INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING
Int. J. Adapt. Control Signal Process. 2010; **00**:1–21
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/acs

Unifying adaptive control with the nonlinear PI methodology: Designs for unknown strict-feedback nonlinear systems with nonsmooth actuator nonlinearities

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SUMMARY

In this paper we extend the nonlinear PI control methodology within an adaptive control framework. An adaptive nonlinear PI controller is proposed for output tracking of strict-feedback nonlinear systems with non-smooth actuator nonlinearities and unknown control directions. The current approach relaxes the standard assumption of known bounds for the associated system nonlinearities made in earlier nonlinear PI schemes. New theoretical boundedness results have been proved that enable the successful combination of backstepping and linear parametric approximators with the nonlinear PI approach and ensure semi-global approximate tracking of the output to some reference trajectory. Following recent extensions of the nonlinear PI method to strict-feedback systems, the intermediate virtual control laws are derived through suitable integral equations. Simulation results are also presented in the paper that verify our theoretical analysis.

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KEY WORDS: adaptive control; nonlinear PI; backstepping; actuator nonlinearity

1. INTRODUCTION

The nonlinear PI method was originally proposed by Ortega, Astolfi and Barabanov in [1] as an alternative approach to parameter adaptation. The central idea of this technique revolves around the general immersion and invariance methodology [2]. Specifically, a stable error equation is defined with a perturbation term that has at least one root and the nonlinear PI gains are designed so that trajectories converge towards the root [1], [3].

In certain control applications there are cases of systems for which the control directions may be unknown. These include for example the uncalibrated visual servoing problem [4] and auto-pilot design of ships [5]. Also, actuation sign errors have been observed in attitude control of micro-satellites [6]. The unknown control direction problem has been mainly addressed with the use of the

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so called Nussbaum gain methodology [7]. This approach has received increased popularity over the years as it provides a design tool for general classes of systems and can be combined successfully with parameter adaptation algorithms [8]-[23].

On the other hand, the nonlinear PI method provides an effective alternative strategy to the unknown control direction problem (section 6 in [1]). Compared to Nussbaum gains, it was shown that the nonlinear PI technique is more robust to certain classes of unmodelled dynamic perturbations (see Example 8 of [24], Section 6.1 of [1] and [25], [26] for details). Extensions to regulation of strict feedback nonlinear systems have been considered recently in [27]. Applications to consensus problems for systems with unknown high frequency gains and switching topologies have been also proposed [28].

In practice, non-smooth nonlinearities such as deadzone, backlash and hysteresis are commonly present in actuators and can significantly downgrade overall system performance. Adaptive inversion [29]-[35] or linearly parameterized models with disturbance-like modelling errors [36]-[42] have been employed in the literature to design controllers that alleviate the undesirable effects of input nonlinearities. Several applications have also been considered such as the synchronization control problem of chaotic systems with input nonlinearities [43]-[44].

In this paper we propose for the first time an adaptive control extension of the nonlinear PI methodology. The solution is provided for a class of uncertain strict-feedback nonlinear systems with non-smooth input nonlinearities and unknown control directions. We adopt a linear time-varying approximate model derived in [20] to describe a general class of non-smooth input nonlinearities. The main contributions of this work are the following:

- We propose a new variant of the nonlinear PI method that *allows for parameter adaptation*. The new adaptive approach relaxes the standard assumption made in earlier nonlinear PI schemes [27], [1] of known bounds for the associated system nonlinearities. This extension is not trivial and is made possible through the use of a new technical lemma (Lemma 1).
- We generalize the method to reference trajectory tracking which was left as an open problem in Remark 9 of [1]. This is achieved by employing as an argument in the nonlinear PI terms *the square of the output of a deadzone with input the tracking error* and deadzone levels that are determined by the desired tracking accuracy.
- The new approach avoids the explosion of complexity problem since there is no need for explicit calculation of the partial derivatives of the virtual control laws. Suitable upper bounds depending on already known variables are obtained and estimated by the neural networks.
- A new integrator backstepping procedure is considered resulting in virtual control laws which are calculated from suitable integral equations in the spirit of [27].

The rest of the paper is organized as follows. In Section 2 the problem under study is defined and some standard results on linear parametric approximators are revisited. In Section 3 the main technical Lemma of the paper is stated and proved. Using this result, a detailed backstepping design is described in Section 4. Boundedness and approximate trajectory tracking for the proposed backstepping design are proved in Section 5. Finally, simulation results are given in Section 6.

DOI: 10.1002/acs

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Notations

For $\delta > 0$ we denote by $\lfloor x \rceil_\delta$ the output of a symmetric deadzone with input $x \in \mathbb{R}$ and deadzone level δ i.e.

$$\lfloor x \rceil_\delta := \begin{cases} x - \delta \operatorname{sgn}(x), & |x| \geq \delta \\ 0, & |x| < \delta \end{cases} \tag{1}$$

with $\operatorname{sgn}(x)$ the sign of x. From the definition of $\lfloor x \rceil_\delta$ and using the fact that $\operatorname{sgn}(\lfloor x \rceil_\delta) = \operatorname{sgn}(x)$ for all $|x| > \delta$ we obtain the following inequalities that will be employed repeatedly in the subsequent analysis

$$|x| \leq |\lfloor x \rceil_\delta| + \delta \quad \forall x \in \mathbb{R} \tag{2}$$

$$x \lfloor x \rceil_\delta = |x| \, |\lfloor x \rceil_\delta| \geq \delta |\lfloor x \rceil_\delta| \quad \forall x \in \mathbb{R} \tag{3}$$

$$\lfloor x \rceil_\delta^2 \leq x \lfloor x \rceil_\delta \quad \forall x \in \mathbb{R}. \tag{4}$$

Also, the smallest integer larger than or equal to x is denoted by $\lceil x \rceil$.

2.2. Problem Formulation

We consider the class of strict-feedback nonlinear systems with non-smooth actuator nonlinearities

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i) x_{i+1}, \quad y = x_1 \\ \dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n) h(u,t) + d(t) \end{aligned} \tag{5}$$

where $\bar{x}_i := [x_1, \cdots, x_i]^T \in \mathbb{R}^i$, $x := [x_1, \cdots, x_n]^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input and $y \in \mathbb{R}$ is the system output. Functions $f_i : \mathbb{R}^i \to \mathbb{R}$, $g_i : \mathbb{R}^i \to \mathbb{R}$ $(1 \leq i \leq n)$ are the system's unknown smooth nonlinearities while $h(u,t)$ represents the non-smooth actuator nonlinearity and $d(t)$ is some bounded disturbance signal.

Assumption 1
Functions $g_i(\cdot)$ are continuous and there exist unknown positive constants $g_{m,i} > 0$ such that $0 < g_{m,i} \leq |g_i(\bar{x}_i)|$ for all $\bar{x}_i \in \mathbb{R}^i$ $(1 \leq i \leq n)$. Thus, functions $g_i(\cdot)$ have constant but unknown signs (*control directions*).

Assumption 2
[20] The function $h(u,t)$ is described by

$$h(u,t) = m(t)u + l(t) \tag{6}$$

where $m(t)$ is a time-varying bounded function which takes values in the closed interval $I = [m^-, m^+]$ with $0 \notin I$ and $l(t)$ is a time varying bounded function, satisfying $\|l(t)\| \leq l^*$ for an unknown constant l^*.

Remark 1
Assumption 1 is plausible since presupposing that all g_i are far from zero is a controllability

DOI: 10.1002/acs

condition for system (5). Also, as proved in detail in [20] several models of typical non-smooth actuator nonlinearities such as dead-zone, backlash, and hysteresis nonlinearities satisfy Assumption 2.

Assumption 3
The perturbation term $d(t)$ represents an unknown bounded disturbance, i.e. it holds true that $\|d(t)\| \leq d^*$ for some unknown constant d^*.

Assumption 4
The reference trajectory $y_d(t)$ and its time derivative are bounded, i.e. there exist unknown constants $y_{M0}, y_{M1} > 0$ such that $|y_d(t)| \leq y_{M0}$, $|\dot{y}_d(t)| \leq y_{M1}$ for all $t \geq 0$.

Our design objective is to impose approximate output tracking i.e. to select a control law such that $\lim_{t\to\infty} |y(t) - y_d(t)| \leq \delta_1$ for an arbitrary continuously differentiable reference trajectory $y_d(t)$ and tracking accuracy $\delta_1 > 0$.

Remark 2
The main goal of the paper is the extension of the nonlinear PI approach within an adaptive control framework which is to the best of the authors' knowledge a new and unsolved challenging theoretical problem. We note that the tracking control problem under study can also be addressed with the use of Nussbaum functions as in [20], [22], [52]. Thus, the proposed approach can be seen as a valid new alternative method since, compared to the standard Nussbaum gain technique, the nonlinear PI approach has improved robustness properties with respect to certain types of unmodelled dynamics (for details see Example 8 in [24], section 6.3 of [1], [25] and [26]).

2.3. *Linear Parametric Approximators*

It is well-known [47], [48],[49] that any continuous nonlinear function $f_0 : \mathbb{R}^n \to \mathbb{R}$ can be approximated within some compact set $\Omega \subset \mathbb{R}^n$ with arbitrary precision by a suitably chosen linear parametric approximator (LPA) such as Radial Basis Functions (RBFs) or High Order Neural Networks (HONNs). Specifically we can write

$$f_0(x) = \theta^{*T}\Phi(x) + \varepsilon(x) \quad \forall x \in \Omega \subset \mathbb{R}^n \tag{7}$$

where $\Phi(x) \in \mathbb{R}^\ell$ the basis vector, ℓ is the number of nodes in the hidden layer, $\varepsilon(x)$ is the optimal approximation error and $\theta^* \in \mathbb{R}^\ell$ is the optimal weight vector defined by

$$\theta^* := \arg\min_{\theta\in\mathbb{R}^\ell} \sup_{x\in\Omega} \| f_0(x) - \theta^T\Phi(x)\|. \tag{8}$$

According to the universal approximation property of LPAs [49] the optimal approximation error norm $\varepsilon_M := \sup_{x\in\Omega} \|\varepsilon(x)\|$ can become arbitrarily small if we select a sufficiently large number of nodes ℓ in the hidden layer and an appropriate basis vector in order to ensure a complete covering of the approximation region.

DOI: 10.1002/acs

3. MAIN LEMMA

Our results are based on the following boundedness Lemma which is a generalization of Lemma 1 of [27]. This Lemma is central for the analysis of the adaptive backstepping nonlinear PI schemes proposed in Section 4.

Lemma 1
Consider the continuous differentiable nonnegative functions $V_i, L_i : [0,t_f) \to \mathbb{R}_+$, the continuous functions $Q_i : [0,t_f) \to \mathbb{R}$ $(1 \le i \le n)$ and $P_i, \bar{P}_i : [0,t_f) \to \mathbb{R}$ defined by

$$P_i(t) := V_i(t) + L_i(t) + \lambda_i \int_0^t V_i(s)ds - \varepsilon_{i+1}\lambda_{i+1} \int_0^t V_{i+1}(s)ds \tag{9}$$

$$\bar{P}_i(t) := P_i(t) + Q_i(t) \qquad (1 \le i \le n) \tag{10}$$

with $\varepsilon_i, \lambda_i > 0$. We also adopt the notation $V_{n+1} \equiv 0$ to ensure uniformity. Let n continuous functions $g_i : [0,\infty) \to \mathbb{R}$ for which there exist positive constants $g_{m,i} > 0$ such that $0 < g_{m,i} \le |g_i(t)|$ for all $t \in [0,\infty)$. If there exist nonnegative functions $\Theta_i : [0,t_f) \to \mathbb{R}_+$ such that

$$\frac{d\bar{P}_i}{dt} \le [\eta_i^* + g_i(t)\kappa_i(P_i)]\Theta_i(t) \tag{11}$$

$$\int_{t_1}^{t_2} \Theta_i(t)dt \ge c_i|Q(t_2) - Q(t_1)| \qquad \forall 0 \le t_1 \le t_2 < t_f \tag{12}$$

with constants $c_i > 0$, $\eta_i^* \ge 0$, $\kappa_i(P_i) := \varphi_i(P_i^2)\cos(P_i)$ and $\varphi_i(\cdot)$ a class $\mathcal{K}_\infty$ function then P_i is upper bounded on $[0,t_f)$. Moreover, if $\varepsilon_i < 1$ for all $i = 1,\cdots,n$ then all $V_i, L_i, \int_0^t V_i(s)ds$ are bounded on $[0,t_f)$.

Proof
The *upper boundedness* of P_i can be proved by using a contradiction argument. If we assume that P_i has no upper bound then due to continuity of P_i there exist two sequences of time instants $\{t_{i,1k}\}$, $\{t_{i,2k}\}$ defined by

$$t_{i,2k} = \inf\left\{t \in [0,t_f) | P_i(t) = 2k\pi + \frac{\pi}{2}\mathrm{sgn}(g_i) + \frac{3\pi}{4}\right\} \tag{13}$$

$$t_{i,1k} = \sup\left\{t \in [0,t_{2k}) | P_i(t) = 2k\pi + \frac{\pi}{2}\mathrm{sgn}(g_i) + \frac{\pi}{4}\right\} \tag{14}$$

such that

$$g_i(t)\kappa_i(P_i(t)) \le -\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2),\ \forall t \in [t_{i,1k}, t_{i,2k}]. \tag{15}$$

Integrating (11) over $[t_{i,1k}, t_{i,2k}]$ we obtain

$$\bar{P}_i(t_{i,2k}) - \bar{P}_i(t_{i,1k}) \le \int_{t_{i,1k}}^{t_{i,2k}} [\eta_i^* + g_i(t)\kappa_i(P_i)]\Theta_i(t)dt. \tag{16}$$

Using now property (15) the above inequality yields

$$\bar{P}_i(t_{i,2k}) - \bar{P}_i(t_{i,1k}) \le -\left[\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2) - \eta_i^*\right] \int_{t_{i,1k}}^{t_{i,2k}} \Theta_i(t)dt. \tag{17}$$

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For sufficiently large $k \geq \left\lceil (1/2\pi)\left(\left(\varphi_i^{-1}(\sqrt{2}\eta_i^*/g_{m,i})\right)^{1/2} + \pi/4\right)\right\rceil$ we have from the inequality above and (12) that

$$\bar{P}_i(t_{i,2k}) - \bar{P}_i(t_{i,1k}) \leq -c_i\left[\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2) - \eta_i^*\right]|Q_i(t_{i,2k}) - Q_i(t_{i,1k})|. \tag{18}$$

Also, from the definition of $\bar{P}_i$ in (10) and (13), (14) we have that

$$\begin{aligned}\bar{P}_i(t_{i,2k}) - \bar{P}_i(t_{i,1k}) &= P_i(t_{i,2k}) - P_i(t_{i,1k}) + Q_i(t_{i,2k}) - Q_i(t_{i,1k}) \\ &= Q_i(t_{i,2k}) - Q_i(t_{i,1k}) + \pi/2 \\ &\geq -|Q_i(t_{i,2k}) - Q_i(t_{i,1k})| + \pi/2.\end{aligned} \tag{19}$$

Combining (18), (19) we obtain

$$\frac{\pi}{2} + \left[c_i\frac{g_{m,i}}{\sqrt{2}}\varphi_i((2k\pi - \pi/4)^2) - c_i\eta_i^* - 1\right]|Q_i(t_{i,2k}) - Q_i(t_{i,1k})| \leq 0 \tag{20}$$

which cannot hold for arbitrarily large k. Specifically, (20) does not hold for

$$k \geq \bar{k}_i := \left\lceil \frac{1}{2\pi}\left[\left(\varphi_i^{-1}\left(\frac{\sqrt{2}}{c_i g_{m,i}}(c_i\eta_i^* + 1)\right)\right)^{1/2} + \frac{\pi}{4}\right]\right\rceil. \tag{21}$$

Thus, considering the integer $k_{i,0} := \inf\{k \in \mathbb{N} | P_i(0) \leq 2k\pi + (\pi/2)\mathrm{sgn}(g_i) + \pi/4\}$ described by

$$k_{i,0} := \left\lceil \frac{1}{2\pi}\left[P_i(0) - \frac{\pi}{4} - \frac{\pi}{2}\mathrm{sgn}(g_i)\right]\right\rceil \tag{22}$$

we have that P_i is upper bounded in $[0, t_f)$ from

$$P_i(t) \leq 2\max\{k_{i,0}, \bar{k}_i\}\pi + \frac{\pi}{2}\mathrm{sgn}(g_i) + \frac{3\pi}{4}. \tag{23}$$

If $\varepsilon_i < 1$ for all $i = 1, \cdots, n$ then using the upper boundedness of all P_i we can sum all (9) to obtain

$$\sum_{i=1}^{n} P_i(t) = \sum_{i=1}^{n}(V_i + L_i) + \sum_{i=1}^{n}(1 - \varepsilon_i)\lambda_i\int_0^t V_i(s)ds \geq 0. \tag{24}$$

The above identity and the boundedness of $\sum_{i=1}^{n} P_i(t)$ yield the boundedness of all $L_i, V_i, \int_0^t V_i(s)ds$ on $[0, t_f)$. $\square$

4. BACKSTEPPING CONTROL DESIGN

In this Section we provide a backstepping design procedure that allows for an application of the boundedness Lemma 1. Approximate output tracking can then be proved using the conclusions of Lemma 1.

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4.1. Step 1

If we define the first error variable $z_1 := x_1 - y_d$ then

$$\dot{z}_1 = f_1(x_1) + g_1(x_1)x_2 - \dot{y}_d. \tag{25}$$

Let α_1 the virtual control to be selected and $z_2 := x_2 - \alpha_1$ the second error variable. For the nonnegative continuous differentiable function $V_1 := (1/2)\lfloor z_1 \rceil_{\delta_1}^2$ with $\delta_1 > 0$ we have

$$\dot{V}_1 = \lfloor z_1 \rceil_{\delta_1} \dot{z}_1 = \lfloor z_1 \rceil_{\delta_1} [f_1(x_1) + g_1(x_1)z_2 - \dot{y}_d + g_1(x_1)\alpha_1]. \tag{26}$$

Using (2), (3) and completing the square we obtain the following bound

$$\begin{aligned} g_1(x_1)z_2\lfloor z_1 \rceil_{\delta_1} &\le |g_1(x_1)||z_2| \left|\lfloor z_1 \rceil_{\delta_1}\right| \\ &\le |g_1(x_1)|(\left|\lfloor z_2 \rceil_{\delta_2}\right| + \delta_2)\left|\lfloor z_1 \rceil_{\delta_1}\right| \\ &\le \delta_2 \left|g_1(x_1)\lfloor z_1 \rceil_{\delta_1}\right| + \frac{1}{2\varepsilon_2\lambda_2} g_1^2(x_1)\lfloor z_1 \rceil_{\delta_1}^2 + \frac{\varepsilon_2\lambda_2}{2}\lfloor z_2 \rceil_{\delta_2}^2 \\ &\le \frac{\delta_2}{\delta_1}|g_1(x_1)|z_1\lfloor z_1 \rceil_{\delta_1} + \frac{1}{2\varepsilon_2\lambda_2} g_1^2(x_1)\lfloor z_1 \rceil_{\delta_1}^2 + \frac{\varepsilon_2\lambda_2}{2}\lfloor z_2 \rceil_{\delta_2}^2 \end{aligned} \tag{27}$$

for some $\delta_2, \lambda_2 > 0$ and $0 < \varepsilon_2 < 1$. From (26) and the above inequality we result in

$$\dot{V}_1 \le \lfloor z_1 \rceil_{\delta_1} [F_1(x_1, z_1) - \dot{y}_d + g_1(x_1)\alpha_1] + \frac{\varepsilon_2\lambda_2}{2}\lfloor z_2 \rceil_{\delta_2}^2 \tag{28}$$

with

$$F_1(x_1, z_1) := f_1(x_1) + \frac{\delta_2}{\delta_1}|g_1(x_1)|z_1 + \frac{1}{2\varepsilon_2\lambda_2} g_1^2(x_1)\lfloor z_1 \rceil_{\delta_1}. \tag{29}$$

Consider now an approximation of the nonlinearity F_1 within some compact set $\Omega_1 \subset \mathbb{R}^2$ by an LPA such that

$$F_1(x_1, z_1) = \theta_1^{*T}\Phi_1(x_1, z_1) + \varepsilon_{a1}(x_1, z_1), \, \forall (x_1, z_1) \in \Omega_1 \tag{30}$$

with $\theta_1^* \in \mathbb{R}^{\ell_1}$ the optimal approximation weight, $\Phi_1(x_1, z_1) \in \mathbb{R}^{\ell_1}$ the regressor vector and $\varepsilon_{a1}(x_1, z_1)$ the approximation error. The error is bounded within Ω_1 i.e. there exists some $\varepsilon_{M1} > 0$ such that $|\varepsilon_{a1}(x_1, z_1)| \le \varepsilon_{M1}$ for all $(x_1, z_1) \in \Omega_1$. In our control law we consider now estimations of the optimal weight norm instead of the whole weight vector. This is an idea originally introduced in [50] and used also in [51] in order to reduce the overall computational cost. If we introduce an estimation β_1 of the optimal weight norm $\|\theta_1^*\|$ and the functions $L_1 := (1/2\gamma_1)\beta_1^2$,

$$P_1 := V_1 + L_1 + \lambda_1 \int_0^t V_1(s)ds - \frac{\varepsilon_2\lambda_2}{2} \int_0^t \lfloor z_2(s) \rceil_{\delta_2}^2 ds \tag{31}$$

with $\gamma_1, \lambda_1 > 0$ then

$$\dot{P}_1 \le \lfloor z_1 \rceil_{\delta_1} \left[\theta_1^{*T}\Phi_1(x_1, z_1) + \varepsilon_{a1}(x_1, z_1) + \frac{\lambda_1}{2}\lfloor z_1 \rceil_{\delta_1} - \dot{y}_d + g_1(x_1)\alpha_1\right] + \frac{1}{\gamma_1}\beta_1\dot{\beta}_1. \tag{32}$$

DOI: 10.1002/acs

Define also the functions $Q_1 := (1/2\gamma_1)(\tilde{\beta}_1^2 - \beta_1^2)$ with $\tilde{\beta}_1 := \beta_1 - \|\theta_1^*\|$ the weight norm estimation error and

$$\bar{P}_1 := P_1 + Q_1. \tag{33}$$

If we select the estimation update law

$$\dot{\beta}_1 = \gamma_1 \|\Phi_1(x_1, z_1)\| |\lfloor z_1 \rceil_{\delta_1}|, \quad \beta_1(0) = 0 \tag{34}$$

then from (32)-(34) and Assumption 4 we have

$$\begin{aligned} \frac{d\bar{P}_1}{dt} &\leq \lfloor z_1 \rceil_{\delta_1} \left[\theta_1^{*T} \Phi_1(x_1, z_1) + \varepsilon_{a1}(x_1, z_1) + \frac{\lambda_1}{2} \lfloor z_1 \rceil_{\delta_1} - \dot{y}_d + g_1(x_1)\alpha_1 \right] + (\beta_1 - \|\theta_1^*\|)\|\Phi_1\| |\lfloor z_1 \rceil_{\delta_1}| \\ &\leq |\lfloor z_1 \rceil_{\delta_1}| \left[\beta_1 \|\Phi_1(x_1, z_1)\| + \varepsilon_{M1} + y_{M1} \right] + \frac{1}{2}\lambda_1 \lfloor z_1 \rceil_{\delta_1}^2 + g_1(x_1) \lfloor z_1 \rceil_{\delta_1} \alpha_1. \end{aligned} \tag{35}$$

Using (3) the above inequality yields

$$\frac{d\bar{P}_1}{dt} \leq \frac{1}{\delta_1} \lfloor z_1 \rceil_{\delta_1} \left[\beta_1 \|\Phi_1(x_1, z_1)\| z_1 + (\varepsilon_{M1} + y_{M1}) z_1 + \frac{1}{2}\delta_1 \lambda_1 \lfloor z_1 \rceil_{\delta_1} + \delta_1 g_1(x_1)\alpha_1 \right]. \tag{36}$$

If we now select the virtual control law

$$\alpha_1 = \kappa_1(P_1)(1 + \|\Phi_1(x_1, z_1)\|^2 + \beta_1^2) z_1 \tag{37}$$

along with (4) and consider the following inequality obtained by completing the square

$$\beta_1 \|\Phi_1(x_1, z_1)\| \leq \frac{1}{2}(\|\Phi_1(x_1, z_1)\|^2 + \beta_1^2) \tag{38}$$

then (36) yields

$$\frac{d\bar{P}_1}{dt} \leq [\eta_1^* + g_1(x_1(t))\kappa_1(P_1)]\Theta_1(t) \tag{39}$$

with

$$\Theta_1(t) := (1 + \|\Phi_1(x_1, z_1)\|^2 + \beta_1^2) z_1 \lfloor z_1 \rceil_{\delta_1} \geq 0 \tag{40}$$

$$\eta_1^* := \max\left\{ \frac{1}{2\delta_1}, \frac{\lambda_1}{2} + \frac{\varepsilon_{M1} + y_{M1}}{\delta_1} \right\}. \tag{41}$$

From the definition of Q_1 and (34), (3), (40) we obtain

$$\begin{aligned} |Q_1(t_1) - Q_1(t_2)| &= \frac{1}{\gamma_1} \|\theta_1^*\| \, |\beta_1(t_2) - \beta_1(t_1)| \\ &= \frac{1}{\gamma_1} \|\theta_1^*\| \left| \int_{t_1}^{t_2} \dot{\beta}_1(s) ds \right| = \int_{t_1}^{t_2} \|\theta_1^*\| \, \|\Phi_1(x_1, z_1)\| \, |\lfloor z_1 \rceil_{\delta_1}| ds \\ &\leq \frac{1}{2\delta_1} \int_{t_1}^{t_2} (\|\theta_1^*\|^2 + \|\Phi_1(x_1, z_1)\|^2) z_1 \lfloor z_1 \rceil_{\delta_1} ds \\ &\leq \frac{\max\{1, \|\theta_1^*\|^2\}}{2\delta_1} \int_{t_1}^{t_2} \Theta_1(s) ds \quad , \forall t_2 \geq t_1. \end{aligned} \tag{42}$$

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DOI: 10.1002/acs

Thus, from (39), (42), conditions (11), (12) of Lemma 1 are true and therefore the function P_1 is upper bounded.

Similarly to [27] we note that (37) defining the virtual control α_1 is actually an integral equation of the form

$$\alpha_1(t) = \kappa_1\Bigg[V_1 + L_1 - \frac{\varepsilon_2\lambda_2}{2}\int_0^t \lfloor x_2(s) - \alpha_1(s)\rceil_{\delta_2}^2 ds \\ + \lambda_1\int_0^t V_1(s)ds\Bigg](1 + \|\Phi_1(x_1,z_1)\|^2 + \beta_1^2)z_1. \tag{43}$$

Alternatively, the virtual control law α_1 can be seen as the output of a first-order dynamical system with state vector

$$\zeta_1 := \frac{\lambda_1}{2}\int_0^t \lfloor z_1(s)\rceil_{\delta_1}^2 ds - \frac{\varepsilon_2\lambda_2}{2}\int_0^t \lfloor z_2(s)\rceil_{\delta_2}^2 ds \tag{44}$$

and input vector $[z_1, \beta_1, x_2]$. This follows from the state and output equations

$$\dot{\zeta}_1 = \frac{\lambda_1}{2}\lfloor z_1\rceil_{\delta_1}^2 - \frac{\varepsilon_2\lambda_2}{2}\left\lfloor x_2 - \kappa_1\left(\frac{1}{2}\lfloor z_1\rceil_{\delta_1}^2 + \frac{1}{2\gamma_1}\beta_1^2 + \zeta_1\right)(1 + \|\Phi_1(x_1,z_1)\|^2 + \beta_1^2)z_1\right\rceil_{\delta_2}^2 \tag{45}$$

$$\alpha_1 = \kappa_1\left(\frac{1}{2}\lfloor z_1\rceil_{\delta_1}^2 + \frac{1}{2\gamma_1}\beta_1^2 + \zeta_1\right)(1 + \|\Phi_1(x_1,z_1)\|^2 + \beta_1^2)z_1. \tag{46}$$

4.2. *Step* i $(2 \le i \le n-1)$:

For the i-th step we define the error variable $z_i := x_i - \alpha_{i-1}$ with dynamics

$$\dot{z}_i = f_i(\overline{x_i}) + g_i(\overline{x_i})z_{i+1} + g_i(\overline{x_i})\alpha_i - \dot{\alpha}_{i-1} \tag{47}$$

where α_i is the i-th virtual control. The following lemma holds true.

Lemma 2

Let the nonlinear system (5), the error variables $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $z_{n+1} \equiv 0$ and the integral terms

$$I_i = \int_0^t \lfloor z_i\rceil_{\delta_i}^2(s)ds. \tag{48}$$

If all virtual control laws are selected to have the form

$$\alpha_i = A_i(\overline{x_i}, \overline{z_i}, \overline{I_{i+1}}, \overline{\beta_i}), \quad (1 \le i \le n-1) \tag{49}$$

where β_i are adaptation parameters defined in each step of the backstepping procedure with the property

$$|\dot{\beta}_i| \le B_i(\overline{x_i}, \overline{z_i}, \overline{I_i}, \overline{\beta}_{i-1}) \tag{50}$$

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DOI: 10.1002/acs

and $\xi_i := [\overline{x}_i, \overline{z}_i, \overline{I}_i, \overline{\beta}_{i-1}]$, $B_i \circ \xi_i \in C^1([t_0, t_f), \mathbb{R}_+)$ then there exist continuous differentiable functions $H_i \circ \xi_{i+1} \in C^1([t_0, t_f), \mathbb{R}_+)$ such that

$$|\dot{\alpha}_i| \leq H_i(\overline{x}_{i+1}, \overline{z}_{i+1}, \overline{I}_{i+1}, \overline{\beta}_i), \quad (1 \leq i \leq n-1). \tag{51}$$

Proof
We will employ an induction argument to prove (51). For $i = 1$ we have from (49), (5) that

$$\dot{\alpha}_1 = \frac{\partial A_1}{\partial x_1}(f_1 + g_1 x_2) + \frac{\partial A_1}{\partial z_1}(f_1 + g_1 x_2 - \dot{y}_d) + \sum_{j=1}^{2} \frac{\partial A_1}{\partial I_j} \lfloor z_j \rceil^2_{\delta_j} + \frac{\partial A_1}{\partial \beta_1}\dot{\beta}_1. \tag{52}$$

Using (50) and completing the square the above inequality yields

$$\begin{aligned} |\dot{\alpha}_1| &\leq \left|\frac{\partial A_1}{\partial x_1}(f_1 + g_1 x_2)\right| + \left|\frac{\partial A_1}{\partial z_1}\right|(|f_1 + g_1 x_2| + y_{M1}) + \sum_{j=1}^{2}\left|\frac{\partial A_1}{\partial I_j}\right| \lfloor z_j \rceil^2_{\delta_j} + \left|\frac{\partial A_1}{\partial \beta_1}\right| B_1 \\ &\leq \frac{1}{2}(\frac{\partial A_1}{\partial x_1})^2 + (f_1 + g_1 x_2)^2 + (\frac{\partial A_1}{\partial z_1})^2 + \frac{y_{M1}^2}{2} + \frac{1}{2}(\frac{\partial A_1}{\partial \beta_1})^2 \\ &\quad + \frac{1}{2}\sum_{j=1}^{2}\left[(\frac{\partial A_1}{\partial I_j})^2 + \lfloor z_j \rceil^4_{\delta_j}\right] + \frac{1}{2}B_1^2 := H_1(\overline{x}_2, \overline{z}_2, \overline{I}_2, \beta_1) \end{aligned} \tag{53}$$

i.e. (51) is valid for $i = 1$. Suppose that (51) holds true for all $j = 1, 2, \ldots, i-1$. We will prove that (51) is also valid for $j = i$. Taking the time derivative of (49) we have that

$$\dot{\alpha}_i = \sum_{j=1}^{i} \frac{\partial A_i}{\partial x_j}\dot{x}_j + \sum_{j=1}^{i} \frac{\partial A_i}{\partial z_j}\dot{z}_j + \sum_{j=1}^{i+1} \frac{\partial A_i}{\partial I_j}\dot{I}_j + \sum_{j=1}^{i} \frac{\partial A_i}{\partial \beta_j}\dot{\beta}_j. \tag{54}$$

From (5) it holds true that $\dot{x}_i = f_i(\overline{x}_i) + g_i(\overline{x}_i)x_{i+1} := h_{x,i}(\overline{x}_{i+1})$ for all $i = 1, \cdots, n-1$. Since we assumed that (51) is true for all $j = 1, \cdots, i-1$ we can write

$$\begin{aligned} |\dot{z}_j| &\leq |h_{x,j}(\overline{x}_{j+1})| + |\dot{\alpha}_{j-1}| \leq h^2_{x,j}(\overline{x}_{j+1}) + H_{j-1}(\overline{x}_j, \overline{z}_j, \overline{I}_j, \overline{\beta}_{j-1}) + \frac{1}{4} \\ &:= h_{z,j}(\overline{x}_{j+1}, \overline{z}_j, \overline{I}_j, \overline{\beta}_{j-1}) \quad \forall j = 1, \cdots, i. \end{aligned} \tag{55}$$

Using (55) and (50) in (54) we obtain

$$\begin{aligned} |\dot{\alpha}_i| &\leq \sum_{j=1}^{i}\left|\frac{\partial A_i}{\partial x_j}h_{x,j}\right| + \sum_{j=1}^{i}\left|\frac{\partial A_i}{\partial z_j}\right| h_{z,j} + \sum_{j=1}^{i+1}\left|\frac{\partial A_i}{\partial I_j}\right| \lfloor z_j \rceil^2_{\delta_j} + \sum_{j=1}^{i}\left|\frac{\partial A_i}{\partial \beta_j}\right| B_j \\ &\leq \frac{1}{2}\sum_{j=1}^{i}\left[(\frac{\partial A_i}{\partial x_j})^2 + (\frac{\partial A_i}{\partial z_j})^2 + (\frac{\partial A_i}{\partial \beta_j})^2 + h^2_{x,j} + h^2_{z,j} + B_j^2\right] \\ &\quad + \frac{1}{2}\sum_{j=1}^{i+1}\left[(\frac{\partial A_i}{\partial I_j})^2 + 1\right]\lfloor z_j \rceil^2_{\delta_j} := H_i(\overline{x}_{i+1}, \overline{z}_{i+1}, \overline{I}_{i+1}, \overline{\beta}_i) \end{aligned} \tag{56}$$

which completes the proof of the lemma. □

DOI: 10.1002/acs

Obviously α_1 defined by (37) is of the form (49). Also, the dynamics of β_1 given by (34) ensure that

$$|\dot{\beta}_1| \le \frac{\gamma_1}{2}(\|\Phi_1(x_1,z_1)\|^2 + \lfloor z_1 \rceil^2_{\delta_1}) := B_1(x_1,z_1,I_1) \tag{57}$$

which is in the form of (50). Assume that the virtual control law α_j and the dynamics of β_j are in the form of (49), (50) respectively for all $j = 1,2,\cdots,i-1$. We will select in the next a virtual control α_i and an estimate β_i for which (49), (50) are also true.

Consider now the function $V_i = \frac{1}{2}\lfloor z_i \rceil^2_{\delta_i}$ with $\delta_i > 0$. The time derivative of V_i is

$$\dot{V}_i = \lfloor z_i \rceil_{\delta_i}[f_i(\bar{x}_i) + g_i(\bar{x}_i)z_{i+1} + g_i(\bar{x}_i)\alpha_i - \dot{\alpha}_{i-1}]. \tag{58}$$

Using Lemma 2 and (3) we have that

$$\dot{V}_i \le \lfloor z_i \rceil_{\delta_i}[f_i(\bar{x}_i) + g_i(\bar{x}_i)z_{i+1} + g_i(\bar{x}_i)\alpha_i + (1/\delta_i)z_i H_{i-1}(\bar{x}_i,\bar{z}_i,\bar{I}_i,\bar{\beta}_{i-1})]. \tag{59}$$

Similarly to the derivation of (27) in step $i=1$ we can prove that $g_i(\bar{x}_i)z_{i+1}\lfloor z_i \rceil_{\delta_i}$ is bounded by

$$g_i(\bar{x}_i)z_{i+1}\lfloor z_i \rceil_{\delta_i} \le \frac{\delta_{i+1}}{\delta_i}|g_i(\bar{x}_i)|z_i\lfloor z_i \rceil_{\delta_i} + \frac{1}{2\varepsilon_{i+1}\lambda_{i+1}}g_i^2(\bar{x}_i)\lfloor z_i \rceil^2_{\delta_i} + \frac{\varepsilon_{i+1}\lambda_{i+1}}{2}\lfloor z_{i+1} \rceil^2_{\delta_{i+1}} \tag{60}$$

for $\delta_{i+1},\lambda_{i+1} > 0$ and $0 < \varepsilon_{i+1} < 1$. If we define $\xi_i := [\bar{x}_i,\bar{z}_i,\bar{I}_i,\bar{\beta}_{i-1}] \in \mathbb{R}^{4i-1}$ and

$$F_i(\xi_i) := f_i(\bar{x}_i) + \frac{\delta_{i+1}}{\delta_i}|g_i(\bar{x}_i)|z_i + \frac{1}{2\varepsilon_{i+1}\lambda_{i+1}}g_i^2(\bar{x}_i)\lfloor z_i \rceil_{\delta_i} + \frac{1}{\delta_i}z_i H_{i-1}(\xi_i) \tag{61}$$

then (59) yields through (60), (61)

$$\dot{V}_i \le \lfloor z_i \rceil_{\delta_i}[F_i(\xi_i) + g_i(\bar{x}_i)\alpha_i] + \frac{\varepsilon_{i+1}\lambda_{i+1}}{2}\lfloor z_{i+1} \rceil^2_{\delta_{i+1}}. \tag{62}$$

Let now an approximation of the nonlinearity $F_i(\xi_i)$ within some compact set $\Omega_i \subset \mathbb{R}^{4i-1}$ by an LPA such that

$$F_i(\xi_i) := \theta_i^{*T}\Phi_i(\xi_i) + \varepsilon_{ai}(\xi_i), \quad \forall \xi_i \in \Omega_i \tag{63}$$

with $\theta_i^* \in \mathbb{R}^{\ell_i}$ the optimal approximation weight, $\Phi_i(\xi_i) \in \mathbb{R}^{\ell_i}$ the regressor vector and $\varepsilon_{ai}(\xi_i)$ the approximation error. The error is bounded within Ω_i i.e. there exists some $\varepsilon_{Mi} > 0$ such that $|\varepsilon_{ai}(\xi_i)| \le \varepsilon_{Mi}$ for all $\xi_i \in \Omega_i$. If we introduce an estimation β_i of the optimal weight norm $\|\theta_i^*\|$, and functions $L_i := (1/2\gamma_i)\beta_i^2$,

$$P_i := V_i + L_i + \lambda_i \int_0^t V_i(s)ds - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2}\int_0^t \lfloor z_{i+1}(s) \rceil^2_{\delta_{i+1}} ds \tag{64}$$

with $\gamma_i > 0$ then we have from (62), (63)

$$\dot{P}_i \le \lfloor z_i \rceil_{\delta_i}[\theta_i^{*T}\Phi_i(\xi_i) + \varepsilon_{ai}(\xi_i) + (\lambda_i/2)\lfloor z_i \rceil_{\delta_i} + g_i(\bar{x}_i)\alpha_i] + \frac{1}{\gamma_i}\beta_i\dot{\beta}_i. \tag{65}$$

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Let us define the functions $Q_i := (1/2\gamma_i)(\tilde{\beta}_i^2 - \beta_i^2)$ where $\tilde{\beta}_i = \beta_i - \|\theta_i^*\|$ and

$$\bar{P}_i := P_i + Q_i. \tag{66}$$

If we select the estimation update law

$$\dot{\beta}_i = \gamma_i \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}|, \quad \beta_i(0) = 0 \tag{67}$$

then

$$|\dot{\beta}_i| \le B_i(\xi_i) := \frac{\gamma_i}{2}\left[\|\Phi_i(\xi_i)\|^2 + \lfloor z_i \rceil_{\delta_i}^2\right] \tag{68}$$

which is of the form (50) of Lemma 2. Also from (65)-(67), (3) we have

$$\frac{d\bar{P}_i}{dt} \le \lfloor z_i \rceil_{\delta_i}\left[\frac{\lambda_i}{2}\lfloor z_i \rceil_{\delta_i} + \frac{\varepsilon_{Mi}}{\delta_i} z_i + g_i(\bar{x}_i)\alpha_i\right] + \beta_i \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}| \tag{69}$$

Completing the square and using (3) we result in

$$\beta_i \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}| \le \frac{1}{2\delta_i}(\beta_i^2 + \|\Phi_i(\xi_i)\|^2)\lfloor z_i \rceil_{\delta_i} z_i. \tag{70}$$

If we choose the virtual control law

$$\alpha_i = \kappa_i(P_i)(1 + |\beta_i|^2 + \|\Phi_i(\xi_i)\|^2) z_i := A_i(\bar{x}_i, \bar{z}_i, \bar{I}_{i+1}, \bar{\beta}_i) \tag{71}$$

which is of the form (49) then we obtain from (69), (70) and (4)

$$\frac{d\bar{P}_i}{dt} \le [\eta_i^* + g_i(\bar{x}_i(t))\kappa_i(P_i)]\Theta_i(t) \tag{72}$$

with

$$\Theta_i(t) := (1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2) z_i \lfloor z_i \rceil_{\delta_i} \tag{73}$$

$$\eta_i^* := \max\left\{\frac{1}{2\delta_i}, \frac{\lambda_i}{2} + \frac{\varepsilon_{Mi}}{\delta_i}\right\}. \tag{74}$$

From the definition of Q_i and (67), (3), (73) it also holds true that

$$\begin{aligned}
|Q_i(t_2) - Q_i(t_1)| &= \frac{1}{\gamma_i}\|\theta_i^*\| \, |\beta_i(t_2) - \beta_i(t_1)| \\
&= \frac{1}{\gamma_i}\|\theta_i^*\| \left|\int_{t_1}^{t_2} \dot{\beta}_i(s) ds\right| = \int_{t_1}^{t_2} \|\theta_i^*\| \|\Phi_i(\xi_i)\| |\lfloor z_i \rceil_{\delta_i}| ds \\
&\le \frac{1}{2\delta_i}\int_{t_1}^{t_2} (\|\theta_i^*\|^2 + \|\Phi_i(\xi_i)\|^2) z_i \lfloor z_i \rceil_{\delta_i} ds \\
&\le \frac{\max\{1, \|\theta_i^*\|\}}{2\delta_i}\int_{t_1}^{t_2} \Theta_i(s) ds \quad \forall t_2 \ge t_1.
\end{aligned} \tag{75}$$

Thus conditions of Lemma 1 hold true and therefore P_i is upper bounded.

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We also note that (71) defining the virtual control α_i is actually an integral equation of the form

$$\alpha_i(t) = \kappa_i \Bigg[V_i + L_i - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2} \int_0^t \lfloor x_{i+1}(s) - \alpha_i(s) \rceil^2_{\delta_{i+1}} ds + \lambda_i \int_0^t V_i(s) ds \Bigg] (1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2) z_i. \tag{76}$$

Similar to the case $i = 1$, the virtual control law α_i can be interpreted as the output of a first-order dynamical system with state

$$\zeta_i := \frac{\lambda_i}{2} \int_0^t \lfloor z_i(s) \rceil^2_{\delta_i} ds - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2} \int_0^t \lfloor z_{i+1}(s) \rceil^2_{\delta_{i+1}} ds \tag{77}$$

and input vector $[z_i, \beta_i, \xi_i^T, x_{i+1}]^T$. This follows from the state-space and output equations given below

$$\dot{\zeta}_i = \frac{\lambda_i}{2} \lfloor z_i \rceil^2_{\delta_i} - \frac{\varepsilon_{i+1}\lambda_{i+1}}{2} \left\lfloor x_{i+1} - \kappa_i \left(\frac{1}{2} \lfloor z_i \rceil^2_{\delta_i} + \frac{1}{2\gamma_i} \beta_i^2 + \zeta_i \right) \left(1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2\right) z_i \right\rceil^2_{\delta_{i+1}} \tag{78}$$

$$\alpha_i = \kappa_i \left(\frac{1}{2} \lfloor z_i \rceil^2_{\delta_i} + \frac{1}{2\gamma_i} \beta_i^2 + \zeta_i \right) \left(1 + \beta_i^2 + \|\Phi_i(\xi_i)\|^2\right) z_i. \tag{79}$$

The complexity in the calculation of the i-th virtual control is in this sense comparable to standard Nussbaum schemes [52]. In [52] for example, the calculation of the i-th virtual control law involves the i-th Nussbaum parameter which is generated by a first-order dynamical system.

4.3. Step n

For the n-th step we define the error variable $z_n := x_n - \alpha_{n-1}$ with dynamics

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n) h(u,t) + d(t) - \dot{\alpha}_{n-1}. \tag{80}$$

Using Assumption 2 eq. (80) can be rewritten in the form

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n) m(t) \alpha_n + g_n(\bar{x}_n) l(t) + d(t) - \dot{\alpha}_{n-1} \tag{81}$$

with $u = \alpha_n$ the control law to be selected.

Consider now the function $V_n = \frac{1}{2} \lfloor z_n \rceil^2_{\delta_n}$ with $\delta_n > 0$. The time derivative of V_n is

$$\dot{V}_n = \lfloor z_n \rceil_{\delta_n} [f_n(\bar{x}_n) + g_n(\bar{x}_n) m(t) \alpha_n + g_n(\bar{x}_n) l(t) + d(t) - \dot{\alpha}_{n-1}]. \tag{82}$$

Using Lemma 2, Assumption 2 and (3) we have that

$$\dot{V}_n \leq \lfloor z_n \rceil_{\delta_n} \left[f_n(\bar{x}_n) + g_n(\bar{x}_n) m(t) \alpha_n + d(t) + \frac{z_n}{\delta_n} \Big(|g_n(\bar{x}_n)| l^* + H_{n-1}(\bar{x}_n, \bar{z}_n, \bar{I}_n, \bar{\beta}_{n-1}) \Big) \right]. \tag{83}$$

Let $\xi_n := [\bar{x}_n, \bar{z}_n, \bar{I}_n, \bar{\beta}_{n-1}] \in \mathbb{R}^{4n-1}$ and the function

$$F_n(\xi_n) = f_n(\bar{x}_n) + \frac{1}{\delta_n} z_n \Big[H_{n-1}(\xi_n) + l^* |g_n(\bar{x}_n)| \Big] \tag{84}$$

and consider an approximation of the nonlinearity $F_n(\xi_n)$ within some compact set $\Omega_n \subset \mathbb{R}^{4n-1}$ by an LPA such that

$$F_n(\xi_n) := \theta_n^{*T} \Phi_n(\xi_n) + \varepsilon_{an}(\xi_n), \quad \forall \xi_n \in \Omega_n \tag{85}$$

with $\theta_n^* \in \mathbb{R}^{\ell_n}$ the optimal approximation weight, $\Phi_n(\xi_n) \in \mathbb{R}^{\ell_n}$ the regressor vector and $\varepsilon_{an}(\xi_n)$ the approximation error. The error is bounded within Ω_n i.e. there exists some $\varepsilon_{Mn} > 0$ such that $|\varepsilon_{an}(\xi_n)| \le \varepsilon_{Mn}$ for all $\xi_n \in \Omega_n$. If we introduce an online estimation $\beta_n(t)$ of the optimal weight norm $\|\theta_n^*\|$, and functions $L_n := (1/2\gamma_n)\beta_n^2(t)$ and

$$P_n := V_n + L_n + \lambda_n \int_0^t V_n(s) ds \tag{86}$$

then we have from (83)-(85)

$$\dot{P}_n \le \lfloor z_n \rceil_{\delta_n} \left[\theta_n^{*T} \Phi_n(\xi_n) + \varepsilon_{an}(\xi_n) + \frac{\lambda_n}{2} \lfloor z_n \rceil_{\delta_n} + g_n(\bar{x}_n) m(t) \alpha_n + d(t) \right] + \frac{1}{\gamma_n} \beta_n \dot{\beta}_n. \tag{87}$$

Using Cauchy-Swartz inequality (87) yields

$$\begin{aligned} \dot{P}_n \le \lfloor z_n \rceil_{\delta_n} & \left[\varepsilon_{an}(\xi_n) + \frac{\lambda_n}{2} \lfloor z_n \rceil_{\delta_n} + g_n(\bar{x}_n) m(t) \alpha_n + d(t) \right] \\ & + \|\theta_n^*\| \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}| + \frac{1}{\gamma_n} \beta_n \dot{\beta}_n. \end{aligned} \tag{88}$$

Define also the functions $Q_n := (1/2\gamma_n)(\tilde{\beta}_n^2 - \beta_n^2)$ where $\tilde{\beta}_n = \beta_n - \|\theta_n^*\|$ is the estimation error and

$$\bar{P}_n := P_n + Q_n. \tag{89}$$

If we select the estimation update law

$$\dot{\beta}_n = \gamma_n \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}|, \quad \beta_n(0) = 0 \tag{90}$$

then from Assumptions 2, 3, (88)-(90) and (3) we obtain

$$\begin{aligned} \frac{d\bar{P}_n}{dt} \le \lfloor z_n \rceil_{\delta_n} & \left[\frac{\lambda_n}{2} \lfloor z_n \rceil_{\delta_n} + (\varepsilon_{Mn} + d^*) \frac{z_n}{\delta_n} \right. \\ & \left. + g_n(\bar{x}_n) m(t) \alpha_n \right] + \beta_n \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}|. \end{aligned} \tag{91}$$

Completing the square and using (3) we have that

$$\beta_n \|\Phi_n(\xi_n)\| |\lfloor z_n \rceil_{\delta_n}| \le \frac{1}{2\delta_n} (\beta_n^2 + \|\Phi_n(\xi_n)\|^2) \lfloor z_n \rceil_{\delta_n} z_n. \tag{92}$$

DOI: 10.1002/acs

Thus, if we choose the control law

$$u = \alpha_n = \kappa_n(P_n)(1+|\beta_n|^2+\|\Phi_n(\xi_n)\|^2)z_n \tag{93}$$

then from (4), (92), eq. (91) yields

$$\frac{d\bar{P}_n}{dt} \leq [\eta_n^* + g_n(\bar{x}_n)m(t)\kappa_n(P_n)]\Theta_n(t) \tag{94}$$

with

$$\Theta_n(t) := (1+|\beta_n|^2+\|\Phi_n(\xi_n)\|^2)z_n\lfloor z_n\rceil_{\delta_n} \tag{95}$$

$$\eta_n^* := \max\left\{\frac{\lambda_n}{2}+\frac{\varepsilon_{Mn}+d^*}{\delta_n}, \frac{1}{2\delta_n}\right\}. \tag{96}$$

It also holds true from the definition of Q_n and (90), (3), (95) that

$$\begin{aligned}|Q_n(t_2)-Q_n(t_1)| &= \frac{1}{\gamma_n}\|\theta_n^*\|\,|\beta_n(t_2)-\beta_n(t_1)|\\ &= \frac{1}{\gamma_n}\|\theta_n^*\|\left|\int_{t_1}^{t_2}\dot{\beta}_n(s)ds\right| = \int_{t_1}^{t_2}\|\theta_n^*\|\|\Phi_n(\xi_n)\||\lfloor z_n\rceil_{\delta_n}|ds\\ &\leq \frac{1}{2\delta_n}\int_{t_1}^{t_2}(\|\theta_n^*\|^2+\|\Phi_n(\xi_n)\|^2)z_n\lfloor z_n\rceil_{\delta_n}ds\\ &\leq \frac{\max\{1,\|\theta_n^*\|\}}{2\delta_n}\int_{t_1}^{t_2}\Theta_n(s)ds \quad \forall t_2\geq t_1.\end{aligned} \tag{97}$$

From (89), (94), (97) conditions (10), (11) ,(12) of Lemma 1 hold true and therefore P_n is upper bounded.

5. BOUNDEDNESS AND TRACKING

Due to the integral terms in the virtual controls and the estimations β_i of the weight norms $\|\theta_i^*\|$ the closed-loop system state vector is of order $3n$. So if we define the augmented state vector $x_{ag} = [x^T, I^T, \beta^T]^T$ then the dynamics of the closed loop system can be written in the form $\dot{x}_{ag} = \bar{f}(x_{ag}, t)$ with $\bar{f}: R^{3n} \to R^{3n}$ a continuous w.r.t. x_{ag} and t vector field. Detailed calculations can verify the locally Lipschitz property with respect to x_{ag} of the vector field $\bar{f}$ in a neighborhood of $[x_0^T \quad 0]^T$. Thus, according to Theorem 3.1 of [45] a unique solution exists within some time interval $[0, t_f)$. From the analysis in section 4 we have proved that (9)-(12) hold true for all $P_i, \bar{P}_i, \Theta_i$ $(1\leq i\leq n)$ defined in (64), (66), (73). If we also choose $\varepsilon_i < 1$ and apply Lemma 1 it is proved that P_i, V_i, L_i, $\int_0^t V_i(s)ds$ are bounded in $[0,t_f)$ $\forall i = 1,\cdots,n$. The boundedness of $V_i, L_i, \int_0^t V_i(s)ds$ ensures that z_i, β_i, I_i are respectively bounded. Then, from $x_1 = z_1 + y_d$ the boundedness of x_1 is deduced. Inductively we can prove that all x_i are bounded. Hence, assuming $x_1,\cdots,x_i$ are bounded we will also prove that x_{i+1} is bounded. This follows directly since, from the boundedness of $\bar{x}_i$ assumption, $\alpha_i = A_i(\bar{x}_i, \bar{z}_i, \bar{I}_{i+1}, \bar{\beta}_i)$ is bounded which in turn yields the boundedness of $x_{i+1} = z_{i+1} + \alpha_i$. Thus, the augmented state vector x_{ag} remains bounded and the final time t_f can be extended to infinity: $t_f = +\infty$ (no finite explosion time).

DOI: 10.1002/acs

We have proved therefore that $\lceil z_i \rfloor_{\delta_i} \in \mathscr{L}_\infty \cap \mathscr{L}_2$ and $x_i, I_i, \beta_i, \alpha_i, u \in \mathscr{L}_\infty$. Then, from (47) we conclude that $\dot{z}_i \in \mathscr{L}_\infty$. Applying now Barbalat's lemma we have that $\lim_{t\to\infty} \lceil z_i(t) \rfloor_{\delta_i} = 0$ which in turn implies $\lim_{t\to\infty} |z_i(t)| \le \delta_i$ and finally $\lim_{t\to\infty} |y(t) - y_d(t)| = \lim_{t\to\infty} |z_1(t)| \le \delta_1$.

We note that the boundedness results of the proposed methodology are semiglobal (which is the standard norm for adaptive LPA control schemes) in the sense that the LPA approximation regions Ω_i should be chosen sufficiently large to include the region $\bar{\Omega}_i := \{\xi_i | \|\xi_i\| \le C_{\xi,i}\}$ wherein the trajectory of $\xi_i(t)$ remains. Previous analysis has shown that there exist some constants $p, C_{x,i}, C_{z,i} > 0$ such that $\sum_{i=1}^n P_i(t) \le p$, $\|\bar{x}_i(t)\| \le C_{x,i}$, $\|\bar{z}_i(t)\| \le C_{z,i}$ for all $t \ge 0$. Since

$$\sum_{i=1}^{n} \frac{1}{2\gamma_i} \beta_i(t)^2 + \sum_{i=1}^{n} \frac{(1-\varepsilon_i)\lambda_i}{2} I_i(t) \le \sum_{i=1}^{n} P_i(t) \le p \tag{98}$$

we have that $\beta_i(t) \le \sqrt{2\gamma_i p} := C_{\beta,i}$, $I_i(t) \le 2p/[(1-\varepsilon_i)\lambda_i] := C_{I,i}$ for all $t \ge 0$ and therefore $\|\xi(t)\| \le \sqrt{C_{x,i}^2 + C_{z,i}^2 + C_{\beta,i}^2 + C_{I,i}^2} := C_{\xi,i}$ for all $t \ge 0$. Thus, for $\Omega_i := \{\xi_i \in \mathbb{R}^{4i-1} | \|\xi_i\| \le \bar{C}_{\xi,i}\}$ one must select $\bar{C}_{\xi,i} \ge C_{\xi,i}$ for the results to be valid.

Thus, we have now proved the following Theorem which is the main result of the paper showing that the backstepping design described in the previous section ensures approximate output tracking and boundedness of all the closed loop signals.

Theorem 1

Consider the strict-feedback nonlinear system described by (5) and a reference trajectory y_d satisfying Assumptions 1-4. If we select the control input according to the equation (93) where the virtual control laws are given by the equations (71) ($\varepsilon_i < 1$), the estimator update laws (67), and the LPA approximation regions $\Omega_i := \{\xi_i \in \mathbb{R}^{4i-1} | \|\xi_i\| \le \bar{C}_{\xi,i}, \bar{C}_{\xi,i} \ge C_{\xi,i}\}$ then all the closed-loop signals are bounded and $\lim_{t\to\infty} |y(t) - y_d(t)| \le \delta_1$.

Remark 3

The proposed control scheme avoids the "explosion of complexity" problem that occurs in standard backstepping control since there is no need to calculate the time derivative of the virtual control laws or it's partial derivatives w.r.t. the state variables [52]. This is achieved due to Lemma 2 where the bounds $|\dot{\alpha}_{i-1}| \le H_{i-1}(\bar{x}_i, \bar{z}_i, \bar{I}_i, \bar{\beta}_{i-1})$ are obtained and the use of LPAs to estimate functions of those bounds (see (61)). This constitutes an alternative more direct approach than dynamic surface control in which the virtual control laws typically pass through first order filters [52].

Remark 4

With respect to the controller parameter selection the following observations can be made after running several simulations with different parameter values. Design parameter δ_1 is predetermined from the desired tracking accuracy of the problem under study. Parameters δ_i $(i \ge 2)$ on the other hand, should be typically chosen bigger in order to avoid large input transients but not extremely big as this would increase significantly the convergence time. Similarly, increasing the adaptation gains γ_i and/or parameters λ_i appears to result in a faster convergence rate at the expense of larger input transients.

DOI: 10.1002/acs

6. SIMULATION STUDY

To verify our theoretical analysis we consider the following second-order nonlinear system

$$
\begin{aligned}
\dot{x_1} &= x_1\cos(x_1) + (1+\sin(x_1)^2)x_2\,, \quad y = x_1 \\
\dot{x_2} &= x_1x_2 - (2\sqrt{x_1^2+x_2^2}+0.5)h(u,t)
\end{aligned} \tag{99}
$$

with a nonsymmetric deadzone nonlinearity

$$
h(u,t) = \begin{cases} (2+\sin t)(1-0.3\sin u)(u-2.5)\,, & \text{if } u > 2.5 \\ 0\,, & \text{if } -1.5 \le u \le 2.5 \\ (2+\sin t)(0.8-0.2\cos u)(u+1.5)\,, & \text{if } u < -1.5 \end{cases}
$$

Our objective is the output y to follow the reference signal $y_d(t) = \sin t$. The applied control to the system is designed according to the procedure described in the previous section. Initially the virtual control a_1 is calculated by

$$
\begin{aligned}
\alpha_1 = \kappa_1 \Bigg[& \frac{1}{2}\lfloor z_1\rceil_{\delta_1}^2 + \frac{1}{2\gamma_1}\beta_1^2 - \frac{\varepsilon_2\lambda_2}{2}\int_0^t \lfloor x_2(s)-\alpha_1(s)\rceil_{\delta_2}^2 ds \\
& + \frac{\lambda_1}{2}\int_0^t \lfloor z_1(s)\rceil_{\delta_1}^2 ds\Bigg](1+\|\Phi_1(x_1,z_1)\|^2+\beta_1^2)z_1
\end{aligned} \tag{100}
$$

with $z_1 = x_1 - y_d$ and β_1 update law

$$
\dot{\beta}_1 = \gamma_1\|\Phi_1(x_1,z_1)\|\,|\lfloor z_1\rceil_{\delta_1}|.
$$

Then, the applied control input u is given by

$$
u = \kappa_2\left[\frac{1}{2}\lfloor z_2\rceil_{\delta_2}^2 + \frac{1}{2\gamma_2}\beta_2^2 + \frac{\lambda_2}{2}\int_0^t \lfloor z_2(s)\rceil_{\delta_2}^2 ds\right](1+\|\Phi_2(x_1,x_2,z_1,z_2,I_1,I_2,\beta_1)\|^2+\beta_2^2)z_2
$$

with $z_2 = x_2 - \alpha_1$, $I_i = \int_0^t \lfloor z_i(s)\rceil_{\delta_i}^2 ds$ $(i = 1,2)$ and β_2 update law

$$
\dot{\beta}_2 = \gamma_2\|\Phi_2(x_1,x_2,z_1,z_2,I_1,I_2,\beta_1)\|\,|\lfloor z_2\rceil_{\delta_2}|.
$$

In our simulation scenario we considered initial conditions $x_1(0) = x_2(0) = 1$ and selected controller parameters $\delta_1 = 0.06$, $\delta_2 = 1$, $\lambda_1 = \lambda_2 = 0.1$, $\gamma_1 = \gamma_2 = 1/2$, $\varepsilon_2 = 0.25$ and functions $\kappa_1(x) = \kappa_2(x) = x^2\cos x$. Two second-order NNs with 6 and 36 neurons respectively were used to implement the regressor vectors $\Phi_1(x_1,z_1), \Phi_2(\xi_2)$ with activation functions $\tanh(\cdot)$. These include terms that are products up to second order of the activated network inputs e.g. $\Phi_1(x_1,z_1) = [1,\tanh(x_1),\tanh(z_1),\tanh^2(x_1),\tanh^2(z_1),\tanh(x_1)\tanh(z_1)]^T$.

For comparison, we have included simulations of the Nussbaum controller described in eq. (67)-(69) of [52] with parameters $k_1 = 3$, $k_2 = 20$, $a_1 = 1$, $a_2 = 0.1$, $\gamma_1 = 1$, $\gamma_2 = 5$ and $\sigma_1 = \sigma_2 = 10^{-3}$.

Simulation results for the proposed adaptive nonlinear PI (ANPI) and the Nussbaum controller of [52] are shown in Figures 1-3. As expected, for both controllers, the output y approximately tracks the reference signal y_d with bounded control inputs u. For the particular selection of control

Int. J. Adapt. Control Signal Process. (2010)
DOI: 10.1002/acs

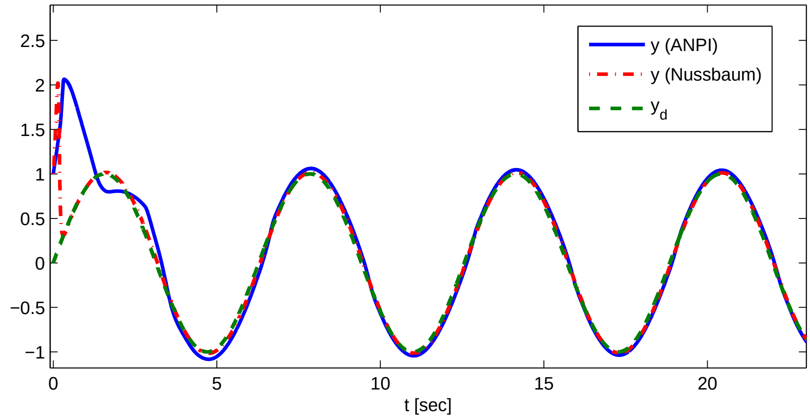

Figure 1. Time responses of the output signal y for the ANPI controller and the Nussbaum controller of [52].

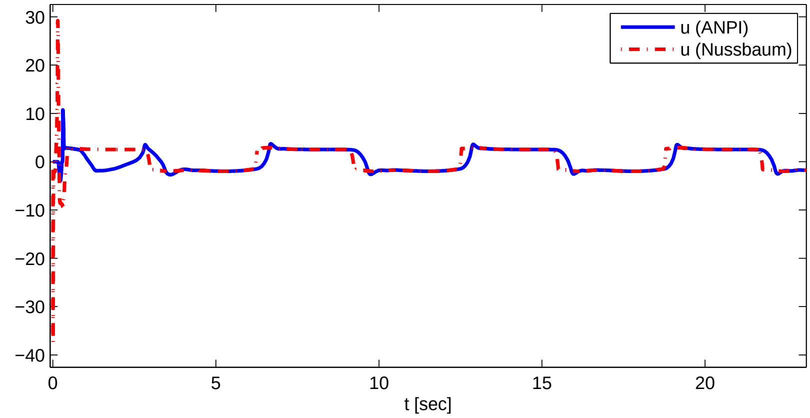

Figure 2. Time response of the control input u for the ANPI controller and the Nussbaum controller of [52].

parameters, the Nussbaum controller yields faster convergence at the expense of higher input transients. Also,both parameter estimates β_1, β_2 remain bounded.

We note that implementing the controller of [52] involves several extra calculations since the partial derivatives $\frac{\partial\alpha_1}{\partial x_1}, \frac{\partial\alpha_1}{\partial x_d}, \frac{\partial\alpha_1}{\partial\zeta_1}, \frac{\partial\alpha_1}{\partial\hat{\theta}_1}$ are needed in this case. For larger system dimensions obtaining those partial derivatives $\frac{\partial\alpha_i}{\partial x_j}$ $(1 \le j \le i)$ becomes increasingly tedious making the implementation extremely difficult. As explained in Remark 3 this problem is avoided with our design.

7. CONCLUSIONS

For the first time, an adaptive control extension of the nonlinear PI methodology has been developed in this work. Specifically, we proposed an adaptive nonlinear PI backstepping LPA controller for strict-feedback SISO nonlinear systems with non-smooth actuator nonlinearities. New theoretical results have been obtained that made possible the analysis of the combined backstepping nonlinear PI method with the adaptive control technique. Using these results we proved that the proposed

DOI: 10.1002/acs

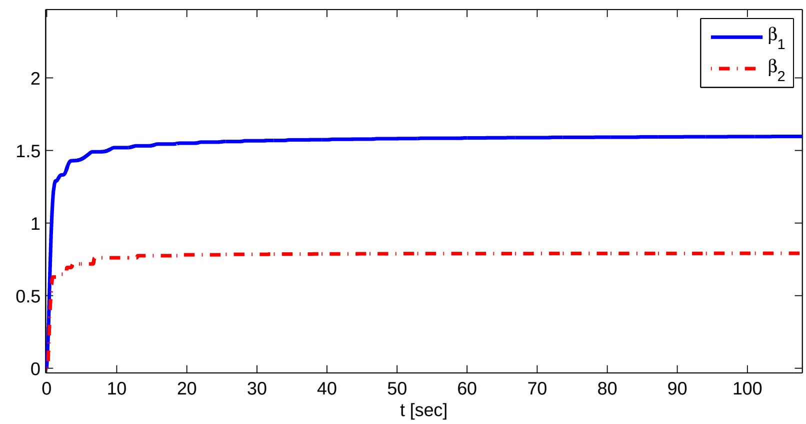

Figure 3. Time response of the adaptation variables β_1, β_2.

controller achieves approximate output tracking and boundedness of all signals in the closed-loop. Future work may consider extensions to MIMO systems and output feedback control.

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DOI: 10.1002/acs