Transient Dynamics of Charged Particles Interacting with Localized Waves of Continuous Spectra

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A modified canonical perturbation method is employed for analyzing the charged particle dynamics as they interact with localized waves with continuous spectrum. In contrast with periodic Hamiltonian models, where the method has already been applied in a multitude of respective systems, the system in hand is inherently aperiodic. The localized waves have the form of amplitude modulated electrostatic fields, ranging from ordinary wave packets to ultrashort pulses. The analytically obtained approximate invariants of the motion contain rich information for the structure of the phase space and the respective distribution functions.

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Wave-particle interaction is one of the most well studied subjects in the physics literature with numerous applications in plasma physics, accelerators, microwave sources, lasers, and other branches of physics. Also, particle dynamics under the presence of electrostatic or electromagnetic waves has been one of the main paradigms, on which the modern theory of nonlinear Hamiltonian dynamics and chaos has been applied [1,2]. However, all previous studies of wave-particle interactions from the point of view of Hamiltonian dynamics have been focused in waves having discrete spectra, namely, periodic waves. The periodicity of the perturbation allows for direct application of results of the Hamiltonian methods as previously obtained in the field of celestial mechanics, where the Hamiltonian formalism has been first applied and where bounded periodic motion and periodic perturbations mostly occur. In these cases the systems are considered as being in steady state and their dynamical features are studied from the point of view of long-time behavior. On the other hand, in many interesting realistic applications of wave-particle interactions, the waves have an amplitude modulated profile in the form of a front or a pulse, and a localized wave, with a continuous spectrum, such as a solitary wave (SW), is involved. Among them we may refer to the rf plasma heating [1] as well as to the investigation of damping of localized waves in plasmas [3]. In the latter the transit time particle acceleration has been considered as the principal dissipation mechanism for the Langmuir soliton collapse. Also, particle dynamics in the case of interactions of short laser pulses with plasmas has several applications in plasma heating, current drive, and diagnostics in fusion devices, while other relevant applications refer to ionospheric modification by rf waves and pulse propagation in space and astrophysical plasmas [4]. As far as the modeling of these interactions is concerned, various approaches, assumptions, and tools have been employed so far, in the scientific literature: Several works are based on the discretization of the spectrum of the wave packets involved [5], while others treat the particle dynamics on the basis of a direct perturbation approach [3,6]. Kinetic-theoretical approaches have also been employed and the Vlasov equation has mainly been used [7] in limiting cases for the ratio of the modulation time scale with the transit time of the particle through a localized wave. Adiabatically modulated or very sharply localized fields fall in this category. Other studies, within the context of the Hamiltonian approach, have also been based on the adiabaticity assumption [8].

In this work we study charged particle dynamics under the presence of one or more electrostatic SWs having different phase and group velocities and propagating in the absence of magnetic field or along a uniform magnetic field, B_0, in a magnetized medium. The forms of the electric field considered have continuous spectra. They range from ordinary wave packets and solitons to ultrashort few-cycle and subcycle transient pulses. It is worth mentioning that, for the latter, the assumption of adiabaticity for the amplitude modulation, adopted in the aforementioned previous works, does not hold. The Hamiltonian formalism is used, providing an insightful context of analysis. The specific Hamiltonian system, apart from being related to the previously mentioned applications, also serves as a paradigm for studying transient dynamics of aperiodic or finite-time systems, in general. Such dynamics also occur in chaotic scattering, where the nonperiodicity of the motion results from the localized form of the potential [9].

The particle dynamics are analyzed on the basis of the canonical perturbation method (CPM) [10]. This method allows us to construct approximate invariants of the motion for the nonintegrable Hamiltonian system describing particle motion. However, the aperiodic character of the Hamiltonian perturbation necessitates a modification of the CPM, in agreement with recent extensions of the KAM theorem for aperiodic perturbations [11]. The resulting invariants contain all the essential information for the strongly inhomogeneous phase space of the system, which is described via appropriate Poincaré surfaces of section. Moreover, it is shown that the aperiodic character of the SW results in chaotic transient momentum variation. The latter depends strongly on the initial particle momentum, in terms of a resonant condition, and also there is a complex dependency on the initial particle position: Neighbor particles having the same initial momentum end up with different momenta after their transition through the SW.

Particle dynamics under the presence of a set of electrostatic SW can be described by the following Hamiltonian:

$$H = \frac{p_z^2}{2} + \sum_n \Phi_n(z - v_{g_n} t) \sin[k_n(z - v_{p_n} t)], \qquad (1)$$

where Φ_n is the profile of the electrostatic potential and v_{g_n} and v_{p_n} are the group and phase velocities of the SW, respectively. Considering the potential part of the Hamiltonian as a perturbation to the free particle motion, and following the standard procedure, according to the CPM ([10], p. 78), we seek an approximate solution of the Hamilton-Jacobi equation by utilizing a near-identity canonical transformation to new variables $(\bar{p}_z, \bar{z})$ for which the new Hamiltonian $\bar{H}$ is a function of the momentum $\bar{p}_z$ alone. To lowest order we obtain the identity transformation, $S_0 = \bar{p}_z z$, while to the first order of perturbation, the transformation can be obtained from the equation:

$$\frac{\partial S_1}{\partial t} + v_z \frac{\partial S_1}{\partial z} = -H_1, \qquad (2)$$

where S_1 is the first order term of the generating function $S(\bar{p}_z, z)$, v_z is the particle velocity, and H_1 is the potential part of the Hamiltonian. In order to solve the linear differential Eq. (2), instead of using the usual Fourier series method [10], which applies for periodic perturbations, the Fourier transform is used, yielding

$$S_1 = \sum_n \frac{e^{i(k_n z - \omega_n t)}}{v_z - v_{g_n}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\bar{\Phi}_n(k)}{k + k_n a_n} e^{ikz} dk + \text{c.c.}, \qquad (3)$$

where $\bar{\Phi}_n$ is the Fourier transform pair of Φ_n and

$$a_n = \frac{v_z - v_{p_n}}{v_z - v_{g_n}}. \qquad (4)$$

Using the convolution property of the Fourier transform and taking properly into account the contribution of the pole of the integrand, S_1 can be written as follows:

$$S_1 = \sum_n i \frac{e^{i[k_n(1-a_n)z - \omega_n t]}}{v_z - v_{g_n}} \int_{-\infty}^{z} \Phi_n(\zeta) e^{ik_n a_n \zeta} d\zeta + \text{c.c.} \qquad (5)$$

The new momentum $\bar{p}_z$ is given to first order by

$$\bar{p}_z = p_z - \frac{\partial S_1}{\partial z}, \qquad (6)$$

where the function S_1 can be evaluated in terms of (p_z, z) within the first order approximation. Since the transformed Hamiltonian is not a function of the new position $\bar{z}$, the new momentum is, thus, an approximate (to first order) invariant of the motion for the perturbed system. Higher order approximations of the invariant of the motion, necessary for increasing perturbation strength, can be easily obtained by utilizing the Lie transforms method [10]. However, in the following it will be shown that, even at this first order approximation, the invariant (6) provides useful information for the structure of the phase space of the system. It is remarkable that the calculation of the first order invariant does not require any assumption on either the scale of the argument (adiabaticity) or the form of Φ_n, provided that the corresponding Fourier integral is well defined. Thus, in the context of the invariant (6), particle interactions with both slowly modulated fields and subcycle pulses can be studied as well as different kinds of amplitude profiles.

First, we investigate particle dynamics under interaction with one SW, having a Gaussian profile of the form

$$\Phi_n(x) = A_n e^{-x^2/(2\sigma_n^2)}, \qquad n = 1. \qquad (7)$$

Using (5) we obtain

$$S_1 = -\sqrt{\frac{\pi}{2}} \frac{\sigma_1 A_1}{v_z - v_{g_1}} e^{-\sigma^2 (k_1 a_1)^2/2} e^{i[k_1(1-a_1)z - \omega_1 t]} \times \left[1 + \text{erf}\left(\frac{z - v_{g_1} t - i a_1 k_1 \sigma^2}{\sqrt{2}\sigma}\right)\right] + \text{c.c.}, \qquad (8)$$

where "erf" is the error function for complex argument. The form of S_1 implies two major consequences for the particle dynamics: (a) the effective strength of the perturbation is proportional to the product of the field amplitude A_1 and the field width σ_1, which is intuitively expected and is in agreement with the time scaling property [8] of the Hamiltonian (1); (b) the presence of the SW affects strongly particles with initial velocities around the resonant velocity given by $a_1 = 0$ or $v_z = v_{p_1}$, within an area, the width of which is determined by the product $\sigma_1 k_1$, as indicated by the exponential term of (8). The width σ_1 of the SW determines its bandwidth and correspondingly the resonant velocity spectrum.

In Fig. 1, numerically and analytically obtained Poincaré surfaces of section, in the extended phase space, are shown, for the case of a SW having $A = 0.005$, $\omega_1 = 1$, $k_1 = 1$, and $\sigma_1 = 1, 10, 30$, respectively. For simplicity we have considered zero group velocity ($v_{g_1} = 0$) of the SW, since the difference between the particle and the group velocity determines only the distance at which the particle enters the SW. The numerical results are obtained through direct numerical integration of the equations of motion. For each value of initial velocity 500 initial positions equally distributed in a range $[-12\sigma_1, -12\sigma_1 + 2\pi]$ are considered,

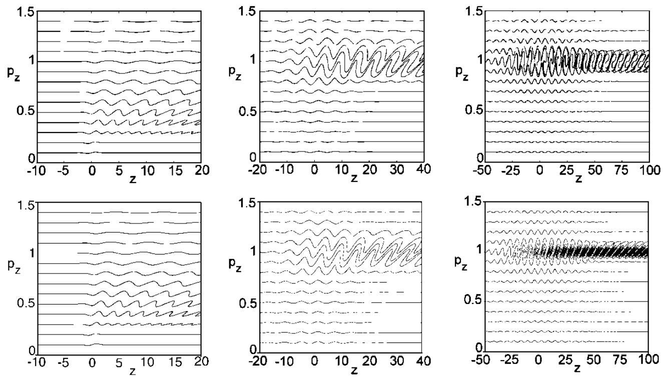

FIG. 1. Numerically (top) and analytically (bottom) obtained Poincaré surfaces of section for interaction with a SW having $A_1 = 0.005$, $\sigma_1 = 1, 10, 30$ (left to right), $\omega_1 = 1$, $v_{p_1} = 1$, $v_{g_1} = 0$.

and (z, v_z) values are recorded at times $t_i = 2\pi i/\omega_1$ ($i = 0, 1, \ldots$). The analytical results are obtained as contour plots ($\bar{p}_z = \text{const}$) of the approximate invariant of the motion through (6) where we have substitute S_1 from (8). In all cases, there is remarkable agreement between the numerical and analytical results.

The phase space is strongly inhomogeneous and, within the resonant areas, particle velocity changes during particle transition through the SW. The particles, after exiting SW, acquire different constant velocities, the values of which depend strongly on their initial position. Thus, after a finite time (particle transition through the pulse) for any two orbits with neighboring initial conditions, their momentum difference remains constant while the position difference grows linearly with time. Outside the resonant areas, particle velocities change slightly during transition and return to their initial values, when the particles exit from the SW. Moreover, the SW width σ_1 is shown to affect both the perturbation strength and the width of the resonant velocities area. It is worth mentioning that for ultrashort SW, such as the subcycle SW shown in Fig. 1 (top), the resonant area is not centered on the phase velocity v_{p_1}, since there are not enough cycles of the carrier wave to affect the particles, a fact that renders the phase velocity meaningless. The increase of the SW width is shown to result in the centering of the resonant area on the phase velocity, as well as in the localization of the strong interaction area.

The extreme values for exiting particle velocities, $v_{z,\text{out}}$, can easily be obtained through the first order invariant, and are given by the

$$v_{z,\text{in}} = v_{z,\text{out}} \pm \sqrt{2\pi}\frac{A_1\sigma_1 k_1(1-a_1)}{v_{z,\text{out}} - v_{g_1}} e^{-\sigma^2(k_1 a_{1,\text{out}})^2/2}. \quad (9)$$

In Figs. 2(a) and 2(b) estimates of maximum and minimum velocity variations $\Delta v_{\text{max,min}} = (v_{z,\text{out}} - v_{z,\text{in}})_{\text{max,min}}$, based on (9), are shown, and compared with numerical results. The width of the resonant area as well as the

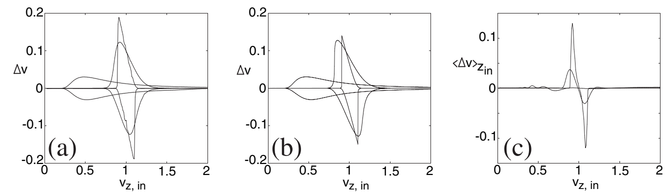

FIG. 2. Numerically (a) and analytically (b) obtained extrema of particle velocity variation Δv for interaction with a SW having $A_1 = 0.005$, $\sigma_1 = 1, 10, 30$, $\omega_1 = 1$, $v_{p_1} = 1$, $v_{g_1} = 0$. Larger values for σ_1 result in stronger localization of the resonant velocity area. (c) Position-averaged particle velocity variation.

strength of the resonance can be estimated through Eq. (9). Because of the sensitivity of the final particle velocity on its initial position and in order to obtain more information for the particle velocity redistribution after interaction with the SW, we numerically compute the average velocity variation, with respect to the initial position $\langle \Delta v \rangle_{z_{\text{in}}}$, which is depicted in Fig. 2(c). It is shown that, in order to have significant average variations, sufficient width of the SW is required (for a given amplitude). Moreover, the initial velocities resulting in extreme average variations are comparable to the ones resulting in extreme variations. Since the average velocity variation is related to the momentum (and energy) exchange between the particles and the SW, one may draw the following conclusion: depending on the value of the initial velocity of a particle beam with respect to the phase velocity, SW amplification or damping may occur.

However, the collective characteristics of particle beams are usually described by the distribution function $F(p_z, z, t)$, which fulfills the Vlasov equation

$$\frac{\partial F}{\partial t} + [F, H] = 0, \quad (10)$$

where [,] are the Poisson brackets. It is well known that, in an integrable system, any function of the invariants of the motion forms a solution of the Vlasov equation. Thus, the approximate invariant of the motion (6) can be used in order to obtain approximate solutions of (10). Setting $\Delta p_z \equiv \partial S_1/\partial z = O(\epsilon)$ and Taylor expanding (6) and $F(\bar{p}_z)$ readily yields

$$F(p_z, z, t) = F_0(p_z) - \frac{\partial F_0}{\partial p_z}\Delta p_z + \frac{1}{2}\frac{\partial}{\partial p_z}\left(\Delta p_z^2 \frac{\partial F_0}{\partial p_z}\right), \quad (11)$$

where we have assumed an initially (at $z_0 - v_g t_0 \rightarrow -\infty$) position-independent (uniform) distribution function. It is worth mentioning that the simple expression for the approximate distribution function given by (11) and (5) has been obtained under no adiabaticity assumption [7], and thus, it is valid for ultrashort pulses as well as solitons and slowly modulated wave packets. Moreover, it can be used for calculations of certain quantities, such as position-averaged momentum variations for any initial momentum

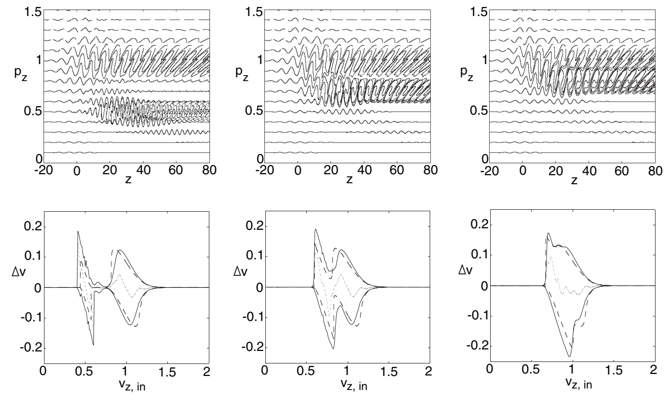

FIG. 3. Interaction with two SWs. (Top) Numerically obtained Poincaré surface of section. (Bottom) Numerically (solid) and analytically (dashed) obtained extrema of velocity variation. The averaged velocity variation is also shown (dotted line). The parameters of the SWs are $A_{1,2} = 0.005$, $\sigma_{1,2} = 10$, $\omega_{1,2} = 1$, $v_{g_1} = 0$, $v_{g_2} = 0.1$, $v_{p_1} = 1$, $v_{p_2} = 0.5, 0.7, 0.8$ (left to right).

distribution. It also generalizes results related to Madey's theorem, which require the assumption of periodicity of the position coordinate [12] that does not hold in our case.

The presence of more than one SW, having different phase velocities, results in multiple resonant areas of strong interaction in the phase space. Depending on the amplitude, width, and phase velocity of each wave, the resonances can be well separated, weakly overlapping, or strongly overlapping, in direct analogy to the case of periodic waves, where the corresponding strong interaction areas are centered on the respective resonant frequencies of the system. In Fig. 3, the cases of particle dynamics under the presence of two SW having $A_{1,2} = 0.005$, $\sigma_{1,2} = 10$, $\omega_{1,2} = 1$, $v_{p_1} = 1$, and $v_{p_2} = 0.5, 0.7, 0.8$, respectively. The group velocities are $v_{g_1} = 0$ and $v_{g_2} = 0.1$, and the nonzero group velocity of the second SW implies that particles with velocities $v_z \leq v_{g_2}$ actually do not transit this SW. The Poincaré surfaces of section, shown in Fig. 3 (top), are obtained numerically, with initial conditions chosen as in Fig. 1. In Fig. 3 (bottom) the extreme values of velocity variations are shown $\Delta v_{\mathrm{max,min}}$, as well as the average velocity variation with respect to the initial position $\langle \Delta v \rangle_{z,\mathrm{in}}$. The resonance overlap is shown to result in merging of the corresponding resonant areas. It is worth mentioning that the analytic results also apply directly to cases where particles interact with aperiodic sequences (series) of multiple SWs, which differ in their initial spatiotemporal positions.

In summary, a modified CPM has been applied in order to study particle dynamics under the presence of one or more localized pulses of an arbitrary profile. For the specific case of Gaussian pulses the resonant structure of the phase space has been investigated through analytically obtained approximate invariants of the motion, which also lead to analytical approximations of the respective distribution function.

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