ACCURATE DETERMINATION OF THE GEOIDAL UNDULATION N

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Abstract

This work, related to the activities of the CERGOP Study Group Geodynamics of the Balkan Peninsula, presents a method for the determination of the variations ΔN and, indirectly, of the geoidal undulation N with an accuracy of a few millimeters. It is based on the determination of the components ξ, η of the deviation of the vertical using modern geodetic instruments (digital total station and GPS receiver).

An analysis of the method is given. Accuracy of the order of 0.01arcsec in the estimated values of the astronomical coordinates Φ and Λ is achieved.

The result of applying the proposed method in an area around Athens is presented. In this test application, a system is used which takes advantage of the capabilities of modern geodetic instruments. The GPS receiver permits the determination of the geodetic coordinates in a chosen reference system and, in addition, provides accurate time information. The astronomical observations are performed through a digital total station with electronic registering of angles and time.

The required accuracy of the values of the coordinates is achieved in about four hours of fieldwork. In addition, the instrumentation is lightweight, easily transportable and can be setup in the field very quickly. Combined with a stream-lined data reduction procedure and the use of up-to-date astronomic data, the values of the components ξ , η of the deviation of the vertical and, eventually, the changes ΔN of the geoidal undulations are determined easily and accurately.

In conclusion, this work demonstrates that it is quite feasible to create an accurate map of the geoid undulation, especially in areas that present large geoid variations and other methods are not capable to give accurate and reliable results.

1. Introduction

The determination of the geoid was for about 70 years (1880-1950) the principal goal of geodesy. Its importance diminished after 1945 with the development of methods for the direct derivation of the physical surface of the earth. However, its determination still remains an essential problem of geodesy. The significance of the geoid has again increased with the establishment of three-dimensional continental and global systems. (Torge, 1991).

The determination of the geoidal undulations N relative to any reference ellipsoid is the main geodetic goal in order to compute orthometric heights H from the geometric heights h, which are determined by GPS measurements.

The orthometric height H is used in all the geodetic works because this height is immediately perceivable on the earth's surface.
The GPS measurements, give the geometric height h and the orthometric height H, may be calculated through the equation $h = H + N$.
The determination of the geoidal undulation, N as it is well known is based either on the variations of the deviation of the vertical or on gravity anomalies Δg. There are several methods for the determination of the geoidal undulation N. These methods are:

- Astrogeodetic leveling.
- Astrogravimetric leveling.
- Geopotential models.

2. Determination of the geoidal undulations N by astrogeodetic leveling.

The determination of the geoidal undulations N is based on the determination of the difference ΔN between two points P and P′. The geoid undulation N of the initial point of a reference ellipsoidal system may be determined via gravimetric measurements and satellite data or may be chosen arbitrarily. The determination of the differences ΔN from point to point allows for the computation of undulations N_i over large areas.
Thus an astrogeodetic map of the geoid for large areas, where the undulations are not smooth may be produced.
For the determination of the differences ΔN, between two points on the earth's surface, by astrogeodetic leveling, it is indispensable to know both the astronomical coordinates Φ,Λ and the geodetic coordinates φ,λ at both points. This allows for the computation of the components ξ,η of the deviation of the vertical of each point.
The geodetic coordinates φ, λ, may be determined by the ubiquitous Global Positioning System (GPS) measurements. The evolution of GPS technology during the last decade, has made feasible the determination of coordinates with an accuracy of the order of $\pm$ 2-3 cm or $\pm$ 0".001.
A big problem, which continues to remain, is the accurate determination of the astronomical coordinates Φ,Λ. The only way to determine Φ, Λ at many points on the earth's surface is by astronomical observations. A system consisting of a modern total station and a GPS receiver has been developed for fast and accurate astronomical observations.

2.1 Determination of the astronomical coordinates.

The astronomical coordinates Φ, Λ can be determined by measurements of horizontal and vertical angles, to several stars combined with UTC time measurements.
The old methods and instruments for the determination of astronomical coordinates Φ, Λ via astronomical observations, require laborious fieldwork for about three nights, in order to achieve first class accurate determinations. Those methods required heavy equipment, skilled observes, and time consuming calculations (Balodimos, 1972a), (Balodimos, 1972b).
The new system, which has been recently developed, consists of a high accuracy digital total station and a GPS receiver, connected by the appropriate cable and software. The GPS receiver provides accurate time information in the form of 1pps (pulse per second) output, which is synchronised with the UTC time to an accuracy of a few microseconds (Trimble, 1990). Also a portable data logger for the recording of the meteorological data is indispensable. This system is compact, low weight and provides the recording of both horizontal and vertical angles to the stars, with an accuracy of $\pm 0.6^{cc}$ or $\pm 0''.2$ (Leica, 1997) and universal time UTC with an accuracy of $\pm$1msec for each measurement.

In order to carry out a first class determination of Φ, Λ, a complete series of observations by this system requires about 4 hours of fieldwork. Also a new way of data reduction has been developed by the method of least squares due to the immediate collection of 2000 to 3000 individual time and angle measurements (Balodimos et al, 2003). The final mean values of Φ, Λ must be corrected for Polar motion and for the curvature of the plumb line (Mueller, 1969).

The selection of the appropriate stars, which will be observed, may be achieved, using a digital planetarium run on the computer (Marriot, 1992-2001), (Lambrou, 2003).

The value of the astronomical latitude Φ from a star observation is computed by (Mueller, 1969):

$$\Phi = \delta \pm z_c \tag{1}$$

where δ is the declination of the star and z_c the vertical angle at its culmination.
According to the Sterneck method the astronomical latitude Φ ought to be determined by pairs of stars, north and south culminating, in order to efface the largest amount of the effect of the astronomical refraction. So the astronomical latitude Φ from a pair of stars is computed by the equation:

$$\Phi = \frac{\Phi_N + \Phi_S}{2} = \frac{\delta_N + \delta_S}{2} + \frac{z_{C_S} - z_{C_N}}{2} \tag{2}$$

Where
z_{C_N}, z_{C_S} = the calculated vertical angles of the culmination for the north and south star
δ_S, δ_N = the corresponding declination.
Φ_N, Φ_S = the corresponding astronomical latitude
The final value of the astronomical latitude $\overline{\Phi}$ is calculated from a sufficient and appropriate number of pairs of stars as a mean value, according to the accuracy required.

The astronomical longitude Λi from a single star i is computed as (Mueller, 1969):

$$\Lambda_i = \alpha_i - t_{c_i} \tag{3}$$

where α_i is the right ascension of the star and t_{c_i} the UTC time of the meridian transit of the star.
The final mean value of the longitude $\overline{\Lambda}$ is calculated from the simplified form of Mayer's formula (Mueller, 1969):

$$\Lambda_i = \overline{\Lambda} + \mathrm{A}_i \cdot \delta\mathrm{A} \tag{4}$$

Where
$\mathrm{A}_i = \pm \frac{\sin z_i}{\cos \delta_i} \begin{bmatrix} +South \\ -North \end{bmatrix}$ is the Mayer's parameter for each star and
δA = the error of the calculated orientation of the total station.

2.1.1 Accuracy of the determination of Φ, Λ

The celestial coordinates α, δ (right ascension, declination) of each star are calculated from the Tycho 2 catalogue with an accuracy of ±7 milliarcseconds (ESA, 1997), (Høg et al., 2000), (Seidelmann et al., 1992).
The vertical angle of the culmination for each star z_{c_i} is determined by fitting a 4th degree polynomial to all horizontal and vertical angle pairs of measurements to each star (Lambrou , 2003), (Balodimos et al., 2003).

The standard error $\sigma_{z_{c_i}}$ is taken from the variance-covariance matrix Vx of the regression for each star. An error of astronomical latitude Φ from a single star is:

$$\sigma_{\Phi_i} = \pm\sqrt{\sigma^2_{Z_{C_i}} + \sigma^2_{\delta_i}} \tag{5}$$

where σ_{δ_i} is the error of the determination of the declination

An a-priori estimation of σ_Φ for one pair of stars, according to equation (2) (Balodimos, 1972a), if $\sigma_{\Phi_N} = \sigma_{\Phi s} = \sigma_{\Phi i}$, is as follows:

$$\sigma_\Phi = \frac{\sigma_{\Phi_i}}{\sqrt{2}} \tag{6}$$

and for ***n*** pairs of stars is

$$\sigma_{\bar{\Phi}} = \frac{\sigma_{\Phi_i}}{\sqrt{2 \cdot n}} \tag{7}$$

For example the a-priori estimation of the rms error for the final value of latitude $\bar{\Phi}$, calculated from 15 pairs of stars, when $\sigma_{\Phi_i} = \pm 0''.1$, is of the order of ± 0".02.

Figure 2 presents the a-priori standard error $\sigma_{\bar{\Phi}}$ of the astronomical latitude Φ determination, for different values of the σ_{Φ_i} from a single star and the number of pairs of stars that will be observed.

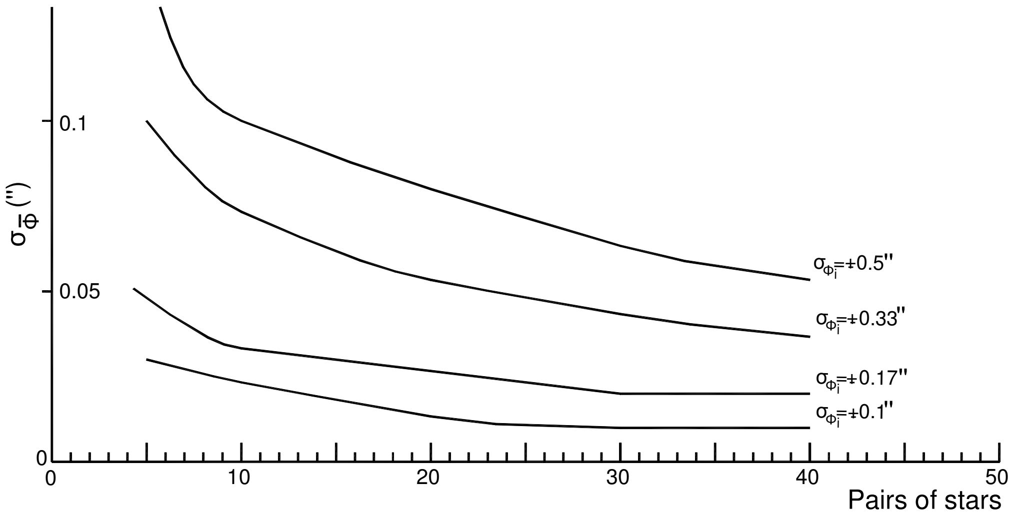

Figure 2. Diagramme of the a -priori error of the determination of Φ relative to the accuracy of the determination from a single star σ_{Φ_i} and the number of pairs of observed stars.

After the observations, the final value $\bar{\Phi}$ of the astronomical latitude Φ is taken as the mean value of the n determined values (Φ_i) from each pair of stars. The a – posteriori error of $\bar{\Phi}$ is calculated by (Mueller, 1969):

$$\sigma_{\bar{\Phi}} = \pm\sqrt{\frac{[\upsilon\upsilon]}{n(n-1)}} \tag{8}$$

The time of culmination t_{C_i} for each star i is determined by fitting a 3rd degree polynomial to all horizontal angles and timing pairs of measurements to each star (Balodimos et al., 2003), (Lambrou, 2003). The standard error $\sigma_{t_{c_i}}$ of this value is taken from of the variance-covariance matrix V_x of the regression for each star. The standard error σ_{Λ_i} of the astronomical longitude Λ_i as determined from a single star will be equal to:

$$\sigma_{\Lambda_i} = \pm\sqrt{\sigma^2_{t_{c_i}} + \sigma^2_{\alpha_i}} \tag{9}$$

where σ_α = the error of the right ascension

An a-priori estimation $\sigma_{\overline{\Lambda}}$, if n stars are observed, will be equal to (Balodimos, 1972a):

$$\sigma_{\overline{\Lambda}} = \frac{\sigma_{\Lambda_i}}{\sqrt{n}} \tag{10}$$

Figure 3 demonstrates the a-priori error of the determination of Λ in connection to the error of its determination from one star (σ_{Λ_i}) and the number of the observed stars (n).

The a -priori estimation of the rms error of the determination of the best estimate value $\overline{\Lambda}$, when $\sigma_{\Lambda_i} = \pm 0''.1$, from 20 stars, is of the order of ± 0".02.

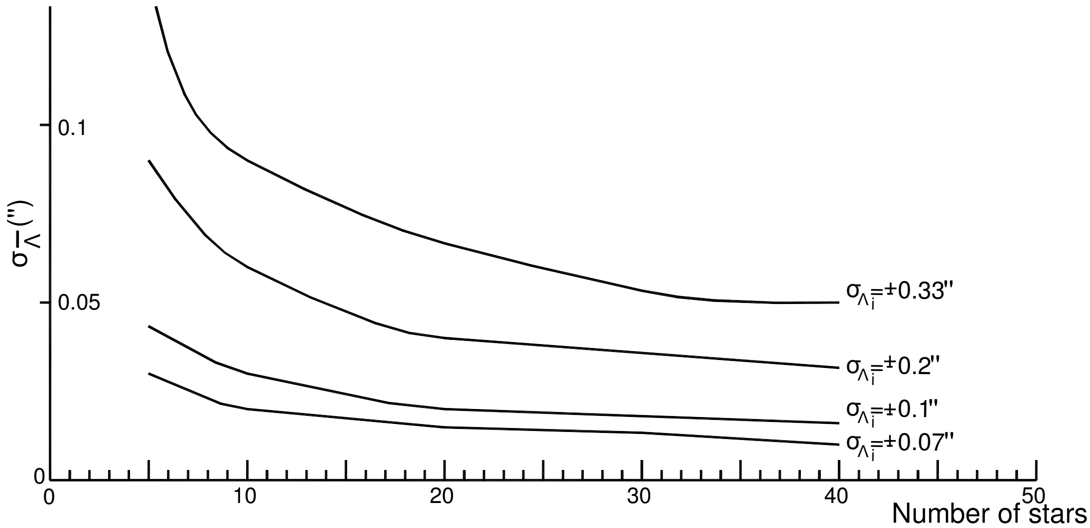

Figure 3. Diagramme of the a – priori error of the determination of Λ relative to the accuracy of the determination from a single star σ_{Λ_i} and the number of the observed stars.

After the measurements the final value $\overline{\Lambda}$ and its a-posteriori standard error of the determination comes out by the use of the least square method according to equation (4).

2.2 Determination of ΔN by astrogeodetic leveling

The deviation of the vertical is a vector composed of two mutually perpendicular components ξ,η. From the spherical triangle (Torge, 1991)

$$\sin\xi = \cos\eta \cdot \sin(\Phi - \phi) \tag{11}$$

$$\sin\eta = \cos\varphi \cdot \sin(\Lambda - \lambda) \tag{12}$$

As ξ , η , $\Phi - \phi$, $\Lambda - \lambda$ are small angles formulas (11) and (12) are universally expressed as:

$$\xi = \Phi - \varphi \tag{13}$$

$$\eta = (\Lambda - \lambda)\cdot\cos\varphi \tag{14}$$

while $\varepsilon = \xi \cdot \cos A + \eta \cdot \sin A$ (15)
is the component of the deviation in azimuth A.
Assuming linear variation of the deviation of the vertical between the endpoints P and P′ one gets:

$$\Delta N_{PP'} = -\left[\frac{\varepsilon_P + \varepsilon_{P'}}{2} \cdot\right] \cdot S_{PP'} \tag{16}$$

Combining the equations (15) and (16) the geoidal undulation difference between point P and P′ is given by the formula:

$$\Delta N_{PP'} = -\left[\frac{\xi_P + \xi_{P'}}{2} \cdot \cos A + \frac{\eta_P + \eta_{P'}}{2} \cdot \sin A\right] \cdot S_{PP'} \tag{17}$$

Where $\xi_P, \eta_P, \xi_{P'}, \eta_{P'} =$ the components of the deviation of the vertical, at each point P, P′.
A = the azimuth of the direction PP′
$S_{PP'}$ = the distance between the two points
Equation (17) gives reliable and accurate results, assuming that the change of the deviation of the vertical is linear between the two points and allows the determination of the geoidal differences for distances about 40Km.

2.2.1 Accuracy of the determination of ΔN

Applying variance-covariance propagation to formula (17) and assuming $\sigma_{\xi_P} = \sigma_{\xi_{P'}} = \sigma_\xi$ and $\sigma_{\eta_P} = \sigma_{\eta_{P'}} = \sigma_\eta$ the standard error of ΔN will be:

$$\sigma_{\Delta N} = \sigma_N = \pm S_{PP'} \cdot \sqrt{\frac{1}{2} \cdot \cos^2 A \cdot \sigma_\xi^2 + \frac{1}{2} \cdot \sin^2 A \cdot \sigma_\eta^2} \tag{18}$$

The error of the determination of the components of the deviation of the vertical σ_ξ, σ_η depends on the accuracy of the astronomical coordinates Φ,Λ and the geodetic coordinates φ, λ. This error can be estimated from formula (13) and (14) as follows:

$$\sigma_\xi = \pm\sqrt{\sigma_\Phi^2 + \sigma_\varphi^2} \tag{19}$$

and

$$\sigma_\eta = \pm \cos\varphi \cdot \sqrt{\sigma_\Lambda^2 + \sigma_\lambda^2} \tag{20}$$

If $\sigma_\Phi = \sigma_\Lambda$ and $\sigma_\varphi = \sigma_\lambda$ then $\sigma_\xi \approx \sigma_\eta$ and equation (18) becomes:

$$\sigma_{\Delta N} = \sigma_N = \pm S_{PP'} \cdot \sqrt{\frac{1}{2} \cdot (\cos^2 A + \cdot \sin^2 A) \cdot \sigma^2{}_\xi} = \pm S_{PP'} \cdot \frac{\sqrt{2}}{2} \cdot \sigma_\xi \tag{21}$$

The following diagram depicts the error in the calculation of ΔN that depends on the error in the calculation of the components ξ,η and the distance between the points.

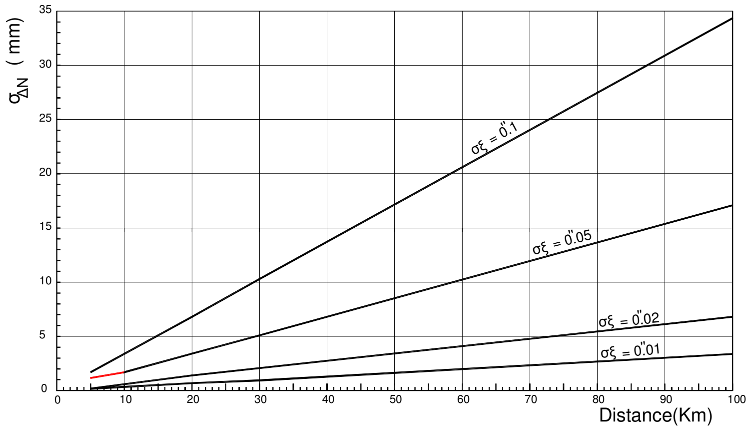

Figure 4. The accuracy of ΔN

3. Application

In Greece, it is well known, that there are large variations in the earth's surface with high mountains, flat areas and many islands. Therefore the worldwide gravimetric geoidal models can not give accurate values for the undulations of the geoid. Sometimes different models may give very different results for the same area, where is a strong need for a reliable and accurate determination of the geoidal undulations.

The proposed method was applied at three stations in the Athens area. Each station was occupied for a total of four hours. The geodetic coordinates φ,λ were given in the ITRF 89 system (epoch 2001) from GPS measurements with an accuracy of the order of ±0″.001 (Lambrou, 2003).

The components ξ, η of the deviation of the vertical in the ITRF 89 system (epoch 2001) as computed using formulas (13), (14) as well as the estimated accuracy of the astronomical coordinates Φ, Λ at each station are presented in table 1.

Table 1. The achieved accuracy of the observations

Point	Number of pairs of stars	σ_Φ (″)	Number of stars	σ_Λ (″)	SYSTEM ITRF 89 (EPOCH 2001)	
					ξ (″)	η (″)
I	13	± 0.013	17	± 0.016	-0.842	-7.290
II	11	± 0.011	22	± 0.016	0.400	2.541
III	12	± 0.015	23	± 0.010	-4.446	-7.148

The differences ΔN of the geoidal undulations between points I, II and III were computed by formula (17) and their accuracy by formula (21).

The same differences ΔN were also computed using one of the most reliable geopotential models, the EGM 96.

The difference ΔΔN between the value of ΔN as computed by the method of astrogeodetic leveling and computed by the model EGM 96 for stations I and III is negligible, but the differences between pillars III and II , II and I are about 40cm. This is a significant difference. The values of ΔN from the two computations, the differences ΔΔN in ΔN, as well as the accuracy achieved by astrogeodetic leveling and the distances between the stations are depicted in table 2.

Table 2. ΔN between stations from two different computations.

	Astrogeodetic	EGM 96	ΔΔN		
From– To	**ΔN(cm)**	**ΔN(cm)**	**(cm)**	**S (Km)**	**$\sigma_{\Delta N}$ (mm)**
I – III	-18.1	-17.0	-1.1	5	0.3
III – II	33.3	76.0	-42.7	21.5	1.0
II - I	-15.2	-59.0	40.8	17	0.8

4. Conclusions

- The system developed consisting of a modern high – precision digital total station connected to a GPS receiver together allows for the determination of the astronomical coordinates Φ, Λ with an accuracy of the order of ±0″.01 combining economy both in instrumentation and labour.
- The accuracy of Φ, Λ and the accuracy of the order of ±0″.001 in the geodetic coordinates φ, λ as achieved by GPS networks allows for the accurate determination of the components of the deviation of the vertical ξ, η.
- With the above mentioned accuracies in ξ, η it is feasible to compute geoidal undulation by astrogeodetic leveling with an accuracy of few millimeters and produce reliable maps of the geoid, even in areas with irregular geoidal undulations, where other methods are not appropriate.

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