Consensus in networks of agents with unknown high-frequency gain signs and switching topology

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***Abstract*—The agreement control problem of single and double-integrator agents with unknown and nonidentical control directions is addressed in this note under switching network topology. Distributed nonlinear PI control laws are proposed which ensure asymptotic consensus among the agents based on a new boundedness lemma and a generalized version of Barbalát's lemma for uniformly piecewise right continuous functions. Theoretical results are verified by simulation studies.**

***Index Terms*—consensus, nonlinear PI, switching, Barbalát lemma.**

I. Introduction

THE consensus problem of multi-agent systems has attracted significant research interest over the last fifteen years [1]- [16]. Several distributed control approaches have been developed for different classes of agent models. These include single-integrator [2], [4], [6], double integrator [9], linear systems [3] and even Euler-Lagrange systems [11]. Results have also been obtained for switching topologies and time delays [2], [5], [4], [6]. The literature on the subject is vast and the interested reader should consult [8], [10], [13], [15] for a more complete list of references.

For certain control problems, the control direction may not be known a priori. Examples are uncalibrated visual servoing [17] and autopilot design of time-varying ships [18]. In [19], Nussbaum proposed a class of nonlinear control gains that resolve this problem. The Nussbaum gain technique allows for general adaptive control designs and numerous applications to different system classes have been developed over the years [20]- [26].

An alternative approach to the unknown control direction problem, the so called nonlinear PI method, was proposed in [27] that includes an extra proportional term in the argument of the control gain function. Derivations in [27]- [30] indicate that the nonlinear PI approach has better robustness properties with respect to certain types of unmodelled dynamics compared to the standard Nussbaum gain technique (see example 8 of [29], section 6.3 of [27], [30] and [31]). Extensions of the nonlinear PI method to strict-feedback nonlinear systems have been also developed recently in [32].

Few results exist in the literature for the combined problem of distributed control design for multi-agent systems with unknown control directions. The first approach by Chen *et al.* [33] proved consensus for agents with unknown but identical control directions using a novel Nussbaum function. This method was also utilized in [34] and [35] for asymptotic regulation. The authors of [37] have also considered the single-integrator consensus problem with identical control directions. The case of unknown and nonidentical control directions was treated in [36] for agents with single-integrator dynamics, in [38] for cooperative output regulation of second-order systems and in [39] for strict-feedback nonlinear systems.

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In this note, we provide a solution for *the consensus problem for single and double-integrator agents with unknown and nonidentical control directions under switching topologies*. To the best of our knowledge this problem has not been studied up to now in the relevant literature [33]- [39]. Distributed nonlinear PI control laws similar to [27], [28], [32] are proposed that ensure asymptotic agreement among the agents. The main technical tools in our analysis are a new boundedness lemma (Lemma 3) and an extension of Barbalát's lemma to uniformly piecewise continuous functions (Lemma 1).

The paper is organized as follows. In Section II we present the main technical lemmas and introduce some prerequisites on graph theory. The consensus distributed control problem is formulated in Section III. In Section IV we state and prove the main results of the paper (Theorems 1 and 2). A simulation example is examined in Section V that verifies our theoretical analysis. Finally some concluding remarks are given in Section VI.

II. Preliminaries

A. Main Lemmas

For a piecewise continuous function, we denote by $\{t_j\}_{j\in I}$ the finite or infinite sequence of discontinuity points with index set $I := \{1, 2, \ldots\} \subseteq \mathbb{N}_+$. For notational convenience we also denote $t_{n+1} := +\infty$ if the set I has finite cardinality $\text{card}(I) = n$.

Definition 1. *Consider a real-valued piecewise right continuous function $f : [0, \infty) \to \mathbb{R}$ and assume $\{t_j\}_{j\in I}$ is the sequence of discontinuity points with $I \subseteq \mathbb{N}$. Function $f(\cdot)$ is called uniformly piecewise right continuous if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$|f(\bar{t}_2) - f(\bar{t}_1)| \leq \epsilon \tag{1}$$

for all $\bar{t}_1, \bar{t}_2 \in [t_j, t_{j+1}), j \in I$ with $|\bar{t}_2 - \bar{t}_1| \leq \delta(\epsilon)$.

The above definition imposes a *uniformity assumption in the uniform continuity of f in each $[t_j, t_{j+1})$ over all $j \in I$.*

A generalization of Barbalát's lemma to piecewise right continuous functions is now stated below.

Lemma 1. *Consider a piecewise right continuous function $\phi : [0, \infty) \to \mathbb{R}$ and let $\{t_j\}_{j\in I}$ be the finite or infinite sequence of discontinuity points with $I \subseteq \mathbb{N}$. Suppose that function $\phi(\cdot)$ is* *uniformly piecewise right continuous and* $\lim_{t\to\infty}\int_0^t \phi(s)ds$ *exists and is finite. If there exists* $\tau > 0$ *such that* $t_{j+1} - t_j > \tau$ *for all* $j \in I$ *then* $\lim_{t\to\infty}\phi(t) = 0$.

Proof. The proof is given in Appendix A. □

Sufficient conditions for uniformly piecewise right continuity are given by the following lemma.

Lemma 2. *Let* $f : [0,\infty) \to \mathbb{R}$ *a piecewise right continuous function. If* f *is also piecewise differentiable with uniformly bounded derivative then* f *is uniformly piecewise right continuous.*

Proof. The proof is straightforward and uses the mean value theorem (MVT). According to MVT since f is continuous in $[t_j, t_{j+1})$ and differentiable in (t_j, t_{j+1}) we have that $f(\bar{t}_2) - f(\bar{t}_1) = f'(t_{12})(\bar{t}_2 - \bar{t}_1)$ for all $\bar{t}_1, \bar{t}_2 \in [t_j, t_{j+1})$ for some $t_{12} \in (\min\{\bar{t}_1, \bar{t}_2\}, \max\{\bar{t}_1, \bar{t}_2\}) \subseteq (t_j, t_{j+1})$. Thus for every $\epsilon > 0$ if we select $\delta(\epsilon) = \epsilon/c > 0$ with $c := \sup_{t\in\cup_{j\in I}(t_j,t_{j+1})} |f'(t)|$ then (1) holds true $\forall \bar{t}_1, \bar{t}_2 \in [t_j, t_{j+1})$ with $|\bar{t}_2 - \bar{t}_1| \le \delta(\epsilon)$. □

The following corollary is obtained from Lemmas 1-2.

Corollary 1. *Consider a piecewise right continuous differentiable function* $\phi : [0,\infty) \to \mathbb{R}$ *and let* $\{t_j\}_{j\in I}$ *be the sequence of discontinuity times. Assume further that* ϕ *has a uniformly bounded derivative and* $\lim_{t\to\infty}\int_0^t \phi(s)ds$ *exists and is finite. If there exists some* $\tau > 0$ *such that* $t_{j+1} - t_j > \tau$ *for all* $j \in I$ *then* $\lim_{t\to\infty}\phi(t) = 0$.

Remark 1. *In [40] a similar result to Corollary 1 appears in the framework of impulsive control with the additional assumption that* $\phi(\cdot)$ *is uniformly bounded. It is noted that Lemma 1 is less restrictive than Corollary 1 and Lemma 1 of [40] since no assumption on the differentiability of* $\phi(\cdot)$ *is imposed. Function* $\phi(t) = W_{0.5,3}(t - \lfloor 2t \rfloor/2)\exp(-t)$ *for example, with* $W_{a,b}(t) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi t)$ *the Weirstrass function and* $\lfloor t \rfloor$ *the largest integer not exceeding* t*, satisfies all conditions of Lemma 1 but fails to meet conditions of Lemma 1 of [40] or Corollary 1.*

The main tool for proving consensus results in this paper is the following boundedness lemma:

Lemma 3. *Let* $M : [0, t_f) \to \mathbb{R}$ *a piecewise right-continuous function and* $S : [0, t_f) \to \mathbb{R}$ *a continuous, piecewise differentiable function such that*

$$\dot{S}(t) = [a_1 + a_2 S(t)\cos(S(t))]\, M(t) \tag{2}$$

with constants $a_1, a_2 \in \mathbb{R}$*. If* $a_2 \neq 0$ *then* $|S(t) - S(0)| \le 2(\pi + |a_1/a_2|)$ *for all* $t \in [0, t_f)$.

Proof. The proof is based on ideas from [27]. The solutions S_e of the algebraic equation $a_1 + a_2 S\cos(S) = 0$ are equilibrium points (e.p.) for the dynamical system (2). These e.p. are spaced within intervals of length less than 2π outside the interval $[-|a_1 a_2^{-1}|, |a_1 a_2^{-1}|]$. Let us define $p_0 := \min\{S \ge 0 | S\cos(S) = -a_1 a_2^{-1}\}$ and the strictly increasing sequence $\{p_k\}_{k=-\infty}^{\infty}$ of all e.p. arranged in increasing order. Then, the e.p. define the real line decomposition $\mathbb{R} = \cup_{k=-\infty}^{\infty} I_k$ with intervals $I_k := [p_k, p_{k+1}]$ and any solution of (2) starting within I_{k^*} for some $k^* \in \mathbb{Z}$ will remain therein for all time. This is due to the fact that the solution cannot pass through the boundaries of those intervals as they are e.p.'s. Thus, for any $S(0) \in \mathbb{R}$ there exists some $k^* \in \mathbb{Z}$ such that $S(0) \in I_{k^*}$ and it holds true that $S(t) \in I_{k^*}$ for all $t \in [0, t_f)$. The length of each I_k is less than $2(\pi + |a_1|/|a_2|)$ and therefore $|S(t) - S(0)| \le 2(\pi + |a_1|/|a_2|)$ for all $t \in [0, t_f)$. □

B. Graph theory

In this section, we revisit basic definitions and results on graph theory [10], [36]. A *weighted directed graph* is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2, \ldots, N\}$ a nonempty finite set of N nodes and $\mathcal{E} = \mathcal{V} \times \mathcal{V}$ an edge set that describes the communication among agents. A sequence of successive edges $\{(i,k), (k,l), ..., (m,j)\}$ is a directed path from node i to node j. A directed graph is *strongly connected* if there is a directed path from node i to node j, for all $i, j \in \mathcal{V}$ with $i \neq j$. A weighted adjacency matrix is a matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, with elements $a_{ii} = 0, \forall i$ and $a_{ij} > 0$, $i \neq j$ if $(i,j) \in \mathcal{E}$ and 0 otherwise.

The in-degree and out-degree of node i are the numbers $d_i = \sum_j a_{ij}$ and $d_i^o = \sum_j a_{ji}$ respectively. Node i is balanced if and only if $d_i = d_i^o$. Also, a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is balanced if and only if all of its nodes are balanced. Matrix $\mathcal{D} = diag\{d_i\} \in \mathbb{R}^{N\times N}$ is called the in-degree matrix and $L = \mathcal{D} - \mathcal{A}$ is the Laplacian matrix of the graph. It is standard that the Laplacian matrix L has a zero eigenvalue associated with the eigenvector $\mathbf{1}_N := [1, 1, \ldots, 1]^T \in \mathbb{R}^N$. For a $N \times N$ matrix A with real eigenvalues we assume in the paper that $\lambda_1(A) \le \cdots \le \lambda_N(A)$ i.e. the eigenvalues are arranged in increasing order.

The following lemma describes the null space of $L + L^T$ for balanced and strongly connected digraphs.

Lemma 4. *[14], [36] If the digraph is balanced and strongly connected, then* $\hat{L} := L + L^T \ge 0$ *and Null*$(\hat{L}) =$ *span*$\{\mathbf{1}_N\}$.

Using the above lemma we can prove the following.

Lemma 5. *For a balanced strongly connected digraph with Laplacian matrix* L *we have that*

$$L + L^T \ge \sigma L^T L \tag{3}$$

with $\sigma := \lambda_2(L + L^T)/\lambda_N(L^T L)$.

Proof. Since $\text{Null}(\hat{L}) = \text{Null}(L^T L) = \text{span}\{\mathbf{1}_N\}$ the singular value decompositions of the two symmetric positive semidefinite matrices are

$$\begin{aligned} L + L^T &= U_1 \Sigma_1 U_1^T \\ L^T L &= U_2 \Sigma_2 U_2^T \end{aligned} \tag{4}$$

with $U_i \in \mathbb{R}^{N\times(N-1)}$ and $\Sigma_i \in \mathbb{R}^{(N-1)\times(N-1)}$ diagonal with positive elements $(i = 1, 2)$. From the orthogonality property of $\begin{bmatrix} U_i & \frac{\mathbf{1}_N}{\sqrt{N}} \end{bmatrix}$ $(i = 1, 2)$ we can write

$$\begin{aligned} L^T L &= \left(U_1 U_1^T + \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}\right) U_2 \Sigma_2 U_2^T \left(U_1 U_1^T + \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}\right) \\ &\leq \lambda_{\max}(\Sigma_2) U_1 U_1^T U_2 U_2^T U_1 U_1^T \\ &= \lambda_{\max}(\Sigma_2) U_1 U_1^T (\mathbb{I}_N - (1/N)\mathbf{1}_N \mathbf{1}_N^T) U_1 U_1^T \\ &= \lambda_{\max}(\Sigma_2) U_1 U_1^T \leq \lambda_{\max}(\Sigma_2) \lambda_{\min}^{-1}(\Sigma_1) U_1 \Sigma_1 U_1^T \\ &= \sigma^{-1}(L + L^T). \qquad \square \end{aligned}$$

III. Problem formulation

We consider either a network of single-integrator agents

$$\dot{x}_i = b_i u_i \qquad (1 \leq i \leq N) \tag{5}$$

or a network of double-integrator agents

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= b_i u_i \qquad (1 \leq i \leq N) \end{aligned} \tag{6}$$

with agent positions $x_i \in \mathbb{R}$, velocities $v_i \in \mathbb{R}$, control inputs $u_i \in \mathbb{R}$ and high frequency gains $b_i \in \mathbb{R}$.

Assumption 1. *All control gains b_i $(i = 1, 2, \ldots, N)$ are unknown, nonzero $(b_i \neq 0)$ with unknown and possibly nonidentical signs.*

To account for the possibility of a switching topology the following assumption is considered.

Assumption 2. *We assume a finite or infinite sequence of switching times $\{t_j\}_{j \in I}$, a set of possible network topologies described by the Laplacian matrices $\{L_\ell\}_{\ell=1}^M$ and a mapping $n : I \to \{1, 2 \cdots, M\}$ such that $L(t) = L_{n(j)}$ for all $t \in [t_j, t_{j+1})$, $j \in I$. We also assume that each L_ℓ is balanced and strongly connected for all $\ell \in \{1, 2, \cdots, M\}$ and there exists some unknown constant τ such that $t_{j+1} - t_j > \tau$ for all $j \in I$.*

The control objective is to design a distributed control law for each agent (5) under Assumptions 1-2 that achieves network consensus in the sense that $\lim_{t\to\infty}[x_i(t) - x_k(t)] = 0$ for all $i, k \in \{1, 2, \ldots, N\}$.

IV. Main results

A. Single-integrator agents

Define $\xi_i(t) := \sum_{k=1}^N a_{ik}(t)(x_i(t) - x_k(t))$ $(1 \leq i \leq N)$ and $\xi := [\xi_1, \cdots, \xi_N]^T$. The main result of the paper for single-integrator agents is stated below.

Theorem 1. *Consider the network of agents* (5) *with switching topology described by Assumption 2 and unknown control directions according to Assumption 1. If we select the distributed control law*

$$u_i(t) = \kappa S_i(t) \cos(S_i(t)) \xi_i(t) \tag{7}$$

with PI term

$$S_i(t) := \frac{1}{2} x_i^2(t) + \lambda \int_0^t x_i(s) \xi_i(s) ds \tag{8}$$

$\kappa, \lambda > 0$, then all x_i, u_i remain bounded and $\lim_{t\to\infty}[x_i(t) - x_k(t)] = 0$ for all $i, k = 1, 2, \ldots, N$.

Proof. Let us write the system (5) with control (7) in vector form notation. To this end, we define the generalized state variables $x_{ag} := [x^T, y^T]^T \in \mathbb{R}^{2N}$ where $x := [x_1, \cdots, x_N]^T \in \mathbb{R}^N$, $y := [y_1, \cdots, y_N]^T \in \mathbb{R}^N$ and

$$y_i := \int_0^t x_i(s) \xi_i(s) ds \quad (1 \leq i \leq N). \tag{9}$$

Then the state equation takes the form

$$\begin{aligned} \dot{x} &= Q(x, y) L(t) x \\ \dot{y}_i &= x^T e_i e_i^T L(t) x \qquad (1 \leq i \leq N) \end{aligned} \tag{10}$$

with

$$Q(x_{ag}) := \operatorname{diag}\{\kappa b_i (x_i^2/2 + \lambda y_i) \cos(x_i^2/2 + \lambda y_i)\}_{i=1}^N \tag{11}$$

and $e_i \in \mathbb{R}^N$ the i-th column of the identity matrix. From (10) the map f of the resulting dynamical system $\dot{x}_{ag} = f(x_{ag}, t)$ is piecewise continuous and locally Lipschitz. Hence a unique continuous solution $x_{ag}(\cdot)$ exists that can be extended over a maximal time interval $[0, t_f)$ as shown in section 8.5 of [42].

For the control law (7) the time derivative of S_i takes the form

$$\dot{S}_i(t) = \Big[\kappa b_i S_i(t) \cos\big(S_i(t)\big) + \lambda\Big] x_i(t) \xi_i(t), \forall t \in [0, t_f). \tag{12}$$

A direct application of Lemma 3 in (12) yields

$$|S_i(t) - S_i(0)| \leq 2\left(\pi + \frac{\lambda}{\kappa |b_i|}\right) \quad \forall t \in [0, t_f).$$

Thus, S_i is bounded in $[0, t_f)$ for all $i = 1, 2, \cdots, N$ and so is their sum $\sum_{i=1}^N S_i$. Define now $\sigma_\ell := \lambda_2(L_\ell + L_\ell^T)/\lambda_N(L_\ell L_\ell^T)$ and $\underline{\sigma} := \min_{1 \leq \ell \leq M} \sigma_\ell$. Summing all S_i and using Lemma 5 we have

$$\begin{aligned} \sum_{i=1}^N S_i &= \frac{1}{2} x^T(t) x(t) + \frac{\lambda}{2} \int_0^t x^T(s)(L(s) + L^T(s)) x(s) ds \\ &\geq \frac{1}{2} x^T(t) x(t) + \frac{\lambda}{2} \underline{\sigma} \int_0^t x^T(s) L^T(s) L(s) x(s) ds \\ &= \frac{1}{2} x^T(t) x(t) + \frac{\lambda}{2} \underline{\sigma} \int_0^t \|\xi(s)\|^2 ds. \end{aligned} \tag{13}$$

Boundedness of $S_i, \sum_{i=1}^N S_i$ and (8), (9), (13) yield boundedness of x, y in $[0, t_f)$. Thus, the whole state vector x_{ag} is bounded and the solution can be extended to $t_f = \infty$ (see Theorem 3.3 in [41] or section 8.5 of [42]).

Hence, from (13) $\xi := Lx \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and also from (5) $\dot{\xi} \in \mathcal{L}_\infty$ except at points t_j $(j \in I)$. Using the generalization of Barbalat's lemma (Corollary 1) we obtain $\lim_{t\to\infty} \xi(t) = 0$.

We will now prove that $\lim_{t\to\infty} \xi(t) = 0$ implies $\lim_{t\to\infty}(x_i(t) - x_k(t)) = 0$ for all $i, k \in \{1, \cdots, N\}$. Let $L_\ell = V_{\ell,1} \Lambda_\ell V_{\ell,2}^T$ the SVD decomposition of L_ℓ where Λ_ℓ is a $(N-1) \times (N-1)$ diagonal matrix with positive elements and $V_{\ell,\nu}$ is a $N \times (N-1)$ matrix with $\begin{bmatrix} V_{\ell,\nu} & N^{-1/2}\mathbf{1}_N \end{bmatrix}$ orthogonal $(\nu = 1, 2;\ \ell = 1, 2, \cdots, M)$.

Then $x(t) = V_{n(j),2}\Lambda_{n(j)}^{-1}V_{n(j),1}^T\xi(t) + (\sum_{k=1}^N x_k/N)\mathbf{1}_N$ for all $t \in [t_j, t_{j+1})$, $j \in I$ and

$$x_i(t) - x_k(t) = (e_i - e_k)^T V_{n(j),2}\Lambda_{n(j)}^{-1}V_{n(j),1}^T\xi(t), \quad \forall t \in [t_j, t_{j+1}), j \in I. \quad (14)$$

Since $\lim_{t\to\infty}\xi(t) = 0$ we obtain from (14) $\lim_{t\to\infty}(x_i(t) - x_k(t)) = 0$ for all $i, k \in \{1, \cdots, N\}$. □

B. Double-integrator agents

For second-order agents (6) we define $\zeta_i := \sum_{k=1}^N a_{ik}(v_i - v_k)$, $\xi_i := \sum_{k=1}^N a_{ik}(x_i - x_k)$, $q_i := v_i + \rho x_i$ and $r_i := \zeta_i + \rho\xi_i$. We also denote $u := [u_1, \cdots, u_N]^T \in \mathbb{R}^N$, $\zeta := [\zeta_1, \cdots, \zeta_N]^T \in \mathbb{R}^N$, $\xi := [\xi_1, \cdots, \xi_N]^T$, $q := [q_1, \cdots, q_N]^T$ and $r := [r_1, \cdots, r_N]^T$. Then, the following theorem holds true.

Theorem 2. *Consider the network of agents* (6) *with switching topology described by Assumption 2 and unknown control directions according to Assumption 1. If we select the distributed control law*

$$u_i(t) = \kappa R_i(t)\cos(R_i(t))\left[\rho v_i(t) + \lambda r_i(t)\right] \quad (15)$$

with PI term

$$R_i(t) := \frac{1}{2}q_i^2(t) + \lambda\int_0^t q_i(s)r_i(s)ds \quad (16)$$

$\kappa, \rho, \lambda > 0$ *then all* x_i, v_i, u_i *remain bounded and* $\lim_{t\to\infty}[x_i(t) - x_k(t)] = 0$, $\lim_{t\to\infty}[v_i(t) - v_k(t)] = 0$ *for all* $i, k = 1, 2, \ldots, N$.

Proof. If we define the generalized state variables $\overline{x}_{ag} := [x^T, v^T, \bar{y}^T]^T \in \mathbb{R}^{3N}$, $z := [x^T, v^T]^T \in \mathbb{R}^{2N}$ where $x := [x_1, \cdots, x_N]^T$, $v := [v_1, \cdots, v_N]^T$, $\bar{y} := [\bar{y}_1, \cdots, \bar{y}_N]^T$ and

$$\bar{y}_i := \int_0^t q_i(s)r_i(s)ds \quad (17)$$

then the state equations take the form

$$\begin{aligned} \dot{z} &= N(x_{ag}, t)z \\ \dot{\bar{y}}_i &= (v + \rho x)^T e_i e_i^T L(t)(v + \rho x)\ (1 \le i \le N) \end{aligned} \quad (18)$$

with

$$N(\bar{x}_{ag}, t) := \begin{bmatrix} 0 & \mathbb{I} \\ \rho\lambda W(\bar{x}_{ag})L(t) & W(\bar{x}_{ag})(\rho\mathbb{I} + \lambda L(t)) \end{bmatrix}$$

and

$$W(\bar{x}_{ag}) := \mathrm{diag}\{\kappa b_i\cos(q_i^2/2 + \lambda\bar{y}_i)\}_{i=1}^N.$$

From (18) the map $\bar{f}$ of the resulting dynamical system $\dot{\bar{x}}_{ag} = \bar{f}(\bar{x}_{ag}, t)$ is piecewise continuous and locally Lipschitz. Hence a unique continuous maximal solution $\bar{x}_{ag}(\cdot)$ exists on some interval $[0, \bar{t}_f)$ (see section 8.5 of [42]).

For the time derivative of R_i we have

$$\dot{R}_i = q_i(t)(b_i u_i(t) + \rho v_i(t) + \lambda r_i(t)),\ \forall t \in [0, \bar{t}_f). \quad (19)$$

Applying the distributed control law (15), eq. (19) yields

$$\dot{R}_i = (1 + \kappa b_i R_i\cos(R_i))[\rho v_i(t) + \lambda r_i(t)]q_i(t), \quad \forall t \in [0, \bar{t}_f). \quad (20)$$

Since $\bar{M}_i(t) := q_i(t)[\rho v_i(t) + \lambda r_i(t)]$ is a piecewise right-continuous function defined on $[0, \bar{t}_f)$ for all $i = 1, \ldots, N$ then from Lemma 3 we have that

$$|R_i(t) - R_i(0)| \le 2\left(\pi + \frac{1}{\kappa|b_i|}\right) \quad \forall t \in [0, \bar{t}_f). \quad (21)$$

Thus, R_i is bounded on $[0, \bar{t}_f)$ for all $i = 1, 2, \cdots, N$ and so is their sum $\sum_{i=1}^N R_i$. Summing all R_i and using Lemma 5 we have

$$\begin{aligned} \sum_{i=1}^N R_i &= \frac{1}{2}\sum_{i=1}^N q_i^2 + \lambda\int_0^t \sum_{i=1}^N q_i(s)r_i(s)ds \\ &= \frac{1}{2}\|q(t)\|^2 + \frac{\lambda}{2}\int_0^t q(s)^T(L(s) + L^T(s))q(s)ds \\ &\ge \frac{1}{2}\|q(t)\|^2 + \frac{\lambda\sigma}{2}\int_0^t \|L(s)q(s)\|^2 ds. \end{aligned} \quad (22)$$

From (22) and the boundedness of all R_i, function $q(t)$ is bounded on $[0, \bar{t}_f)$ and since $\dot{x}(t) = -\rho x(t) + q(t)$ both x, v can be proved bounded on $[0, \bar{t}_f)$. Also from the boundedness of q_i, R_i and (16),(17) $\bar{y}_i$ is bounded in $[0, \bar{t}_f)$. Hence, the whole state vector $\bar{x}_{ag}$ is bounded and the solution can be extended to $\bar{t}_f = \infty$ (see Theorem 3.3 in [41] or section 8.5 of [42]).

Thus, $x, v, R_i, \in \mathcal{L}_\infty$ and from (15),(22) it holds true that $u_i \in \mathcal{L}_\infty$ and $r(t) = L(t)q(t) = \zeta(t) + \rho\xi(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. If we define the matrix $B := \mathrm{diag}\{b_1, \cdots, b_N\}$ and take into account that $u, v \in \mathcal{L}_\infty$ we result in $\dot{r}(t) = L(\dot{v} + \rho\dot{x}) = L(Bu + \rho v) \in \mathcal{L}_\infty$ for all $t \in [0, \infty) \setminus \{t_j\}_{j\in I}$. If we combine this fact and the property $r \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ then a direct application of Corollary 1 yields $\lim_{t\to\infty} r(t) = 0$. Considering again the SVD decompositions of the Laplacian matrices L_ℓ ($\ell = 1, \cdots, M$) we can write from $r(t) = L(t)q(t)$ similarly to the proof of Theorem 1 that $q(t) = V_{n(j),2}\Lambda_{n(j)}^{-1}V_{n(j),1}^T r(t) + (\sum_{k=1}^N q_k(t)/N)\mathbf{1}_N$ for all $t \in [t_j, t_{j+1})$, $j \in I$ and

$$q_i(t) - q_k(t) = (e_i - e_k)^T V_{n(j),2}\Lambda_{n(j)}^{-1}V_{n(j),1}^T r(t), \quad \forall t \in [t_j, t_{j+1}), j \in I. \quad (23)$$

Since $\lim_{t\to\infty} r(t) = 0$ the above identity yields $\lim_{t\to\infty}(q_i(t) - q_k(t)) = 0$ for all $i, k \in \{1, \cdots, N\}$.Therefore for every $\epsilon > 0$ there exist time $T_1(\epsilon) \ge 0$ such that

$$|q_i(t) - q_k(t)| \le \frac{\rho\epsilon}{2} \quad \forall t \ge T_1(\epsilon), \forall i, k \in \{1, \cdots, N\}. \quad (24)$$

From the definition of q_i we have $\dot{x}_i = -\rho x_i + q_i$ that yields for all $t \ge T \ge 0$

$$x_i(t) = e^{-\rho(t-T)}x_i(T) + e^{-\rho t}\int_T^t e^{\rho s}q_i(s)ds.$$

Subtracting two instances of the above identity for indexes i, k and using the triangle inequality we obtain

$$\begin{aligned} |x_i(t) - x_k(t)| \le& e^{-\rho(t-T)}[|x_i(T)| + |x_k(T)|] \\ &+ e^{-\rho t}\int_T^t e^{\rho s}|q_i(s) - q_k(s)|ds \end{aligned} \quad (25)$$

for all $t \ge T \ge 0$, $i, k \in \{1, \cdots, N\}$.

Since $x \in \mathcal{L}_\infty$ there exists $c > 0$ such that $|x_i(t)| \le c$ for all $t \ge 0$, $i \in \{1, \cdots, N\}$. Thus, for all $t \ge T_2(\epsilon) := T_1(\epsilon) +$

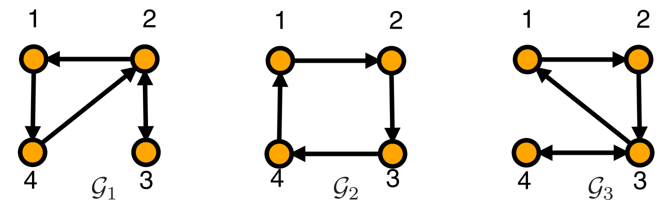

Fig. 1. The 3 graph configurations $\mathcal{G}_i = (\mathcal{V}, \mathcal{E}_i)$ (i=1,2,3).

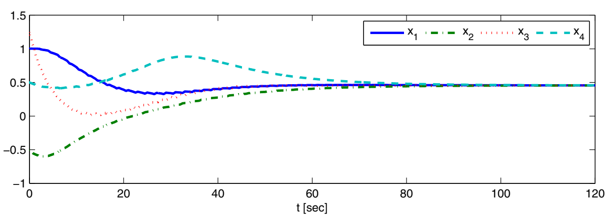

Fig. 2. The agents positions x_i for SI agents $(i = 1, \ldots, 4)$.

$\max\{0, (1/\rho)\ln(4c/\epsilon)\}$ we have from (25) for $T = T_1(\epsilon)$ and (24)

$$|x_i(t) - x_k(t)| \le e^{-\rho t}\int_{T_1(\epsilon)}^{t} e^{\rho s}|q_i(s) - q_k(s)|ds + 2ce^{-\rho(t-T_1(\epsilon))} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad (26)$$

Thus, for every $\epsilon > 0$ there exists some $T_2(\epsilon) \ge 0$ such that $|x_i(t) - x_k(t)| \le \epsilon$ i.e. $\lim_{t\to\infty}[x_i(t) - x_k(t)] = 0$ for all $i, k \in \{1, 2, \cdots, N\}$.

Finally, from the definition of q_i we have that

$$\lim_{t\to\infty}[v_i(t) - v_k(t)] = \lim_{t\to\infty}[q_i(t) - q_k(t)] - \rho\lim_{t\to\infty}[x_i(t) - x_k(t)] = 0$$

for all $i, k \in \{1, 2, \cdots, N\}$. □

V. Simulation Example

We consider a network consisting of four agents with single-integrator (SI) dynamics described by (5) (Case 1) or (b) double-integrator (DI) dynamics described by (6) (Case 2). For both cases we assume initial conditions $x(0) = [1, -0.5, 1.25, 0.5]^T$ and non-identical unknown control gains $b_1 = 1$, $b_2 = -1$, $b_3 = 2$, $b_4 = 1$. For case 2 we also assume $v(0) = [-1, 1, -1, -1]^T$. The network's topology switches between three different balanced and strongly connected graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ shown in Fig. 1. We assume an infinite sequence of switchings that occur in a periodic manner with transitions $\mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to \cdots$. The specific form of the graph at each time instant is

$$\mathcal{G}(t) = \begin{cases} \mathcal{G}_1\,, & \text{if } t \in [0, 0.5)\text{modulo}(2) \\ \mathcal{G}_2\,, & \text{if } t \in [0.5, 1)\text{modulo}(2) \\ \mathcal{G}_3\,, & \text{if } t \in [1, 2)\text{modulo}(2) \end{cases}$$

Control laws (7), (8) and (15), (16) are employed with parameters $\kappa = \lambda = 0.1$, $\rho = 6$ for cases 1 and 2 respectively. Simulations results are shown in Figs. 2-6. As expected, all x_i, v_i, u_i are bounded and asymptotic consensus is achieved for both cases.

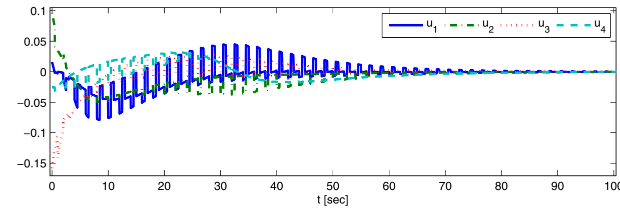

Fig. 3. The control inputs u_i for SI agents $(i = 1, \ldots, 4)$.

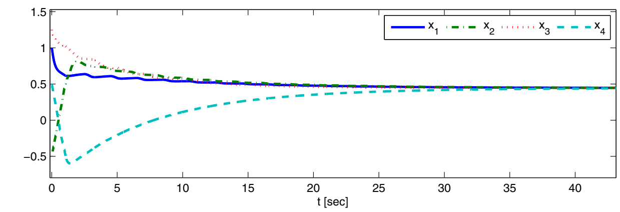

Fig. 4. The agents positions x_i for DI agents$(i = 1, \ldots, 4)$.

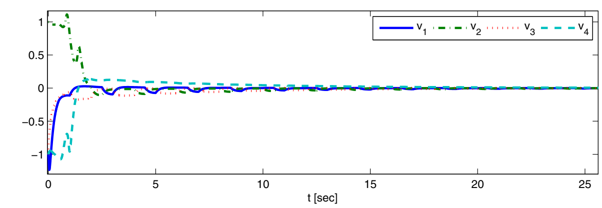

Fig. 5. The agents velocities v_i for DI agents$(i = 1, \ldots, 4)$.

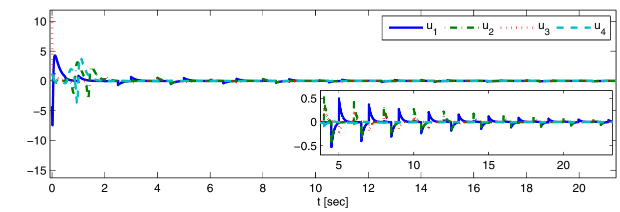

Fig. 6. The control inputs u_i for DI agents $(i = 1, \ldots, 4)$.

VI. Conclusion

We present a solution to the asymptotic consensus problem for agents with nonidentical, unknown control directions and switching topology. Our main tools are a generalization of Barbalát's lemma for uniformly piecewise continuous functions and a new boundedness lemma that can efficiently used with distributed nonlinear PI functions.

Appendix A
Proof of Lemma 1

Proof. Assume the opposite. Then, for some sufficiently small $\epsilon > 0$ there exists a sequence of times $\{T_\sigma\}_{\sigma=1}^{\infty}$ with $\lim_{\sigma\to\infty} T_\sigma = +\infty$ such that $|\phi(T_\sigma)| > \epsilon$ for all $\sigma \in \mathbb{N}$. Obviously $T_\sigma \in [t_{j^*(\sigma)}, t_{j^*(\sigma)+1})$ for some $j^*(\sigma) \in I$ and since $t_{j^*(\sigma)+1} - t_{j^*(\sigma)} > \tau$ we have that either $[T_\sigma, T_\sigma + \tau/2] \subseteq [t_{j^*(\sigma)}, t_{j^*(\sigma)+1})$ or $[T_\sigma - \tau/2, T_\sigma] \subseteq [t_{j^*(\sigma)}, t_{j^*(\sigma)+1})$. Without loss of generality we assume that $[T_\sigma, T_\sigma + \tau/2] \subseteq$

$[t_{j^*(\sigma)}, t_{j^*(\sigma)+1})$. Since $\phi(t)$ is uniformly piecewise right continuous there exists some $\delta(\epsilon) > 0$ such that

$$|\phi(t) - \phi(T_\sigma)| \leq \epsilon/2 \quad \forall t \in [T_\sigma, T_\sigma + \delta(\epsilon)] \subset [t_{j^*(\sigma)}, t_{j^*(\sigma)+1}).$$

Hence,

$$\begin{aligned} |\phi(t)| \geq & |\phi(T_\sigma)| - |\phi(t) - \phi(T_\sigma)| \\ > & \epsilon - \epsilon/2 = \epsilon/2, \qquad \forall t \in [T_\sigma, T_\sigma + \delta(\epsilon)]. \end{aligned} \tag{27}$$

Thus,

$$\left| \int_{T_\sigma}^{T_\sigma + \delta(\epsilon)} \phi(s) ds \right| = \int_{T_\sigma}^{T_\sigma + \delta(\epsilon)} |\phi(s)| ds > \frac{1}{2} \epsilon \delta(\epsilon). \tag{28}$$

If the limit $\lim_{t \to \infty} \int_0^t \phi(s) ds$ exists and is finite then

$$\begin{aligned} \lim_{\sigma \to \infty} \int_{T_\sigma}^{T_\sigma + \delta(\epsilon)} \phi(s) ds = & \lim_{\sigma \to \infty} \int_0^{T_\sigma + \delta(\epsilon)} \phi(s) ds \\ & - \lim_{\sigma \to \infty} \int_0^{T_\sigma} \phi(s) ds = 0. \end{aligned} \tag{29}$$

Inequalities (28) and (29) yield the desired contradiction. □

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