Integrator backstepping with the nonlinear PI method: An integral equation approach

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Abstract

In this paper we extend the nonlinear PI method within an integrator backstepping framework. The main idea of our approach is a novel selection of the virtual control laws through suitable nonlinear integral equations. Using the proposed methodology a new robust regulation control scheme is developed for time-varying strict feedback nonlinear systems with unknown control directions. A simulation study demonstrates the validity of our theoretical results.

Keywords: Backstepping, nonlinear PI, asymptotic regulation.

1. Introduction

Integrator backstepping has been the main approach to control design for nonlinear systems in strict-feedback form [1], [2], [3]. The basic idea of backstepping is to select recursively suitable functions of the state variables as virtual controls for lower dimension subsystems of the overall system. In each step a new virtual control is obtained that is dependent from the designs of the preceding stages. At the final step, a feedback law for the actual control input is obtained, that achieves the original design objective.

For systems with unknown control directions a class of nonlinear gains called Nussbaum gains have been proposed in [4]. Nussbaum gains have been successfully combined with the backstepping methodology in [5] to solve the adaptive regulation problem for strict-feedback nonlinear systems with unknown control directions. In [6], Nussbaum gains have also been employed for asymptotic robust regulation of time-varying strict feedback nonlinear systems. Since then, numerous applications of Nussbaum gains to different nonlinear system classes have been proposed (see for example [7]-[14]).

Georgiou and Smith [15] revealed some limitations of the Nussbaum methodology. Particularly, they showed lack of robustness with respect to external disturbances and unmodelled dynamics. The so called λ-tracking approach [16], [17] handles effectively the robustness problem with respect to bounded disturbances. However, no solution has been found for the problem of robustness to unmodelled dynamics.

In [18], Ortega, Astolfi and Barabanov introduced the nonlinear PI control method that provides an alternative solution to the unknown control direction design problem (section 6 of [18]). Their suggestion is a control law of the form $u = z\cos(z)x$ where the argument z of the control gain $z\cos(z)$ is some form of PI square error e.g. $z = (1/2)x^2 + \lambda \int_0^t x^2(s)ds$ [19]. In their paper they showed improved robustness to unmodelled dynamics compared to Nussbaum gain based controllers. This issue was further researched in [20], [21] wherein sufficient conditions for robustness to parasitic dynamics have been proved for some classes of nonlinear systems. It is therefore important to extend the nonlinear PI method to more general classes of nonlinear systems.

To the best of the author's knowledge, *the nonlinear PI approach has not been extended before within an integrator backstepping framework* due to important technical difficulties. In this paper, we provide a solution for this problem and design a control scheme for asymptotic robust regulation of time varying strict feedback nonlinear systems with unknown control directions. This is by no means a straightforward task and is achieved via a novel selection of the virtual control laws through suitable integral equations.

The paper is organized as follows. In Section 2 the asymptotic robust regulation problem for time varying strict feedback nonlinear systems with unknown control directions is revisited. Our main technical lemma is introduced in Section 3. Then, a backstepping design is presented in Section 4 that enables the application of our technical lemma. The main result of the paper is stated in Section 5 that proves asymptotic regulation and boundedness for the closed-loop system. Simulations demonstrating our theoretical results are carried out in Section 6. We close with some concluding remarks in Section 7.

Notations. Throughout this paper we adapt the following notations:

- $\mathcal{L}_\infty$ denotes the class of all bounded continuous functions.
- $\mathcal{L}_2$ denotes the class of all square integrable continuous functions.

- $\mathcal{K}_\infty$ denotes the class of all strictly increasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\lim_{x\to+\infty} \varphi(x) = +\infty$.
- δ_{ij} is the Kronecker delta function with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.
- $\lceil x \rceil$ denotes the smallest integer larger than or equal to x.

2. Problem Formulation

Consider the asymptotic regulation problem of the strict feedback nonlinear system of the form

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i, t) + g_i(t)x_{i+1}\,(1 \leq i \leq n-1) \\ \dot{x}_n &= f_n(\bar{x}_n, t) + g_n(t)u \end{aligned} \tag{1}$$

with $x := [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ the state vector, $u \in \mathbb{R}$ the control input and $\bar{x}_i := [x_1, \ldots, x_i]^T$. The following assumptions are considered.

Assumption 1. *[6] The origin is an equilibrium point of* (1) *and there exist functions $f_{ij}(\bar{x}_i, t)$ such that the unknown nonlinearities f_i can be written as $f_i(\bar{x}_i, t) = \sum_{j=1}^{i} x_j f_{ij}(\bar{x}_i, t)$. For the functions $f_{ij}(\bar{x}_i, t)$ we assume that there exist some unknown constants $\gamma_{ij} \geq 0$ and known continuous functions $\rho_{ij}(\cdot)$ $(1 \leq i \leq n; 1 \leq j \leq i)$ such that $f_{ij}^2(\bar{x}_i, t) \leq \gamma_{ij}\rho_{ij}(\bar{x}_i)$ for all $\bar{x}_i \in \mathbb{R}^i$ and $t \in [0, \infty)$ $(1 \leq i \leq n; 1 \leq j \leq i)$.*

Assumption 2. *Functions $g_i(\cdot)$ are continuous and there exist unknown positive constants $g_{i1}, g_{i2} > 0$ such that $0 < g_{i1} \leq |g_i(t)| \leq g_{i2}$ for all $t \in [0, \infty)$. Thus, the unknown functions $g_i(\cdot)$ (control directions) have constant but unknown sign.*

The control objective is to select u so that all signals remain bounded and $\lim_{t\to\infty} x_i(t) = 0$ for all $i = 1, 2, \cdots, n$.

Remark 1. *The motivation for our work is not to solve the specific problem per se but rather to extend the nonlinear PI method within an integrator backstepping framework, i.e. to construct a nonlinear PI backstepping methodology. Besides, an alternative solution to the particular control problem has been given in [6] using Nussbaum gains.*

3. Main Technical Lemma

Our results are based on the following Lemma.

Lemma 1. *Consider n continuous differentiable nonnegative functions $V_i : [0, t_f) \to \mathbb{R}_+$ $(1 \le i \le n)$ and the functions $P_i : [0, t_f) \to \mathbb{R}$ defined by*

$$P_i(t) := V_i(t) + \lambda_i \int_0^t V_i(s)ds - \sum_{\substack{j=1 \\ j \neq i}}^{i+1} \epsilon_j \lambda_j \int_0^t V_j(s)ds \qquad (1 \le i \le n) \tag{2}$$

with $\epsilon_i, \lambda_i > 0$ and the notation $V_{n+1} \equiv 0$ to ensure uniformity. Let also n continuous functions $g_i : [0, \infty) \to \mathbb{R}$ for which there exist positive constants $g_{i1}, g_{i2} > 0$ such that $0 < g_{i1} \le |g_i(t)| \le g_{i2}$ for all $t \in [0, \infty)$. If there exist nonnegative functions $\Theta_i : [0, t_f) \to \mathbb{R}_+$ such that

$$\frac{dP_i}{dt} \le [\eta_i^* + g_i(t)\kappa_i(P_i)]\Theta_i(t) \tag{3}$$

with constants $\eta_i^* \ge 0$, $\kappa_i(P_i) := \varphi_i(P_i^2)\cos(P_i)$ and $\varphi_i(\cdot)$ a class $\mathcal{K}_\infty$ function then all P_i are upper bounded on $[0, t_f)$.

Moreover, if $\epsilon_i(n + 1 - i - \delta_{i1}) < 1$ with δ_{ij} the Kronecker delta function then all V_i, $\int_0^t V_i(s)ds$ are bounded on $[0, t_f)$.

Proof. Inequality (3) ensures the *upper boundedness* of P_i by $p_{mi} := 2k_i^*\pi + (\pi/2)(1 + \text{sgn}(g_i))$ with $k_i^* = \max\{\lceil (1/2\pi)P_i(0) \rceil, \lceil (1/2\pi)\sqrt{\varphi_i^{-1}(\eta_i^*/g_{i1})} \rceil\}$. This follows directly from $P_i(0) \le p_{mi}$ and the fact that, whenever $P_i(t) = p_{mi}$, we have

$$\frac{dP_i}{dt} \le -\Big\{|g_i|\varphi_i\Big[(2k_i^*\pi)^2\Big] - \eta_i^*\Big\}\Theta_i \le 0 \tag{4}$$

i.e. $P_i(t)$ is nonincreasing whenever $P_i(t) = p_{mi}$ and therefore P_i cannot grow larger than p_{mi} . We emphasize that by definition (2), P_i is not necessarily nonnegative and therefore the existence of a lower bound of P_i cannot be directly deduced.

If all P_i are upper bounded then their sum is also upper bounded i.e. there exists some constant $\beta > 0$ such that

$$\sum_{i=1}^{n} P_i(t) \le \beta. \tag{5}$$

The sum of all P_i can be written equivalently from (2) as

$$\sum_{i=1}^{n} P_i(t) = \sum_{i=1}^{n} V_i(t) + \sum_{i=1}^{n} \lambda_i \int_0^t V_i(s)ds - \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{i+1} \epsilon_j \lambda_j \int_0^t V_j(s)ds$$

$$= \sum_{i=1}^{n} V_i(t) + \sum_{i=1}^{n} \left[1 - (n+1-i-\delta_{i1})\epsilon_i\right] \lambda_i \int_0^t V_i(s)ds \qquad (6)$$

Hence, if a sufficiently small ϵ_i is chosen such that $\epsilon_i = \nu_i/(n+1-i-\delta_{i1})$ with $0 < \nu_i < 1$ then from (5), (6) we have

$$0 \leq \sum_{i=1}^{n} V_i(t) + \sum_{i=1}^{n} (1-\nu_i)\lambda_i \int_0^t V_i(s)ds \leq \beta \qquad (7)$$

i.e. V_i, $\int_0^t V_i(s)ds$ are bounded on $[0, t_f)$. □

In the next section we propose a backstepping design that specifically aims to achieve (3) in every step of the algorithm.

4. Nonlinear PI Backstepping Design

4.1. Step $i = 1$:

In a standard backstepping manner we define the first error variable $z_1 := x_1$, the virtual control law α_1 to be designed and the error variable $z_2 := x_2 - \alpha_1$. Then, from (1) and Assumption 1 we have

$$\dot{z}_1 = f_{11}(x_1, t)z_1 + g_1(t)\alpha_1 + g_1(t)z_2. \qquad (8)$$

In accordance with the nonlinear PI approach [18], [19] we also define the nonnegative PI function

$$V_{01} := (1/2)z_1^2 + \lambda_1 \int_0^t z_1^2(s)ds \qquad (9)$$

with $\lambda_1 > 0$. The derivative of V_{01} takes the form

$$\dot{V}_{01} = z_1^2 f_{11}(x_1, t) + \lambda_1 z_1^2 + g_1(t)z_1\alpha_1 + g_1(t)z_1 z_2. \qquad (10)$$

Using the inequality

$$g_1(t)z_1 z_2 \leq (1/4\epsilon_2\lambda_2)g_1^2(t)z_1^2 + \epsilon_2\lambda_2 z_2^2 \qquad (11)$$

for some $\epsilon_2, \lambda_2 > 0$ and Assumptions 1-2 we have

$$\dot{V}_{01} \leq z_1^2[\gamma_{11}\rho_{11}(x_1) + \lambda_1 + 1/4 + (1/4\epsilon_2\lambda_2)g_{12}^2] + g_1(t)\alpha_1 z_1 + \epsilon_2\lambda_2 z_2^2. \quad (12)$$

Typically the PI function V_{01} is used as an argument in the trigonometric functions of the virtual control law (see [18], [19]). Instead, we define

$$P_1 := V_{01} - \epsilon_2\lambda_2 \int_0^t z_2^2(s)ds \quad (13)$$

and select the virtual control law

$$\alpha_1 := \alpha_{g1} x_1 \quad (14)$$

with gain function

$$\alpha_{g1} := \kappa_1(P_1)(1 + \rho_{11}(x_1)) \quad (15)$$

to obtain from (12)-(15)

$$\dot{P}_1 \leq [\eta_1^* + g_1(t)\kappa_1(P_1)](1 + \rho_{11}(x_1))x_1^2 \quad (16)$$

with $\eta_1^* := \max\{\gamma_{11}, \lambda_1 + 1/4 + g_{12}^2/(4\epsilon_2\lambda_2)\}$, $\kappa_1(P_1) := \varphi_1(P_1^2)\cos(P_1)$ and $\varphi_1(\cdot)$ a continuous differentiable class $\mathcal{K}_\infty$ function. The above inequality is in the form of (3) with P_1 similar to (2).

If we closely look at the virtual control law (14), (15) we observe that α_1 is actually given by an integral equation of the form

$$\alpha_1(t, x_1) = \kappa_1\left[V_{01}(t, x_1) - \epsilon_2\lambda_2 \int_0^t [x_2(s) - \alpha_1(s, x_1(s))]^2 ds\right](1 + \rho_{11}(x_1))x_1. \quad (17)$$

Let us define now $\zeta_1 := \lambda_1 \int_0^t z_1^2(s)ds - \epsilon_2\lambda_2 \int_0^t z_2^2(s)ds$. Then, from (17) the ζ_1 dynamics are

$$\dot{\zeta}_1 = \lambda_1 x_1^2 - \epsilon_2\lambda_2\left[x_2 - \kappa_1\left(\frac{x_1^2}{2} + \zeta_1\right)(1 + \rho_{11}(x_1))x_1\right]^2 \quad (18)$$

and α_1 is

$$\alpha_1 = \kappa_1\left(\frac{x_1^2}{2} + \zeta_1\right)(1 + \rho_{11}(x_1))x_1. \quad (19)$$

From (18), (19) one can see that the virtual control α_1 is the output of a first order nonlinear dynamic system (dynamic virtual control law) with state ζ_1 and input x_1, x_2.

4.2. Step i ($2 \leq i \leq n-1$)*:*

For the i-th step we define the error variable $z_{i+1} := x_{i+1} - \alpha_i = x_{i+1} - \alpha_{gi} z_i$ with α_i the i-th virtual control.

Lemma 2. *Let the nonlinear system* (1)*. If Assumption 1 holds true then for the error variables* $z_1 = x_1$*,* $z_i = x_i - \alpha_{g,i-1}(\bar{x}_{i-1}, t) z_{i-1}$*,* $z_{n+1} \equiv 0$ *and the control input* $u = \alpha_{gn}(\bar{x}_n, t) z_n$ *we have that*

$$\dot{z}_i = \sum_{k=1}^{i} M_{ik}(\bar{x}_i, t) z_k + g_i z_{i+1} \; (1 \leq i \leq n) \tag{20}$$

with

$$M_{ii}(\bar{x}_i, t) := f_{ii}(\bar{x}_i, t) + g_i(t)\alpha_{gi}(\bar{x}_i, t) - g_{i-1}(t)\alpha_{g,i-1}(\bar{x}_{i-1}, t) \tag{21}$$

$$\begin{aligned} M_{i,i-1}(\bar{x}_i, t) :=& f_{i,i-1}(\bar{x}_i, t) + \alpha_{g,i-1}(\bar{x}_{i-1}, t)[f_{i,i}(\bar{x}_i, t) - M_{i-1,i-1}(\bar{x}_{i-1}, t)] \\ & - \sum_{j=1}^{i-1} \frac{\partial \alpha_{g,i-1}}{\partial x_j}(f_j(\bar{x}_j, t) + g_j(t) x_{j+1}) - \frac{\partial \alpha_{g,i-1}}{\partial t} \end{aligned} \tag{22}$$

$$\begin{aligned} M_{i,j}(\bar{x}_i, t) :=& f_{i,j}(\bar{x}_i, t) + \alpha_{gj}(\bar{x}_j, t) f_{i,j+1}(\bar{x}_i, t) \\ & - \alpha_{g,i-1}(\bar{x}_{i-1}, t) M_{i-1,j}(\bar{x}_{i-1}, t) \qquad (1 \leq j \leq i-2). \end{aligned} \tag{23}$$

Furthermore, there exist unknown constants $c_{ij} \geq 0$ *and* $\bar{M}_{ij}(\bar{x}_i, t) \geq 0$ *known continuous functions such that*

$$M_{ij}^2(\bar{x}_i, t) \leq c_{ij} \bar{M}_{ij}(\bar{x}_i, t) \quad (1 \leq i \leq n; 1 \leq j \leq i) \tag{24}$$

with

$$\bar{M}_{i,i}(\bar{x}_i, t) := \rho_{i,i}(\bar{x}_i) + \alpha_{g,i}^2(\bar{x}_i, t) + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) \tag{25}$$

$$\begin{aligned} \bar{M}_{i,i-1}(\bar{x}_i, t) :=& \rho_{i,i-1}(\bar{x}_i) + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t)[\rho_{i,i}(\bar{x}_i) + \bar{M}_{i-1,i-1}(\bar{x}_{i-1}, t)] \\ & + \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{g,i-1}}{\partial x_j}\right)^2 \left(\sum_{i_1=1}^{j} x_{i_1}^2 \sum_{i_2=1}^{j} \rho_{j,i_2}(\bar{x}_j) + x_{j+1}^2 \right) + \left(\frac{\partial \alpha_{g,i-1}}{\partial t}\right)^2 \end{aligned} \tag{26}$$

$$\begin{aligned} \bar{M}_{i,j}(\bar{x}_i, t) :=& \rho_{i,j}(\bar{x}_i) + \alpha_{gj}^2(\bar{x}_j, t)\rho_{i,j+1}(\bar{x}_i) \\ & + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t)\bar{M}_{i-1,j}(\bar{x}_{i-1}, t) \qquad (1 \leq j \leq i-2). \end{aligned} \tag{27}$$

Proof. The proof is given in the appendix. □

Taking (20), (21) into account the time derivative of the nonnegative function $V_{0i} := (1/2)z_i^2 + \lambda_i \int_0^t z_i^2(s)ds$ can be written as

$$\dot{V}_{0i} = \big[f_{ii}(\bar{x}_i, t) - g_{i-1}(t)\alpha_{g,i-1}(\bar{x}_{i-1}, t) + \lambda_i + g_i(t)\alpha_{gi}(\bar{x}_i, t)\big] z_i^2 + \sum_{j=1}^{i-1} M_{ij}(\bar{x}_i, t) z_i z_j + g_i(t) z_i z_{i+1} \tag{28}$$

with $\lambda_i > 0$. Completing the square, the last two terms in the right hand side of (28) are bounded by

$$\sum_{j=1}^{i-1} M_{ij}(\bar{x}_i, t) z_i z_j \leq \sum_{j=1}^{i-1} \epsilon_j \lambda_j z_j^2 + \sum_{j=1}^{i-1} \frac{1}{4\epsilon_j \lambda_j} M_{ij}^2(\bar{x}_i, t) z_i^2 \tag{29}$$

$$g_i(t) z_i z_{i+1} \leq \epsilon_{i+1} \lambda_{i+1} z_{i+1}^2 + \frac{g_i^2(t)}{4\epsilon_{i+1}\lambda_{i+1}} z_i^2 \tag{30}$$

with $\epsilon_j, \lambda_j > 0$ for all $j = 1, 2, \cdots, n$. Similarly,

$$f_{ii}(\bar{x}_i, t) \leq f_{ii}^2(\bar{x}_i, t) + 1/4 \tag{31}$$

$$-g_{i-1}(t)\alpha_{g,i-1}(\bar{x}_{i-1}, t) \leq \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + g_{i-1}^2(t)/4. \tag{32}$$

Using (29)-(32) in (28) we result in

$$\dot{V}_{0i} \leq \Bigg[g_i(t)\alpha_{gi}(\bar{x}_i, t) + \eta_i + f_{ii}^2(\bar{x}_i, t) + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + \sum_{j=1}^{i-1} \frac{M_{ij}^2(\bar{x}_i, t)}{4\epsilon_j \lambda_j} \Bigg] z_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{i+1} \epsilon_j \lambda_j z_j^2 \tag{33}$$

with $\eta_i := \lambda_i + (1 + g_{i-1,2}^2)/4 + g_{i2}^2/(4\epsilon_{i+1}\lambda_{i+1})$. If we now define

$$P_i := V_{0i} - \sum_{\substack{j=1 \\ j\neq i}}^{i+1} \epsilon_j \lambda_j \int_0^t z_j^2(s) ds \tag{34}$$

and select a virtual control law α_i

$$\alpha_i := \alpha_{gi} z_i \tag{35}$$

with gain

$$\alpha_{gi} := \kappa_i(P_i)\left[1 + \rho_{ii}(\bar{x}_i) + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + \sum_{j=1}^{i-1} \frac{\bar{M}_{ij}(\bar{x}_i, t)}{4\epsilon_j \lambda_j}\right] \tag{36}$$

then (24), (33)-(36) yield

$$\frac{dP_i}{dt} \leq [\eta_i^* + g_i(t)\kappa_i(P_i)]\left[1 + \rho_{ii}(\bar{x}_i) + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + \sum_{j=1}^{i-1} \frac{\bar{M}_{ij}(\bar{x}_i, t)}{4\epsilon_j \lambda_j}\right] z_i^2 \tag{37}$$

where $\eta_i^* := \max\{1, \eta_i, \gamma_{ii}, c_{i1}, \cdots, c_{i,i-1}\}$, $\kappa_i(P_i) := \varphi_i(P_i^2)\cos(P_i)$ and $\varphi_i(\cdot)$ a continuous differentiable class $\mathcal{K}_\infty$ function. Note that α_i defined by (35),(36) satisfies the following integral equation

$$\begin{aligned}\alpha_i =& \kappa_i\left[V_{0i} - \sum_{j=1}^{i-1} \epsilon_j \lambda_j \int_0^t z_j^2(s)ds - \epsilon_{i+1}\lambda_{i+1} \int_0^t (x_{i+1}(s) - \alpha_i(s))^2 ds\right]\left[\rho_{ii}(\bar{x}_i)\right.\\ &\left. + 1 + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + \sum_{j=1}^{i-1} \frac{\bar{M}_{ij}(\bar{x}_i, t)}{4\epsilon_j \lambda_j}\right] z_i,\end{aligned} \tag{38}$$

(37) is in the form of (3) and P_i of (34) is in the form of (2) as desired.

Let us define now

$$\zeta_i := \lambda_i \int_0^t z_i^2(s)ds - \sum_{\substack{j=1\\ j\neq i}}^{i+1} \epsilon_j \lambda_j \int_0^t z_j^2(s)ds. \tag{39}$$

Then, from (35) the ζ_i dynamics are

$$\begin{aligned}\dot{\zeta}_i =& \lambda_i z_i^2 - \sum_{j=1}^{i-1} \epsilon_j \lambda_j z_j^2 - \epsilon_{i+1}\lambda_{i+1}\left\{x_{i+1} - \kappa_i\left(\frac{z_i^2}{2} + \zeta_i\right)\left[\rho_{ii}(\bar{x}_i)\right.\right.\\ &\left.\left. + 1 + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + \sum_{j=1}^{i-1} \frac{\bar{M}_{ij}(\bar{x}_i, t)}{4\epsilon_j \lambda_j}\right] z_i\right\}^2 \ (1 \leq i \leq n-1)\end{aligned} \tag{40}$$

and α_i is

$$\alpha_i = \kappa_i\left(\frac{z_i^2}{2} + \zeta_i\right)\left[\rho_{ii}(\bar{x}_i) + 1 + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + \sum_{j=1}^{i-1} \frac{\bar{M}_{ij}(\bar{x}_i, t)}{4\epsilon_j \lambda_j}\right] z_i. \tag{41}$$

From (40), (41) one can see that the virtual control α_i is the output of a first order nonlinear dynamic system (dynamic virtual control law) with state ζ_i and input $\bar{x}_{i+1}, \bar{\zeta}_{i-1}$.

4.3. Step $i = n$:

If a control law of the form $u = \alpha_{gn}(\bar{x}_n, t) z_n$ is selected with $\alpha_{gn}(\bar{x}_n, t)$ to be defined then from (20), (21) the derivative of the nonnegative function $V_{0n} := (1/2) z_n^2 + \lambda_n \int_0^t z_n^2(s) ds$ takes the form

$$\dot{V}_{0n} = \left[f_{nn}(\bar{x}_n, t) - g_{n-1}(t) \alpha_{g,n-1}(\bar{x}_{n-1}, t) + \lambda_n + g_n(t) \alpha_{gn}(\bar{x}_n, t) \right] z_n^2 + \sum_{j=1}^{n-1} M_{nj}(\bar{x}_n, t) z_n z_j. \tag{42}$$

Utilizing now inequalities (29), (31), (32) for $i = n$ we have

$$\dot{V}_{0n} \leq \Bigg[g_n(t) \alpha_{gn}(\bar{x}_n, t) + \eta_n + f_{nn}^2(\bar{x}_n, t) + \alpha_{g,n-1}^2(\bar{x}_{n-1}, t) + \sum_{j=1}^{n-1} \frac{M_{nj}^2(\bar{x}_n, t)}{4 \epsilon_j \lambda_j} \Bigg] z_n^2 + \sum_{j=1}^{n-1} \epsilon_j \lambda_j z_j^2 \tag{43}$$

with $\eta_n := \lambda_n + (1 + g_{n-1,2}^2)/4$. If we now define

$$P_n := V_{0n} - \sum_{j=1}^{n-1} \epsilon_j \lambda_j \int_0^t z_j^2(s) ds \tag{44}$$

and select the control input

$$u := \alpha_{gn} z_n \tag{45}$$

with gain

$$\alpha_{gn} := \kappa_n(P_n) \left[1 + \rho_{nn}(\bar{x}_n) + \alpha_{g,n-1}^2(\bar{x}_{n-1}, t) + \sum_{j=1}^{n-1} \frac{\bar{M}_{nj}(\bar{x}_n, t)}{4 \epsilon_j \lambda_j} \right] \tag{46}$$

then (24), (43)-(46) yield

$$\dot{P}_n \leq [\eta_n^* + g_n(t) \kappa_n(P_n)] \left[1 + \rho_{nn}(\bar{x}_n) + \alpha_{g,n-1}^2(\bar{x}_{n-1}, t) + \sum_{j=1}^{n-1} \frac{\bar{M}_{nj}(\bar{x}_n, t)}{4 \epsilon_j \lambda_j} \right] z_n^2 \tag{47}$$

where $\eta_n^* := \max\{1, \eta_n, \gamma_{nn}, c_{n1}, \cdots, c_{n,n-1}\}$. Inequality (47) is in the form of (3) and P_n of (44) is in the form of (2) as desired.

Let us define now

$$\zeta_n := \lambda_n \int_0^t z_n^2(s)ds - \sum_{j=1}^{n-1} \epsilon_j \lambda_j \int_0^t z_j^2(s)ds \tag{48}$$

with dynamics

$$\dot{\zeta}_n = \lambda_n z_n^2 - \sum_{j=1}^{n-1} \epsilon_j \lambda_j z_j^2. \tag{49}$$

Then, the control input u can be written as

$$u = \kappa_n \left(\frac{z_n^2}{2} + \zeta_n \right) \left[1 + \rho_{nn}(\bar{x}_n) + \alpha_{g,n-1}^2(\bar{x}_{n-1}, t) + \sum_{j=1}^{n-1} \frac{\bar{M}_{nj}(\bar{x}_n, t)}{4\epsilon_j \lambda_j} \right] z_n. \tag{50}$$

From (49), (50) one can see that the control input u is the output of a first order nonlinear dynamic system (dynamic control law) with state ζ_n and input $\bar{x}_n, \bar{\zeta}_{n-1}$.

5. Boundedness and Regulation

The analysis in the previous section showed that if the virtual control laws and the control input are chosen by (35), (36), (45), (46) then (16), (37), (47) hold true. From these inequalities and the definitions of P_i regulation and boundedness results can be proved based on Lemma 1. Thus,we are ready to prove the main result of the paper.

Theorem 1. *Consider the strict-feedback time-varying nonlinear system described by* (1) *and satisfying Assumptions 1-2. If the control input* (45), (46) *is selected with virtual control laws given by the integral equations* (17), (38), $\bar{M}_{ij}$ *calculated recursively from* (25)-(27) *and* $\epsilon_i(n+1-i-\delta_{i1}) < 1$ *then all the closed-loop signals are bounded and* $\lim_{t\to\infty} x_i(t) = 0$ *for all* $i = 1, 2, \cdots, n$.

Proof. Due to the integral terms in the virtual controls the closed-loop system state is of order $2n$. Specifically, the augmented state vector is $x_{ag} := [x^T \quad \zeta^T]$ with $\zeta := [\zeta_1, \ldots, \zeta_n]^T$ and ζ_i is the integral term in the argument of $\kappa(\cdot)$ for each α_i given by (39),(48) with dynamics described by (18), (40), (49). Considering now the definitions of α_i, u given by (19), (41), (50) we can prove that $\alpha_i = \alpha_i(\bar{x}_i, \bar{\zeta}_i)$ $(1 \leq i \leq n-1)$ and $u = u(\bar{x}_n, \bar{\zeta}_n)$. Thus, the closed-loop system dynamics can be written in the form $\dot{x}_{ag} = \tilde{f}(t, x_{ag})$ with $\tilde{f} : [0, \infty) \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ some continuous vector field. Detailed calculations can verify the locally Lipschitz property with respect to x_{ag} of the vector field $\tilde{f}$ in a neighborhood of $[x_0^T \quad 0]^T$. Thus, according to Theorem 3.1 of [22] a unique solution exists within some time interval $[0, t_f)$.

From the analysis in Section 4 inequalities (16), (37), (47) hold true for P_i $(1 \leq i \leq n)$ defined by (13), (34), (44) and nonnegative functions $\Theta_i(\cdot)$ defined by

$$\Theta_i(t) := \left[1 + \rho_{ii}(\bar{x}_i) + \alpha_{g,i-1}^2(\bar{x}_{i-1}, t) + \sum_{j=1}^{i-1} \frac{\bar{M}_{ij}(\bar{x}_i, t)}{4\epsilon_j \lambda_j}\right] z_i^2. \qquad (51)$$

Selecting ϵ_i sufficiently small so that $\epsilon_i(n+1-i-\delta_{i1}) < 1$ we can directly apply Lemma 1 to prove that $V_{0i}, P_i, z_i, \int_0^t z_i^2(s)ds$ are bounded in $[0, t_f)$ $(1 \leq i \leq n)$. From the definition (39), the integral terms ζ_i are also bounded in $[0, t_f)$. We can now prove sequentially that all x_i, α_i are bounded in $[0, t_f)$. This is obvious for $i = 1$ since $x_1 = z_1$ and $\alpha_1 = \alpha_1(x_1, \zeta_1)$. Using now the identity $x_{i+1} = z_{i+1} + \alpha_i(\bar{x}_i, \bar{\zeta}_i)$ recursively we can prove that all $x_i, \alpha_i, \bar{M}_{ij}$ are bounded in $[0, t_f)$ $(1 \leq i \leq n; 1 \leq j \leq i)$. Due to boundedness of the whole augmented state vector x_{ag} there is no finite explosion time and the final time t_f can now be extended to infinity, i.e. $t_f = +\infty$ (see Theorem 3.3 in [22] or section 8.5 of [23]).

We have proved therefore that $z_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $x_i, \zeta_i, \alpha_i, u \in \mathcal{L}_\infty$. Then, from (20) we have that $\dot{z}_i \in \mathcal{L}_\infty$. Barbalat's lemma (Lemma 8.2 in [22]) can now be employed to prove that $\lim_{t\to\infty} z_i(t) = 0$. Since $\alpha_i = \alpha_{gi} z_i$ and $\alpha_{gi} \in \mathcal{L}_\infty$ we also have $\lim_{t\to\infty} \alpha_i(t) = 0$ and $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty}(z_i(t) + \alpha_{i-1}(t)) = 0$ for all $i = 1, 2, \cdots, n$. □

Remark 2. *The influence of the controller parameters ϵ_i, λ_i on the control law magnitude and the overall performance is rather complex since they both appear in the definition of P_i in* (34) *but also in the $\bar{M}_{ij}/(4\epsilon_j\lambda_j)$ terms in the definition of α_i given by* (38). *Thus, careful tuning of the parameters through trial and error is needed for optimal performance.*

6. Simulation Studies

Example 1: Let the following second-order nonlinear system

$$\begin{aligned}\dot{x}_1 =& \theta_1(t)x_1^2 \sin\left[\theta_1(t)x_1\right] + g_1(t)x_2 \\ \dot{x}_2 =& \theta_2(t)x_1x_2 + g_2(t)u\end{aligned} \tag{52}$$

with $g_1(t) = 1.5 + 0.5\sin(t)$, $g_2(t) = -4 + 1.5\cos(5t)$, $\theta_1(t) = \sin(t)$, $\theta_2(t) = 2 + \cos(t)$ and initial conditions $x_1(0) = -1$, $x_2(0) = 2.5$. From (52) the bound functions $\rho_{11}(x_1) = \rho_{21}(\bar{x}_2) = 0.1x_1^2$ and $\rho_{22}(\bar{x}_2) = 0$ are selected. The backstepping nonlinear PI controller (nPI) is applied with continuous differentiable $\mathcal{K}_\infty$ functions $\varphi_1(P_1^2) := \ln(1 + P_1^2)$, $\varphi_2(P_2^2) := [\ln(1 + P_2^2)]^{1/5}$ and parameters $\lambda_1 = \lambda_2 = 1$ and $\epsilon_1 = \epsilon_2 = 0.1$.

The results are compared with the Nussbaum gains control scheme of [6] for which the parameters $d_1 = 10^{-3}$, $d_2 = 5 \times 10^{-5}$, $\gamma_1 = 0.1$, $\gamma_2 = 5 \times 10^{-2}$ are considered. Small values for the parameters d_1, d_2 are selected via trial and error to avoid extremely high control input values. Functions ψ_1, ψ_2 are described in [6]. The simulation results are shown in Fig. 1-3. One can see from Fig. 1, 2 that both schemes achieve asymptotic regulation with comparable performance. However, with respect to control effort, the nonlinear PI outperforms the Nussbaum gain (NG) approach. When the latter is applied, a large peak in the control input occurs (Fig. 3) that results in a faster response in x_2 (Fig. 2).

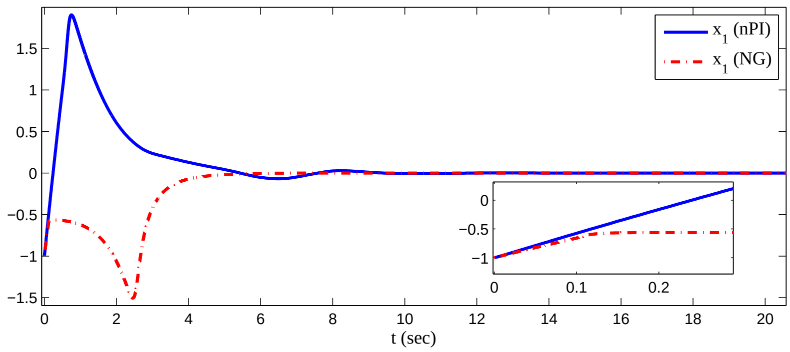

Figure 1: State $x_1(t)$ response for controllers nPI and NG.

In order to test the controllers' robustness to dynamic uncertainties we further assume first order dynamic perturbations on the system output.

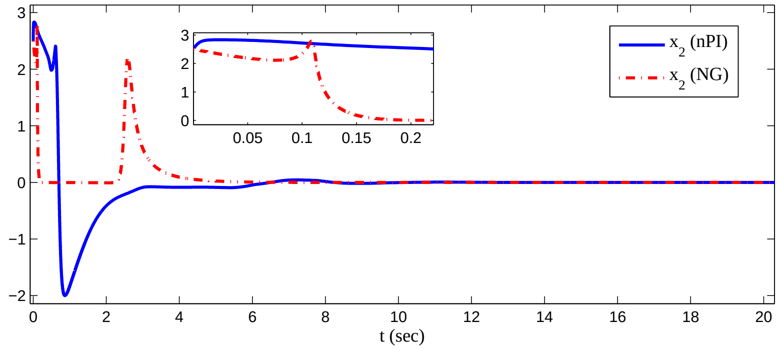

Figure 2: State $x_2(t)$ response for controllers nPI and NG.

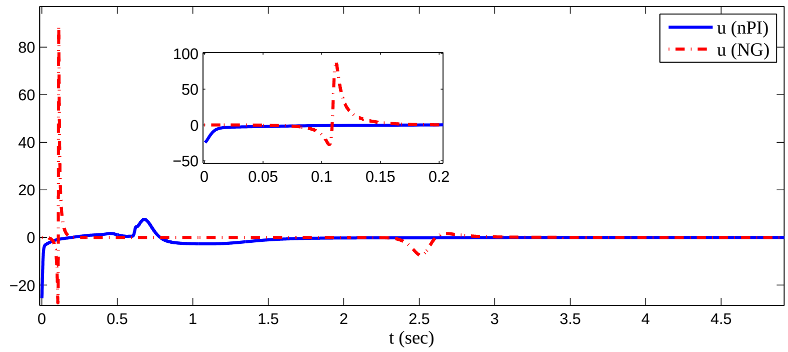

Figure 3: Control input $u(t)$ response for controllers nPI and NG.

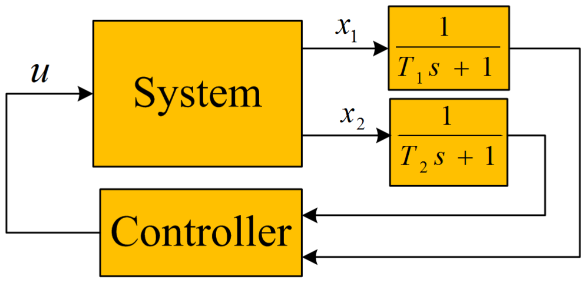

Figure 4: Unmodelled dynamic perturbations on the system output.

These are shown in Fig. 4 and we consider the time constants $T_1 = T_2 = 0.3$. Simulation results for the two controllers (nPI and NG) are given in Fig. 5, 6. Asymptotic regulation is still achieved by the nonlinear PI controller (Fig.

5) while the NG controller yields an unbounded time response (Fig. 6).

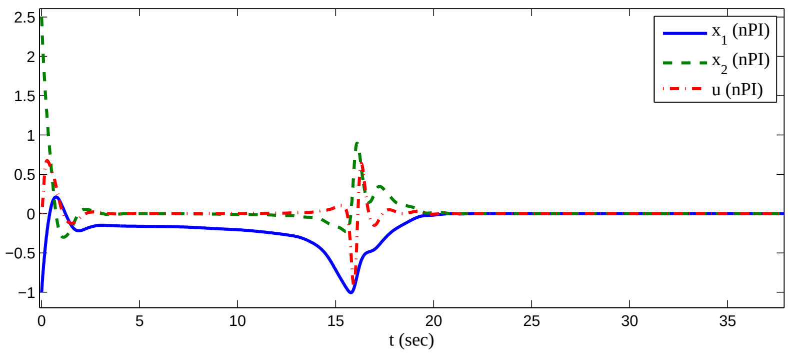

Figure 5: Time responses for x_1, x_2, u for the nPI controller under dynamic perturbations.

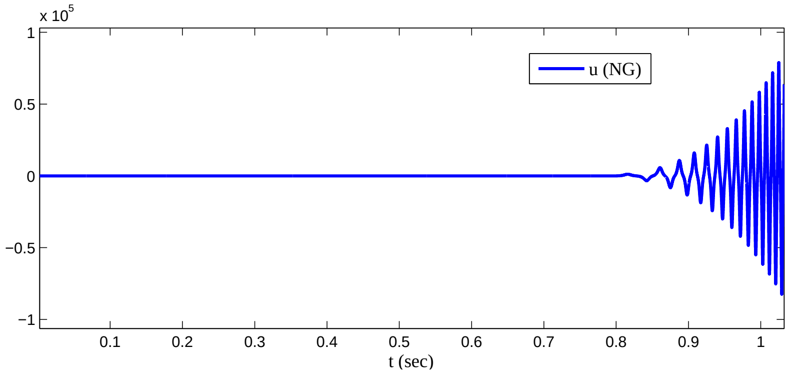

Figure 6: Control input response for the NG controller under dynamic perturbations.

Example 2 (Ship steering): To further test the effectiveness of the proposed backstepping control we consider the Norrbin model for ship steering [24], [25] relating the rudder angle δ to the heading ψ given by

$$T\ddot{\psi} + \dot{\psi} + \alpha\dot{\psi}^3 = K\delta \tag{53}$$

with T the time constant, K the rudder gain and α the Norrbin coefficient determined via a spiral test. Parameters T, K, α vary considerably with the operating conditions and are assumed unknown. System (53) can be written equivalently as

$$\dot{x}_1 = x_2 \tag{54}$$

$$\dot{x}_2 = f(x_2) + gu \tag{55}$$

with $x_1 := \psi$, $x_2 := \dot{\psi}$, $f(x_2) = -(1/T)(x_2 + \alpha x_2^3)$ and $g := k/T$. In our simulation scenario, we assume initial conditions $[x_1(0), x_2(0)] = [30^0, 0^0 s^{-1}]$ and constant values $T = 21s$, $K = 0.23s^{-1}$, $\alpha = 0.3s^2$ obtained from identification results of a frigate at a speed of $U = 12m/s$ [25]. The proposed nonlinear PI control scheme is applied with $\rho_{11} = \rho_{21} = 0$, $\rho_{22}(x_2) = 1 + x_2^4$, $\varphi_1(P_1^2) = \ln(1 + P_1^2)$, $\varphi_2(P_2^2) = [\ln(1 + P_2^2)]^{1/5}$ and parameters $\epsilon_1 = \epsilon_2 = 0.5$, $\lambda_1 = \lambda_2 = 0.4$. The simulation results are shown in Fig. 7, 8. The proposed nonlinear PI effectively regulates the ship heading by suitably adjusting the rudder angle as expected by the theoretical analysis.

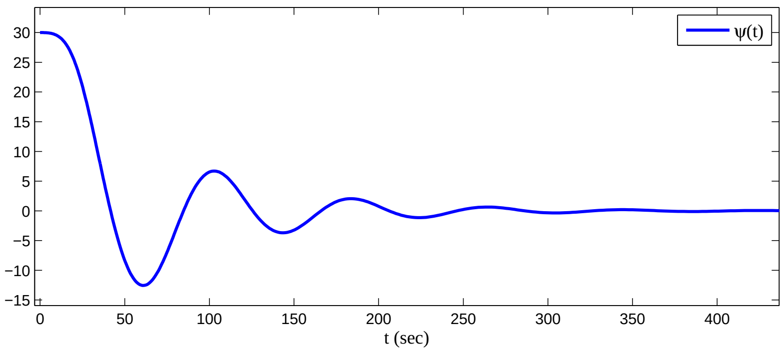

Figure 7: The heading angle time response $\psi(t)$ of the ship in degrees.

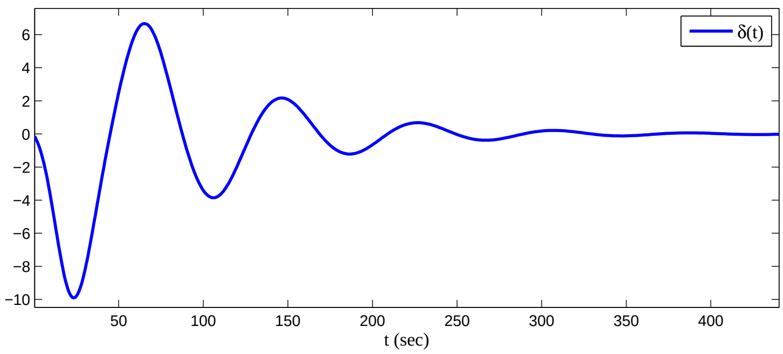

Figure 8: The rudder angle $\delta(t)$ time response.

7. Conclusion

In this paper, a solution is proposed for the backstepping design problem with the nonlinear PI method. A distinct feature of our approach is that the virtual control laws are provided from suitably selected integral equations. Future work may address extensions within an adaptive control framework.

Appendix A. Proof of Lemma 2

Proof. The proof of (20) is based on an induction argument. For $i = 1$ (20) is obviously true from (8). Assume now that (20) holds true for all $i = 1, \ldots, k-1$. We will prove its validity for $i = k$. Since $z_k := x_k - \alpha_{g,k-1}(\bar{x}_{k-1}, t)z_{k-1}$ its derivative takes the form

$$\begin{aligned} \dot{z}_k = f_k(\bar{x}_k, t) + g_k(t)[z_{k+1} + \alpha_{gk}(\bar{x}_k, t)z_k] - \alpha_{g,k-1}\dot{z}_{k-1} \\ - \left[\sum_{j=1}^{k-1} \frac{\partial \alpha_{g,k-1}}{\partial x_j}(f_j(\bar{x}_j, t) + g_j(t)x_{j+1}) + \frac{\partial \alpha_{g,k-1}}{\partial t}\right] z_{k-1} \end{aligned} \tag{A.1}$$

From Assumption 1 we have

$$\begin{aligned} f_k(\bar{x}_k, t) &= \sum_{j=1}^{k} x_j f_{kj}(\bar{x}_k, t) = \sum_{j=1}^{k} [z_j + \alpha_{g,j-1}(\bar{x}_{j-1}, t)z_{j-1}] f_{kj}(\bar{x}_k, t) \\ &= f_{kk}(\bar{x}_k, t)z_k + \sum_{j=1}^{k-1} [f_{kj}(\bar{x}_k, t) + \alpha_{gj}(\bar{x}_j, t) f_{k,j+1}(\bar{x}_k, t)] z_j. \end{aligned} \tag{A.2}$$

Using (A.2) in (A.1) and assuming (20) true for $i = k-1$ the proof of (20) for $i = k$ is obtained directly with M_{kj} given by the recursive eq. (21)-(23).

With respect to the upper bounds of $M_{ij}^2(\bar{x}_i, t)$ we have that

$$M_{11}^2(x_1, t) = [f_{11}(\bar{x}_1, t) + g_1(t)\alpha_{g1}(\bar{x}_1, t)]^2 \leq c_{11}\bar{M}_{11}(x_1, t) \tag{A.3}$$

with $c_{11} := 2\max\{\gamma_{11}, g_{12}^2\}$ and $\bar{M}_{11}(x_1, t) := \rho_{11}(x_1) + \alpha_{g1}^2(x_1, t)$. Assume now that $M_{ij}^2(\bar{x}_i, t) \leq c_{ij}\bar{M}_{ij}(\bar{x}_i, t)$ for $i = 1, \cdots, k$ and $j = 1, \cdots, i$. We will prove that $M_{k+1,j}^2(\bar{x}_{k+1}, t) \leq c_{k+1,j}\bar{M}_{k+1,j}(\bar{x}_{k+1}, t)$ for $j = 1, \cdots, k+1$.

From the definitions of $M_{ii}, \bar{M}_{ii}$ in (21), (25) and Cauchy-Schwarz (C-S) inequality we have

$$\begin{aligned} M_{k+1,k+1}^2(\bar{x}_{k+1}, t) \leq & 3f_{k+1,k+1}^2(\bar{x}_{k+1}, t) + 3g_{k+1,2}^2\alpha_{g,k+1}^2(\bar{x}_{k+1}, t) \\ & + 3g_{k2}^2\alpha_{g,k}^2(\bar{x}_k, t) \leq c_{k+1,k+1}\bar{M}_{k+1,k+1}(\bar{x}_{k+1}, t) \end{aligned} \tag{A.4}$$

with $c_{k+1,k+1} := 3\max\{\gamma_{k+1,k+1}, g^2_{k+1,2}, g^2_{k,2}\}$. Similarly from the definitions (22), (26) and C-S inequality we obtain

$$\begin{aligned} M^2_{k+1,k} \le 2(k+2)\Bigg\{ & f^2_{k+1,k}(\bar{x}_{k+1},t) + \alpha^2_{g,k}[f^2_{k+1,k+1}(\bar{x}_{k+1},t) + M^2_{k,k}(\bar{x}_k,t)] \\ & + \sum_{j=1}^{k}\left(\frac{\partial\alpha_{g,k}}{\partial x_j}\right)^2\left(\sum_{i_1=1}^{j} x^2_{i_1}\sum_{i_2=1}^{j} f^2_{j,i_2}(\bar{x}_j,t) + g^2_{j2}x^2_{j+1}\right) + \left(\frac{\partial\alpha_{g,k}}{\partial t}\right)^2\Bigg\} \\ & \le c_{k+1,k}\bar{M}_{k+1,k}(\bar{x}_{k+1},t) \end{aligned} \tag{A.5}$$

with

$$c_{k+1,k} := 2(k+2)\max\left\{c_{k,k}, \max_{\substack{1\le r\le k \\ 1\le s\le r}}\{g^2_{r2},\gamma_{rs}\}, \gamma_{k+1,k}, \gamma_{k+1,k+1}\right\}. \tag{A.6}$$

Finally for $M^2_{k+1,j}(\bar{x}_{k+1},t)$ for $j \in \{1,\cdots,k-1\}$ we have from (23), (27) and C-S inequality

$$\begin{aligned} M^2_{k+1,j}(\bar{x}_{k+1},t) \le & 3f^2_{k+1,j}(\bar{x}_{k+1},t) + 3\alpha^2_{gj}(\bar{x}_j,t)f^2_{k+1,j+1}(\bar{x}_{k+1},t) \\ & + 3\alpha^2_{g,k}(\bar{x}_k,t)M^2_{k,j}(\bar{x}_k,t) \\ & \le c_{k+1,j}\bar{M}_{k+1,j}(\bar{x}_{k+1},t) \qquad (1 \le j \le k-1) \end{aligned} \tag{A.7}$$

with $c_{k+1,j} := 3\max\{\gamma_{k+1,j}, \gamma_{k+1,j+1}, c_{k,j}\}$. □

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