point, and f stand for the final, or mirror, point.

Magnetic moment (the first adiabatic invariant) $^{8:167}$

$$\mu = \frac{mv_\perp^2(t)}{2B(t)} = \text{constant}$$

Force on particle in magnetic fields where $\nabla B/B \ll 1$ $^{8:171}$

$$F_\| \approx -\mu \nabla_\| B$$

Velocity in terms of velocity space pitch angle $^{8:174}$

$$v_{\perp i} = v_0 \sin\theta \qquad v_{\| i} = v_0 \cos\theta$$

Conservation of energy $^{8:174}$

$$\frac{1}{2}m\left(v_{\perp i}^2 + v_{\| i}^2\right) = \frac{1}{2}mv_{\perp f}^2 = U_{\text{total}} = \text{constant}$$

Mirroring condition $^{8:175}$

$$\sin^2\theta_c = \frac{U_{\perp i}}{U_{\text{total}}} = \frac{v_{\perp i}^2}{v_{\perp f}^2} = \frac{B_{min}}{B_{max}}$$

Fraction of trapped particles (Maxwellian distribution) $^{8:176}$

$$\mathcal{F}_{\text{trapped}} = \frac{1}{n}\int_{\theta_c}^{\pi-\theta_c} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty \mathcal{F}_{\text{Maxwellian}}(v)v^2 dv$$

where n is the total number of particles in the distrubition function

4.2 Binary Coulomb Collisions

r is the relative distance
$\mathbf{v}_1$ and $\mathbf{v}_2$ are particle velocities in the lab frame
$\mathbf{V}$ and $\mathbf{v}$ are the center of mass and relative velocities
v_0 and b_0 are the initial relative velocity and impact parameter
χ is the scattering angle in the center of mass frame
$\dot{x}$ is the time derivative of quantity x

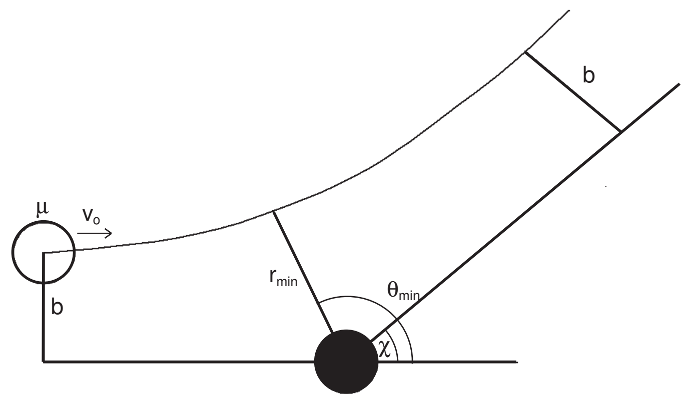
Force between 2 charged particles

$$\mathbf{F} = -\nabla\left(\frac{q_1 q_2}{4\pi\epsilon_0 r}\right)$$

Transformation to center of mass frame $^{8:186}$

$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2} \qquad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$$

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_2\mathbf{v}}{m_1+m_2} \qquad \mathbf{v}_2 = \mathbf{V} - \frac{m_1\mathbf{v}}{m_1+m_2}$$



Schematic of two body collision in the reduced mass frame.

Reduced mass [8: 186]

$$m_\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Conservation of energy [8: 187]

$$\frac{1}{2} m_\mu v^2 + \frac{q_1 q_2}{4\pi\epsilon_0 r} = E_0 = \frac{1}{2} m_\mu v_0^2 = \text{constant}$$

Conservation of angular momentum [8: 187]

$$m_\mu \mathbf{r} \times \mathbf{v} = \mathbf{L}_0 = -m_\mu b v_0 = \text{constant}$$

Transformation to cylindrical coordinates [8: 188]

$$\mathbf{r} = r\hat{\boldsymbol{r}} \qquad \mathbf{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

Ordinary differential equation for unknown r(t) [8: 188]

$$\dot{r} = \mp v_0 \left(1 - 2\frac{b_{90}}{r} - \frac{b^2}{r^2}\right)^{1/2}$$

Solution in terms of scattering angle χ [8: 190]

$$\tan(\frac{\chi}{2}) = \frac{b_{90}}{b} = \frac{q_1 q_2}{4\pi\epsilon_0 m_\mu v_0^2 b}$$

Impact parameter for 90 degree collision [8: 188]

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0 m_\mu v_0^2}$$

4.3 Single Particle Collisions with Plasma

x is an incident test particle; y is a target plasma particle. Q_{xy} are quantities depending on particle x incident upon particle y.

Total loss in test particle linear momentum [8: 192]

$$\frac{d}{dt}(m_x v_x) = -(\Delta m_x v_x) n_y \sigma v_x$$
$$= -\int (\Delta m_x v_x) f_i(\mathbf{v}_y)|\mathbf{v}_x - \mathbf{v}_y| b\, db\, d\alpha\, d^3v$$

Definition of test particle collision frequency [8: 193]

$$\frac{d}{dt}(m_x v_x) \equiv -\nu_{xy}(m_x v_x)$$

Test particle collision frequency [8: 193]

$$\nu_{xy}(v_x) = \frac{1}{m_x v_x} \int (\Delta m_x v_x) f_y(\mathbf{v}_y)|\mathbf{v}_x - \mathbf{v}_y| b\, db\, d\alpha\, d^3v$$

4.3.1 Collision Frequencies

Approximated expressions hold only for $v_e \sim v_{The} \gg v_{Thi}$

Electron-ion [8: 197]

$$\nu_{ei} = \left(\frac{e^4 n_i \ln\Lambda}{4\pi\epsilon_0^2 m_e m_\mu}\right) \frac{1}{v_e^3 + 1.3 v_{Th_i}^3} \approx \frac{e^4 n_i \ln\Lambda}{4\pi\epsilon_0^2 m_e^2 v_e^3}$$
$$\approx 8.06 \times 10^5 \frac{n_i \ln\Lambda}{v_e^3} \quad [s^{-1}]$$

Electron-electron [8: 197]

$$\nu_{ee} = \left(\frac{e^4 n_e \ln\Lambda}{2\pi\epsilon_0^2 m_e^2}\right) \frac{1}{v_e^3 + 1.3 v_{Th_e}^3} \quad [s^{-1}]$$

Ion-ion [8: 197]

$$\nu_{ii} = \left(\frac{e^4 n_i \ln\Lambda}{2\pi\epsilon_0^2 m_i^2}\right) \frac{1}{v_i^3 + 1.3 v_{Th_i}^3} \quad [s^{-1}]$$

Ion-electron [8: 197]

$$\nu_{ie} = \left(\frac{e^4 n_e \ln\Lambda}{4\pi\epsilon_0^2 m_e m_i}\right) \frac{1}{v_i^3 + 1.3 v_{Th_e}^3} \quad [s^{-1}]$$

Collision frequency scalings [8: 197]

$$\nu_{ee} \sim \nu_{ei} \qquad \nu_{ii} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \nu_{ei} \qquad \nu_{ie} \sim \left(\frac{m_e}{m_i}\right) \nu_{ei}$$

4.3.2 Collision Times

Electron-ion collision time [11:5]

$$\tau_{ei} = \frac{12\pi^{3/2}\epsilon_0^2 m_e^{1/2} T_e^{3/2}}{\sqrt{2} n_i Z_i^2 e^4 \ln\Lambda} = 1.09 \times 10^{16} \frac{T_{\text{e, keV}}^{3/2}}{Z_i^2 n \ln\Lambda} \quad [s]$$

Ion-ion collision time [11:5]

$$\tau_{ii} = \frac{12\pi^{3/2}}{2^{1/2}} \frac{\epsilon_0^2 m_i^{1/2} T_i^{3/2}}{n_i Z_i^4 e^4 \ln\Lambda_i} = 4.67 \times 10^{17} \left(\frac{m_i}{m_p}\right)^{1/2} \frac{T_{\text{i, keV}}^{3/2}}{Z_i^4 n \ln\Lambda_i} \quad [s]$$

An ion collision time is sometimes defined to be $\tau_i = \sqrt{2}\tau_{ii}$
Ion-impurity collision time [11:177]

$$\tau_{iI} = \frac{12\pi^{3/2}}{2^{1/2}} \frac{m_i^{1/2} T_i^{3/2} \epsilon_0^2}{n_I Z_i^2 Z_I^2 e^4 \ln\Lambda} = 4.67 \times 10^{17} \left(\frac{m_i}{m_p}\right)^{1/2} \frac{T_{\text{i, keV}}^{3/2}}{Z_i^2 Z_I^2 n \ln\Lambda_i} \quad [s]$$

Electron-to-ion energy transfer time

$$R_{\text{ei}} = \frac{\frac{3}{2} n \left(T_e - T_i\right)}{\left(\frac{m_i}{2m_e}\tau_e\right)}$$

Thermal equilibration frequency (rate of species a equilibrating to species b) [12:34]

$$\nu_{ab}^{Th} = 1.8 \times 10^{-19} \frac{(m_a m_b)^{1/2} Z_a^2 Z_b^2 n_b \ln\Lambda}{(m_a T_b + m_b T_a)^{3/2}} \quad [\text{s}^{-1}]$$

For ions and electrons with $T_e \approx T_i = T$, $\nu_{ei} n_e = \nu_{ie} n_i$ [12:34]

$$\nu_{ei}^{Th} = 3.2 \times 10^{-15} \frac{Z^2 \ln\Lambda}{(m_i/m_p T^{3/2})} \quad [\text{m}^3\text{s}^{-1}]$$

4.4 Particle Beam Collisions with Plasmas

In this section, plasma density is written as $n_{20} = n/10^{20}$

Beam-electron collision frequency [8: 202–203]

$$\nu_{be}(v_b) = \left(\frac{Z_b^2 e^4 n_e \ln\Lambda}{4\pi\epsilon_0^2 m_e m_b}\right)\frac{1}{v_b^3 + 1.3 v_{The}^3}$$
$$\approx \frac{1}{3(2\pi)^{3/2}}\frac{Z_b^2 e^4 m_e^{1/2} n_e \ln\Lambda}{\epsilon_0^2 m_b T_e^{3/2}} \qquad (v_{The}^3 \gg v_b^3 \text{ only})$$
$$= 100\frac{n_{20}}{T_{\mathrm{keV}}^{3/2}} \qquad [\mathrm{s}^{-1}] \qquad \text{(alpha heating only)}$$

Beam-ion collision frequency [8: 203]

$$\nu_{bi}(v_b) = \left(\frac{Z_b^2 e^4 n_i \ln\Lambda}{4\pi\epsilon_0^2 m_\mu m_b}\right)\frac{1}{v_b^3 + 1.3 v_{Th_i}^3}$$
$$\approx \frac{1}{4\pi}\frac{Z_b^2 e^4 n_i \ln\Lambda}{\epsilon_0^2 m_\mu m_b v_b^3} \qquad (v_{Thi}^3 \ll v_b^3 \text{ only})$$
$$= 0.94\frac{n_{20}}{(E_{\mathrm{beam}})_{\mathrm{MeV}}^{3/2}} \qquad [\mathrm{s}^{-1}] \qquad \text{(alpha heating only)}$$

When $\nu_{be} = \nu_{bi}$, $v_b = v_{crit}$ and the beam changes the plasma particle it preferentially damps upon. For $v_b > v_{crit}$, the beam damps on plasma electrons; for $v_b < v_{crit}$, the beam damps on plasma ions. The critical beam energy is calculated as $E_c = 1/2 m_{beam} v_{beam}^2 = 14.8(m_{beam}/m_i^{2/3})T_e$. For a 15 keV plasma, the critical energy corresponds to 660 keV.

A high energy beam entering a plasma has energy behavior (purely slowing down on electrons) [26: 249]

$$E_b = E_{b,0}\left[e^{-3t/\tau_{se}} - \left(\frac{E_c}{E_{b,0}}\right)^{3/2}\left(1 - e^{-3t/\tau_{se}}\right)\right]^{2/3}$$

where $E_{b,0}$ is the initial beam energy and τ_{se} is the slowing down time assuming $v_b < v_{The}$ [26: 246]

$$\tau_{se} = \frac{3(2\pi)^{3/2}\epsilon_0^2 m_b T_e^{3/2}}{m_e^{1/2} n e^4 \ln\Lambda}$$

Chapter 5

Plasma Parameters and Definitions

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

e is the elementary electric charge
Z is atomic (proton) number
m is the particle mass
e and i subscripts refer to electrons and ions, respectively
B is the magnetic field
n is the particle density; $n_{20} = n/10^{20}$
T is temperature; $T_{\text{keV}} = T$ in units of kiloelectron-volts

5.1 Single Particle Parameters

Thermal speed [m/s] [5: 5]

$$v_{The} = \left(\frac{2T_e}{m_e}\right)^{1/2} \qquad v_{Thi} = \left(\frac{2T_i}{m_i}\right)^{1/2}$$

Plasma frequencies [radians/s] [8: 135]

$$\omega_{pe} = \left(\frac{n_e e^2}{m_e \epsilon_0}\right)^{1/2} \qquad \omega_{pi} = \left(\frac{n_i (Z_i e)^2}{m_i \epsilon_0}\right)^{1/2}$$

Cyclotron frequencies [radians/s] [8: 134]

$$\Omega_e = \frac{eB}{m_e} \qquad \Omega_i = \frac{Z_i eB}{m_i}$$

Gyro radii [m] [8: 134]

$$\rho_{Le} = \frac{(2m_e T_e)^{1/2}}{eB} \qquad \rho_{Li} = \frac{(2m_i T_i)^{1/2}}{Z_i eB}$$

Single particle parameters as functions of magnetic field B [T], density n_{20} [m^{-3}], and temperature T [keV] [3]

Particle	Plasma frequencies $\left[\frac{\omega}{f}\right]$	Cyclotron frequencies $\left[\frac{\omega}{f}\right]$	Gyro radii [m]
electron	$5.641 \times 10^{11}\ n_{20}^{1/2}$ rad/s $89.779\ n_{20}^{1/2}$ GHz	$1.759 \times 10^{11}\ B$ rad/s $28.000\ B$ GHz	$1.066\times10^{-4}\frac{T_{\mathrm{keV}}^{1/2}}{\mathrm{B}}$
proton	$1.316 \times 10^{10}\ n_{20}^{1/2}$ rad/s $2.094\ n_{20}^{1/2}$ GHz	$9.579 \times 10^{7}\ B$ rad/s $15.241\ B$ MHz	$4.570\times10^{-3}\frac{T_{\mathrm{keV}}^{1/2}}{\mathrm{B}}$
deuteron	$9.312 \times 10^{9}\ n_{20}^{1/2}$ rad/s $1.482\ n_{20}^{1/2}$ GHz	$4.791 \times 10^{7}\ B$ rad/s $7.626\ B$ MHz	$6.461\times10^{-3}\frac{T_{\mathrm{keV}}^{1/2}}{\mathrm{B}}$
triton	$7.609 \times 10^{9}\ n_{20}^{1/2}$ rad/s $1.211\ n_{20}^{1/2}$ GHz	$3.200 \times 10^{7}\ B$ rad/s $5.092\ B$ MHz	$7.906\times10^{-3}\frac{T_{\mathrm{keV}}^{1/2}}{\mathrm{B}}$
helion	$7.610 \times 10^{9}\ n_{20}^{1/2}$ rad/s $1.211\ n_{20}^{1/2}$ GHz	$6.400 \times 10^{7}\ B$ rad/s $10.186\ B$ MHz	$3.952\times10^{-3}\frac{T_{\mathrm{keV}}^{1/2}}{\mathrm{B}}$
alpha	$6.605 \times 10^{9}\ n_{20}^{1/2}$ rad/s $1.051\ n_{20}^{1/2}$ GHz	$4.822 \times 10^{7}\ B$ rad/s $7.675\ B$ MHz	$4.554\times10^{-3}\frac{T_{\mathrm{keV}}^{1/2}}{\mathrm{B}}$

5.2 Plasma Parameters

Debye length [8:125]

$$\frac{1}{\lambda_D^2} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} = \frac{e^2 n_0}{\epsilon_0 T_e} + \frac{e^2 n_0}{\epsilon_0 T_i}$$

$$\lambda_D \approx \left(\frac{\epsilon_0 T_e}{e^2 n_0}\right)^{1/2} = 2.35 \times 10^{-5} \left(\frac{T_{\mathrm{keV}}}{n_{20}}\right)^{1/2} \qquad [\mathrm{m}]$$

Debye-shield ion potential (spherical coordinates) [26:37]

$$V = \frac{e}{4\pi\epsilon_0 r} e^{-\sqrt{2}r/\lambda_D}$$

Volume of a Debye sphere [3]

$$\mathcal{V}_D = \frac{4}{3}\pi\lambda_D^3$$

Plasma parameter [8:133]

$$\Lambda_D = \mathcal{V}_D n_0 = \frac{4}{3}\pi \left(\frac{\epsilon_0 T_e}{e^2 n_0}\right)^{3/2} n_0 \approx 5.453 \times 10^6 \frac{T_{\mathrm{keV}}^{3/2}}{n_{20}^{1/2}}$$

Effective plasma charge [8:56]

$$Z_{\mathrm{eff}} = \frac{\sum\limits_{\text{all ions}} n_j Z_j^2}{\sum\limits_{\text{all ions}} n_j Z_j} = \frac{1}{n_e} \sum_{\text{all ions}} n_j Z_j^2$$

5.3 Plasma Speeds

In this section, ρ_0 is the mass density of the plasma, p_0 is the plasma pressure, and γ is the adiabatic index.

Alfvén speed [8: 314]

$$v_a = (B_0^2/\mu_0\rho_0)^{1/2}$$

Sound speed [5: 96,67]

$$c_{\mathrm{s}} \approx \left(\frac{ZT_{\mathrm{e}} + \gamma T_{\mathrm{i}}}{m_{\mathrm{i}}}\right)^{1/2}$$

where

$$\gamma = 1 \text{ (isothermal flow)}$$

$$\gamma = \frac{2+N}{N} \text{ (N is the number of degrees of freedom)}$$

Plasma mach number [5: 298]

$$M \equiv \frac{v_{\mathrm{plasma}}}{c_s}$$

5.4 Fundamentals of Maxwellian Plasmas

General Maxwellian velocity distribution function [21: 64–65]

$$\mathcal{F}_M(v_x, v_y, v_z) = \mathcal{F}_M(\mathbf{v}) = C \exp\left(-\frac{bm}{2}\left[(v_x - a_x)^2 + (v_y - a_y)^2 + (v_z - a_z)^2\right]\right)$$

where a_x, a_y, a_z, b, and c are constants. If $a_x = a_y = a_z = 0$ we have an (ordinary) Maxwellian; otherwise we have a *drifting* Maxwellian where the drift (or mean) velocity is $\mathbf{v}_{dr} = (a_x, a_y, a_z)$.

Ordinary Maxwellian velocity distribution function for a plasma [21: 64–65]

$$\mathcal{F}_M(\mathbf{v}) = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m}{2T}\left(v_x^2 + v_y^2 + v_z^2\right)\right)$$

Definition of temperature in a Maxwellian plasma [21: 66]

$$\frac{3}{2}nT \equiv n\left\langle \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)\right\rangle \equiv \int \mathcal{F}_M(\mathbf{v})\left(\frac{1}{2}\left(v_x^2 + v_y^2 + v_z^2\right)\right) d\mathbf{v}$$

Total number density of particles in a Maxwellian plasma [21: 66]

$$n = \int\limits_{\substack{\text{all}\\ \text{velocity}\\ \text{space}}} \mathcal{F}_M(\mathbf{v})\, d\mathbf{v} = 4\pi \int_0^\infty w^2 \mathcal{F}_M(w)\, dw$$

where we have transformed to spherical coordinates such that $d\mathbf{v} = w^2 sin\theta \, dw \, d\theta \, d\phi$ and $w = (v_x^2 + v_y^2 + v_z^2)^{1/2}$.

Average (thermal) particle speed in a Maxwellian plasma [21:67]

$$\bar{c} \equiv \frac{1}{n} \int_0^\infty w \mathcal{F}_M(w) \, dw = \left(\frac{8T}{\pi m} \right)^{1/2}$$

Thermal particle flux in a single dimension x for a Maxwellian plasma [21:67]

$$\Gamma \equiv \int_0^\infty \int_0^\infty \int_0^\infty v_x \, \mathcal{F}_M(\mathbf{v}) \, dv_x \, dv_y \, dv_z = \frac{1}{4} n \bar{c}$$

5.5 Definition of a Magnetic Fusion Plasma

A fusion plasma is defined as an electrically conducting ionized gas that is dominated by collective effects and that magnetically confines its composing particles. If L is the macroscopic length scale of the plasma, ω_{transit} is 1 over the time required for a particle to cross the plasma, and v_T is the thermal particle velocity, the criteria to be a fusion plasma are: [8:136]

Required condition	Physical consequence
$\lambda_D \ll L$	Shielding of DC electric fields
$\omega_{pe} \gg \omega_{\text{transit}} = v_{The}/L$	Shielding of AC electric fields
$\Lambda_D \gg 1$	Collective effects dominate
$\rho_{Li} \ll L$	Magnetic confinment of particle orbits
$\Omega_i \gg v_{Thi}/L$	Particle gyro orbits dominate free streaming

5.6 Fundamental Plasma Definitions

5.6.1 Resistivity

Plasma resistivity of an unmagnetized plasma [5:179,183]

$$\eta = \frac{m_e}{n_e e^2 \tau_c} \approx 5.2 \times 10^{-5} \frac{Z_{\text{eff}} \ln \Lambda}{T_{\text{eV}}^{3/2}} \quad [\Omega \cdot \text{m}]$$

Spitzer resistivity of a singly charged *unmagnetized* plasma [26:71]

$$\eta_s = 0.51 \frac{m_e}{n_e e^2 \tau_e} = 0.51 \frac{m_e^{1/2} e^2 \ln \Lambda}{3 \epsilon_0^2 (2\pi T_e)^{3/2}}$$
$$\approx 1.65 \times 10^{-9} \frac{\ln \Lambda}{T_{\text{e, keV}}^{3/2}} \quad [\Omega \cdot \text{m}]$$

Spitzer resistivity of a singly charged *magnetized* plasma [26:71]

$$\eta_{\text{s},||\text{ to B}} = \eta_s \qquad\qquad \eta_{\text{s},\perp\text{ to B}} = 1.96\eta_s$$

Spitzer resistivity of a plasma with impurities [26:72]

$$\eta_s = Z_{\text{eff}}\eta_s$$

Spitzer resistivity of a pure non-hydrogenic plasma of charge Z [26:72]

$$\eta(Z) = N(Z)Z\eta_s \quad \text{where} \quad \begin{cases} N = 0.85 \text{ for } Z = 2 \\ N = 0.74 \text{ for } Z = 4 \end{cases}$$

5.6.2 Runaway Electrons

Volumetric runaway electron production rate [26:74]

$$R = \frac{2}{\sqrt{\pi}}\frac{n}{\tau_{\text{se}}}\left(\frac{E}{E_D}\right)^{1/2} exp\left[-\frac{E_D}{4E} - \left(\frac{2E_D}{E}\right)^{1/2}\right]$$

where the Driecer electric field, E_D is [26:74]

$$\begin{aligned} E_D &= \frac{ne^3 \ln\Lambda}{4\pi\epsilon_0^2 m_e v_{The}^2} \\ &\approx 4.582 \times 10^6 \frac{n \ln\Lambda}{v_{The}^2} \quad \text{[V/m]} \end{aligned}$$

and the electron slowing down time, τ_{se}, for $v_e \gg v_{The}$ is [26:74]

$$\begin{aligned} \tau_{se} &= \frac{4\pi\epsilon_0^2 m_e^2 v_e^3}{ne^4 \ln\Lambda} \\ &\approx 1.241 \times 10^{-6} \frac{v_e^3}{n \ln\Lambda} \quad \text{[s]} \end{aligned}$$

Relativistic runaway electron limit (Connor-Hastie limit) [26:74]

$$\begin{aligned} E &< \frac{ne^3 \ln\Lambda}{4\pi\epsilon_0^2 m_e c^2} \\ &\approx 4.645 \times 10^{-53} \frac{n \ln\Lambda}{m_e} \end{aligned}$$

Chapter 6

Plasma Models

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

$\mathbf{u}$ is the plasma flow velocity
$\mathbf{v}$ is the particle velocity vector
$\mathbf{a}$ is the particle acceleration vector
m is the particle mass
α is a general particle
n_α is the number density of particle α
ρ is the plasma mass density
p is the plasma pressure
a and R_0 are the minor and major radius of a toroidal plasma
κ is the plasma elongation
e is the fundamental charge unit
i and e subscripts refer to the ions and electrons, respectively

6.1 Kinetic

The kinetic Vlasov equation [7: 10]

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \nabla_v f = \left(\frac{df}{dt}\right)_c$$

The kinetic Vlasov equation (collisionless, maxwellian plasmas)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0$$

where [8: 46]

$$f = \frac{1}{(\pi v_{Th})^{3/2}} \exp\left(-v^2/v_{Th}^2\right)$$

6.2 Two Fluids

Continuity equation [7:15]

$$\left(\frac{dn_\alpha}{dt}\right)_\alpha + n_\alpha \nabla \cdot \mathbf{v}_\alpha = 0$$

Momentum equation [7:15]

$$n_\alpha m_\alpha \left(\frac{d\mathbf{v}}{dt}\right)_\alpha - q_\alpha n_\alpha \left(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}\right) + \nabla \cdot \overleftrightarrow{\mathbf{P}}_\alpha = \mathbf{R}_\alpha$$

Energy equation [7:15]

$$\frac{3}{2} n_\alpha \left(\frac{dT_\alpha}{dt}\right)_\alpha + \overleftrightarrow{\mathbf{P}}_\alpha : \nabla \mathbf{v}_\alpha + \nabla \cdot \mathbf{h}_\alpha = Q_\alpha$$

Maxwell's equations [7:15]

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(Zen_i \mathbf{v}_i - en_e \mathbf{v}_e\right) + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0}\left(n_i - n_e\right)$$

$$\nabla \cdot \mathbf{B} = 0$$

where

Convective derivative [7:14]

$$\left(\frac{d}{dt}\right) \equiv \frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla$$

Heat generated by unlike collision [7:13]

$$Q_\alpha \equiv \int \frac{1}{2} m_\alpha w_\alpha^2 C_{\alpha\beta}\, d\mathbf{w}$$

Mean momentum transfer from unlike particles [7:14]

$$\mathbf{R}_\alpha \equiv \int m_\alpha \mathbf{w} C_{\alpha\beta}\, d\mathbf{w}$$

Heat flux due to random motion [7:14]

$$\mathbf{h}_\alpha \equiv \frac{1}{2} n_\alpha m_\alpha \left\langle w^2 \mathbf{w} \right\rangle$$

Temperature [7:14]

$$T_\alpha \equiv p_\alpha / n_\alpha$$

Anisotropic part of pressure tensor [7:14]

$$\overleftrightarrow{\mathbf{\Pi}}_\alpha \equiv \overleftrightarrow{\mathbf{P}}_\alpha - p_\alpha \overleftrightarrow{\mathbf{I}}$$

Pressure tensor [7:14]

$$\overleftrightarrow{\mathbf{P}}_\alpha \equiv n_\alpha m_\alpha \left\langle \mathbf{w}\mathbf{w} \right\rangle$$

Scalar pressure [7:14]

$$p_\alpha \equiv \frac{1}{3} n_\alpha m_\alpha \left\langle w^2 \right\rangle$$

Collision operator [7:11]

$$\left(\frac{\partial f_\alpha}{\partial t} \right)_c = \sum_\beta C_{\alpha\beta}$$

Relations involving the collision operators C_{ij} [7:11]

(a) $\int C_{ee}\, d\mathbf{u} = \int C_{ii}\, d\mathbf{u} = \int C_{ei}\, d\mathbf{u} = \int C_{ie}\, d\mathbf{u} = 0$

(b) $\int m_e \mathbf{u} C_{ee}\, d\mathbf{u} = \int m_i \mathbf{u} C_{ii}\, d\mathbf{u} = 0$

(c) $\int \frac{1}{2} m_e u^2 C_{ee}\, d\mathbf{u} = \int \frac{1}{2} m_i u^2 C_{ii}\, d\mathbf{u} = 0$

(d) $\int \left(m_e \mathbf{u} C_{ei} + m_i \mathbf{u} C_{ie} \right) d\mathbf{u} = 0$

(e) $\int \frac{1}{2} \left(m_e u^2 C_{ei} + m_i u^2 C_{ie} \right) d\mathbf{u} = 0$

6.3 One Fluid

Taking the two fluid equation, we assume $m_e \to 0$ and $n_i = n_e \equiv n$ so that [7:17]

$$\begin{aligned} \rho &= m_i n \\ \mathbf{v} &= \mathbf{v}_i \\ \mathbf{v}_e &= \mathbf{v} - \mathbf{J}/en \end{aligned}$$

Additional simplifications $p = nT = p_e + p_i$ and $T = T_e + T_i$ lead to the one-fluid equations [7:18]

Continuity of mass equation [7:18]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

Continuty of charge equation [7:18]

$$\nabla \cdot \mathbf{J} = 0$$

Momentum equation [7:18]

$$\rho \frac{d\mathbf{v}}{dt} - \mathbf{J} \times \mathbf{B} + \nabla p = -\nabla \cdot (\overleftrightarrow{\mathbf{\Pi}}_i + \overleftrightarrow{\mathbf{\Pi}}_e)$$

Force balance [7:19]

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en} \left(\mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \overleftrightarrow{\mathbf{\Pi}}_e \mathbf{R}_e \right)$$

Equations of state [7:19]

$$\frac{d}{dt} \left(\frac{p_i}{\rho^\gamma} \right) = \frac{2}{3\rho^\gamma} \left(Q_i - \nabla \cdot \mathbf{h}_i - \overleftrightarrow{\mathbf{\Pi}}_i : \nabla \mathbf{v} \right)$$

$$\frac{d}{dt} \left(\frac{p_e}{\rho^\gamma} \right) = \frac{2}{3\rho^\gamma} \left[Q_e - \nabla \cdot \mathbf{h}_e - \overleftrightarrow{\mathbf{\Pi}}_e : \nabla \left(\mathbf{v} - \frac{\mathbf{J}}{en} \right) \right] + \frac{1}{en} \mathbf{J} \cdot \nabla \left(\frac{p_e}{\rho^\gamma} \right)$$

Maxwell's equation (low frequency limit) [7:19]

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

6.4 Magnetohydrodynamics (MHD)

MHD scalings [8:247]

length : $a >> \rho_{Li} >> [\rho_{Le} \sim \lambda_{De}]$
frequency : $\bar{\nu}_{e1} << v_{Thi}/a << \omega_{ci} << [\omega_{ce} \sim \omega_{pe}]$
velocity : $v_{Thi} \sim v_a << v_{The} << c$

where a is the tokomak minor radius.

The MHD equations

Continuity of mass[8:252]

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$$

Momentum equation[8: 252]

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p$$

Ohm's law[8: 252]

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \text{ (ideal MHD)}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta_{||}\mathbf{J} \text{ (resistive MHD)}$$

Equation of state[8: 252]

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

Maxwell's equation (low frequency limit)[8: 252]

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

where $\eta_{||}$ is the parallel resistivity.

6.4.1 Frozen-in Magnetic Field

In ideal MHD, it is found that [8: 300]

$$\frac{d\Phi}{dt} = 0$$

where $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux.

6.5 MHD Equilibria

The generalized MHD equilibrium equations [8: 261]

$$\mathbf{J} \times \mathbf{B} = \nabla p \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \nabla \cdot \mathbf{B} = 0$$

MHD relations for linear plasma devices

Case	Current Relation	Equilibrium Relation
θ-pinch [8:265]	$\mu_0 J_\theta = -\frac{dB_z}{dr}$	$\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0}\right) = 0$
z-pinch [8:267]	$\mu_0 J_z = \frac{1}{r}\frac{d(rB_\theta)}{dr}$	$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$
screw pinch [8:269]	$\mu_0 \mathbf{J} = -\frac{dB_z}{dr}\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{d(rB_\theta)}{dr}\hat{\mathbf{z}}$	$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$

6.6 Grad-Shafranov Equation

The Grad-Shafranov equation is the solution of the ideal MHD equations in two dimensions. In this section, A_ϕ is the toroidal component of the vector potential. [7:110–111]

$$\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

where

$$\mathbf{B} = \frac{1}{R}\nabla\psi \times \hat{\boldsymbol{\phi}} + F/R\hat{\boldsymbol{\phi}}$$

$$\mu_0 \mathbf{J} = \frac{1}{R}\frac{dF}{d\psi}\nabla\psi \times \hat{\boldsymbol{\phi}} - \frac{1}{R}\Delta^*\psi\hat{\boldsymbol{\phi}}$$

where

$$\Delta^*\psi \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2}$$

$$\psi = \frac{\psi_p}{2\pi} = RA_\phi = \frac{1}{2\pi}\int \mathbf{B}_p \cdot d\mathbf{A}$$

$$F(\psi) = RB_\phi$$

Full Solutions to the Grad-Shafranov Equation

Geometry	Solution
High β Noncircular Tokamak [7:148]	$\frac{A}{4}r^2 + \frac{C}{8}r^3\cos(\theta) + \sum_{m=0}^{\infty} H_m r^m \cos(m\theta)$
Spherical Tokamak [7:163]	$-\frac{A}{2}Z^2 + \frac{C}{8}R^4 + c_1 + c_2 R^2 + c_3\left(R^4 - 4R^2Z^2\right)$

6.7 MHD Stability

This section deals with how a plasma behaves when it is perturbed slightly from equilibrium. Therefore, most physical parameters have an equilibrium value (subscript 0) with an added perturbed value (subscript 1). We also define the displacement vector $\boldsymbol{\xi} = \int \tilde{v}_1$. [8: 311]

$$\tilde{\rho}_1 = -\nabla \cdot \left(\rho_0 \tilde{\boldsymbol{\xi}}\right)$$
$$\tilde{p}_1 = -\tilde{\boldsymbol{\xi}} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \tilde{\boldsymbol{\xi}}$$
$$\tilde{\mathbf{Q}} \equiv \tilde{\mathbf{B}}_1 = \nabla \times \left(\tilde{\boldsymbol{\xi}} \times \mathbf{B}_0\right)$$
$$\rho \frac{\partial^2 \tilde{\boldsymbol{\xi}}}{\partial t^2} = \mathbf{F}(\tilde{\boldsymbol{\xi}})$$
$$\mathbf{F}(\tilde{\boldsymbol{\xi}}) = \mathbf{J} \times \tilde{\mathbf{B}}_1 + \tilde{\mathbf{J}}_1 \times \mathbf{B} - \nabla \tilde{p}_1$$
$$\mathbf{F}(\tilde{\boldsymbol{\xi}}) = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \tilde{\mathbf{Q}} + \frac{1}{\mu_0}(\nabla \times \tilde{\mathbf{Q}}) \times \mathbf{B} + \nabla(\tilde{\boldsymbol{\xi}} \cdot \nabla p + \gamma p \nabla \cdot \tilde{\boldsymbol{\xi}})$$

Then assuming all pertubed quantities $Q_1 = Q_1 \exp\left[-i\left(\omega t - \mathbf{k} \cdot \mathbf{r}\right)\right]$, we can find expressions for the first order terms by letting $\frac{\partial}{\partial t} \to \omega$ and $\nabla \to \mathbf{k}$.

6.7.1 Variational Formulation

A different way of looking at the stability problem is given by [7: 250]

$$\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi})}$$

where

$$\begin{aligned} \delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) &= -\frac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) d\mathbf{r} \\ &= -\frac{1}{2} \int \boldsymbol{\xi}^* \cdot [\frac{1}{\mu_0}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \\ &\quad \times \mathbf{Q} + \nabla(\gamma p \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla p)]\, d\mathbf{r} \\ K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) &= \frac{1}{2} \int \rho |\boldsymbol{\xi}|^2 d\mathbf{r} \end{aligned}$$

An equilibrium is stable if $\delta W \geq 0$. The energy principle can be evaluated simply in two cases: a conducting wall directly in contact to the plasma ($\mathbf{n} \cdot \boldsymbol{\xi}_\perp(r_{wall}) = 0$) or with a vacuum region next to the plasma. A vacuum region is more realistic than an adjacent conducting wall, so the variational principle becomes $\delta W = \delta W_F + \delta W_S + \delta W_V$, where F, S, V refer to fluid, surface, and vacuum, respectively. [7: 261]

$$\delta W_F = \frac{1}{2}\int_F d\mathbf{r}\left[\frac{|\mathbf{Q}|^2}{\mu_0} - \boldsymbol{\xi}_\perp^* \cdot (\mathbf{J}\times\mathbf{Q}) + \gamma p|\nabla\cdot\boldsymbol{\xi}|^2 + (\boldsymbol{\xi}_\perp\cdot\nabla p)\nabla\cdot\boldsymbol{\xi}_\perp^*\right]$$

$$\delta W_S = \frac{1}{2}\int_S dS|\mathbf{n}\cdot\boldsymbol{\xi}_\perp|^2\mathbf{n}\cdot\left\|\nabla\left(p+\frac{B^2}{2\mu_0}\right)\right\|$$

$$\delta W_V = \frac{1}{2}\int_V d\mathbf{r}\frac{|\hat{\mathbf{B}}_1|^2}{\mu_0}$$

$$\mathbf{n}\cdot\hat{\mathbf{B}}_1|_{r_w} = 0$$

$$\mathbf{n}\cdot\hat{\mathbf{B}}_1|_{r_w} = \mathbf{n}\cdot\nabla\times\left(\boldsymbol{\xi}_\perp\times\hat{\mathbf{B}}\right)|_{r_p}$$

where $\|A\|$ refers to the jump in A from the vacuum to the plasma.

6.8 Stability of the Screw Pinch

The general screw pinch stability with a vacuum region is given by two different expressions. For internal modes, [7: 293]

$$\delta W = \frac{2\pi^2 R_0}{\mu_0}\int_0^a \left(f\xi'^2 + g\xi^2\right) dr$$

with $\xi(a) = 0$. For external modes, [7: 293]

$$\delta W = \frac{2\pi^2 R_0}{\mu_0}\left\{\int_0^a \left(f\xi'^2 + g\xi^2\right) dr + \left[\left(\frac{krB_z - mB_\theta}{k_0^2 r^2}\right) rF + \frac{r^2\Lambda F^2}{|m|}\right]_a \xi_a^2\right\}$$

where ξ_a is arbitrary and

$$f = \frac{rF^2}{k_0^2}$$

$$g = 2\frac{k^2}{k_0^2}\left(\mu_0 p\right)' + \left(\frac{k_0^2 r^2 - 1}{k_0^2 r^2}\right) rF^2 + 2\frac{k^2}{rk_0^4}\left(kB_z - \frac{mB_\theta}{r}\right) F$$

$$\Lambda = -\frac{|m|K_a}{kaK_a'}\left[\frac{1-(K_b' I_a)/(I_b' K_a)}{1-(K_b' I_a')/(I_b' K_a')}\right]$$

$$F = \mathbf{k}\cdot\mathbf{B} = \frac{mB_\theta}{r} + kB_z$$

6.8.1 Suydam's Criterion

For stability in a screw pinch [7: 298]

$$\frac{rB_z^2}{\mu_0}\left(\frac{q'}{q}\right)^2 + 8p' > 0$$

Chapter 7

Transport

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

e is the elementary electric charge
Z is the atomic (proton) number
m is the particle mass
n is the number density; $n_{20} = n/10^{20}$
$\mathbf{v}$ is the particle velocity vector
ρ is the plasma mass density
T is the plasma temperature
χ is the thermal diffusivity
q is the heat flux
$||$ and $\perp$ indicate parallel and perpendicular to $\mathbf{B}$
a and R_0 are the minor and major radii of a toroidal plasma
S is a general source term

7.1 Classical Transport

7.1.1 Diffusion Equations in a 1D Cylindrical Plasma

If the following approximations are made to the MHD single fluid equations 8: 451−453

(a) neglect inertial terms: $\rho\frac{\partial \mathbf{v}}{\partial t} = 0$

(b) split resistivity into components: $\eta\mathbf{J} \rightarrow \eta_\perp\mathbf{J}_\perp + \eta_{||}\mathbf{J}_{||}$

(c) equate electron and ion temperature: $T_e \approx T_i \equiv T$

(d) introduce the low-β tokamak expansion: $B_z(r,t) = B_0 + \delta B_z(r,t)$ where B_0 is a constant and $\delta B_z \ll 1$

then a short calculation leads a set of diffusion-like transport equations

Particles [8: 453]

$$\frac{\partial n}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD_n\left(\frac{\partial n}{\partial r} + \frac{n}{T}\frac{\partial T}{\partial r} + \frac{2\eta_\parallel}{\beta_p\eta_\perp}\frac{n}{rB_\theta}\frac{\partial(rB_\theta)}{\partial r}\right)\right] \qquad D_n = \frac{2nT\eta_\perp}{B_0^2}$$

where the poloidal beta is $\beta_p = 4\mu_0 nT/B_\theta^2 \sim 1$

Temperature [8: 453]

$$3n\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rn\chi\frac{\partial T}{\partial r}\right) + S$$

Magnetic field [8: 453]

$$\frac{\partial(rB_\theta)}{\partial t} = r\frac{\partial}{\partial r}\left(\frac{D_B}{r}\frac{\partial(rB_\theta)}{\partial r}\right) \qquad D_B = \frac{\eta_\parallel}{\mu_0}$$

7.1.2 Classical Particle Diffusion Coefficients

Classical particle diffusion results from coulomb collisions. No shift in the center of mass occurs for like-particle collisions; therefore, $D = 0$ for *like*-particles and only *unlike*-particle collisions lead to particle diffusion, implying $D_n^{electrons} = D_n^{ions}$.

Net momentum electron-ion exchange collision frequency [8: 217]

$$\bar{\nu}_{ei} = \sqrt{\frac{2}{\pi}}\frac{\omega_{pe}}{\Lambda}\ln\Lambda \approx 1.8\times 10^5\frac{n_{20}}{T_{\mathrm{keV}}^{3/2}}$$

Random walk diffusion coefficient [8: 462]

$$D_{\mathrm{rw}} = 4\frac{\bar{\nu}_{ei}m_eT_e}{e^2B_0^2} \sim \frac{r_{Le}^2}{\bar{\tau}_{ei}}$$

Fluid model diffusion coefficient [8: 463]

$$D_{\mathrm{fm}} = \frac{2nT\eta_\perp}{B_0^2} = 2\frac{\bar{\nu}_{ei}m_eT_e}{e^2B_0^2}$$

Braginskii diffusion coefficient [8: 463]

$$D_{\mathrm{Brag}} = 2.0\times 10^{-3}\frac{n_{20}}{B_0^2T_{\mathrm{keV}}^{1/2}} \qquad [\mathrm{m}^2/\mathrm{s}]$$

7.1.3 The Collision Operator

In this section, Einstein summation notation ($A_aB_a = \sum_a A_aB_a$) is used, and $\mathbf{u} \equiv \mathbf{v} - \mathbf{v}'$, and d^3v' is a differential volume element in velocity space. The Fokker-Planck form of the collision operator [11: 29]

$$C_{ab}(f_a, f_b) = \frac{\partial}{\partial v_k}\left[A_k^{ab}f_a + \frac{\partial}{\partial v_l}(D_{kl}^{ab}f_a)\right]$$

where [11:28–29]

$$A_k^{ab} \equiv \left(1 + \frac{m_a}{m_b}\right) L^{ab} \frac{\partial \phi_b}{\partial v_k}$$

$$D_{kl}^{ab} \equiv -L^{ab} \frac{\partial^2 \psi_b}{\partial \nu_k \partial v_l}$$

$$L^{ab} \equiv \left(\frac{Z_a Z_b e^2}{m_a \epsilon_0}\right) \ln \Lambda$$

$$\phi_b(\mathbf{v}) \equiv -\frac{1}{4\pi} \int \frac{1}{u} f_b(\mathbf{v}') d^3 v'$$

$$\psi_b(\mathbf{v}) \equiv -\frac{1}{8\pi} \int u f_b(\mathbf{v}') d^3 v'$$

Rosenbluth form of the collision operator (general) [11:30]

$$C_{ab}(f_a, f_b) = -\frac{Z_a^2 Z_b^2 e^4 \ln \Lambda}{8\pi \epsilon_0^2 m_a} \frac{\partial}{\partial v_k} \int U_{kl} \left[\frac{f_a(\mathbf{v})}{m_b} \frac{\partial f_b(\mathbf{v}')}{\partial v_l'} - \frac{f_b(\mathbf{v}')}{m_a} \frac{\partial f_a(\mathbf{v})}{\partial v_l}\right] d^3 v'$$

where [11:29]

$$U_{kl} \equiv \frac{u^2 \delta_{kl} - u_k u_l}{u^3}$$

Rosenbluth form of the collision operator (Maxwellian plasma) [11:37]

$$C_{ab}(f_a, f_{b0}) = \nu_D^{ab} \mathcal{L}(f_a) + \frac{1}{v^2} \frac{\partial}{\partial v} \left[v^3 \left(\frac{m_a}{m_a + m_b} \nu_s^{ab} f_a + \frac{1}{2} \nu_{||}^{ab} v \frac{\partial f_a}{\partial v}\right)\right]$$

where $x_\alpha = v / v_{Th\alpha}$ and [11:38]

$$\mathcal{L}(f_a) \equiv \frac{1}{2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f_a}{\partial \theta}\right) + \frac{1}{sin^2 \theta} \frac{\partial f_a}{\partial \phi}\right]$$

$$G(x) \equiv \frac{\phi(x) - x\phi'(x)}{2x^2} = \begin{cases} \dfrac{2x}{3\sqrt{\pi}} & x \to 0 \\ \dfrac{1}{2x^2} & x \to \infty \end{cases}$$

$$\nu_D^{ab}(v) = \hat{\nu}_{ab} \frac{\mathrm{erf}(x_b) - G(x_b)}{x_a^3}$$

$$\nu_s^{ab}(v) = \hat{\nu}_{ab} \frac{2T_a}{T_b} \left(1 + \frac{m_b}{m_a}\right) \frac{G(x_b)}{x_a}$$

$$\nu_{||}^{ab}(v) = 2\hat{\nu}_{ab}\frac{G(x_b)}{x_a^3}$$

$$\hat{\nu}_{ab} = \frac{n_b Z_a^2 Z_b^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_a^2 v_{Tha}^3}$$

Krook collision operator [11: 84]

$$C(f) = \nu(f_0 - f)$$

where ν is chosen as some characteristic collision time and f_0 is a base distribution frequency, which is often chosen to be a Maxwellian.

7.1.4 Classical Thermal Diffusivities

Random walk thermal diffusivities [8: 464]

$$\chi_i = \frac{1}{4}\frac{v_{Thi}^2}{\Omega_i^2\bar{\tau}_{ii}} \sim \frac{r_{Li}^2}{\bar{\tau}_{ii}}$$

$$\chi_e = \frac{1}{4}\frac{v_{The}^2}{\Omega_e^2\bar{\tau}_{ee}} \sim \frac{r_{Le}^2}{\bar{\tau}_{ee}}$$

Braginskii Thermal Diffusivity Coefficients (50%-50% D-T plasma) [8: 465]

$$\chi_i = 0.10\frac{n_{20}}{B_0^2 T_k^{1/2}} \qquad [\mathrm{m}^2/\mathrm{s}]$$

$$\chi_e = 4.8\times 10^{-3}\frac{n_{20}}{B_0^2 T_{\mathrm{keV}}^{1/2}} \qquad [\mathrm{m}^2/\mathrm{s}]$$

7.2 Neoclassical Transport

The following formulae are only valid in the so-called “banana regime” of tokamak confinement devices, where a significant fraction of confined particles undergo magnetic mirroring due to inhomogeneity of the magnetic fields.

7.2.1 Passing Particles

Half-transit time [8: 480]

$$\tau_{1/2} = \frac{l}{v_\|} = \frac{\pi R_0 q}{v_\|}$$

Drift Velocity [8: 481]

$$\mathbf{v}_D = \frac{m_i}{eB}\left(v_\|^2 + \frac{v_\perp}{2}\right)\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B} \approx \frac{1}{\Omega_i R_0}\left(v_\|^2 + \frac{v_\perp}{2}\right)(\hat{\boldsymbol{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)$$

7.2.2 Trapped Particles

In this section, q is the tokamak safety factor.

The minimum (inboard) and maximum (outboard) magnetic fields [8:484]

$$B_{\min} = B_0 \frac{R_0}{R_0 + a} \qquad B_{\max} = B_0 \frac{R_0}{R_0 - a}$$

Trapped particle condition [8:484]

$$\frac{v_\parallel^2}{v^2} < 1 - \frac{B_{\min}}{B_{\max}} = 1 - \frac{R_0 - a}{R_0 + a} \approx 2 \frac{a}{R_0}$$

Fraction of trapped particles (maxwellian distribution $\mathcal{F}_{\mathrm{M}}$) [8:485]

$$\mathcal{F}_{\text{trapped}} = \frac{1}{n} \int_{\theta_c}^{\pi - \theta_c} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty \mathcal{F}_{\mathrm{M}}(v) v^2 dv = \cos\theta_c \approx \left(\frac{2a}{R_0} \right)^{1/2}$$

Half-bounce time[8:487]

$$\tau_{1/2} \approx \frac{l}{v_\parallel} \approx \frac{2l}{v_\parallel} \approx \frac{2\pi R_0 q}{v_\parallel}$$

Full-bounce frequency [8:487]

$$\omega_B = \frac{v_\parallel}{2R_0 q}$$

Mean square step size (random walk model) [8:488]

$$(\Delta l)^2 = \langle (\Delta r)^2 \rangle = 4 \frac{|v_D|^2}{\omega_B^2} \langle \cos^2 \theta_0 \rangle = 2 \frac{q^2 v^4}{\Omega_i^2 v_\parallel^2}$$

Mean step size (averaged over velocity) [8:488]

$$(\Delta l)^2 \approx 3 \left(q^2 \frac{R_0}{a} \right) \frac{v_{Thi}^2}{\Omega_i^2} \sim \left(q^2 \frac{R_0}{a} \right) \rho_{Li}^2$$

where it has been assumed that

$$v^2 \sim \frac{3T}{m} = \frac{3}{2} v_{Thi}^2 \qquad v_\parallel^2 \approx \frac{a}{R_0} v^2 \sim \frac{a}{2R_0} 3 v_{Thi}^2$$

7.2.3 Trapped Particle Neoclassical Transport Coefficients

Random walk model neoclassical diffusion coefficient [8:489]

$$D_n^{NC} = f \frac{\langle (\Delta r)^2 \rangle}{\tau_{\text{eff}}} = 5.2 q^2 \left(\frac{R_0}{a} \right)^{3/2} \left(\frac{2 m_e T_e}{e^2 B_0^2 \bar{\tau}_{ei}} \right) = 5.2 q^2 \left(\frac{R_0}{a} \right)^{3/2} D_n^{CL} \quad [\mathrm{m}^2/\mathrm{s}]$$

Neoclassical diffusion coefficient (Rosenbluth, Hazeltine, and Hinton) [8:489]

$$D_n^{NC} = 2.2 q^2 \left(\frac{R_0}{a} \right)^{3/2} D_n^{CL}$$

Thermal diffusivities (Rosenbluth, Hazeltine, and Hinton) [8: 489]

$$\chi_i^{NC} = 0.68q^2 \left(\frac{R_0}{a}\right)^{3/2} \chi_i^{CL} = 0.068q^2 \left(\frac{R_0}{a}\right)^{3/2} \left(\frac{n_{20}}{B_0^2 T_{\mathrm{keV}}^{1/2}}\right) \quad [\mathrm{m}^2/\mathrm{s}]$$

$$\chi_e^{NC} = 0.89q^2 \left(\frac{R_0}{a}\right)^{3/2} \chi_e^{CL} = 4.3 \times 10^{-3} q^2 \left(\frac{R_0}{a}\right)^{3/2} \left(\frac{n_{20}}{B_0^2 T_{\mathrm{keV}}^{1/2}}\right) \quad [\mathrm{m}^2/\mathrm{s}]$$

7.2.4 Transport regime criteria

Definition of collisionality [11: 149]

$$\nu^* \equiv \nu q R / v_{Th}$$

The banana regime [11: 149]

$$\nu q R / v_T \ll \epsilon^{3/2}$$

The plateau regime [11: 149]

$$\epsilon^{3/2} \ll \nu q R / v_T \ll 1$$

The Pfirsch-Schluter regime [11: 149]

$$\nu q R / v_T \ll 1$$

Chapter 8

Plasma Waves

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

E and **B** are the electric and magnetic fields, respectively
$\hat{\mathbf{b}}$ is a unit vector in the direction of **B**
$||$ and $\perp$ indicate parallel and perpendicular to $\hat{\mathbf{b}}$
k is the wave vector
ω_{pi} is the plasma frequency for particle i
Ω_i is the cyclotron frequency for particle i
$\tilde{\text{X}}$ implies that X can be a complex number

8.1 Cold Plasma Electromagnetic Waves

Starting from Maxwell's equation, linearize $Q = \widetilde{Q}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$, $\mathbf{J} = \overleftrightarrow{\boldsymbol{\sigma}}\cdot\mathbf{E}$
22: 8

$$\mathbf{n} \times \mathbf{n} \times \mathbf{E} + \left(\overleftrightarrow{\mathbf{I}} + \frac{i\overleftrightarrow{\boldsymbol{\sigma}}}{\epsilon_o \omega}\right) \cdot \mathbf{E} = 0$$

where $\mathbf{n} = c\mathbf{k}/\omega$

By combining this equation with the momentum conservation equations, $\mathbf{J} = \sum_j n_j q_j v_j$, setting $p = 0$, and setting collisionality to 0, it is possible to solve for σ. Plugging in σ and then writing in tensor form with Stix notation,

$$\begin{pmatrix} S - n_{||}^2 & -iD & n_\perp n_{||} \\ iD & S - n^2 & 0 \\ n_\perp n_{||} & 0 & P - n_\perp^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

where 22:7

$$S = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2}$$
$$D = \sum_j \frac{\Omega_j}{\omega} \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2}$$
$$P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}$$
$$R = S + D = 1 - \sum_j \frac{\omega_{pj}^2}{\omega\left(\omega + \Omega_j\right)}$$
$$L = S - D = 1 - \sum_j \frac{\omega_{pj}^2}{\omega\left(\omega - \Omega_j\right)}$$

For a wave propagating at an angle θ to $\hat{\mathbf{b}}$ 22:8

$$An^4 - Bn^2 + C = 0$$
$$A = S\sin^2\theta + P\cos^2\theta$$
$$B = RL\sin^2\theta + PS\left(1 + \cos^2\theta\right)$$
$$C = PRL$$

The equation can be rearranged, conveniently obtaining a handy mnemonic ("P Nar Nal Snarl Nap") 22:9

$$\tan^2(\theta) = -\frac{P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

For a wave traveling at an angle θ to $\hat{\mathbf{b}}$, the dispersion relation is given by the Appleton-Hartree equation 22:38

$$n^2 = 1 - \frac{2\omega_{pe}^2\left(\omega^2 - \omega_{pe}^2\right)}{2\omega^2\left(\omega^2 - \omega_{pe}^2\right) - \Omega_e^2\omega^2\sin^2\theta \pm \Omega_e\omega^2\Sigma}$$

$$\Sigma = \left[\Omega_e^2\sin^4\theta + 4\omega^4\left(1 - w_{pe}^2/\omega^2\right)^2\cos^2\theta\right]^{1/2}$$

Polarization of a wave 22:10

$$\frac{iE_x}{E_y} = \frac{n^2 - S}{D}$$

8.1.1 Common Cold Plasma Waves

Defining special frequencies 5:127 22:29

[1]Cutoff/Resonances listed for single ion species plasma

Name	Dispersion Relation[5:145]	Type	Resonance	Cutoff [1]
Light Wave	$n^2 = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}$	$\mathbf{B}_0 = 0$	NA	ω_{pe}
O wave	$n^2 = P$	$\mathbf{n} \perp \mathbf{B}_0$	NA	ω_{pe}
X wave	$n^2 = RL/S$	$\mathbf{n} \perp \mathbf{B}_0$	ω_{uh}, ω_{lh}	ω_r, ω_l
R wave	$n^2 = R$	$\mathbf{n}\vert\vert\mathbf{B}_0$, right-handed	Ω_e	ω_r
L wave	$n^2 = L$	$\mathbf{n}\vert\vert\mathbf{B}_0$, left-handed	Ω_i	ω_l

$$\omega_R = \frac{\Omega_e + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}}{2}$$

$$\omega_L = \frac{-\Omega_e + \sqrt{\Omega_e^2 + 4\omega_{pe}^2}}{2}$$

$$\omega_{uh}^2 = \omega_p^2 + \Omega_e^2$$

$$\omega_{lh}^2 = \Omega_i^2 + \frac{\omega_{pi}^2}{1 + \omega_{pe}^2/\Omega_e^2}$$

For waves that are almost one of the O, X, L, R waves, with a small angle with respect to the proper propogation direction, we get the following dispersion relations. In these equations, $\alpha = \omega_{pe}^2/\omega^2$.[19]

Wave Name	Dispersion Relation
QT-O	$n^2 \simeq \frac{1-\alpha}{1-\alpha\cos^2\theta}$
QT-X	$n^2 \simeq \frac{(1-\alpha)^2\omega^2 - \Omega_e^2 \sin^2\theta}{(1-\alpha)\omega^2 - \Omega_e^2\sin^2\theta}$
QL-L	$n^2 \simeq 1 - \frac{\alpha\omega}{\omega + \Omega_e\cos\theta}$
QL-R	$n^2 \simeq 1 - \frac{\alpha\omega}{\omega - \Omega_e\cos\theta}$

8.2 Electrostatic Waves

These are waves with no perturbed magnetic field. The dispersion relation can be derived by combining Gauss's law, momentum conservation, and continuity equations. It is important to note that $\mathbf{k}||\mathbf{E}$. Let k lie in the (x,z) plane, where z points the direction of $\hat{\mathbf{b}}$.[19]

$$1 - \sum_j \left(\frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} \frac{k_x^2}{k^2} + \frac{\omega_{pj}^2}{\omega^2} \frac{k_z^2}{k^2} \right) = 0$$

Electrostatic wave solutions

Solution for k	Dispersion relation	Solution	Type
$k_x = 0$	$1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} = 0$	$\omega^2 = \sum_j \omega_{pj}^2$	Plasma oscillations
$k_z = 0$	$1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} = 0$	$\omega = \omega_{uh}$	Upper hybrid
		$\omega = \omega_{lh}$	Lower hybrid
$k_x \neq 0$, $k_z \neq 0$	$\omega^2 = \frac{\omega_{uh}^2}{2} \pm \frac{\omega_{uh}^2}{2}\left(1 - \frac{4\Omega_e^2\omega_{pe}^2\cos^2\theta}{\omega_{uh}^4}\right)^{1/2}$		Trivelpiece-Gould

8.3 MHD Waves

Using the MHD formulation describing plasmas, one can define a perturbation and analyze what waves propogate in a plasma. The matrix equation is given below, where $v_a = B_0/\sqrt{\mu_0\rho_0}$ and $v_s = \sqrt{\gamma p_0/\rho_0}$. [8: 314]

$$\begin{pmatrix} \omega^2 - k_{||}^2 v_a^2 & 0 & 0 \\ 0 & \omega^2 - k^2 v_a^2 - k_\perp^2 v_s^2 & -k_\perp k_{||} v_s^2 \\ 0 & -k_\perp k_{||} v_s^2 & \omega^2 - k_{||}^2 v_s^2 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix} = 0$$

MHD wave solutions

Solution	Solution ($\beta \ll 1$)	Wave type
$\omega^2 = k_{||}^2 v_a^2$		Shear Alfvén [8: 314]
$\omega^2 = \frac{k^2}{2}\left(v_a^2 + v_s^2\right)\left[1 \pm (1-\alpha)^{1/2}\right]$ where $\alpha = 4\frac{k_{||}}{k^2}\frac{v_s^2 v_a^2}{(v_s^2+v_a^2)^2}$	$\omega^2 \approx (k_\perp^2 + k_{||}^2)v_a^2$	Compressional [8: 315] Alfvén
	$\omega^2 \approx k_{||}^2 v_s^2$	Sound [8: 316]

8.4 Hot Plasma

The hot plasma dispersion relations are calculated using the Vlasov equation and including finite Larmor orbit effects. Hot plasma effects are also characterized by electrostatic and electromagnetic waves.

8.4.1 Electrostatic

The dispersion relation (general plasmas)[19]

$$\epsilon(\omega, k) = 1 + \sum_j \chi_j(\omega, k) = 0$$

$$\chi_j = \frac{\omega_{pj}^2}{k^2}\frac{2\pi}{n_{0j}}\int_{-\infty}^{\infty} dv_{||}\int_0^{\infty} v_\perp dv_\perp \sum_m \frac{J_m^2(\frac{k_\perp v_\perp}{\Omega_j})\left[k_{||}\frac{\partial f_{0j}}{\partial v_{||}} + \frac{m\Omega_j}{v_\perp}\frac{\partial f_{0j}}{\partial v_\perp}\right]}{\omega - m\Omega_j - k_{||}v_{||}}$$

where the number subscripts refer to the linearization order.

The dispersion relation (Maxwellian plasmas)

$$\epsilon = 1 + \sum_j \frac{1}{k^2\lambda_{Dj}^2}\left[1 + \zeta_{0j}\sum_{m=-\infty}^{m=\infty}\Gamma_m(b_j)Z(\zeta_{mj})\right]$$

where

$$Z(\zeta_{mj}) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{dv_{||}e^{v_{||}^2/v_{Thj}^2}}{v_{||} - \left(\frac{\omega - m\Omega_j}{k_{||}}\right)} \qquad b_j = k_\perp^2\rho_{Lj}^2$$

$$\zeta_{mj} = (w - m\Omega_j)/(k_{||}v_{Thj}) \qquad \Gamma_m(b_j) \equiv I_m(b_j)\exp(-b_j)$$

This function can be evaluated as a power or an asymptotic series.

$|\zeta_m| \ll 1$ (kinetic limit)

$$ReZ(\zeta_m) \approx -2\zeta_m\left[1 - \frac{2}{3}\zeta_m^2 + \frac{4}{15}\zeta_m^4 + ...\right] \text{(power series)}$$

$|\zeta_m| \gg 1$ (fluid limit)

$$ReZ(\zeta_m) \approx -\frac{1}{\zeta_m}\left[1 + \frac{1}{2\zeta_m^2} + \frac{3}{4\zeta_m^4} + ...\right] \text{(asymptotic)}$$

Γ can be expanded as well as

$$\Gamma_0 \approx 1 - b + \frac{3}{4}b^2 + O(b^3)$$
$$\Gamma_1 \approx \frac{b}{2}(1 - b) + O(b^3)$$
$$\Gamma_2 \approx b^2/8 + O(b^3)$$
$$\Gamma_3 \approx O(b^3)$$

8.4.2 Electromagnetic

The dispersion relation in the electromagnetic limit[19]

$$\begin{pmatrix} K_{xx} - n_z^2 & K_{xy} & K_{xz} + n_x n_z \\ K_{yz} & K_{yy} - n_x^2 - n_z^2 & K_{yz} \\ K_{zx} + n_x n_z & K_{zy} & K_{zz} - n_z^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$K_{xx} = 1 + \sum_{j=i,e} \sum_{l=-\infty}^{\infty} \frac{l^2 \Omega^3}{k_\perp^2} \int dv^2 J_l^2(\lambda) P_l$$

$$K_{xy} = -K_{yx} = i \sum_j \sum_{l=-\infty}^{\infty} \frac{l\Omega^2}{k_\perp} \int dv^2 v_\perp J_l(\lambda) J_l'(\lambda) P_l$$

$$K_{xz} = \sum_j \sum_{l=-\infty}^{\infty} \frac{l\Omega^2}{k_\perp} \int dv^2 v_{||} J_l^2(\lambda) Q_l$$

$$K_{yy} = 1 + \sum_j \sum_{l=-\infty}^{\infty} \Omega \int dv^2 \left[v_\perp J_l'(\lambda) \right]^2 P_l$$

$$K_{yz} = -i \sum_j \sum_{l=-\infty}^{\infty} \Omega \int dv^2 v_\perp v_{||} J_l(\lambda) J_l'(\lambda) Q_l$$

$$K_{zx} = \sum_j \sum_{l=-\infty}^{\infty} \frac{l\Omega^2}{k_\perp} \int dv^2 v_{||} J_l^2(\lambda) P_l$$

$$K_{zy} = i \sum_j \sum_{l=-\infty}^{\infty} \Omega \int dv^2 v_\perp v_{||} J_l(\lambda) J_l'(\lambda) P_l$$

$$K_{zz} = 1 + \sum_j \sum_{l=-\infty}^{\infty} \Omega \int dv^2 v_{||}^2 J_l^2(\lambda) Q_l$$

$$\int dv^2 \equiv 2 \int_{-\infty}^{\infty} dv_{||} \int_0^{\infty} v_\perp dv_\perp$$

$$P_l = 2\pi \frac{\omega_{pj}^2}{\omega \Omega_j} \frac{\left[\frac{\partial f_{0j}}{\partial v_\perp^2} + \frac{k_{||} v_{||}}{\omega} \left(\frac{\partial f_{0j}}{\partial v_{||}^2} - \frac{\partial f_{0j}}{\partial v_\perp^2} \right) \right]}{\omega - l\Omega_j - k_z v_{||}}$$

$$Q_l = 2\pi \frac{\omega_{pj}^2}{\omega \Omega_j} \frac{\left[\frac{\partial f_{0j}}{\partial v_{||}^2} - \frac{l\Omega_j}{\omega} \left(\frac{\partial f_{0j}}{\partial v_{||}^2} - \frac{\partial f_{0j}}{\partial v_\perp^2} \right) \right]}{\omega - l\Omega_j - k_z v_{||}}$$

where $\lambda = |k_\perp v_\perp / \Omega|$ and $\partial f / \partial v^2 \equiv \partial f / \partial \left(v^2 \right)$

Dispersion relation evaluation for an isotropic Maxwellian plasma [5:276]

$$K_{xx} = 1 + \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-b_j}}{b_j} \zeta_0 \sum_{n=-\infty}^{\infty} n^2 I_n(b_j) Z(\zeta_n)$$

$$K_{xy} = -K_{yz} = i \sum_j \pm \frac{\omega_p^2}{\omega^2} e^{-b_j} \zeta_0 \sum_{n=-\infty}^{\infty} n \left[I_n(b_j) - I_n'(b_j) \right] Z(\zeta_n)$$

$$K_{xz} = K_{zx} = \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-b_j}}{(2b_j)^{1/2}} \zeta_0 \sum_{n=-\infty}^{\infty} n I_n(b_j) Z'(\zeta_n)$$

$$K_{yy} = 1 + \sum_j \frac{\omega_{pj}^2}{\omega^2} \frac{e^{-b_j}}{b_j} \zeta_0 \sum_{n=-\infty}^{\infty} \left(n^2 I_n(b_j) + 2b_j^2 \left[I_n(b_j) - I_n'(b_j) \right] \right) Z(\zeta_n)$$

$$K_{yz} = -K_{zy} = -i \sum_j \pm \frac{\omega_{pj}^2}{\omega^2} \left(\frac{b_j}{2} \right)^{1/2} e^{-b_j} \zeta_0 \sum_{n=-\infty}^{\infty} \left[I_n(b_j) - I_n'(b_j) \right] Z'(\zeta_n)$$

$$K_{zz} = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} e^{-b} \zeta_0 \sum_{n=-\infty}^{\infty} I_n(b_j) \zeta_n Z'(\zeta_n)$$

Chapter 9

Nuclear Physics

In this chapter, all units are SI with the exception of: temperature and energy, which are defined in the historical units of eV (electron-volts); cross sections, which are defined in the historical units of barn; and Bosch-Hale reaction rates, which are given in cubic centimeters per second.

e is the elementary electric charge
Z is the number of nuclear protons
N is the number of nuclear neutrons
A is the number of nucleons (N+Z)
m is the particle mass
n is the particle number density
$\mathbf{v}$ is the particle velocity vector
E is the particle kinetic energy

9.1 Fundamental Definitions

Nuclear reaction notation [15: 378–381]

$$\begin{matrix} a + X \Rightarrow b + Y \\ \text{or} \\ X(a,b)Y \end{matrix} \quad \text{where} \quad \left\{ \begin{array}{ll} \left. \begin{array}{l} a = \text{bombarding particle} \\ X = \text{target nucleus} \end{array} \right\} & \text{entrance channel} \\ \left. \begin{array}{l} b = \text{ejected particle(s)} \\ Y = \text{product nucleus} \end{array} \right\} & \text{exit channel} \end{array} \right.$$

X(a,b)Y	general nuclear reaction
X(a,a)X	elastic scattering
X(a,a')X*	inelastic scattering
X(n,n)X	neutron elastic scattering
X(n,n')X*	neutron inelastic scattering
X(n,2n)Y	neutron multiplication
X(n,γ)Y	neutron capture
X(n,f)Y	neutron-induced fission

Nuclear mass [15:65]

$$m = Zm_p + Nm_n - E_B/c^2$$

Nuclear binding energy [15:68]

$$E_B(A, Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A, Z)$$

where the value of the coefficients a in MeV is [15:68]

$a_v = 15.5 \qquad a_s = 16.8 \qquad a_c = 0.72 \qquad a_a = 23 \qquad a_p = 34$ MeV

$$\delta(A, Z) = \begin{cases} a_p A^{-3/4} & \text{(Z,N even)} \\ 0 & \text{(A odd)} \\ -a_p A^{-3/4} & \text{(Z,N odd)} \end{cases}$$

Nuclear reaction Q-value [15:381]

$$Q = \left((m_a + m_X) - (m_b + m_Y)\right) c^2 = E_f - E_i$$

Reaction threshold energy ($Q < 0$) [15:382]

$$E_{\text{thresh}} = -Q \frac{m_Y + m_b}{m_y + m_b - m_a}$$

9.2 Nuclear Interactions

$A \equiv m_A/m_N$ is approximately the nucleus atomic mass number
i and f refer to initial and final, respectively
θ is the angle between the a and b trajectory in the lab frame
θ_{cm} is the angle between the a and b trajectory in the center of mass frame

Reaction Q-value from kinematics [15:384]

$$Q = E_b \left(1 + \frac{m_b}{m_Y}\right) - E_a \left(1 - \frac{m_a}{m_Y}\right) - 2 \left(\frac{m_a m_b}{m_Y^2} E_a E_b\right)^{1/2} \cos\theta$$

Ejected particle energy: X(a,b)Y [15:382]

$$E_y^{1/2} = \zeta \frac{(m_a m_b E_a)^{1/2}}{m_Y + m_b} \pm \frac{\left(m_a m_b E_a \zeta^2 + (m_Y + m_b)[m_Y Q + (m_Y - m_a)E_a]\right)^{1/2}}{m_Y + m_b}$$

Maximum kinetic energy transfer fraction: X(a,a)X [3]

$$\left.\frac{E_f}{E_i}\right|_{\theta=0} = \frac{4A_1A_2}{(A_1 + A_2)^2}$$

9.2.1 Charged Particle Interactions

Stopping power of heavy charged particles (Bethe formula) [15:194]

$$\frac{dE}{dx} = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi z^2 N_A Z\rho}{m_e c^2 \beta^2 A}\left[\ln\left(\frac{2m_e c^2\beta^2}{I}\right) - \ln\left(1-\beta^2\right) - \beta^2\right]$$

where $\beta c = v$ and ze are the incident particle's speed and charge;
ρ is the mass density of the stopping medium;
N_A is Avogadro's number;
I is mean excitation of atomic electrons, typically taken as $I \approx 10Z$.

Collisional stopping power of electrons [15:196]

$$\left(\frac{dE}{dx}\right)_c = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{2\pi N_A Z\rho}{m_e c^2\beta^2 A}\left[\ln\frac{E_i(E_i+m_ec^2)^2\beta^2}{2I^2 m_e c^2} + \left(1-\beta^2\right) - \left(2\sqrt{1-\beta^2} - 1 + \beta^2\right)\ln 2 + \frac{1}{8}\left(1-\sqrt{1-\beta^2}\right)^2\right]$$

Radiative stopping power of electrons ($\gtrsim$ 1 MeV) [15:196]

$$\left(\frac{dE}{dx}\right)_r = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{Z^2 N_A\left(E_i + m_e c^2\right)\rho}{137 m_e^2 c^4 A}\left[4\ln\frac{2\left(E_i+m_ec^2\right)}{m_ec^2} - \frac{4}{3}\right]$$

9.2.2 Neutron Interactions

Ejected neutron energy: X(n,n)X [15:448]

$$E_{n,f} = \frac{A^2 + 2A\cos\theta_{\text{cm}} + 1}{(A+1)^2}$$

Maximum kinetic energy transfer fraction: X(n,n)X [15:448]

$$\left.\frac{E_{n,f}}{E_{n,i}}\right|_{\theta=0} = \left(\frac{A-1}{A+1}\right)^2$$

Neutron lethargy ($E_n \lesssim$ 10 MeV) [15:450]

$$\xi = 1 + \frac{(A-1)^2}{2A}\ln\left(\frac{A-1}{A+1}\right)$$

Number of collisions required to change neutron energies [15:450]

$$N = \frac{\ln(E_i/E_f)}{\xi}$$

9.2.3 Gamma Interactions

Klein-Nishina cross section for Compton scattering [15: 201]

$$\frac{d\sigma}{d\Omega} = r_0^2 \left[\frac{1}{1+\alpha\left(1-\cos\theta\right)}\right]^3 \left[\frac{1+\cos^2\theta}{2}\right] \left[1+\frac{\alpha^2\left(1-\cos\theta\right)}{\left(1+\cos^2\theta\right)\left[1+\alpha\left(1-\cos\theta\right)\right]}\right]$$

where ($\alpha = E_\gamma/mc^2$) and ($r_0 = e^2/4\pi\epsilon_0 mc^2 = 2.818$ fm) is the classical electron radius.

Energy shift from Compton scattering [15: 201]

$$E_f = \frac{E_i}{1+\left(\frac{E_i}{mc^2}\right)\left(1-\cos\theta\right)}$$

9.3 Cross Section Theory

A reaction cross section $\sigma(E_a)$, or more simply σ, is a measure of the probability that reaction $X(a,b)Y$ will occur. Consider a beam of a particle with current I_a directed onto X target nuclei with an areal density of N_X per unit area. By observing the energy and angular distribution, $e(E_b)$ and $r(\theta,\phi)$ respectively, of the ejected particle b into a solid angle $d\Omega$ then the doubly differential cross section can be determined, which is the probability of observing particle b at in solid angle $d\Omega$ with energy E_b. [15: 392–394]

Doubly differential cross section [15: 392–393]

$$\frac{d\sigma}{d\Omega\, dE_b} = \frac{r(\theta,\phi)\; e(E_b)}{4\pi I_a N_X}$$

Differential energy cross section [15: 393]

$$\frac{d\sigma}{dE_b} = \int \frac{d\sigma}{d\Omega\, dE_b} d\Omega$$

Differential angular cross section [15: 393]

$$\frac{d\sigma}{d\Omega} = \int \frac{d\sigma}{d\Omega\, dE_b}\, dE_b$$

Reaction cross section [15: 393]

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int_0^{2\pi}\int_0^{\pi} \frac{d\sigma}{d\Omega}\, \sin\theta\, d\theta\, d\phi$$

Total cross section [15: 393]

$$\sigma_T = \sum_{i=0}^{\text{all reactions}} \sigma_i$$

The Sommerfeld parameter [23: 5]

$$\eta = \frac{Z_a Z_X e^2}{\hbar v}$$

Astrophysical S-factor [23: 5]

$$S(E) = \sigma(E) E \exp(2\pi\eta)$$

9.4 Reaction Rate Theory

For a thermonuclear plasma, the volumetric reaction rate $\mathcal{R}$ (also known as *thermal reactivity*) describes the number of reactions occuring per unit volume per unit time. In thermonuclear fusion plasmas, $\mathcal{R}$ is obtained by integrating the energy-dependent cross section, $\sigma(v)$, over the distribution functions of the participating species. [26: 5–8]

Partial volumetric reaction rate [26: 5–6]

$$\mathcal{R}_p = \sigma(v) v f_1(\mathbf{v}_1) f_2(\mathbf{v}_2) \qquad \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$$

Total volumetric reaction rate for general $f(v)$ [26: 6]

$$\mathcal{R} = \iint \sigma(v) v f_1(\mathbf{v}_1) f_2(\mathbf{v}_2)\, d^3v_1\, d^3v_2$$

Total volumetric reaction rate for Maxwellian $f(v)$) [26: 6]

$$\mathcal{R} = n_1 n_2 \langle\sigma v\rangle \text{ where } \begin{cases} \langle\sigma v\rangle = \left(\dfrac{8\mu^3}{\pi T^3}\right)^{1/2} \dfrac{1}{m_1^2} \displaystyle\int \sigma(E) E e^{-\mu E/m_1 T}\, dE \; [\mathrm{m^3 s^{-1}}] \\ \mu = \dfrac{m_1 m_2}{m_1 + m_2} \qquad\qquad E = \dfrac{1}{2} m v^2 \end{cases}$$

9.5 Nuclear Reactions for Fusion Plasmas

All data from the ENDF/B-VII nuclear data libraries. [4]

Abbreviations: n=neutron, p=^{1}H, d=^{2}H, t=^{3}H, h=^{3}He, α =^{4}He

	Reactants		Products (kinetic energy in MeV)	Branching ratio	Q-value (MeV)
1.	d + t	$\longrightarrow$	α(3.52) + n(14.07)	1.00	17.59
2.	d + d	$\longrightarrow$	t(1.01) + p(3.02)	0.50	4.03
		$\longrightarrow$	h(0.82) + n(2.45)	0.50	3.27
3.	d + h	$\longrightarrow$	α(3.67) + p(14.68)	1.00	18.35
4.	t + t	$\longrightarrow$	α + 2n	1.00	11.33
5.	h + t	$\longrightarrow$	α + p + n	0.51	12.10
		$\longrightarrow$	α(4.77) + d(9.54)	0.43	14.32
		$\longrightarrow$	^{5}He(1.87) + p(9.34)	0.06	11.21
6.	p + ^{6}Li	$\longrightarrow$	α(1.72) + h(2.30)	1.00	4.02
7.	p + ^{7}Li	$\longrightarrow$	2 α	0.20	17.35
		$\longrightarrow$	^{7}Be + n	0.80	-1.64
8.	d + ^{6}Li	$\longrightarrow$	2 α	1.00	22.37
9.	p + ^{11}B	$\longrightarrow$	3 α	1.00	8.62

9.6 Nuclear Reactions for Fusion Energy

All data from the ENDF/B-VII nuclear data libraries. [4]

Abbreviations: n=neutron, t=^{3}H, B=tritium breeding, M=neutron multiplication

	Reaction	Q-value [MeV]	Purpose	σ(0.025 eV) [barn]	σ(14.1 MeV) [barn]
1.	^{6}Li(n,t)^{4}He	4.78	B	978	0.03
2.	^{6}Li(n,2nα)^{6}Li	-3.96	M	-	0.08
3.	^{7}Li(n,2n)^{6}Li	-7.25	M	-	0.03
4.	^{7}Li(n,2nα)^{3}H	-8.72	B/M	-	0.02
5.	^{9}Be(n,2n)^{8}Be	-1.57	M	-	0.48
6.	^{204}Pb(n,2n)^{203}Pb	-8.39	M	-	2.22
7.	^{206}Pb(n,2n)^{205}Pb	-8.09	M	-	2.22
8.	^{207}Pb(n,2n)^{206}Pb	-6.74	M	-	2.29
9.	^{208}Pb(n,2n)^{207}Pb	-7.37	M	-	2.30

9.7 Fusion Cross Section Parametrization

The Bosch-Hale parametrization of the fusion reaction cross section [2]

$$\sigma(E_{\text{keV}}) = \frac{S(E)}{E\exp(B_G/\sqrt{E})} \quad \text{[millibarn]}$$

where

$$S(E_{\text{keV}}) = \frac{A_1 + E(A_2 + E(A_3 + E(A_4 + EA_5)))}{1 + E(B_1 + E(B_2 + E(B_3 + EB_4)))}$$

Bosch-Hale parametrization coefficients for several fusion reactions[2]

	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
B_G [keV]	31.3970	31.3970	34.3827	68.7508
A_1	5.3701×10^{4}	5.5576×10^{4}	6.927×10^{4}	5.7501×10^{6}
A_2	3.3027×10^{2}	2.1054×10^{2}	7.454×10^{8}	2.5226×10^{3}
A_3	-1.2706×10^{-1}	-3.2638×10^{-2}	2.050×10^{6}	4.5566×10^{1}
A_4	2.9327×10^{-5}	1.4987×10^{-6}	5.200×10^{4}	0.0
A_5	-2.5151×10^{-9}	1.8181×10^{-10}	0.0	0.0
B_1	0.0	0.0	6.380×10^{1}	-3.1995×10^{-3}
B_2	0.0	0.0	-9.950×10^{-1}	-8.5530×10^{-6}
B_3	0.0	0.0	6.981×10^{-5}	5.9014×10^{-8}
B_4	0.0	0.0	1.728×10^{-4}	0.0
Valid Range [keV]	0.5<E<4900	0.5<E<5000	0.5<E<550	0.3<E<900

Tabulated Bosch-Hale cross sections [millibarns] [2]

E (keV)	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
3	2.445×10^{-4}	2.513×10^{-4}	9.808×10^{-3}	1.119×10^{-11}
4	2.093×10^{-3}	2.146×10^{-3}	1.073×10^{-1}	1.718×10^{-9}
5	8.834×10^{-3}	9.038×10^{-3}	5.383×10^{-1}	5.199×10^{-8}
6	2.517×10^{-2}	2.569×10^{-2}	1.749×10^{0}	6.336×10^{-7}
7	5.616×10^{-2}	5.720×10^{-2}	4.335×10^{0}	4.373×10^{-6}
8	1.064×10^{-1}	1.081×10^{-1}	8.968×10^{0}	2.058×10^{-5}
9	1.794×10^{-1}	1.820×10^{-1}	1.632×10^{1}	7.374×10^{-5}
10	2.779×10^{-1}	2.812×10^{-1}	2.702×10^{1}	2.160×10^{-4}
12	5.563×10^{-1}	5.607×10^{-1}	6.065×10^{2}	1.206×10^{-3}
15	1.178×10^{0}	1.180×10^{0}	1.479×10^{2}	7.944×10^{-3}
20	2.691×10^{0}	2.670×10^{0}	4.077×10^{2}	6.568×10^{-2}

9.8 Fusion Reaction Rate Parametrization

The Bosch-Hale parametrization of the volumetric reaction rates [2]

$$\langle\sigma v\rangle = C_1 \cdot \theta \cdot \sqrt{\frac{\xi}{m_\mu c^2 T^3_{\mathrm{i,\ keV}}}}\, e^{-3\xi} \qquad \begin{array}{c}[\mathrm{cm^3\ s^{-1}}]\\ \textit{or}\\ \times 10^{-6}\ [\mathrm{m^3\ s^{-1}}]\end{array}$$

where

$$\theta = T \Big/ \left(1 - \frac{T(C_2 + T(C_4 + TC_6))}{1 + T(C_3 + T(C_5 + TC_7))}\right) \qquad \xi = \left(\frac{B_G^2}{4\theta}\right)^{1/3}$$

Bosch-Hale parametrization coefficients for volumetric reaction rates [2]

	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
B_G [keV$^{1/2}$]	31.3970	31.3970	34.3827	68.7508
$m_\mu c^2$ [keV]	937 814	937 814	1 124 656	1 124 572
C_1	5.43360×10^{-12}	5.65718×10^{-12}	1.17302×10^{-9}	5.51036×10^{-10}
C_2	5.85778×10^{-3}	3.41267×10^{-3}	1.51361×10^{-2}	6.41918×10^{-3}
C_3	7.68222×10^{-3}	1.99167×10^{-3}	7.51886×10^{-2}	-2.02896×10^{-3}
C_4	0.0	0.0	4.60643×10^{-3}	-1.91080×10^{-5}
C_5	-2.96400×10^{-6}	1.05060×10^{-5}	1.35000×10^{-2}	1.35776×10^{-4}
C_6	0.0	0.0	-1.06750×10^{-4}	0.0
C_7	0.0	0.0	1.36600×10^{-5}	0.0
Valid range (keV)	$0.2<T_i<100$	$0.2<T_i<100$	$0.2<T_i<100$	$0.5<T_i<190$

Tabulated Bosch-Hale reaction rates [m^{3}s^{-1}] [2]

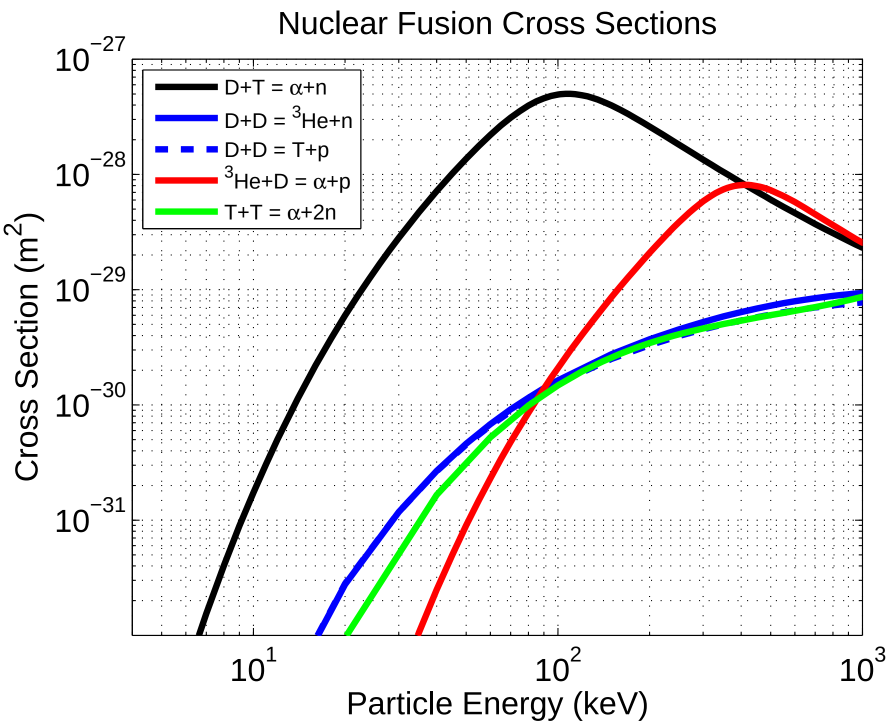
T (keV)	^{2}H(d,n)^{3}He	^{2}H(d,p)^{3}H	^3H(d,n)^{4}He	^{3}He(d,p)^{4}He
1.0	9.933×10^{-29}	1.017×10^{-28}	6.857×10^{-27}	3.057×10^{-32}
1.5	8.284×10^{-28}	8.431×10^{-28}	6.923×10^{-26}	1.317×10^{-30}
2.0	3.110×10^{-27}	3.150×10^{-27}	2.977×10^{-25}	1.399×10^{-29}
3.0	1.602×10^{-26}	1.608×10^{-26}	1.867×10^{-24}	2.676×10^{-28}
4.0	4.447×10^{-26}	4.428×10^{-26}	5.974×10^{-24}	1.710×10^{-27}
5.0	9.128×10^{-26}	9.024×10^{-26}	1.366×10^{-23}	6.377×10^{-27}
8.0	3.457×10^{-25}	3.354×10^{-25}	6.222×10^{-23}	7.504×10^{-26}
10.0	6.023×10^{-25}	5.781×10^{-25}	1.136×10^{-22}	2.126×10^{-25}
12.0	9.175×10^{-25}	8.723×10^{-25}	1.747×10^{-22}	4.715×10^{-25}
15.0	1.481×10^{-24}	1.390×10^{-24}	2.740×10^{-22}	1.175×10^{-24}
20.0	2.603×10^{-24}	2.399×10^{-24}	4.330×10^{-22}	3.482×10^{-24}

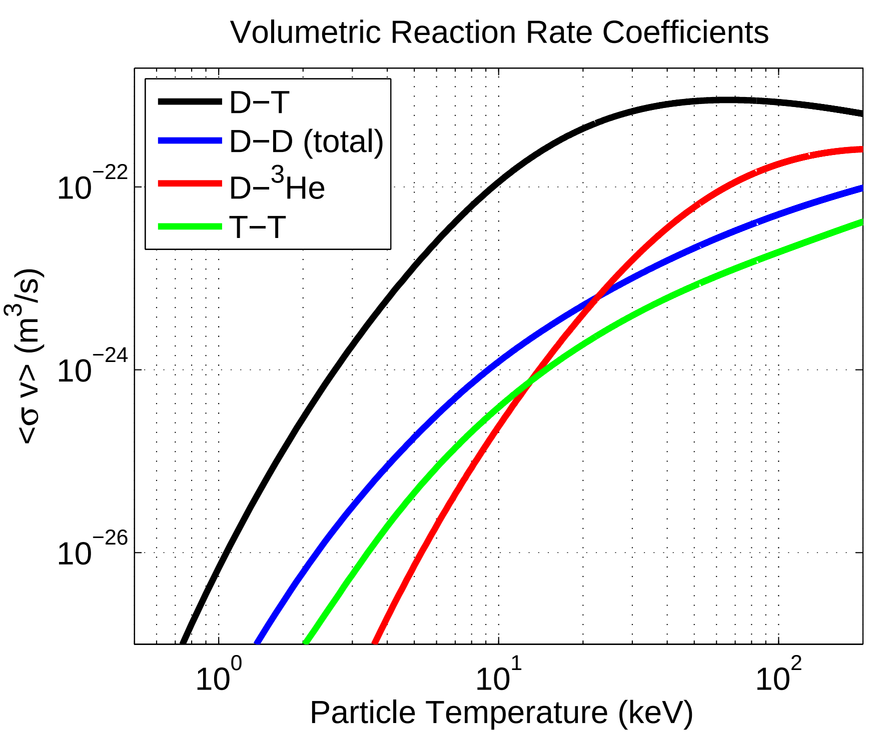
Approximate DT volumetric reaction rate ($10 \lesssim T$ [keV] $\lesssim 20$) [26:7]

$$\langle\sigma v\rangle_{\mathrm{DT}} = 1.1 \times 10^{-24} T^2_{\mathrm{keV}} \qquad [\mathrm{m^3\ s^{-1}}]$$

9.9 Cross Section and Reaction Rate Plots

Data from the ENDF/B-VII nuclear data libraries [4] is plotted directly below and used to calculate the volumetric reaction rate coefficients (thermal reactivity) [3].





Chapter 10

Tokamak Physics

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

e is the fundamental charge unit
Z is the number of nuclear protons
R_0 is the major radius of a toroidal plasma
a and b are the horizontal and vertical minor radii of a toroidal plasma
    Note: $a = b$ for circular cross sections, and a is used by convention
R and r denote lengths in the major and minor radii, respectively
θ and ϕ are the poloidal and toroidal angular coordinates, respectively
B_x is the magnetic field in direction x
B_{xa} is the magnetic field evaluated at the plasma edge in direction x
I_p is the toroidal plasma current
p is the plasma pressure
v is velocity of the plasma

10.1 Fundamental Definitions

Inverse aspect ratio [26:117]

$$\epsilon = \frac{a}{R_0}$$

Plasma elongation [26:741]

$$\kappa = \frac{b}{a}$$

Plasma triangularity [26:741]

$$\delta = \frac{(c + d)/2}{a}$$

where c, d are distances to the top of the plasma and the x-point, respectively, from the plasma center.

Large aspect ratio expansion ($\epsilon \ll 1$) [8:280]

$$\frac{1}{R} \approx \frac{1}{R_0}\left(1 - \frac{r}{R_0}\cos\theta\right)$$

Surface area of a torus

$$S_{\mathrm{c-torus}} = 4\pi^2 a R_0 \qquad \text{(Circular cross section)} \; [14:24]$$

$$S_{\mathrm{e-torus}} = 8\pi a R_0 E(k) \approx 4\pi^2 a R_0 \left(\frac{1+\kappa^2}{2}\right)^{1/2} \qquad \text{(Elliptical cross section)} \; [27]$$

Volume of a torus

$$V_{\mathrm{c-torus}} = 2\pi^2 a^2 R_0 \qquad \text{(Circular cross section)} \; [14:24]$$

$$V_{\mathrm{e-torus}} = 2\pi^2 a^2 \kappa R_0 \qquad \text{(Elliptical cross section)} \; [27]$$

MHD toroidal plasma volume [7:112]

$$V(\psi) = \pi R_0 \int_0^{2\pi} d\theta r^2 \left[1 + \frac{2}{3}\left(\frac{r}{R_0}\right)\cos\theta\right]$$

Volume averaged plasma pressure [3]

$$\langle p \rangle = \frac{1}{V}\int_{volume} p \, d\tau$$

Toroidal plasma beta [7:71]

$$\beta_t = \frac{2\mu_0 \langle p \rangle}{B_{\phi a}^2}$$

Poloidal plasma beta [7:71]

$$\beta_p = \frac{2\mu_0 \langle p \rangle}{B_{\theta a}^2} = \frac{8\pi^2 a^2 \kappa^2 \langle p \rangle}{\mu_0 I_p^2}$$

Radial electric field in a rotating toroidal plasma [3]

$$\mathrm{E}_r \approx \mathrm{v}_\phi \mathrm{B}_\theta - \mathrm{v}_\theta \mathrm{B}_\phi + \frac{1}{Z_i e n}\nabla p$$

10.2 Magnetic Topology

Toroidal magnetic field for plasma confinement [3]

$$B_\phi \approx \frac{B_\phi(r) R_0}{R} \approx \frac{B_\phi(r)}{1 - \epsilon\cos\theta} \quad \text{(valid for } \epsilon \ll 1\text{)}$$

Poloidal magnetic field for plasma confinement [3]

$$B_\theta \approx \frac{\mu_0 I_p(r)}{2\pi r}$$

Safety factor (general) [26:111]

$$q(r) = \frac{\text{\# of toroidal field line orbits at r}}{\text{\# of poloidal field line orbits at r}}$$

Safety factor for cylindrical plasma (r, θ, z) [26:112]

$$q(r)_{\text{cyl}} = \frac{rB_\phi(r)}{RB_\theta(r)} = \frac{2\pi r^2 B_\phi(r)}{\mu_0 I_p(r) R}$$

Safety factor for toroidal plasma (R, θ, ϕ) [8:288]

$$\begin{aligned} q(r^*)_{\text{tor}} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{rB_\phi}{RB_\theta}\,d\theta \\ &= \frac{r_0 B_\phi(r_0)}{R_0 B_\theta(r_0)\left(1 - r_0^2/R_0^2\right)^{1/2}} \end{aligned}$$

where the flux surfaces $r^* = r_0$ are circles.

Safety factor at the edge for toroidal plasma (R, θ, ϕ) [8:387]

$$q_* \equiv \frac{2\pi a^2 B_0}{\mu_0 R_0 I_p} = \frac{\pi k}{4E(k)(\beta/\epsilon)^{1/2}}$$

where $E(k) \approx \left[k^2 + \pi^2/4(1-k^2)\right]$ is the complete elliptic integral of the second kind and the definition of k is given by

$$\frac{B_i^2}{B_0^2} \equiv 1 - \frac{2\mu_0 p}{B_0^2} + \frac{4\epsilon\mu_0 p}{k^2 B_0^2}\left(2 - k^2\right)$$

Approximate edge safety factor for a large aspect ratio toroidal plasma [8:414]

$$q_* \approx \frac{2\pi B_0 a^2}{\mu_0 R_0 I_p}\left(\frac{1+\kappa^2}{2}\right)$$

Magnetic shear [8:408]

$$s = \frac{r}{q}\frac{dq}{dr}$$

10.3 Magnetic Inductance

Definition of magnetic inductance [8: 281]

$$\frac{1}{2}LI^2 \equiv \int_{\text{volume}} \frac{B^2}{2\mu_0}\, d\tau$$

Normalized inductance per unit length [dimensionless] [8: 281]

$$\ell \equiv \frac{L/2\pi R_0}{\mu_0/4\pi} = \frac{2L}{\mu_0 R_0}$$

Internal inductance of a toroidal plasma [8: 281]

$$L_i = \frac{8\pi R_0}{I_p^2} \int_0^a \frac{B_\theta^2}{2\mu_0}\, r dr = \frac{\mu_0 R_0 \langle B_\theta^2 \rangle}{2B_{\theta a}^2}$$

$$\ell_i = \frac{\langle B_\theta^2 \rangle}{B_\theta^2(a)}$$

External inductance of a toroidal plasma [8: 281]

$$L_e = \frac{8\pi R_0}{I_p^2} \int_a^\infty \frac{B_\theta^2}{2\mu_0}\, r dr = \mu_0 R_0 \left(\ln \frac{8R_0}{a} - 2 \right)$$

$$\ell_e = 2 \ln \frac{8R_0}{a} - 4$$

10.4 Toroidal Force Balance

Equation of toroidal force balance [8: 279]

$$\int \hat{\mathbf{R}} \cdot (\mathbf{J} \times \mathbf{B} - \nabla p)\, d\tau = 0$$

where

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = \frac{R_0}{R} \frac{\partial B_\phi}{\partial r} \hat{\boldsymbol{\theta}} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{R_0}{R} r B_\theta \right) \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{R}} \cdot \mathbf{J} \times \mathbf{B} =$$

$$-\cos\theta \left[\frac{R_0^2}{R^2} \frac{\partial}{\partial r} \left(\frac{B_\phi^2}{2\mu_0} \right) + \frac{R_0 B_\theta}{\mu_0 r R} \frac{\partial}{\partial r} \left(\frac{R_0}{R} r B_\theta \right) \right] - \frac{B_{\text{vert}}}{\mu_0 r} \frac{\partial}{\partial r} \left(\frac{R_0}{R} r B_\theta \right)$$

The hoop force [8: 280]

$$\mathbf{F}_{\text{hoop}} = \frac{I_p^2}{2} \frac{\partial}{\partial R} (L_i + L_e)\, \hat{\mathbf{R}} = 2\pi^2 a^2 (\ell_i + \ell_e + 2) \frac{B_{\theta a}^2}{2\mu_0}\, \hat{\mathbf{R}}$$

$$= \frac{\mu_0 I_p^2}{2} \left(\ln \frac{8R}{a} - 1 + \frac{\ell_i}{2} \right) \hat{\mathbf{R}}$$

The tire tube force [8: 280]

$$\mathbf{F}_{\text{tire}} = 2\pi^2 a^2 \langle p \rangle \,\hat{\mathbf{R}} = \frac{\mu_0 I_p^2 \beta_p}{4}\,\hat{\mathbf{R}}$$

The 1/R force [8: 280]

$$\mathbf{F}_{1/\text{R}} = 2\pi^2 a^2 \left(\frac{B_{\phi a}^2}{2\mu_0} - \frac{\langle B_\phi^2 \rangle}{2\mu_0} \right) \hat{\mathbf{R}} = \frac{\mu_0 I_p^2}{4} (\beta_p - 1)\,\hat{\mathbf{R}}$$

where $\langle p \rangle = \frac{1}{2\mu_0}\left(B_{\phi a}^2 - \langle B_\phi^2 \rangle + B_{\theta a}^2 \right)$
have been used.

The vertical field force on toroidal plasma ring [8: 282]

$$\mathbf{F}_{\text{vert}} = -2\pi R_0 B_{\text{vert}} I_p \,\hat{\mathbf{R}}$$

The vertical magnetic field required to balance toroidal forces [8: 282]

$$B_{\text{vert}} = \frac{\mathbf{F}_{\text{hoop}} + \mathbf{F}_{\text{tire}} + \mathbf{F}_{1/\text{R}}}{2\pi R_0 I_p} = \frac{\epsilon}{4} B_{\theta a} \left(\ell_e + \ell_i + 2 + \frac{2\mu_0 \langle p \rangle}{B_{\phi a}^2} + \frac{B_{\phi a}^2 - \langle B_\phi^2 \rangle}{B_{\theta a}^2} \right) = \frac{\mu_0 I_p}{4\pi R_0} \left(\ln \frac{8R}{a} - \frac{3}{2} + \frac{\ell_i}{2} + \beta_p \right)$$

Shafranov shift of the plasma center [7: 2129]

$$\Delta = \frac{b^2}{2R_0} \left[\left(\beta_p + \frac{l_i - 1}{2} \right) \left(1 - \frac{a^2}{b^2} \right) + \ln \frac{b}{a} \right] - \frac{B_{vert}}{B_{\theta 1}(b)}$$

10.5 Plasma Para- and Dia-Magnetism

Global pressure balance equation in a screw pinch [8: 270]

$$\langle p \rangle = \frac{1}{2\mu_0} \left(B_{za}^2 - \langle B_z^2 \rangle + B_{\theta a}^2 \right)$$

can be rearranged to give

$$\beta_p = \frac{B_{za}^2 - \langle B_z^2 \rangle}{B_{\theta a}^2} + 1$$

Diamagnetic : $\beta_p < 1$ Paramagnetic : $\beta_p > 1$

10.6 MHD Stability Limits

Using experimental data from a wide variety of tokamaks, empirical scalings for critical tokamak instabilities have been constructed. Units: current in MA, length in m, magnetic field in T, and density in $n_{20} = n/10^{20}$.

(a) Beta limits (no plasma shaping)

$$\beta_t \leq \beta_L \frac{I}{aB_\phi}$$

$\beta_L = 0.028$ Troyon kink limit - no wall[24]

$\beta_L = 0.044$ Sykes ballooning limit - no wall

$\beta_L = 0.06$ kink - ideal conducting wall

(b) Definition of β_N [26: 347]

$$\beta_N \equiv \beta_t[\%] \, \frac{aB_\phi}{I_p[\text{MA}]}$$

(c) The Greenwald (or Density) Limit [26: 377]

$$n_{20} \leq n_G = \frac{I_P[\text{MA}]}{\pi a^2}$$

10.7 Tokamak Heating and Current Drive

(a) Ohmic plasma heating

The neo-classical resistivity approximation is [8: 538]

$$\eta_{||} = \frac{1}{\left[1 - (r/R_0)^{1/2}\right]^2} \eta_{||}^{\text{Spitzer}}$$

Plugging in for the current as $J_{||} = E_0/\eta_{||}$ [8: 539]

$$P_\Omega = \left(\frac{5.6 \times 10^{-2}}{1 - 1.31\epsilon^{1/2} + 0.46\epsilon} \right) \left(\frac{R_0 \left(I[\text{MA}]\right)^2}{a^2 \kappa T_{keV}^{3/2}} \right) [\text{MW}]$$

(b) Neutral beam plasma heating [26: 246–248]

$$P = m_b \frac{ne^4 \ln\Lambda}{2\pi\epsilon_0^2 m_b^2} \left(\frac{2m_e^{1/2} E_b}{3(2\pi)^{1/2} T_e^{3/2}} + \frac{m_b^{3/2}}{2^{3/2} m_i E_b^{1/2}} \right)$$

where E_b is the energy of the beam.

The critical beam energy when the ions and the electrons are heated equally by the beam is

$$E_c = 14.8 \frac{A_b}{A_i^{2/3}} T_e$$

10.7.1 Current Drive

(a) Inductive current

This current is driven via the central solenoid. The current distribution is calculated through the use of Faraday's law and $J_{||} = E_0/\eta_{||}$. Total current normally has to be measured in order to normalize the distrubition of current density.

(b) Bootstrap current

Bootstrap current is the self-generated current drive in the plasma from trapped and passing electrons in the plasma.

The exact form of the bootstrap current density is given by [8: 496]

$$j_B = -4.71 q \left(\frac{R_0}{r}\right)^{1/2} \frac{T}{B_0} \left[\frac{\partial n}{\partial r} + 0.04 \frac{n}{T} \frac{\partial T}{\partial r}\right]$$

The total bootstrap fraction is given by [8: 496]

$$f_B \approx -1.18 \frac{\partial}{\partial r}(\ln n + 0.04 \ln T) / \frac{\partial}{\partial r}(\ln r B_\theta) \left(\frac{r}{R_0}\right)^{1/2} \beta_p \sim \epsilon^{1/2} \beta_p$$

(c) Neutral beam current drive

By positioning a neutral beam in the tangential direction, it is possible to drive both rotation and current. Neutral beam current drive efficiency scales as (at $E_b = 40 A_b T_e$)[6]

$$I[\mathrm{A}]/P[\mathrm{W}] \approx \frac{0.06 T_e}{n_{20} R Z_b}(1 - Z_b/Z_{\mathrm{eff}})$$

(d) Lower hybrid current drive

Currently one of the most used current drive mechanisms is the lower hybrid system. It launches a wave that Landau damps on the fast electron population and preferentially drives electrons in one direction. [8: 623]

$$I[\mathrm{A}]/P[\mathrm{W}] = 1.17/\left(n_{||}^2 R_0 n_{20}\right)$$

There exists an accessibility condition for the waves which forces an increase in the launched $n_{||}$ [22: 100]

$$n_{||}^2 > \left(S^{1/2} + \left|\frac{D^2}{P}\right|^{1/2}\right)^2$$

where S, P, and D are defined in Chapter 8.
Because LHCD relies on Landau damping, there is an additional constraint on the $n_{||}$: Landau damping dominates at $n_{||}^c \gtrsim 7.0/T_{\mathrm{keV}}^{1/2}$ [9]

(e) Fast Magnetosonic wave current drive

Allows peaked on-axis profiles and has the following current drive efficiency[6]

$$I[\mathrm{A}]/P[\mathrm{W}] = 0.025\frac{T_{\mathrm{keV}}}{n_{20}R_0}$$

10.8 Empirical Scaling Laws

10.8.1 Energy Confinement Time Scalings

Goldston auxiliary heated tokamak scaling (l refers to the plasma size $\sim a$) [26: 152]

$$\tau_E \sim B_p^2 l^{1.8}/nT$$

The ITER-89 L-Mode (ITER89-P) [26: 740]

$$\tau_E = 0.048 I_M^{0.85} R_0^{1.2} a^{0.3} \kappa^{0.5} \bar{n}_{20}^{0.1} B_0^{0.2} A^{0.5} P_M^{-0.5} \quad [\mathrm{s}]$$

The ITER-98 L-Mode[17]

$$\tau_E = 0.023 I_M^{0.96} B_T^{0.03} n_{19}^{0.40} M^{0.20} R^{1.83} \epsilon^{-0.06} \kappa^{0.64} P_{MW}^{-0.73} \quad [\mathrm{s}]$$

The ITER-98 (IPB98[y,2]); ELMy H-mode[17]

$$\tau_E = 0.0562 I_M^{0.93} B_T^{0.15} n_{19}^{0.41} M^{0.19} R^{1.97} \epsilon^{0.58} \kappa^{0.78} P_{MW}^{-0.69} \quad [\mathrm{s}]$$

Scaling for linear regime energy transport[17]

$$\tau_E = 0.07 n_{20} q \kappa^{0.5} a R^2 \quad [\mathrm{s}]$$

Critical density of linear to saturated regime[17]

$$n_{20} = 0.65 A_i^{0.5} B_T q^{-1} R^{-1}$$

10.8.2 Plasma Toroidal Rotation Scaling

Plasma toroidal rotation (Rice) scaling[20]

$$\Delta V_{tor} \propto \Delta W/I_p$$

The 2010 multi-machine scaling database found that (with v_a being the Alfvén speed)[20]

$$v/v_a = 0.65\beta_T^{1.4} q_j^{2.3}$$

where $q_j = 2\pi\kappa a^2 B/\mu_0 R I_p$

10.8.3 L-H Mode Power Scalings

The ITPA empirical scaling law for the L to H mode transition power threshold[16]

$$P_{\text{L-H}}[\text{MW}] = 2.15e^{\pm 0.107} n_{20}^{0.782\pm 0.037} B_{\text{T}}^{0.772\pm 0.031} a^{0.975\pm 0.08} R^{0.999\pm 0.101}$$

10.9 Turbulence

Fundamental definitions [26: 422–424]

$$L_n = n/\nabla n \qquad L_T = T/\nabla T$$

$$b = k_\theta^2 \rho_i^2 \qquad \eta_j = L_{nj}/L_{Tj}$$

$$\epsilon = m_j v^2/2T_j$$

The diamagnetic drift velocity [26: 420]

$$\mathbf{v}_{dj} = \frac{\mathbf{B} \times \nabla p_j}{q_j n_j B^2}$$

Diamagnetic frequency [26: 421]

$$\omega_{*j} = -\frac{k_y T_j}{eBn}\frac{dn}{dr}$$

Ion Larmor radius evaluated at the sound speed[25]

$$\rho_S \equiv c_s/\Omega_i$$

Normalized Larmor radius

$$\rho_* \equiv \rho_S/a$$

10.9.1 General Drift Wave Turbulence

Mixing length estimate[25]

$\tilde{n}^{\text{rms}}/n_0 \sim 1/k_\perp L_\text{n}$

Density fluctuations and plasma potential correlation[25]

$\tilde{n}/n_0 \approx \left(e\tilde{\phi}/kT_e\right)(1 - i\delta)$

where δ is the dissipation of the electron momentum to the background plasma.

Time averaged electrostatic turbulent flux of particles $\tilde{\Gamma}$, momentum $\overleftrightarrow{\mu}$, and heat $\tilde{Q}$[25]

$$\tilde{\Gamma} = -\frac{\langle \tilde{n}\nabla\tilde{\phi}\rangle \times \bar{\mathbf{B}}}{\mathrm{B}^2} + \langle \tilde{n}\tilde{v}_{||}\rangle \mathbf{B}/\mathrm{B}$$

$$\overleftrightarrow{\mu} = \left\langle \left(-\frac{\nabla\tilde{\phi} \times \bar{\mathbf{B}}}{\mathrm{B}^2} + \tilde{v}_{||}\mathbf{B}/\mathrm{B}\right)\left(-\frac{\nabla\tilde{\phi} \times \mathbf{B}}{B^2} + \tilde{v}_{||}\mathbf{B}/\mathrm{B}\right)\right\rangle$$

$$\tilde{Q} \equiv \frac{5}{2}\bar{n}\bar{T}\left[\frac{1}{\bar{T}}\left(-\frac{\langle \tilde{T}\nabla\tilde{\phi}\rangle \times \bar{\mathbf{B}}}{\mathrm{B}^2} + \langle \tilde{T}\tilde{v}_{||}\rangle \mathbf{B}/\mathrm{B}\right) + \frac{1}{\bar{n}}\left(-\frac{\langle \tilde{n}\nabla\tilde{\phi}\rangle \times \bar{\mathbf{B}}}{\mathrm{B}^2} + \langle \tilde{n}\tilde{v}_{||}\rangle \mathbf{B}/\mathrm{B}\right)\right]$$

where fluctuating values are marked by a tilde and $\langle\ \rangle$ is a time average.

Time averaged momentum and and energy fluxes due to fluctuating magnetic fields[25]

$$\overleftrightarrow{\mu}^{EM} = \frac{\langle \tilde{B}\tilde{B}\rangle}{\mu_0\bar{n}M_i}$$

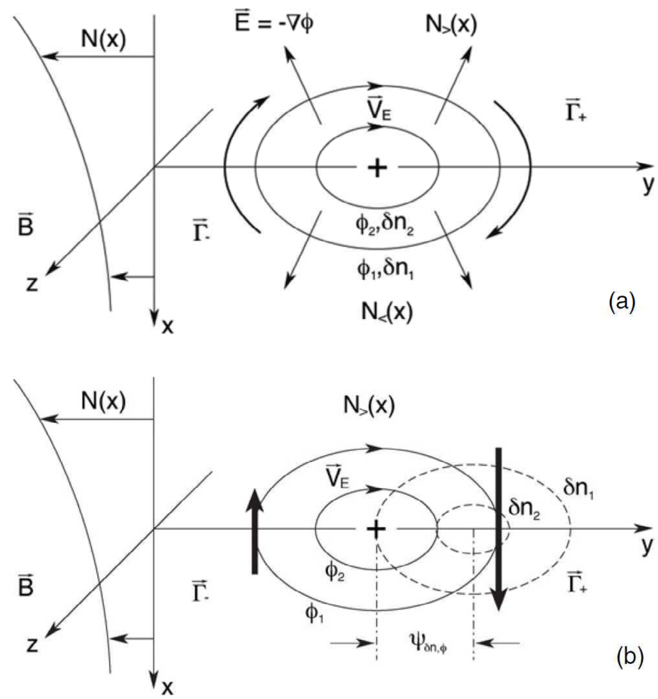
$$\tilde{Q}^{EM} = \frac{\langle \tilde{q}_{||e}\tilde{B}\rangle}{\bar{B}}$$

10.9.2 General Drift Tubulence Characteristics

Perpendicular drift wave turbulence is characterized by ρ_S, with $k_{||} \ll k_\perp$ and $k_\perp\rho_S$ depending on dissipative mechanism, linear free energy source, and nonlinear energy transfer.

Ion thermal gradient (ITG) turbulence occurs when $\eta_i > \eta_\text{crit} \sim 1$ and has the following approximate characteristics[25]

$k_\perp\rho_s \sim 0.1 - 0.5$



(a) Fluctuations without parallel electron dissipation. (b) Fluctuations with finite electron dissipation. Figure from Tynan et al 2009. Copyright 1999 by the American Physical Society.

$$R/L_{T_i} > R/L_{T_i}|_{\text{crit}} \sim 3 - 5$$

$$v_{ph} \sim v_{di}$$

Trapped electron mode (TEM) instabilities occurs at approximately $k_\perp \rho_S \sim 1$. At higher wavenumbers the TEM transitions into the electron thermal gradient (ETG) instability with $\eta_e > \eta_{\text{crit}} \sim 1$ and the following approximate characteristics[25]

$$k_\perp \rho_s \sim 1 - 10$$

$$R/L_{T_e} > R/L_{T_e}|_{\text{crit}} \sim 3 - 5$$

10.9.3 Passing Particle Instabilities

In this section, it is assumed that $k_{||} v_{The} >> \omega >> k_{||} v_{Thi}$, such that electrons respond to the electrostatic potential. Also, the frequency of the magnetic curvature drifts is assumed to be [26: 422]

$$\omega_{di} = 2L_n \omega_{*i}/R \ll \omega$$

The passing particle dispersion relation [26: 424]

$$\left[\rho_i^2\frac{\partial^2}{\partial x^2} - \left(\frac{L_n/R}{b^{1/2}(T_e/T_i)q\Omega}\right)^2\left(\frac{\partial}{\partial\theta} + ik_\theta sx\right) - \frac{2R/L_n}{(T_e/T_i)\Omega}\left(\cos\theta + \frac{i\sin\theta}{k_\theta}\frac{\partial}{\partial x}\right) - \left(\frac{\Omega-1}{(T_e/T_i)\Omega + (1+\eta_i)} + b\right)\right]\tilde{\phi} = 0$$

where x is the distance from the reference mode rational surface m = nq(r) and $\tilde{\phi}$ is the perturbed electrostatic potential.

Ion thermal gradient (ITG, eta-i, η_i) toroidal frequency [26: 428]

$$\omega_{\rm ITG} \approx (\eta_i\omega_{*i}\omega_{di})^{1/2}$$

ITG critical instability limit [26: 429]

$$\eta_{ic} = \begin{cases} 1.2 & R/L_n < (R/L_n)_{\rm crit} \\ \frac{4}{3}\left(1+\frac{T_i}{T_e}\right)(1+2s/q)\,R/L_n & R/L_n > (R/L_n)_{\rm crit} \end{cases}$$

where

$$(R/L_n)_{\rm crit} = \frac{0.9}{(1+T_i/T_e)(1+2s/q)}$$

Electron thermal gradient (ETG, η_e mode) dispersion relation with $T_i \approx T_e$ [26: 429]

$$-\frac{k_{||}^2 v_{The}^2}{\omega^2}\left(1 - \frac{\omega_{*e}}{\omega}(1+\eta_e)\right) + 1 + \frac{\omega_{*e}}{\omega} = 0$$

If $\eta_e \gg 1$ then there is an unstable mode with [26: 429] $\omega \approx (-k_{||}^2 v_{The}^2\eta_e\omega_{*e})^{1/3}$

10.9.4 Trapped Particle Modes

The collisionless trapped particle dispersion relation [26: 432]

$$\frac{1}{\sqrt{2\epsilon}}\left(\frac{1}{T_i} + \frac{1}{T_e}\right) = \frac{1}{T_i}\frac{\omega - \omega_{*i}}{\omega - \bar{\omega}_{di}} + \frac{1}{T_e}\frac{\omega - \omega_{*e}}{\omega - \bar{\omega}_{de}}$$

where

$$\bar{\omega}_{dj} = \frac{\omega_{dj}}{2}\left[\left(\frac{v_{||}}{v_{Thj}}\right)^2 + \left(\frac{v_\perp}{2v_{Thj}}\right)^2\right]\{\cos\theta + \frac{k_r}{k_\theta}\sin\theta\}$$

This dispersion relation gives rise to the trapped ion mode if $\nu_{\rm eff} = \nu_j/\epsilon > \omega_{dj}$ and has growth/frequency [26: 433–434]

$$\omega = \frac{\sqrt{2\epsilon}}{1+T_e/T_i}\omega_{*e} - i\frac{\nu_i}{\epsilon} + i\frac{\epsilon^2}{(1+T_e/T_i)^2}\frac{\omega_{*e}^2}{\nu_e}$$

Parameter	Approximate range in k_θ (cm^{-1})	Approximate length scale(cm)	Mode
	< 1	60	ITG
	< 2	1	ITG, TEM
Density fluctuations ($\tilde{n}$)	< 7	1-10	ITG, TEM
	3-12	1	TEM
	> 20	1, 20	ETG
Temperature fluctuations ($\tilde{T}_e$)	< 1	1	ITG
Flows, GAMs, ZF	< 1	1	ITG
$\tilde{n_e}\tilde{T}_e$ cross phase	< 1	1	ITG

This mode has the largest imaginary part if $\nu_e \approx \epsilon^{3/2}\omega_{*e}$.

The TEM can be calculated due to the trapped particle dispersion relation. The mode is driven by trapped electron collisions and electron temperature gradients. [26: 434–435]

If $\nu_{\text{eff}} \gg \omega_{*e}$ then the growth rate is

$$\gamma \approx \epsilon^{3/2}\frac{\omega_{*e}^2}{\nu_e}\eta_e$$

Chapter 11

Tokamak Edge Physics

In this chapter, all units are SI with the exception of temperature, which is defined in the historical units of eV (electron-volts).

e is the elementary electric charge
q is the total particle charge
Z is the particle atomic (proton) number
T is the plasma temperature
n is the plasma number density
p is the plasma pressure
e and i refer to electrons and ions, respectively
a and R_0 are the minor and major radii of a toroidal plasma

11.1 The Simple Scrape Off Layer (SOL)

The simple SOL model describes 1D plasma flow from the core plasma to material boundary surfaces for limited or diverted plasma along the toroidal magnetic topology. By assuming a high degree of collisionality (ν_*), fluid approximations for plasma flow are valid and the neoclassical effects on particle orbits due to toroidal magnetic topology can be safely ignored.

Parallel SOL connection length (rail/belt limiters, poloidal divertors) [21:17]

$$L_{||} \approx \pi R q$$

Particle time in the SOL (simple 1D model) [21:20]

$$t_{\mathrm{dwell}} \approx L_{||}/c_{\mathrm{s}}$$

SOL width (simple 1D model) [21:23]

$$\lambda_{\mathrm{SOL}} \approx \left(D_{\perp} L_{||}/c_{\mathrm{s}}\right)^{1/2}$$

where $D_{\perp}$ is the anomalous diffusion coefficient.

Conservation of pressure in the SOL [21:47]

$$p_e + p_i + mnv^2 = \text{constant}$$

Plasma density variation along the SOL [21:47]

$$n(x) = \frac{n_0}{1 + M(x)^2}$$

where n_0 is the density at the 'top' of the SOL and $M = v/c_s$ is the plasma mach number.

Electrons follow a Boltzmann distribution in the SOL [21:28]

$$n = n_0 \exp\left(eV/T_e\right)$$

SOL particle sources: ionization (i) and cross-field transport (t) [21:35–40]

$$S_p = S_{p,i} + S_{p,t} = n_{\text{plasma}} n_{\text{neutrals}} \langle \sigma v \rangle_i + D_\perp n / \lambda_{\text{SOL}}^2$$

where $\langle \sigma v \rangle_i \equiv \langle \sigma v \rangle_i (T_e, Z)$ is the ionization rate coefficient.

Particle flux density in the SOL at the sheath edge (se) [21:47]

$$\Gamma_{se} = \frac{1}{2} n_0 c_s$$

where n_0 is the density outside the pre-sheath.

Electric field through SOL required to satisfy the Bohm Criterion [21:48]

$$V_{se} = -0.7 \frac{T_e}{e}$$

Floating sheath voltage [21:79]

$$V_s = 0.5 \frac{T_e}{e} \ln \left[2\pi \frac{m_e}{m_i} \left(Z + \frac{T_i}{T_e} \right) \right]$$

Debye sheath width [21:27]

$$\lambda_{\text{Debye}} \approx \left(\frac{\epsilon_0 T_e}{n_e e^2} \right)^{1/2}$$

11.2 Bohm Criterion

The Bohm Criterion is derived from conservation of energy ($1/2 m_i v^2 = -eV$) and particle conservation ($n_i v = \textit{constant}$). In an unmagnetized plasma it sets the SOL plasma exit velocity into the sheath edge (se). In magnetized plasma, it sets the SOL plasma exit velocity parallel to the magnetic field, after which the ions become demagnetized and perpendicularly enter the sheath; electrons remain magnetized. [21:61–98] .

Bohm Criterion (assuming $T_i = 0$) [21:73]

$$v_{\rm se} \geq \left(\frac{T_e}{m_i}\right)^{1/2} = c_s$$

Bohm Criterion (general form) [21:76]

$$\int_0^\infty \frac{f^i_{\rm se}(v)\,dv}{v^2} \leq \frac{m_i}{T_e}$$

11.3 A Simple Two Point Model For Diverted SOLs

Diverted plasmas can obtain significant ΔT along the SOL, resulting in divertor temperatures less than 10 eV. The SOL can be approximated using a two point model: point 1 is the outboard midplane entrance to the SOL ("upstream" or "u") and point 2 is the divertor terminus of the SOL ("target" or "t"). It is assumed that upstream density, n_u, and the heat flux into the SOL, $q_{||}$, are control parameters; upstream and target temperatures, T_u and T_t, as well as plasma density in front of the target, n_t, are subsequently determined.

11.3.1 Definitions

Dynamic and static pressure [21:435]

$$p = nT\left(1 + M^2\right) \quad \text{where} \quad \begin{cases} M_{\rm u}^2 \ll 1 \\ M_{\rm t}^2 \approx 1 \text{ (Bohm Criterion)} \end{cases}$$

Heat conduction parallel to magnetic field [21:187]

$$q_{\|,\,\rm cond} = -k_0 T^{5/2}\frac{dT}{dx} \quad \text{where} \quad \begin{cases} k_{\rm e,0} \approx 2000 \;\; [\text{W m}^{-1}\text{ eV}^{7/2}] \\ k_{\rm i,0} \approx 60 \;\; [\text{W m}^{-1}\text{ eV}^{7/2}] \end{cases}$$

Sheath heat flux transmission coefficient at a biased surface [21:652]

$$\gamma = 2.5\frac{T_i}{ZT_e} - \frac{eV}{T_e} + 2\left[2\pi\frac{m_e}{m_i}\left(Z + \frac{T_i}{T_e}\right)\right]^{-1/2}\exp\left(\frac{eV}{T_e}\right) + \chi_i$$

where $T_e \neq T_i$, χ_i is the electron-ion recombination energy, and no secondary electrons emitted.

11.3.2 Fundamental Relations

SOL pressure conservation [21: 224]

$$2n_tT_t = n_uT_u$$

SOL power balance [21: 224]

$$T_u^{7/2} = T_t^{7/2} + \frac{7q_{||}L}{2k_0}$$

SOL heat flux limited to sheath heat flux [21: 224]

$$q_{||} = \gamma n_t T_t c_{st} \qquad \approx 7 \quad \text{(D-D plasma, floating surface)}$$

11.3.3 Consequences

Upstream SOL temperature [21: 226]

$$T_u \approx \left(\frac{7q_{||}L}{2k_0}\right)^{2/7} \qquad \text{assuming that } T_t^{7/2} \ll T_u^{7/2}$$

→ T_u is independent of n_u
→ T_u is insensitive to parameter changes due to the 2/7 power
→ $q_{||}$ is extremely sensitive to T_u due to the 7/2 power

Target SOL temperature [21: 227]

$$T_t \approx \frac{2m_i}{\gamma^2e^2}\frac{q_{||}^{10/7}}{(Lk_0)^{4/7}n_u^2}$$

→ T_t is proportional to $\frac{1}{n_u^2}$

Target SOL density [21: 227]

$$n_T = \frac{n_u^3}{q_{||}^2}\left(\frac{7q_{||}L}{2k_0}\right)^{6/7}\frac{\gamma^2e^2}{4m_i}$$

→ n_T is proportional to n_u^3

Chapter 12

Tokamak Fusion Power

In this chapter, all units are SI with the exception of temperature and energy, which are defined in the historical units of eV (electron-volts).

n is the plasma density; $n_{20} = n/10^{20}$; $n_0 = n_e \approx n_i$
T is the plasma temperature; $T_{\text{keV}} = T$ in units of kiloelectron-volts
p is the plasma pressure
ν is the radialprofile peaking factor
P is a power density
S is a total power
U is a total energy
E is the nuclear reaction energy gain
e is the elementary electric charge
q is the total particle charge
Z is the particle atomic (proton) number
e and i subscripts refer to electrons and ions, respectively
D and T refer to deuterium and tritium, respectively
κ is the plasma elongation; κ=b/a
a,b, and R_0 are the 2 minor and major radii of a toroidal plasma
V is the volume of the plasma

12.1 Definitions

Fusion power density [26:8]

$$P_{\text{fusion}} = n_D n_T \langle\sigma v\rangle_{DT} E_{\text{fusion}} = \frac{1}{4} n_e^2 \langle\sigma v\rangle_{DT} E_{\text{fusion}}$$

Fusion power [26:22]

$$\begin{aligned} S_{\text{total}} &= \frac{\pi}{2} E_{\text{fusion}} \int\limits_{\substack{\textit{plasma}\\ \textit{cross}\\ \textit{section}}} n^2 \langle\sigma v\rangle_{DT} R \, dS \\ &= \frac{0.15}{2\nu+1} Rab \left(\frac{n}{10^{20}}\right)^2 T_{\text{keV}}^2 \qquad \text{[MW]} \end{aligned}$$

where it has been assumed that:

Pressure profile:	$nT = \hat{n}\hat{T}\left(1 - \frac{r^2}{\tilde{a}^2}\right)^\nu$
Reaction rate:	$\langle\sigma v\rangle_{\mathrm{DT}} \approx 1.1 \times 10^{-24} T_{\mathrm{keV}}^2$
Plasma cross section:	$\tilde{a} = (ab)^{1/2}$

Alpha power density [26: 10]

$$P_\alpha = n_D n_T \langle\sigma v\rangle_{DT} E_\alpha = \frac{1}{4} n_e^2 \langle\sigma v\rangle_{DT} E_\alpha \approx (1/5) P_{\mathrm{fusion}}$$

Neutron power density [26: 10]

$$P_{\mathrm{neutron}} = n_D n_T \langle\sigma v\rangle_{DT} E_{\mathrm{neutron}} = \frac{1}{4} n_e^2 \langle\sigma v\rangle_{DT} E_{\mathrm{neutron}} \approx (4/5) P_{\mathrm{fusion}}$$

Ohmic heating power density [26: 240]

$$P_{\mathrm{ohmic}} \approx \eta J_{\mathrm{plasma}}^2$$

Stored energy in confined plasma [26: 9]

$$W = \int 3nT\, d\tau = 3\langle nT\rangle V$$

Definition of energy confinement time [26: 9]

$$\tau_E = \frac{\text{Stored energy in the confined plasma}}{\text{Power lost from the confined plasma}} = \frac{W}{S_{\mathrm{loss}}}$$

Power loss from a confined plasma due to conduction [26: 9–10]

$$P_{\mathrm{conduction}} = \frac{3nT}{\tau_E}$$

Power loss from a confined plasma due to bremsstrahlung radiation [26: 227–228]

$$P_{\mathrm{bremsstrahlung}} \approx (5.35 \times 10^{-37}) Z^2 n_e n_i T_{\mathrm{keV}}^{1/2} \quad [\mathrm{W\ m^{-3}}]$$

12.2 Power Balance in a D-T Fusion Reactor

Confined fusion plasmas are not in thermal equilibrium, and, therefore, power must be balanced in a steady-state tokamak reactor. Power that is lost from the confined plasma due to conduction, radiation and other mechanisms must be continuously replenished by alpha particle and auxilliary heating mechanisms.

$$0 = (P_{\mathrm{alpha}} + P_{\mathrm{auxilliary}}) - (P_{\mathrm{conduction}} + P_{\mathrm{bremsstrahlung}} + \ldots)$$

where

$$P_{\mathrm{auxilliary}} = P_{\mathrm{ohmic}} + P_{\mathrm{ICH}} + P_{\mathrm{ECH}} + P_{\mathrm{neutral\ beam}} + \ldots$$

12.2.1 Impurity Effects on Power Balance

The fractional impurity densities $f_j = n_j/n_0$ in the plasma core cause:

(a) Modified quasi-neutrality balance [26: 36]

$$n_e = n_D + n_T + \sum_j Z n_j$$

(b) Increased radiated power loss [3]

$$P_{\text{bremsstrahlung}} \approx (5.35 \times 10^{-37}) n_e^2 T_{\text{keV}}^{1/2} Z_{\text{eff}} \quad [\text{W m}^{-3}]$$

(c) Dilution of fusion fuel [3]

$$P_{\text{alpha}} = \frac{1}{4} n_e^2 (1 - \sum_j f_j Z_j)^2 \langle \sigma v \rangle E_\alpha$$

12.2.2 Metrics of Power Balance

The physics gain factor for D-T plasma [26: 12]

$$Q_{\text{phys}} = \frac{\frac{1}{4} n_e^2 \langle \sigma v \rangle E_{\text{fusion}} \cdot V_{\text{plasma}}}{P_{\text{heating}}} = \frac{5 P_\alpha}{P_{\text{heating}}}$$

where

(a) Q_{phys}=1 is break even

(b) Q_{phys} >5 is a burning plasma

(c) $Q_{\text{phys}} = \infty$ is an ignited plasma

The engineering gain factor [3]

$$Q_{\text{eng}} = \frac{P^{\text{out}}_{\text{electricity}}}{P^{\text{in}}_{\text{electricity}}}$$

12.3 The Ignition Condition (or Lawson Criterion)

The ignition condition describes the minimum values for density (n), temperature (T), and energy confinement time (τ_E) that are required for a confined plasma to reach ignition. Ignition is defined as $P_{\text{alpha}} > P_{\text{loss}}$, where $P_{\text{auxilliary}} = 0$. [26: 10–15] For a given temperature, T, the following equations describe the minimum $n\tau_E$ required to reach ignition under different assumptions:

(a) $P_{\text{alpha}} = P_{\text{conduction}}$ [26:10–11]

$$n\tau_E = \frac{12kT}{\langle\sigma v\rangle E_\alpha}$$

Using $\langle\sigma v\rangle_{DT} \approx 1.1 \times 10^{-24} T_{\text{keV}}^2$ and $E_\alpha = 3.5\,\text{MeV}$ [3]:

$$nT\tau_E \gtrsim 3 \times 10^{21}\ \text{m}^{-3}\ \text{keV s}$$

(b) $P_{\text{alpha}} = P_{\text{conduction}} + P_{\text{bremsstrahlung}}$ [3]:

$$n\tau_E = \frac{12kT}{\langle\sigma v\rangle E_\alpha - 2.14 \times 10^{-36} T_{\text{keV}}^{1/2}}$$

(c) $P_{\text{alpha}} = P_{\text{conduction}} + P_{\text{bremsstrahlung}}$ with alpha impurities $f_\alpha = n_\alpha / n_e$ [3]:

$$n\tau_E = \frac{12kT}{(1 - 2f_\alpha)^2 \langle\sigma v\rangle E_\alpha - (1 + 2f_\alpha) 2.14 \times 10^{-36} T_{\text{keV}}^{1/2}}$$

(d) $P_{\text{alpha}} = P_{\text{conduction}} + P_{\text{bremsstrahlung}}$ with impurity densities $f_j = n_j / n_e$ [3]:

$$n\tau_E = \frac{12kT}{(1 - \sum_j f_j Z_j)^2 \langle\sigma v\rangle E_\alpha - (2.14 \times 10^{-36}) Z_{\text{eff}} T_{\text{keV}}^{1/2}}$$

Chapter 13

Tokamaks of the World

- Comprehensive list of all major tokamaks with parameters
`www.tokamak.info`
- ASDEX-U (Garching, Germany)
`http://www.ipp.mpg.de/ippcms/eng/for/projekte/asdex/techdata.html`
- Alcator C-Mod (Cambridge, USA)
`http://www.psfc.mit.edu/research/alcator/`
- DIII-D (San Diego, USA)
`https://fusion.gat.com/global/Home`
- EAST (Hefei, China)
`http://english.hf.cas.cn/ic/ip/east`
- FTU (Frescati, Italy)
`http://www.efda.org/eu_fusion_programme/machines-ftu_i.htm`
- Ignitor (Kurchatov, Russia)
`http://www.frascati.enea.it/ignitor`
- ITER (Cadarache, France)
`http://www.iter.org/mach`
- JET (Culham, United Kingdom)
`http://www.jet.efda.org/jet/jets-main-features`
- J-TEXT (Wuhan, China)
`http://www.jtextlab.com/EN/SortInfo.aspx?sid=43`
- JT-60SA (Naka, Japan)
`http://www-jt60.naka.jaea.go.jp/english/figure-E/html/figureE_jt60sa_5.html`
- KSTAR (Daejeon, S. Korea)
`http://www.pppl.gov/kstar/html/about_kstar.html`
- SST-1 (Gandhinagar, India)
`http://www.ipr.res.in/sst1/SST1parameters.html`

- T-10 (Kurchatov, Russia)

- T-15U (Kurchatov, Russia)
  `www.toodlepip.com/tokamak/t15-ft_p7-3.pdf`
- TCV (Lausanne, Switzerland)
  `https://crppwww.epfl.ch/tcv`
- TEXTOR (Jülich, Germany)
  `http://www2.fz-juelich.de/ief/ief-4//textor_en`
- TFTR (Princeton, USA)
  `http://w3.pppl.gov/tftr/info/tftrparams.html`
- Tore Supra (Cadarache, France)
  `http://www-fusion-magnetique.cea.fr/gb/index.html`

Tokamak	R (m)	a (m)	ϵ	B_ϕ [a] (T)	I_p (MA)	κ	δ	Vp (m^{-3})	Pulse (s)	Config[b]	PFC[c]	ICRH (MW)	ECRH (MW)	LHCD (MW)	Beam (MW)
ASDEX-U	1.65	0.5	0.30	3.1	1.4	1.8	0.4	14	10	SN	C/W	6	4	-	20
C-Mod	0.67	0.22	0.33	8	2	1.7	0.6	1.0	2.5	L/SN/DN	M	6	-	1	-
DIII-D	1.66	0.67	0.40	2.2	3	2.6	1.0	12	5	SN/DN	C	5	6	-	20
EAST	1.70	0.40	0.24	3.5 SC	0.5	2	0.5	13	1000	DN/SN	C	3	0.5	4	-
FTU	0.93	0.3	0.32	8	1.6	1.7	0.55	3	1.5	L	SS/M/W	0.5	1.3	2.5	-
Ignitor[d]	1.32	0.47	0.36	13	11	1.83	0.4	11	10	L	M	(18-24)	-	-	-
ITER[e]	6.2	2	0.32	5.3 SC	17	1.86	0.5	837	400	SN	Be/C/W	20	20	-	33
JET	2.96	0.96	0.32	3.8	5	1.7	0.33	200	92	L/SN/DN	Be/W	-	7	-	34
JT-60SA[f]	3.16	1.02	0.32	2.7 SC	5.5	1.83	0.57	132	100	DN	C	-	7	-	34
J-TEXT	1.05	0.26	0.25	3	0.4	1	-	1.4	0.3	L/SN/DN	C	-	-	-	-
KSTAR	1.8	0.5	0.28	3.5	2.0	2.0	0.8	18	20	DN	C	6	-	1.5	8
SST-1[g]	1.1	0.2	0.18	3C	0.22	1.9	0.8	2	1000	DN	C	1.5	0.2	1	0.8
T-10	1.5	0.36	0.24	5	0.8	1	-	4	1	L	SS/C	-	2	-	-
T-15	2.43	0.42	0.17	3.5 SC	1	1.47	0.25	12	1000	SN	C	-	7	4	9
TCV	0.88	0.25	0.28	1.4	1.2	2.8	-0.7–1	3	4	L/SN/DN	SS/C	-	4.5	-	-
Textor	1.75	0.47	0.27	2.8	0.8	1	-	8	10	L/SN	SS/C	4	1	-	4
TFTR[h]	2.52	0.87	0.35	5.6	2.7	1	-	6	10	L	C	12.5	-	-	39.5
Tore Supra	2.25	0.7	0.31	4.5 SC	1.7	1	-	22	390	L	C	9	2.4	5	1.7

[a] Magnet coils are conducting unless denoted as superconducting (SC)
[b] Plasma configuration: L = limited; SN = diverted single null; DN = diverted double null
[c] Plasma Facing Components: Be = beryllium; C = CFC/graphite; M = molybdenum; SS = stainless steel; W = tungsten
[d] Proposed; construction not begun
[e] Construction begun; first plasma predicted 2019
[f] Construction begun: first plasma predicted 2014
[g] Construction begun: first plasma predicted 2012
[h] Decommissioned in 1997

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