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A CONVENIENT METHOD FOR ACCURATE HEIGHT DIFFERENCES DETERMINATION

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ABSTRACT

This paper proposes a convenient and easy to perform method for the determination of accurate orthometric height differences. The **A**ccurate **T**rigonometric **H**eighting (ATH) is based on the trigonometric heighting. The ATH rejects all the disadvantages of trigonometric heighting, such as the measurement of the height of the instrument and the target, and the errors caused by the geodetic refraction and the curvature of the Earth.

A reflectorless total station is indispensable for the measurements, as the method can be applied either between accessible or inaccessible points.

The precision of the method varies between ±0.5mm to ±1.5mm, is analogous to the spirit leveling and depends mainly on the accuracy of the zenith angles measurement.

The ATH is described in detail and also, an application is presented in order to elevate its effectiveness and to certify its accuracy.

KEYWORDS: Trigonometric heighting, reflectorless total station, geodetic refraction.

1. INTRODUCTION

The determination of accurate orthometric height differences between two points still remains cumbersome. Spirit levelling is very time consuming, even using digital levels. Additionally, it needs the establishment of benchmarks at convenient and accessible positions. Trigonometric heighting is applied between accessible points only but its precision is reduced due to geodetic refraction and the measurement of the height of the instrument and the target. Even if the simultaneous and reciprocal trigonometric heighting technique is applied the measurement of the height of the instrument and the target is needed.

According to the trigonometric heighting, the height difference ΔH_{AB} between two points A and B on the Earth surface is calculated by the equation:

$$\Delta H_{AB} = D_{AB} \cdot \cos z_{AB} + (1-K) \cdot \frac{D_{AB}^2}{2R} \cdot \sin^2 z_{AB} + i - j \qquad (1)$$

where:

ΔH_{AB} the orthometric height difference between A and B
D_{AB} the measured slope distance
z_{AB} the measured zenith angle at A
R the Earth's radius
K the refraction coefficient
i height of instrument at A
j height of target at B

In order to eliminate all the errors and to measure accurate height differences between inaccessible points, the method of the **A**ccurate **T**rigonometric **H**eighting (ATH) is proposed [Lambrou, 2007]. The height difference between points which situated on special positions as gaps on the Earth, high inclination surfaces, rough surfaces, high structures, trigonometric network points, can be determined.

The evolution of the technology in modern total stations provides reflectorless distance measurements on all the surfaces. This capacity is proved indispensable.

The accurate measurements of the zenith angles and the distances, the independence of the method from the Earth's curvature, the geodetic refraction and the measurement of the instrument and the target height, make the ATH accurate enough of the order of the spirit levelling.
The ATH gives the possibility for quick, accurate and easy determinations. It is a simple and applicable method between accessible or inaccessible points, when accurate orthometric height differences are requested.

2. THE METHOD

A tripod bearing a reflectorless total station is put close (some meters) to point A in order to eliminate the effect of the Earth's curvature and the refraction (fig .1). The zenith angle (z) and the slope distance (D) from the center of the instrument towards the point A are measured. Then, the height difference ΔH_{1A} between the point 1, where the total station axes coincide and the point A, can be calculated by:

$$\Delta H_{1A} = D_{1A} \cdot \cos z_{1A} + (1-K) \cdot \frac{D_{1A}^2}{2 \cdot R} \cdot \sin^2 z_{1A} + i_1 - j_A \qquad (2)$$

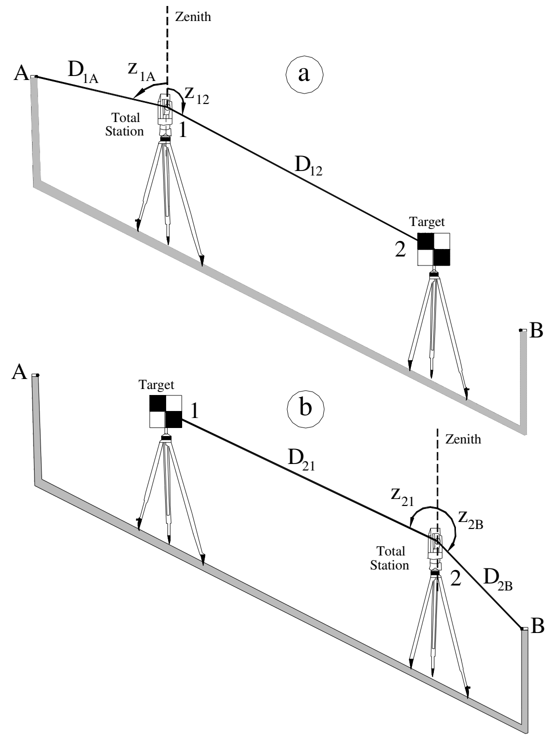

Fig. 1: The application of the accurate trigonometric heighting with two instrument settings.

As the distance D_{1A} is approximately 10m, the factor $(1-K)\cdot\frac{D_{1A}^2}{2\cdot R}\cdot\sin^2 z_{1A}$ is practically zero. Also, as the height difference is calculated from the center of the total station directly to the point A $j_A=0$ and $i_1=0$. Then, eq. (2) becomes:

$$\Delta H_{1A} = D_{1A}\cdot\cos z_{1A} \tag{3}$$

Another tripod bearing a target or a prism is put at a convenient position close to the point B. The zenith angle z_{12} and the slope distance D_{12} are measured (fig. 1a).
The height difference (fore) between the center of the total station, point 1 ($i_1=0$), and the center of the prism, point 2 ($j_2=0$), is calculated by:

$$\Delta H_{12} = D_{12}\cdot\cos z_{12} + (1-K)\cdot\frac{D_{12}^2}{2\cdot R}\cdot\sin^2 z_{12} + i_1 - j_2 = D_{12}\cdot\cos z_{12} + (1-K)\cdot\frac{D_{12}^2}{2\cdot R}\cdot\sin^2 z_{12} \tag{4}$$

Then, the total station and the target change their positions reciprocally above the tribraches, as quickly as possible, in order to avoid the change of the refraction coefficient K in this time interval. Also both tripod and tribrach systems must be stable during the change. Afterwards, the zenith angle z_{21} and the slope distance D_{21} are measured (fig. 1b).

So, a second measurement (back), of the previous height difference was carried out.

$$\Delta H_{21} = D_{21}\cdot\cos z_{21} + (1-K)\cdot\frac{D_{21}^2}{2\cdot R}\cdot\sin^2 z_{21} \tag{5}$$

Also, the measurements of the z_{2B} and D_{2B} towards point B are carried out from this position. The height difference between the center of the total station, point 2 ($i_2 = 0$), and the point B ($j_B = 0$), is calculated correspondingly to eq. (2) and (3) as follows:

$$\Delta H_{2B} = D_{2B} \cdot \cos z_{2B} + (1-K) \cdot \frac{D_{2B}^2}{2R} \cdot \sin^2 z_{2B} + i_2 - j_B = D_{2B} \cdot \cos z_{2B} \qquad (6)$$

The total height difference between the points A and B is calculated by the eq. (7).

$$\Delta H_{AB} = \Delta H_{2B} - \Delta H_{1A} + \frac{\Delta H_{12} + \Delta H_{21}}{2} =$$

$$= D_{2B} \cos z_{2B} - D_{1A} \cdot \cos z_{1A} + \frac{\left[D_{12} \cdot \cos z_{12} + (1-K) \cdot \frac{D_{12}^2}{2 \cdot R} \cdot \sin^2 z_{12} \right] - \left[D_{21} \cdot \cos z_{21} + (1-K) \cdot \frac{D_{21}^2}{2 \cdot R} \cdot \sin^2 z_{21} \right]}{2}$$

$$= D_{2B} \cos z_{2B} - D_{1A} \cdot \cos z_{1A} + \frac{D_{12} \cdot \cos z_{12} - D_{21} \cdot \cos z_{21}}{2} \qquad (7)$$

It is obvious that the height differences ΔH_{12} and ΔH_{21} have opposite sign and they would be almost the same. So, the factor $(1-K) \cdot \frac{D_{12}^2}{2R} \cdot \sin^2 z_{12}$ is effaced as the curvature of the Earth has equal value in both equations and also between both measurements, fore and back a small time interval is mediated and therefore the refraction coefficient K remains constant.
If the distance between the points A and B is longer, then more intermediate stations are needed and the calculation formula for the height difference ΔH_{AB} becomes:

$$\Delta H_{AB} = \Delta H_{nB} - \Delta H_{1A} + \sum_{i=2}^{n} \frac{\Delta H_{i-1,i} + \Delta H_{i,i-1}}{2} \qquad (8)$$

where n the number of the intermediate instrument settings.

3. ERROR ANALYSIS

If it was carried out only one instrument setting then $ÄÇ_{ÁÂ} = ÄÇ_{1Â} - ÄÇ_{1Á} \rightarrow ó_{\Delta H_{AB}} = \pm\sqrt{ó_{\Delta H_{1B}}^2 + ó_{\Delta H_{1A}}^2}$

- If $ó_{\Delta H_{1A}} = ó_{\Delta H_{1B}} = ó_{\Delta H}$ then $ó_{\Delta H_{AB}} = \pm\sqrt{2 \cdot ó_{\Delta H}^2}$, where $ó_{\Delta H} = \sqrt{\cos^2 z \cdot ó_D^2 + D^2 \cdot \sin^2 z \cdot ó_z^2}$
- if $ó_D = \pm 3\text{mm}$, $ó_z = \pm 3^{cc}$ and D=200m then $ó_{\Delta H_{AB}} = \pm 1\text{mm}$.

By applying the variance-covariance law in eq. (8), the total error in the determination of the ΔH_{AB} for i = 2 and $ó_{\Delta H_{i,i-1}} = ó_{\Delta H_{i-1,i}} = ó_{\Delta H_i}$ is calculated by:

$$ó_{\Delta H_{AB}} = \pm\sqrt{ó_{\Delta H_{nB}}^2 + ó_{\Delta H_{1A}}^2 + \left(\frac{1}{2}\right)^2 \cdot ó_{\Delta H_{i-1,i}}^2 + \left(\frac{1}{2}\right)^2 \cdot ó_{\Delta H_{i,i-1}}^2} = \pm\sqrt{ó_{\Delta H_{nB}}^2 + ó_{\Delta H_{1A}}^2 + \frac{1}{2} \cdot ó_{\Delta H_i}^2} = \pm\sqrt{2 \cdot ó_{\Delta H}^2 + \frac{1}{2} \cdot ó_{\Delta H_i}^2} \qquad (9)$$

- If they were carried out n settings and $\sigma_{\Delta H_{i-1,i}} = \sigma_{\Delta H_{i,i-1}}$ then

$$\sigma_{\Delta H_{AB}} = \pm\sqrt{2\cdot\sigma^2_{\Delta H} + \frac{n-1}{2}\cdot\sigma^2_{\Delta H_i}} \qquad (10)$$

where:

n number of the intermediate instrument settings (n≥2)

$\sigma_{\Delta H_{A1}} = \sigma_{\Delta H_{nB}} = \sigma_{\Delta H}$ the error in the determination of the ΔH at the initial and final instrument setting.

$\sigma_{\Delta H_{i,i-1}} = \sigma_{\Delta H_{i-1,i}} = \sigma_{\Delta H_i}$ the error in the determination of each intermediate ΔH.

By using a total station of angular accuracy $\pm 3^{cc}$ (±1″) and distance accuracy ±3 mm for distances about 100 m and four instrument settings, the expected height difference precision may reach ±1.5 mm.

On the other hand, by using a total station of angular and distance accuracy of $\pm 15^{cc}$ (±5″) and ±5 mm respectively, then the accuracy of the height difference determination will be about ±5 mm.

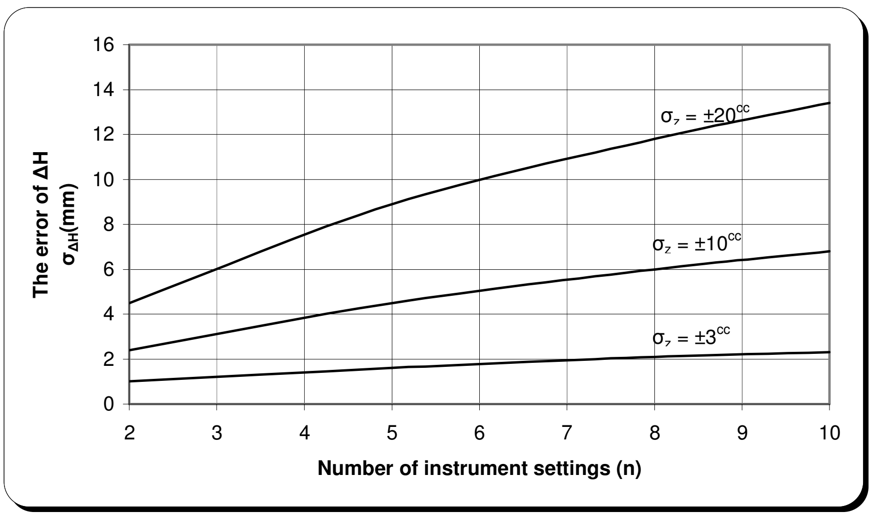

Fig. 2: The error of the height difference determination related to the number of the instrument settings (n) and the error in the zenith angle.

Figure 2 is proved that the achieved accuracy mainly depends on the accuracy of the measured zenith angles.

4. DISCUSSION

The ATH is independent of:

- *The geodetic refraction coefficient* K. The value of each height difference comes out as the mean value of two measurements which are carried out practically, simultaneously. The quantity of the error in the height difference due to geodetic refraction is equal in both calculated height differences but its sign is opposite (minus or plus), so when the mean value is calculated by the eq. (7) or (8) this factor is erased.

  Even under the hypothesis that the coefficient of geodetic refraction K changes in such a short time interval between both measurements, the remaining error depends on the change of the K.

The error of the height difference determination related to the change of the refraction coefficient δK during the measurement illustrated in figure 3.

As it comes out by the diagram (fig. 3) this error depends on the value of the zenith angle and the distance. In most cases is smaller than 0.3mm, namely insignificant.

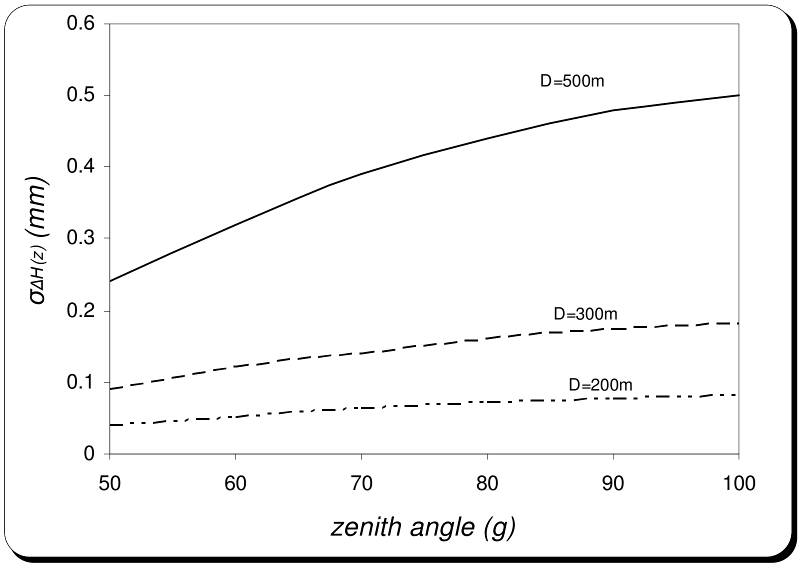

Fig. 3 The error of the height difference determination related to the change of the refraction coefficient K (δ_K=±0.05), during the measurement.

- *The curvature of the surface of the earth*. The measurements were carried out reciprocally in both directions. This error is also effaced via the extraction of the mean value of each height difference between two points as the factor $\frac{D^2}{2R} \cdot \sin^2 z$ has opposite sign, as follows the sign of the height difference. The same as was previously describe.

- *The measurement of the height of the instrument and the height of the target.*

It is obviously clear from the description of the method that the measurement of the height of the instrument and the height of the target is not needed. Each intermediate height difference is measured between the center of the instrument namely the point where their axes coincide and the center of the target. By using the ATH is carried out a ″flying″ levelling traverse, where the instrument and the targets are put on ordinary positions without centering as it was done at the trigonometric heighting.

Each intermediate calculated height difference ΔH_{ATH} (fig. 4) is different from the height difference between the corresponding points on the ground ΔH_G.
The measurement of both heights causes significant error of the order of few millimeters at the trigonometric heighting method and some times contain gross errors done by the observer.

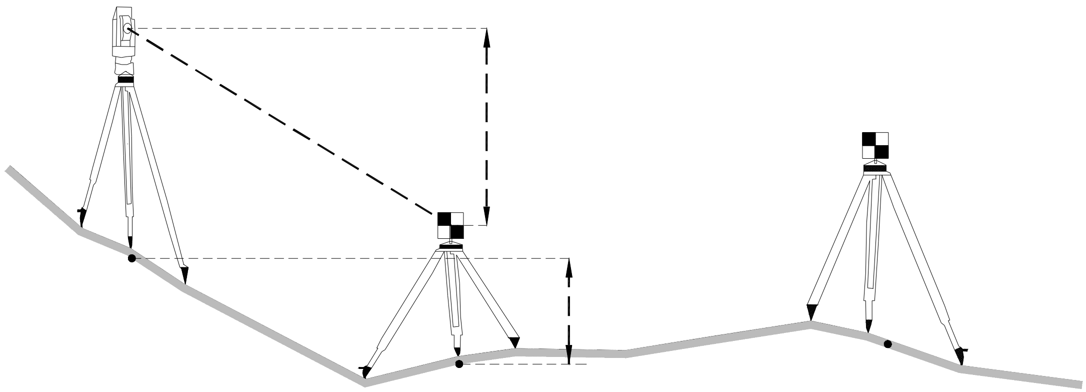

Fig. 4 The height difference ΔH_{ATH} between the intermediate instrument settings is referring to the center of the instrument and the center of the target.

5. APPLICATION

A height network was measured in order to check and prove the benefits of the ATH. The network consists of 10 points (fig.5), which are marked by benchmarks.

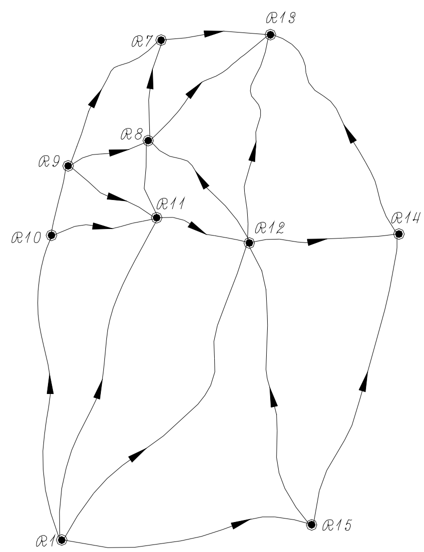

Fig. 5: The height network

Twenty height differences are measured using the ATH. The point R1 was kept stable with orthometric height 192.419m. The distances between the points vary from 100m to 700m. The number of the instrument stations needed for the connection of the benchmarks varies from 2 to 8 and the distance covered by each station varies from 50 m to 150 m, depending on the relief of the Earth surface and the existing buildings.

A reflectorless total station Leica TCR303 is used, which has an angular accuracy of $\pm 10^{cc}$ ($\pm 3''$) and a distance accuracy of ± 3 mm.

The same height differences had been measured by using a Leica NA3003 digital level for first order height determination via double run levelling [Chronis 2003]. These measurements are used in order to compare the results.

Table 1 contains the results of the measurements as well as the time that was needed for each levelling. Also in the Table 2 the final calculated orthometric heights and their uncertainties by both methods are presented.

ΔH	ATH (m)	Time (min)	Spirit levelling (m)	Time (min)
R_8 - R_7	4.547	15	4.5464	30
R_8 - R_9	-2.785	15	-2.7876	25
R_8 - R_{11}	3.536	20	3.5325	40
R_9 - R_{10}	2.825	30	2.8280	25
R_8 - R_{10}	0.043	20	0.0402	25
R_9 - R_7	7.330	30	7.3346	30
R_{10} - R_{11}	3.495	20	3.4921	40
R_1 - R_{15}	2.580	30	2.5819	60
R_1 - R_{10}	-9.221	40	-9.2200	120
R_{12} - R_{11}	-9.206	30	-9.2072	70
R_{13} - R_7	-6.599	25	-6.6004	70
R_{13} - R_8	-11.144	30	-11.145	80
R_{15} - R_{14}	9.101	35	9.1006	85
R_{14} - R_{13}	-9.795	35	-9.7940	80
R_{12} - R_{13}	-1.596	40	-1.5944	80
R_{12} - R_{14}	8.201	35	8.2021	80
R_{12} - R_{15}	-0.897	35	-0.8969	70
R_1 - R_{11}	-5.725	40	-5.7270	120
R_{12} - R_8	-12.738	35	-12.7387	90
R_1 - R_{12}	3.479	30	3.4778	120

Table 1. The results of the measurements

Point	H (ATH) (m)	σ_H (mm)	H (levelling) (m)	σ_H (mm)
R7	187.704	±1.2	187.705	±0.8
R8	183.158	±1.0	183.159	±0.6
R9	180.374	±1.2	180.371	±0.8
R10	183.199	±0.9	183.199	±0.6
R11	186.693	±0.9	186.691	±0.6
R12	195.898	±0.8	195.898	±0.6
R13	194.303	±1.1	194.305	±0.7
R14	204.099	±1.1	204.100	±0.7
R15	194.999	±1.0	195.000	±0.6

Table 2. The final results.

As it comes out, it was needed about 22 hours for the network measurement using double run levelling as it was needed about 10 hours using the ATH. The observed differences between 0 mm to 3 mm in the heights of the network points are acceptable as they are under the limit of the achieved uncertainty in both networks according to the variance – covariance law for confidence level 95%.

6. CONCLUSIONS

- The ATH is quick, easy and convenient for application as performing double run measurements simultaneously and reciprocally saving time and labour.
- The calculated height difference is free of:
  - The measurement of the height of the instrument and the height of the target.
  - The different manufactured height of the instrument and the target.
  - The geodetic refraction coefficient K.
  - The curvature of the earth.
- The main differences of the ATH from the well known surveying technique of simultaneous reciprocal trigonometric heighting are:
  - The use of reflectorless total station in order to determine directly height differences between accessible and inaccessible points.
  - The method is free of the measurement of the height of the instrument and the height of the target, namely is free of their errors.
  - The tripods are put on ordinary positions and no centering is necessary. So the followed way is improved reducing the time needed.
- The height differences that are calculated between the points of a 3D network via this method can participate as another type of independent observation equations in a least square adjustment.
- The application of the ATH proves that the achieved accuracy in the heights determination is of the same order with the spirit levelling.
- The total time needed for the network measurement by ATH is eliminated about 50%, in comparison to the spirit levelling, especially in the long distance levellings.
- The mean accuracy of the method is of the order of ±1.5mm depending mainly on the class of the used total station but it can be reach the ± 0.5mm.

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