LOCAL GEOID MODELLING VIA THE GEOMETRIC INTERPOLATION METHOD

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Abstract

The accurate geoid determination remains still a difficult task for the geodesists. For this purpose several methods and satellite, gravimetric or land – measured data may be used. The need for this definition is significant as it is the unique opportunity to transform the easy derived ellipsoidal (geometric) heights via GPS measurements to orthometric heights, which will be used in the surveying applications.
The geometric interpolation method allows an accurate determination of the local geoid model, even at the centimeter precision level, based on known either orthometric and ellipsoidal heights of certain points or geoid undulation differences ΔN between certain points. The accuracy of the derived local geoid model depends on the number of the availiable points, the quality of the orthometric and ellipsoidal heights or the differences ΔN used and the smoothness of the geoid surface at the concrete area.
This paper attempts to make an approximation of the geoid surface over a small area of a few km^2. Two surfaces, the plane and the ellipsoid, have been tested for the best fitting via the least square method as adequate number of known points was available. Furthermore, independent checks were applied to ensure the quality of the arisen model. The evaluation of the results and the achieved accuracy allows us to come up to some very useful conclusions about the success of the applied geometric interpolation method and indicates the applications where the computed model can be used.

1. Introduction

The facility that GPS measurements furnish to the surveying applications has a great disadvantage that remains quite difficult to remove. This is the determination of the geometric height h of the earth surface points above the ellipsoid of the geocentric World Geodetic System 1984 (WGS ′84). The geometric height has no physical meaning, as it is not conceivable by anyone, if the point is placed either by the seaside, or at the top of a big mountain. Thus, the transformation of the geometric heights to orthometric ones is absolutely necessary. The well-known formula $H = h – N$ must be used. The geoid undulation N may be determined by astrogeodetic observations, gravity measurements and satellite methods. The value of the geoid undulation N varies from -100m in Indian Ocean to +80m at the North Atlantic Ocean with rms ±30m. Therefore, the global geoid modelling is a very serious and useful effort, as it is the only way to derive information about the global geoid shape. As we do not have sufficient number of accurate measurements worldwide and additionally the geoid – ellipsoid undulation is not smooth enough, the given values of the geoid undulation N by these models may be different at a concrete area even some meters from the real N values. Consequently, the heights transformation by using the geoid undulation values N from a global earth model is not always successful and sometimes leads to big errors. For this reason, over of few square kilometers areas with large geoid alterations, a local geoid model must be calculated. The adequate accuracy for the most land surveys of the cm level may be achieved via the geometric interpolation method. If the indispensable data, namely the geoid undulation differences ΔN or the orthometric and geometric heights for efficient number of points were available, then a linear interpolation, a plane or a second order polynomial surface may successfully approximate the geoid's surface.

2. Geometric geoid modelling.

The best method for the calculation of an accurate geoid model over limited earth areas is to use together gravity measurements, ellipsoidal heights derived via GPS measurements and spirit leveling observations, which provide orthometric heights. The above procedure gives more accurate results, which cover high accuracy field surveys, especially when large alteration of the geoid undulation is observed.

If gravimetric data were available, for the examined area, they might be used simultaneously with the geometric data in order to remove any localized biases that the geometrically calculated geoid model used to have. Since gravimetric data are not available over the most areas on earth surface, other equivalent approaches may be followed. A regional geoid model may be produced via an approximation by a linear interpolation or by a flat surface or by a low order polynomial surface. The best choice between these similar geometric approximation methods depends on the available data.

The arisen question is how precisely such a model can approximate the geoid. Unfortunately there is not a unique answer. That depends on the size of the test area and its relief, the order of the geoid alterations, the local gravity anomalies, the number and the precision of the known orthometric and geometric heights or the precision that the value ΔN is calculated, which mainly influences the achieved accuracy.

By linear interpolation (Featherstone W.E, et. al, 1998) applied to estimate the geoid – ellipsoid undulation, if both the difference ΔN_{ij} and the geometric height difference Δh_{ik} between the points i and k are known, the orthometric height at the intermidiate point k between i and j can be calculated by using the formula:

$$H_k = H_i + \Delta h_{ik} - \frac{S_{ik}}{S_{ij}} \cdot \Delta N_{ij} \tag{1}$$

where S_{ik} and S_{ij} the distances between the points as they are given by the GPS measurements.

The geoid undulation difference ΔN_{ij} between two points may be calculated if the geometric and orthometric heights of the points are known. Otherwise, the value of ΔN_{ij} can be easily calculated via the method of the astrogeodetic levelling, according to the formula:

$$\Delta N_{ij} = -\left[\frac{\xi_i + \xi_j}{2} \cdot \cos A + \frac{\eta_i + \eta_j}{2} \cdot \sin A \right] \cdot S_{ij} \tag{2}$$

Where ξ_i , ξ_j , η_i , η_j the components in meridian and in prime vertical respectively, of the deviation of the vertical, at each point.

A the azimuth of the direction ij

S_{ij} the distance between the two points i, j.

The values of ξ and η may be calculated - if the astronomical coordinates Φ, Λ and the geodetic coordinates φ, λ are known-by the following well known equations:

$$\xi = \Phi - \phi \tag{3}$$

$$\eta = (\Lambda - \lambda) \cdot \cos\varphi \tag{4}$$

A new methodology has been developed (Lambrou E., 2003), (Balodimos D.-D., et al, 2003) for astrogeodetic observations by using a new system, which consists of a high accuracy digital total station and a GPS receiver. According to this methodology the astronomical coordinates Φ, Λ of each point can be calculated after four hours observations at an adequate precision, which allows the estimation of the ΔN value between two points by an accuracy of a few millimeters (Lambrou

E., et al 2003). The geodetic coordinates φ, λ as well as the Δh values are given by the GPS measurements. If the above-mentioned methodology is applied, then according to the formula (1), the knowledge of the orthometric height of only one point is needed.

The precision of the ellipsoidal heights h or the ellipsoidal height differences Δh, which are determined by using the GPS, effected by errors, depends on the method used for the determination (static, kinematic, RTK), the geometry of the satellites, the length of the measured baselines, the satellites broadcast ephemeris, the ionospheric effect and the multipaths. Special attention must be paid to the measurement of the antenna height. The error in the measurement or the registration of the antenna height is a very common mistake that may cause error of decates of centimeters in the final calculated geometric height. The GPS positioning can provide maximum accuracy of ellipsoidal height differences of the order of the centimeter if relative positioning is used by code and carrier phase observations (Featherstone W.E, etal, 1998).

The precision of the orthometric heights H depends on the measurement method used. Today the spirit levelling using digital levels and bar – code staves provides an accuracy of a few mm in the calculated heights. Also, by using ordinary mechanical levels, an accuracy of 1 – 2 cm may be achieved. Furthermore, a special trigonometric heighting method may be applied for points that are located in long distances or they are inaccessible via the spirit levelling. By the presupposition that simultaneous and reciprocal sightings are carried out, the effects of the curvature of the earth and the geodetic refraction are completely effaced. Moreover, by using modern accurate total stations, the achieved accuracy reaches a few millimeters.

Therefore, for a few square kilometers area, if the orthometric and the geometric heights of least three points were known, then it would be possible to have a local approximation of the geoid undulation N be calculated by a plane or an ellipsoidal surface. If more points were known then the best fitting surface would be determined via the least square method application.

The approximation by a plane model done under hypothesis of the linear geoid slope. So will be done by a constant tilt plane. For this case the mean plane equation is:

$$\mathrm{N_i} = \mathrm{a} \cdot \mathrm{x_i} + b \cdot \mathrm{y_i} + c \tag{5}$$

Whence $\mathrm{N_i}$ the geoid undulation at each known point i

a, b, c the parameters of the plane and

x_i, y_i are the coordinates of each point i

In case three points are known, three similar equations are formed to solve the parameters a, b and c. In matrix form the solution is like (Collier P.A, etal, 1997):

$$\begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \end{bmatrix} = \begin{bmatrix} \mathrm{x_i} & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathrm{N_1} \\ \mathrm{N_2} \\ \mathrm{N_3} \end{bmatrix} \tag{6}$$

In the preferable case of more known points in the examined area an over determined solution will be applied, which gives the possibility to check the final result and to evaluate the reliability of the determined planar model. The residuals of a least square solution, namely the standard error $\sigma_{o,}$ indicate the quality of the result and give information about the accuracy of the calculated model. Correspondly, a second order ellipsoidal surface may be tested to approximate better the test area. An ellipsoidal surface will be given by the following equation:

$$\mathrm{a_1} \cdot \mathrm{x_i^2} + b_1 \cdot \mathrm{y_i^2} + c_1 \cdot z_i^2 = 1 \tag{7}$$

The equation (7) may be written as follows:

$$z_i^2 = -\frac{\mathrm{a_1}}{c_1} \cdot \mathrm{x_i^2} - \frac{b_1}{c_1} \cdot \mathrm{y_i^2} + \frac{1}{c_1} \tag{8}$$

Assuming that $z_i^2 = \mathrm{N}_i^2$, $\mathrm{A} = -\frac{\mathrm{a_1}}{c_1}$, $\mathrm{B} = -\frac{b_1}{c_1}$, $C = \frac{1}{c_1}$ then the equation (8) is tranformed:

$$\mathrm{N_i^2} = \mathrm{A} \cdot \mathrm{x_i^2} + \mathrm{B} \cdot \mathrm{y_i^2} + \mathrm{C} \tag{9}$$

In any case, as in all modeling techniques, the result ought to be checked for its accuracy. A quality assurance control must be applied after the determination of the appropriate surface. For this purpose, two or more known points are needed to check if the proposed surface, namely the determined equation, is valid over the whole test area.

3. Application

An experimental application was carried out in a test site on a Greek - island at Aegean Sea. The network consists of 14 points and the site covers an area of about 2Km ∗ 2Km (fig.1).

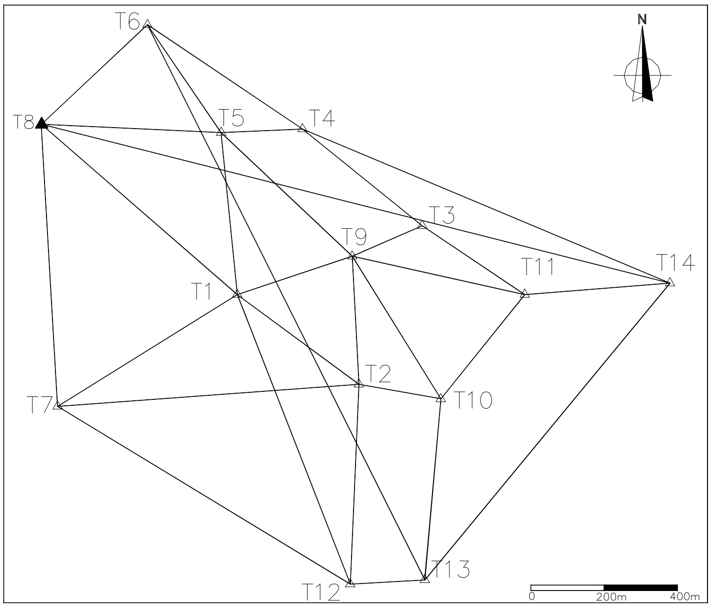

Figure 1. The network.

The maximum height difference between the points is of the order of 150m since the elevations vary from 3m to 157m up to the mean sea level. The coordinates of the network points were determined by GPS measurements (Trimble, 1992). There were twenty-eight GPS bases measured between the points.

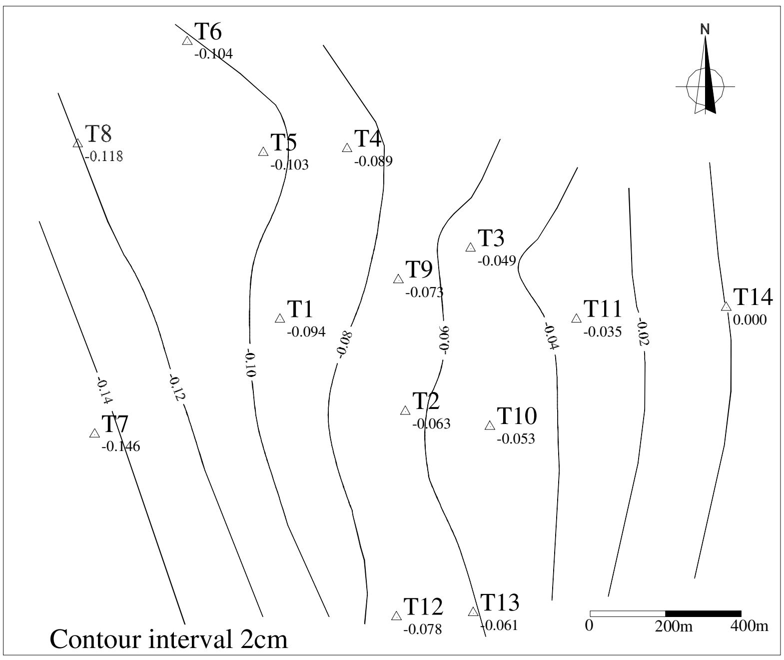

Figure 2. The direct – calculated geoid feature.

One point was held fixed as its coordinates x, y at the Hellenic Geodetic Reference System ′87 were known. In figure 1, the fixed point is marked by a solid triangle.
The orthometric height differences of the points were determined by spirit leveling or by trigonometric heighting using accurate total stations. The orthometric heights uncertainties vary from ±5mm to ±1cm. The geometric heights of the network points were obtained by the GPS solution under the assumption that the orthometric height of point T_{14} coincided with its geometric one. Consequently, the geoid undulation N is equal to zero at point T_{14}.
Figure 2 illustrates the direct geoid undulations N as calculated using the original heights data. The relative geoid undulation between the network points range up to 15cm.
Therefore, it was decided to approximate the geoid surface via geometrical interpolation, by using either a plane equation or an ellipsoidal equation.
Five points of the network were selected for the equations determination, enough to cover all the test area. The points used were T_1, T_4, T_7, T_{11}, T_{13}. In this case, an over determined solution was reached by means of the least square method as more equations than the unknown parameters were formed.
The following plane equation was reached after applying equation (5):

$$N_i = 8.4504 \cdot 10^{-5} \cdot x_i + 2.2519 \cdot 10^{-5} \cdot y_i - 55.8351 \qquad (10)$$

The standard deviation of the solution was $\sigma_0 = \pm 6mm$.
The second approximation that was carried out, via the ellipsoidal surface was as follows:

$$N_i^2 = -1.3943 \cdot 10^{-11} \cdot x_i^2 - 1.5938 \cdot 10^{-13} \cdot y_i^2 + 6.8874 \qquad (11)$$

The standard deviation of this solution was $\sigma_0 = \pm 3mm$.
New values of the geoid undulation N are calculated according to the above equations (10) and (11), for the remaining 9 points of the network and their orthometric heights H are calculated via the fundamental equation H = *h* – N. This calculation serves as a quality control of each approximation.

Point	*Orthometric height H (m)*	*Geometric height h (m)*	*$H^I = h - N^I$ by the plane model (m)*	*$H - H^I$ (cm)*	*$H^{II} = h - N^{II}$ by the ellipsoidal model (m)*	*$H - H^{II}$ (cm)*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*
T_1	**90.201**	**90.107**	**90.208**	**0.7**	**90.204**	**0.3**
T_2	58.128	58.065	58.139	1.1	58.147	1.9
T_3	95.921	95.872	95.930	0.9	95.932	1.1
T_4	**140.605**	**140.516**	**140.601**	**0.4**	**140.607**	**0.2**
T_5	94.680	94.577	94.681	0.1	94.685	0.5
T_6	115.489	115.385	115.505	1.6	115.505	1.6
T_7	**119.544**	**119.398**	**119.541**	**0.3**	**119.546**	**0.2**
T_8	156.860	156.742	156.887	2.7	156.880	2.0
T_9	58.535	58.462	58.536	0.1	58.543	0.8
T_{10}	41.588	41.535	41.590	0.2	41.593	0.5
T_{11}	**37.671**	**37.636**	**37.671**	**0**	**37.669**	**0.2**
T_{12}	50.561	50.483	50.560	0.1	50.571	1.0
T_{13}	**3.969**	**3.908**	**3.968**	**0.1**	**3.970**	**0.1**
T_{14}	3.605	3.605	3.606	0.1	3.612	0.7

Table 1. Orthometric and geometric heights of the test network points via direct and model calculations.

The table 1 illustrates the direct calculated orthometric and geometric heights of the points in columns 1 and 2 respectively. Also it depicts the orthometric heights H^I calculated using the N^I value for each point as they come out according to the equation (10), which namely is the plane approximation. The column 5 includes the orthometric heights H^{II} calculated via the ellipsoidal surface approximation. Finally, columns 4 and 6 refer to the differences $H – H^I$ and $H – H^{II}$ between the initial value of each point orthometric height H and the derived value, for each point orthometric height by the GPS according to both the different models (H^I, H^{II}).
In this table the points used for both the models calculation appear in bold letters.
The comparison between the columns 4 and 6 of the table 1 show that the differences via the plane approximation range from 0cm to 2.7cm as, the differences via the ellipsoidal approximation range from 0.1cm to 2cm. It is also easy to see that many values in the column 4 are smaller than the corresponding ones in column 6, which means that the ellipsoidal approximation has larger bias at most points, thus presenting a worst fitting surface. Consequently, the plane determined via the equation (10) fits much better the geoid surface in the concrete area.
In any case the above-derived differences of about 1cm according to both models approximation are of the level of the precision that the direct calculated heights H and *h* have.

4. Concluding Remarks

- Today, as the GPS survey campaigns spread more and more, the need for the determination of orthometric heights via the provided geodetic ones by the GPS, has increased.
- The selection of the procedure that will be applied in order to transform the geometric heights to orthometric ones depends on the available data.
- Today the astrogeodetic levelling method is feasible to apply as the modern digital geodetic instruments (total station and GPS) provide an easy and accurate determination of the Φ, Λ, φ, and λ coordinates. Moreover the orthometric height of only one point in the area and the geometric heights differences Δh between the points are the needed data.
- In an area where the orthometric heights of at least three points were known and their geometric heights differences were calculated via GPS measurements, the feature of the geoid undulation N can be approximated under the assumption that the orthometric height of one point coincided with its geometric height, in case that the absolute value of N is unknown.
- A quality assurance control must be applied to the calculated geoid surface via other points in the test area, whose orthometric heights are obtained by spirit leveling or special trigonometric heighting.
- It has been proved by the application that both surface models approximate the concrete area quite-well. The ellipsoidal surface has bigger biases. Both are acceptable according to the expected accuracy of cm order caused by the original data.
- The planar approximation seems to be adequate for a few square kilometers areas.
- The final derived geoid model precision is in accordance with the precision with which the original heights data H and *h*, had been calculated.
- The planar approximation at an adequate order of precision of ±1cm is sufficient for the most ordinary surveying and cadastrale works.

References

1. Balodimos D.-D., Korakitis R., Lambrou E., Pantazis G. *Fast and accurate determination of astronomical coordinates φ, λ and azimuth, using a total station and gps receiver,* Survey Review, Vol 37, No290, October 2003, p.p 269 -275.
2. Collier P.A, Croft M.J., *Heights from GPS in an engineering environment*, Survey reviews 34, 263 and 264, January – April 1997.
3. Dodson A.H., *GPS for height determination*, Survey Reviews 33, 256, April 1995.

4. Featherstone W.E., Dentith M.C., Kirby J.F., *Strategies for the accurate determination of orthometric heights from GPS,* Survey Review 34, 267, January 1998.
5. Lambrou E., *Astrogeodetic Observations using Digital Geodetic Instruments*, NTUA, School of Rural and Surveying Engineering (In Greek), Doctoral Thesis, 2003.
6. Lambrou E., Pantazis G., Balodimos D.– D., *Accurate determination of the geoidal undulation N,* Reports on Geodesy, No 1 (64), 2003, Warsaw University of Technology.
7. Trimble Navigation, *Users manual for 4600 LS*, 1992.

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