Kinetic Theory for Distribution Functions of Wave-Particle Interactions in Plasmas

Y. Kominis,[1] A. K. Ram,[2] and K. Hizanidis[1]

[1]*School of Electrical and Computer Engineering, National Technical University of Athens, Association EURATOM-Hellenic Republic, Zographou GR-15773, Greece*
[2]*Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

The evolution of a charged particle distribution function under the influence of coherent electromagnetic waves in a plasma is determined from kinetic theory. For coherent waves, the dynamical phase space of particles is an inhomogeneous mix of chaotic and regular orbits. The persistence of long time correlations between the particle motion and the phase of the waves invalidates any simplifying Markovian or statistical assumptions—the basis for usual quasilinear theories. The generalized formalism in this Letter leads to a hierarchy of evolution equations for the reduced distribution function. The evolution operators, in contrast to the quasilinear theories, are time dependent and nonsingular and include the rich phase space dynamics of particles interacting with coherent waves.

The presence of coherent electromagnetic waves and their interaction with charged particles are ubiquitous phenomena in plasmas that are encountered in space as well as in laboratory fusion devices. The waves modify the distribution function of the charged particles which, in turn, through Maxwell's equations, modify the electromagnetic fields. The wave-particle interactions can, for example, saturate the growth of an instability in space plasmas, or change the current profile in a fusion device.

The evolution of a particle distribution function is usually described by the quasilinear theory (QLT) leading to a velocity (action) diffusion equation in which the wave-particle interactions are included through the diffusion operator [1]. It is assumed that the particles continuously interact with electromagnetic waves and that their motion is randomized, with respect to the phase of the wave, after one period of the wave. This is akin to the Markovian assumption used, for example, in studying Brownian motion. The motion is then characterized by completely uncorrelated particle orbits, phase mixing, loss of memory, and ergodicity. These statistical properties lead to an important advantage—the long time behavior of particle dynamics is the same as that after one interaction time with the wave. However, there is one significant drawback. The diffusion coefficient is singular, with a Dirac delta function singularity [1]. Difficulties related to the numerical implementation of the singularities are usually "cured" by considering a continuous spectrum and/or strongly chaotic dynamics [1]. The Markovian assumption is contrary to the dynamical behavior of particles interacting with coherent waves [2]. The particle phase space is a mix of chaotic and coherent motion with islands of coherent motion embedded within chaotic regions. Also, the phase space is bounded and near the boundaries, or near islands, particles can get stuck and undergo coherent, correlated, motion for times very much longer than an interaction time. Even when the amplitude of the waves is assumed

to be impractically large so that the entire phase is chaotic, as in the standard map, the QLT fails to give an appropriate description of the evolution of the distribution function [3]. The persistence of long time correlations invalidates the Markovian assumption [4–6]. Furthermore, in practice, particles do not continuously interact with the same spectrum of waves, either because the waves evolve in time or because the waves are spatially confined. Particles undergoing multiple transits are likely to drift away from the location where the previous interaction took place. This occurs in tokamaks where the radio frequency waves used for heating and current drive are localized over part of the plasma. In our derivation of the kinetic evolution equations for the distribution function we include the rich and complex phase space dynamics of the particles without resorting to any simplifying statistical assumptions that are not valid in most physical situations of interest involving wave-particle interactions.

The Hamiltonian for the particle dynamics

$$H(\mathbf{J}, \boldsymbol{\theta}, t) = H_0(\mathbf{J}) + \epsilon H_1(\mathbf{J}, \boldsymbol{\theta}, t) \tag{1}$$

consists of two parts: the integrable part $H_0(\mathbf{J})$ that is a function of the constants of the motion $\mathbf{J}$ of a particle moving in a prescribed equilibrium field [7,8], and

$$H_1(\mathbf{J}, \boldsymbol{\theta}, t) = \sum_{\mathbf{m} \neq 0} A_{\mathbf{m}}(\mathbf{J}) e^{i(\mathbf{m} \cdot \boldsymbol{\theta} - \omega'_{\mathbf{m}} t)}, \tag{2}$$

which includes electromagnetic waves and any perturbations to the equilibrium field. $\boldsymbol{\theta}$ are the angles canonically conjugate to the actions $\mathbf{J}$ and t is time. The complex frequency $\omega'_{\mathbf{m}} = \omega_{\mathbf{m}} + i\gamma_{\mathbf{m}}$ allows for steady state ($\gamma_{\mathbf{m}} = 0$), growing ($\gamma_{\mathbf{m}} > 0$), or damped ($\gamma_{\mathbf{m}} < 0$) waves. In the guiding center approximation for an axisymmetric tokamak equilibrium, the three actions are the magnetic moment, the canonical angular momentum, and the toroidal flux enclosed by a drift surface. The respective conjugate angles are the gyrophase, azimuthal angle, and poloidal

angle [7]. ϵ is an ordering parameter indicating that the effect of H_1 is perturbative.

The time evolution of any well-behaved function $f(\mathbf{z}, t)$ of $\mathbf{z}(t) = (\mathbf{J}(t), \boldsymbol{\theta}(t))$ from an initial time t_0 to time t is given by $f(\mathbf{z}(t; t_0), t) = S_H(t; t_0) f(\mathbf{z}_0, t_0)$ where $S_H(t; t_0)$ is the time evolution operator. The derivation of $S_H(t; t_0)$ is equivalent to solving the equations of motion. An appropriate way to determine $S_H(t; t_0)$ is to transform to a new set of canonical variables $\mathbf{z}' = (\mathbf{J}', \boldsymbol{\theta}')$ using an operator $T(\mathbf{z}, t)$. The transformation is such that the new Hamiltonian $K(\mathbf{z}')$ leads to a time evolution operator $S_K(t; t_0)$ that can be readily determined. A particularly useful transformation is one for which K is a function of $\mathbf{J}'$ only. Then $\mathbf{J}'$ are constants of the motion and $S_K(t; t_0)$ evolves the angles $\boldsymbol{\theta}'$ so that $f(\mathbf{z}'(t; t_0), t) = S_K(t; t_0) f(\mathbf{z}'_0, t_0) = f(\mathbf{J}'_0, \boldsymbol{\theta}'_0 + \Delta\boldsymbol{\theta}')$, where $\Delta\boldsymbol{\theta}' = \int_{t_0}^{t} \omega_K(\mathbf{J}'_0, s) ds$ and $\omega_K(\mathbf{J}'_0, t) = \nabla_{\mathbf{J}'_0} K(\mathbf{J}'_0, t)$.

The operator $T(\mathbf{z}, t)$ is determined using the Lie transform theory: $T = e^{-L}$ where $Lf = [w, f]$. The Poisson bracket is defined as $[a, b] = \nabla_{\boldsymbol{\theta}} a \cdot \nabla_{\mathbf{J}} b - \nabla_{\mathbf{J}} a \cdot \nabla_{\boldsymbol{\theta}} b$. The function $w(\mathbf{z})$ is the Lie generator. The Lie transform theory generates canonical transformations such the operator T commutes with any function of $\mathbf{z}$ [9]. This important property implies that the evolution of $f(\mathbf{z}, t)$ can be evaluated by transforming to $\mathbf{z}'$, applying the time evolution operator $S_K(t; t_0)$ to the transformed function, and then transforming back to $\mathbf{z}$. Thus [9],

$$f(\mathbf{z}(t; t_0), t) = T(\mathbf{z}_0, t_0) S_K(t; t_0) T^{-1}(\mathbf{z}_0, t_0) f(\mathbf{z}_0, t_0), \quad (3)$$

where $T^{-1} = e^{L}$ is the inverse operator.

For physical systems of interest described by (1) and (2), it is unlikely that T can be completely determined. However, for nearly integrable systems, the Lie transform theory can be applied to determine T perturbatively as a power series in ϵ [9]. The old Hamiltonian H, the new Hamiltonian K, the transformation operator T, and the Lie generator w are expressed as a power series in ϵ: $X(\mathbf{z}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n X_n(\mathbf{z}, t)$, where X represents any of the variables H, K, T, L, w [9]. Here w_0 is chosen so that T_0 is the identity transformation I. Through second order, the transformations T and T^{-1} are $T_0 = I$, $T_1 = -L_1$, $T_2 = -\frac{1}{2} L_2 + \frac{1}{2} L_1^2$ and $T_0^{-1} = I$, $T_1^{-1} = L_1$, $T_2^{-1} = \frac{1}{2} L_2 + \frac{1}{2} L_1^2$, respectively. The generating functions are given by

$$\frac{\partial w_n}{\partial t} + [w_n, H_0] = n(K_n - H_n)$$

$$- \sum_{m=1}^{n-1} (L_{n-m} K_m - m T_{n-m}^{-1} H_m). \quad (4)$$

The left-hand side of Eq. (4) is the total time derivative of w_n along the unperturbed orbits obtained from H_0. So w_n is determined by integrating along these orbits. In order to eliminate the dependence of the new Hamiltonian on $\boldsymbol{\theta}$, we impose the condition that K_n's are either functions of the new actions only or constants. Then,

$$w_1 = -\sum_{\mathbf{m} \neq 0} A_{\mathbf{m}}(\mathbf{J}) e^{i[\mathbf{m} \cdot (\boldsymbol{\theta} - \omega_0(\mathbf{J})t)]} \frac{e^{i\Omega'_{\mathbf{m}}(J)t} - e^{i\Omega'_{\mathbf{m}}(J)t_0}}{i\Omega'_{\mathbf{m}}(\mathbf{J})}, \quad (5)$$

where we have set $K_1 = 0$ and $\Omega'_{\mathbf{m}}(\mathbf{J}) = \mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{J}) - \omega'_{\mathbf{m}}$ with $\boldsymbol{\omega}_0(\mathbf{J}) = \nabla_{\mathbf{J}} H_0$ being the frequency vector of the unperturbed system. Similarly, we can set $K_2 = 0$ and derive an equation for w_2.

The Lie generators in the finite time interval $[t_0, t]$ lead to $w_n(\mathbf{z}_0, t_0) = 0$ and, consequently, $T(\mathbf{z}_0, t_0) = I$. Since $K_n = 0$, $(n = 1, 2)$, the evolution operator S_K is the evolution in time along the unperturbed orbits given by H_0. Thus, $S_K = S_{K_0} = S_{H_0}$. The time evolution of $f(\mathbf{z}, t)$ in Eq. (3) from $t = t_1$ to $t = t_2$ is given by

$$f(\mathbf{z})_{t_2} = T^{-1}(\mathbf{z}_{t_1} + \Delta\mathbf{z}, t_2) f(\mathbf{z})_{t_1}, \quad (6)$$

where $f(\mathbf{z})_t = f(\mathbf{z}(t))$ with $\Delta\mathbf{z}$ being evaluated along unperturbed orbits. Equation (6) is a functional mapping which maps f at time $t = t_1$ to f at time $t = t_2$. If we choose $f = \mathbf{z}$, i.e., f is the set composed of the dynamical variables, the mapping (6) gives a near-symplectic mapping for the evolution of $\mathbf{z}$ [10]. When $f(\mathbf{z})$ is chosen to be the particle distribution function, Eq. (6) is an approximation to the original Vlasov (Liouville) equation to the same order as the operator T^{-1}. Equation (6) is an iterative scheme for the time evolution of f in the same way as symplectic [11], or near-symplectic [10], mappings are for the evolutions of particle orbits. The accuracy of the mapping depends on an effective perturbation strength which is proportional to ϵ as well as to the time step $\Delta t = t_2 - t_1$ [11]. Thus, Eq. (6) applies to any perturbation strength provided that the time step is sufficiently small to control the accuracy of the mapping.

For a particle distribution function $f(\mathbf{J}, \boldsymbol{\theta})$, let us define a function $F(\mathbf{J}, \boldsymbol{\theta}_s)$, where $\boldsymbol{\theta}_s$ is a subset of $\boldsymbol{\theta}$. F is obtained from f by averaging over the angles $\bar{\boldsymbol{\theta}}$ which are not in the set $\boldsymbol{\theta}_s$, i.e., $\bar{\boldsymbol{\theta}} = \boldsymbol{\theta} - \boldsymbol{\theta}_s$. Then, from (6),

$$F(\mathbf{J}, \boldsymbol{\theta}_s)_{t_2} = \langle T^{-1}(\mathbf{J}, \boldsymbol{\theta}_s + \Delta\boldsymbol{\theta}_s, t_2) \rangle_{\bar{\boldsymbol{\theta}}} F(\mathbf{J}, \boldsymbol{\theta}_s)_{t_1}, \quad (7)$$

where $\langle \ldots \rangle_{\bar{\boldsymbol{\theta}}}$ denotes averaging over $\bar{\boldsymbol{\theta}}$. Here the operator $T^{-1}(\mathbf{z}, t)$, averaged over $\bar{\boldsymbol{\theta}}$, acts on a function of $\mathbf{J}$ and $\boldsymbol{\theta}_s$. From the second order expansion, and the fact that all functional dependencies on $\boldsymbol{\theta}$ are periodic with respect to $\boldsymbol{\theta}$, differentiating Eq. (7) yields

$$\frac{\partial F(\tilde{\mathbf{z}}, t)}{\partial t} = \nabla_{\tilde{\mathbf{z}}}[\mathbf{D}(\tilde{\mathbf{z}}, t) \nabla_{\tilde{\mathbf{z}}} F(\tilde{\mathbf{z}}, t_0)] + \mathbf{C}(\tilde{\mathbf{z}}, t) \nabla_{\tilde{\mathbf{z}}} F(\tilde{\mathbf{z}}, t_0)$$

$$(8)$$

with $\tilde{\mathbf{z}} = (\mathbf{J}, \boldsymbol{\theta}_s)$,

$$\mathbf{D}(\tilde{\mathbf{z}}, t) = \frac{1}{2} \frac{\partial}{\partial t}$$
$$\times \begin{pmatrix} \langle (\nabla_{\boldsymbol{\theta}} w_1)^2 \rangle_{\bar{\boldsymbol{\theta}}} & -\langle (\nabla_{\mathbf{J}} w_1)(\nabla_{\boldsymbol{\theta}} w_1) \rangle_{\bar{\boldsymbol{\theta}}} \\ -\langle (\nabla_{\mathbf{J}} w_1)(\nabla_{\boldsymbol{\theta}} w_1) \rangle_{\bar{\boldsymbol{\theta}}} & \langle (\nabla_{\mathbf{J}} w_1)^2 \rangle_{\bar{\boldsymbol{\theta}}} \end{pmatrix},$$
$$(9)$$

$$\mathbf{C}(\tilde{\mathbf{z}}, t) = \frac{\partial}{\partial t}(\langle \nabla_{\boldsymbol{\theta}}(w_1 + w_2/2)\rangle_{\bar{\boldsymbol{\theta}}}, \langle \nabla_{\mathbf{J}}(w_1 + w_2/2)\rangle_{\bar{\boldsymbol{\theta}}}),$$
(10)

where we have set $t_1 = t_0$ as the initial time and $t_2 = t$ as the running time. $\mathbf{D}$ is usually referred to as the diffusion tensor and $\mathbf{C}$ as the friction vector. Since, on the right-hand side, F depends on the initial time t_0, Eq. (8) is not a usual Fokker-Planck (FP) type of equation. Substituting $f = \mathbf{z}$ in Eq. (6) gives $\mathbf{C} = \lim_{\Delta t \to 0}\langle(\Delta \mathbf{z})\rangle_{\bar{\boldsymbol{\theta}}}/\Delta t$, and $\mathbf{D} = \lim_{\Delta t \to 0}\langle(\Delta \mathbf{z})(\Delta \mathbf{z})\rangle_{\bar{\boldsymbol{\theta}}}/2\Delta t$ where $(\Delta \mathbf{z})$ is the variation of $\mathbf{z}$. This form of $\mathbf{C}$ and $\mathbf{D}$ is similar to the usual quasilinear diffusion coefficients [12].

If the Lie transform technique is carried out to higher orders in ϵ, there appear higher order derivatives of F in the right-hand side of Eq. (8) [13]. This is analogous to the Kramers-Moyal expansion of the master equation in stochastic processes [12].

Since Lie operators acting on any function of the dynamical variables can be commuted through the function to act directly on the dynamical variables, the evolution of the particle distribution function is related to single particle dynamics. The Lie generating functions which determine the structure of $\mathbf{D}$ and $\mathbf{C}$ in (9) and (10) are related to approximate invariants of the particle dynamics, when Eqs. (4) are solved in the infinite time interval. The level curves of these approximate invariants provide the structure of the phase space that appears in Poincare surfaces of section, including resonant islands and KAM (Kolmogorov-Arnold-Moser) curves [2]. Consequently, all essential information for the resonant structure of the dynamical phase space is included in the formalism. The inhomogeneity of the phase space manifests itself in the diffusion tensor $\mathbf{D}$ through w_n. Thus, the topology of all of phase space is in Eq. (8).

If we do not average over any of the angles, Eq. (8) is the evolution equation for the complete distribution function. The sequential averaging of one angle at a time generates a hierarchy of evolution equations for the appropriately angle-averaged distribution function. In each step of this hierarchy the dimension of the phase space for the distribution function is reduced. While each angle variable varies more rapidly than its canonically conjugate action variable, it may not necessarily evolve faster than the time for wave-particle interactions. For example, in a tokamak plasma the particle gyration angle is averaged over since it corresponds to the fastest time scale. However, the poloidal or toroidal angles of the particle vary more slowly and can be included in the hierarchical description [8]. The averaging process does not affect the accuracy of the perturbation theory. The elements of $\mathbf{D}$ in (9) can be analytically evaluated even when we include all the canonical angles. The physical consequences associated with averaging over one or more angles can be determined by the change in each element of $\mathbf{D}$.

The averaging over all angles leads to an evolution equation [Eq. (8)] where the distribution function depends on the actions only. Then $\mathbf{C} = \mathbf{0}$ and $\mathbf{D}$ is completely determined by w_1 in Eq. (5). So, to second order in ϵ, the time evolution equation for the action distribution function depends only on the first order effects in particle dynamics. This result is akin to Madey's theorem for wave-particle interactions in microwave sources [14]. Besides an explicit form for the diffusion tensor and the friction vector, our procedure provides the self-adjoint form of the evolution equation for the distribution function for particles whose motion is described by a Hamiltonian system with arbitrary number of degrees of freedom [15]. In action space,

$$\mathbf{D}(\mathbf{J}, t) = \sum_{\mathbf{m} \neq 0} \frac{\mathbf{mm}|A_{\mathbf{m}}(\mathbf{J})|^2 e^{2\gamma_{\mathbf{m}}t}}{\Omega_{\mathbf{m}}(\mathbf{J})^2 + \gamma_{\mathbf{m}}^2}$$
$$\times \left\{ \gamma_{\mathbf{m}}[1 - e^{-\gamma_{\mathbf{m}}t}\cos(\Omega_{\mathbf{m}}(\mathbf{J})t)] \right.$$
$$\left. + \Omega_{\mathbf{m}}(\mathbf{J})e^{-\gamma_{\mathbf{m}}t}\sin(\Omega_{\mathbf{m}}(\mathbf{J})t) \right\},$$
(11)

where $\Omega_{\mathbf{m}}(\mathbf{J}) = \mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{J}) - \omega_{\mathbf{m}}$ and $\mathbf{mm}$ is a dyadic. Given this $\mathbf{D}$, Eq. (8) reduces to a FP equation when $\nabla_{\mathbf{J}}F(\mathbf{J}, t) \simeq \nabla_{\mathbf{J}}F(\mathbf{J}, t_0)$. Physically, this implies that the evolution of F occurs over times that are shorter than the relaxation time for F. In contrast to the traditional QLT [1], our kinetic evolution equation has a time-dependent tensor $\mathbf{D}$ which does not distinguish between resonant and nonresonant particles. Also, again in contrast to QLT, our formalism is the same for growing or damped waves. In the vicinity of resonances given by $\Omega_{\mathbf{m}} = 0$, $\mathbf{D}$ is continuous and nonsingular even when $\gamma_{\mathbf{m}} = 0$. The width of the resonance decreases with time. The time-dependent $\mathbf{D}$ is similar to the "running diffusion tensor" discussed by Balescu [16]. However, there is one significant difference. The $\mathbf{D}$ obtained above depends on the dynamical actions and includes inhomogeneous resonant structure of the phase space. Balescu's tensor is independent of actions and applies to a Markovian-type of chaotic phase space.

In the limit $t \to \infty$, and for $\gamma_{\mathbf{m}} = 0$, Eq. (11) leads to the time-independent quasilinear diffusion tensor $\mathbf{D}_{\mathbf{ql}}(\mathbf{J}) = \sum_{\mathbf{m} \neq 0}\mathbf{mm}|A_{\mathbf{m}}(\mathbf{J})|^2\delta(\Omega_{\mathbf{m}}(\mathbf{J}))$, where δ is Dirac's delta function. The long time limit is justified only for statistically random, or Markovian, processes. [2]. The singular delta function excludes short time transient effects and is difficult to implement numerically. Importantly, the asymptotic time limit results in a time-irreversible FP equation while the time-dependent $\mathbf{D}$ in (11), being an odd function of time, leads to a time-reversible evolution equation.

We illustrate the differences between our formalism and the usual QLT for a one dimensional unperturbed particle Hamiltonian $H_0(J) = J^2/2$. The initial distribution function is assumed to be a Maxwellian $F_0(J) = (1/\sqrt{2\pi}) \times \exp(-J^2/2)$, and the perturbing field is a Gaussian wave packet with $\gamma_m = 0$. In Fig. 1 we plot $D(J, t)$, from Eq. (11), as a function of J and t. It is evident that as $t \to \infty$, $D(J, t)$ approaches, as expected, the quasilinear
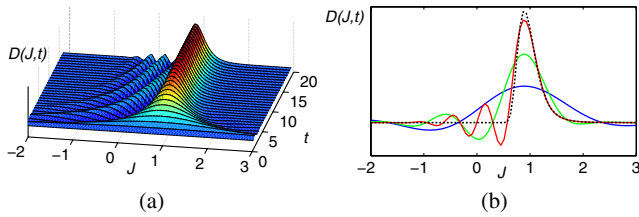
FIG. 1 (color online). (a) $D(J, t)$, from (11), as a function of J and t for a continuous spectrum of waves, $A_m = 0.003 \exp[-(m-1)^2/(2 \times 0.3^2)]$, $\omega_m = 1$, and $\gamma_m = 0$. (b) $D(J, t)$ as a function of J for $t = 3\pi/4$ (blue), $3\pi/2$ (green), 3π (red), and ∞ (black dashed). The black dashed curve corresponds to the quasilinear diffusion coefficient.

form. The difference between our model and the usual QLT becomes clear when we look at the evolution of the angle-averaged $F(J, t)$. In Fig. 2(a) we plot the evolution of $F(J, t)$ from its initial Maxwellian state for $D(J, t)$ given in Eq. (11). The corresponding evolution for a time-independent quasilinear diffusion coefficient is plotted in Fig. 2(b). The time-dependent D of Eq. (11) leads to early time effects that persist for all times. These effects are not at all present in the QLT result. Consequently, the long time behavior of the two distribution functions differs significantly. The implication of this difference is very important. If we take the limit $t \to \infty$ of Eq. (8), then this limit cannot be commuted through the derivative on the right hand side. Otherwise, the long time behavior in Fig. 2(a) would have been the same as in Fig. 2(b). Thus, the usual QLT is incapable of accounting for diffusive effects at early times which affect the long time behavior of the particle distribution function.

In conclusion, we have derived a hierarchy of evolution equations for distribution functions of particles interacting with coherent waves in a plasma. The derivation does not make any Markovian or statistical assumptions, either for the particle dynamics or for the wave spectra, that are the crux of the standard quasilinear theories. The final kinetic equation in the hierarchy, obtained by averaging over all the canonical angles, is an evolution equation for the distribution function in the action space. The diffusion operator in this equation is nonsingular and time-dependent, and includes the inhomogeneity of the dynamical phase space composed of chaotic motion and correlated
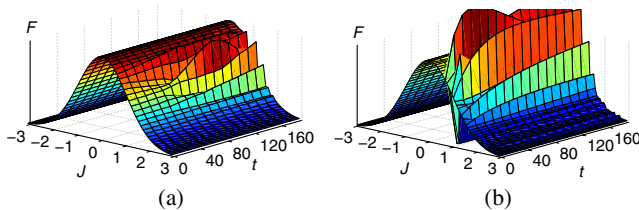
motion. Moreover, in contrast to the quasilinear theory, our formalism is capable of describing transient effects. The asymptotic behavior of the distribution function obtained from our theory differs markedly from that of quasilinear theory. Since our formalism is quite general, the kinetic equation can include physical processes that cannot be implemented in the standard quasilinear approach.

FIG. 2 (color online). Evolution of $F(J)$ obtained from (a) Eq. (8) using Eq. (11); (b) Eq. (8) using quasilinear $D_{\rm ql}$.

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