Power dependent soliton location and stability in complex photonic structures

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Abstract: The presence of spatial inhomogeneity in a nonlinear medium results in the breaking of the translational invariance of the underlying propagation equation. As a result traveling wave soliton solutions do not exist in general for such systems, while stationary solitons are located in fixed positions with respect to the inhomogeneous spatial structure. In simple photonic structures with monochromatic modulation of the linear refractive index, soliton position and stability do not depend on the characteristics of the soliton such as power, width and propagation constant. In this work, we show that for more complex photonic structures where either one of the refractive indices (linear or nonlinear) is modulated by more than one wavenumbers, or both of them are modulated, soliton position and stability depends strongly on its characteristics. The latter results in additional functionality related to soliton discrimination in such structures. The respective power (or width / propagation constant) dependent bifurcations are studied in terms of a Melnikov-type theory. The latter is used for the determination of the specific positions, with respect to the spatial structure, where solitons can be located. A wide variety of cases are studied, including solitons in periodic and quasiperiodic lattices where both the linear and the nonlinear refractive index are spatially modulated. The investigation of a wide variety of inhomogeneities provides physical insight for the design of a spatial structure and the control of the position and stability of a localized wave.

OCIS codes: (190.6135) Spatial solitons; (190.4420) Nonlinear optics, transverse effects in; (190.4720) Optical nonlinearities of condensed matter.

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1. Introduction

Electromagnetic waves propagating in periodically structured dielectric materials, such as Photonic Crystals (PCs), share many properties with electron waves in ordinary semiconductors [1, 2] and matter-wave realizations of a Bose-Einstein Condensate (BEC) in optical lattices [3, 4]. The periodicity leads to the appearance of a band structure of the respective underlying linear system. This structure along with a corroborating nonlinear response is responsible for a multitude of effects such as modulation instability, Bloch oscillations and soliton generation, to name a few. Spatially localized structures in optical lattices is a subject of intense theoretical and experimental efforts in our days [5].

Periodicity in the structure implies periodicity in the material properties, linear and nonlinear. In the PCs the effective linear refractive index is a function of both the wavelength and the structural parameters of the unit lattice. The light is linearly guided by the differences in the effective refractive index of the lattice elements. The linear material properties can be controlled either by choosing a suitable configuration for the lattice or dynamically in the sense of optically induced waveguide arrays and lattices [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Along this direction, lattice solitons in photonic structures with periodically varying linear refractive index have been extensively investigated either by studying the original continuous model, consisting of the NonLinear Schrodinger Equation (NLSE) describing soliton propagation [17], or by utilizing approximate discrete models [17, 18]. On the other hand, recent advances in fabrication techniques have made possible the fabrication of photonic structures with rapidly varying nonlinear refractive index [19], as well. Soliton formation and propagation in such structures has been studied mostly for cases where the nonlinear refractive index varies along the propagation distance, and, as an analog of dispersion management, is usually referred as nonlinearity management [20, 21, 22, 23]. However, the effect of a transversely varying nonlinear refractive index has been only recently studied theoretically for the case of a homogeneous [24] as well as a spatially modulated [25, 26, 27, 28] linear refractive index. Cases where both linear and nonlinear refractive index are spatially modulated have been also studied for waveguide arrays consisting of interlaced linear and nonlinear waveguides, and analytical soliton solutions have been found [29, 30, 31]. The emphasis of the theoretical research has been mainly focused on achieving high localization and investigating the mobility of localized waves [26, 32, 33, 34, 35, 36, 37, 38, 39]. While localization and mobility is a common goal in all the diverse physical systems exhibiting lattice solitons, not enough emphasis has been given so far to the location of the localized states within the periodic spatial structure and its controllability.

In this work we study lattice soliton formation and stability in a configuration of planar geometry where both the linear and the nonlinear refractive index are inhomogeneous with respect to the transverse dimension, with their spatial dependence being of a generic form describing either periodic or quasiperiodic structures [40]. It is well known that the presence of a transverse inhomogeneity results in the breaking of the translational invariance of the system. Therefore, localized states can be no longer located in any position within the photonic structure meaning

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that there exist only specific positions within the structure where solitons can be formed. The simple case of a lattice with a sinusoidally varying linear refractive index, is well studied and solitons have been shown to exist in the high- and low- index positions, refereed as on-site and off-site solitons, respectively [41, 42], with the former being stable and the latter being unstable for all values of the propagation constants [43].

However, it is shown that, for the case of a superlattice (where solely the linear refractive index is modulated by several wavenumbers) the determination of the position of a localized state with respect to the underlying structure as well as its stability is not a trivial task. The case of incommensurate wavenumbers, describing quasi periodic lattices, is also more complex. More importantly, both the position and the stability type of the localized states depend on the respective propagation constant and the power. Furthermore, it is shown that the appropriate modulation of the nonlinear refractive index can be used for altering either the stability of the localized states or their positions, in comparison to a case where only the linear refractive index is modulated. In all cases investigated in this work it is shown that additional complexity of the photonic structure results in additional functionality allowing for soliton discrimination with respect to their characteristics, due to underlying power (or width / propagation constant) dependent bifurcations. The aforementioned features are investigated by utilizing the Melnikov's theory [44, 45] resulting in an analytic expression from which both the position and stability type of a localized state are determined. It is worth mentioning that such an expression allows for the consideration of a large variety of different quite complex configurations and it can be used for providing physical intuition for the designing of specific structures possessing desirable properties.

The paper is organized as follows: The model and its stationary solutions along with the application of the Melnikov's method are presented in Section II. The cases of a periodically and a quasiperiodically modulated linear refractive index is investigated in Section III, while the case where both the linear and the nonlinear refractive indices are spatially modulated is investigated in Section IV. The summary of the results and the conclusions are given in Section V.

2. Model and stationary solutions

Soliton propagation in transversely inhomogeneous planar media is described by a one-dimensional NLSE with coefficients depending on the transverse coordinate

$$i\frac{\partial \psi}{\partial z}+\frac{\partial^2 \psi}{\partial x^2}+2|\psi|^2\psi+\varepsilon\left[n_0(x)\psi+n_2(x)|\psi|^2\psi\right]=0 \qquad (1)$$

where x is the transverse coordinate normalized to x_0, z is the propagation distance normalized to $z_0 = 2kx_0^2$, and ψ is the electric field ampltude normalized to $I_0^{1/2}$ with $I_0 = (n_2kk_0x_0^2)^{-1}$. The functions $n_0(x)$ and $n_2(x)$ describe the transverse variation of the linear and the nonlinear refractive index (potential), respectively. The functions $n_i(x), i = 0,2$ can be of any form describing periodic or quasiperiodic lattices. A normalized propagation distance $z = 100$, corresponds to an actual propagation length of $10.7-24.3mm$, for the case of a nonlinear material of AlGaAs type ($n = 3.34$, $n_2 = 1.5\times10^{-13}cm^2/W$), when the transverse coordinate is normalized to $X_0 = 2-3\mu m$.
The stationary solutions of Eq. (1) have the form

$$\psi(x,z) = u(x)e^{i\beta z} \qquad (2)$$

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with $u(x)$ a real function describing the transverse wave profile and β the propagation constant. The corresponding stationary equation is the following:

$$\frac{d^2u}{dx^2} - \beta u + 2u^3 + \varepsilon \left[n_0(x)u + n_2(x)u^3 \right] = 0 \tag{3}$$

This equation corresponds to a one-degree of freedom dynamical system with Hamiltonian

$$H = \frac{p^2}{2} - \beta \frac{q^2}{2} + \frac{q^4}{2} + \varepsilon \left[n_0(x) \frac{q^2}{2} + n_2(x) \frac{q^4}{2} \right] \tag{4}$$

with $(q, p) = (u, du/dx)$ being the canonical variables. The system is nonautonomous and non-integrable due to the explicit dependence of the Hamiltonian on the transverse coordinate x (playing the role of "time"), which expresses the inhomogeneity of the medium. In the following, we consider the $x-$ dependent terms of the Hamiltonian as a first order perturbation of the remaining Hamiltonian considered as of zero-order. Moreover, we consider the case $\beta > 0$ for which a homoclinic solution of the unperturbed system exist, corresponding to the stationary soliton solution of the homogeneous system. This case ($\beta > 0$) correspond to the semi-infinite gap of the band structure of a periodic medium. However, note that a Melnikov function similar to the one obtained in this work has also been obtained in the study of gap solitons emerging from the transmission band edges inside finite gaps [43]. The homoclinic solution is formed by the smooth join of the stable and unstable manifolds of the hyperbolic saddle fixed point located at the origin of the phase space. Therefore, this closed curve is filled with nontransverse homoclinic points (defined as points of intersection between the stable and unstable manifolds). This highly degenerate structure is expected to break under perturbation and perhaps yield transverse homoclinic orbits or no homoclinic orbits at all [44, 45]. The latter is directly related to the breaking of the translational invariance of unperturbed NLSE, due to spatial inhomogeneity as in Eq. (1). Since the existence of stationary soliton solutions of the perturbed system is directly related to the existence of such homoclinic orbits, the latter is of crucial importance for our study.

In order to study such homoclinic bifurcations, Melnikov's theory can be utilized for the study of periodic and dissipative perturbations [44, 45]. According to this theory, the existence of transverse intersections between the stable and unstable manifolds is provided by the simple zeros of the so-called Melnikov's function, which is related to the distance between the two manifolds in a Poincare surface of section.

The unperturbed part of the system described by Eq.(4) has the homoclinic solution (for $\beta > 0$)

$$(q_0(x), p_0(x)) = \left(\pm\sqrt{\beta}\,\mathrm{sech}[\sqrt{\beta}(x - x_0)], \mp\beta\,\mathrm{sech}[\sqrt{\beta}(x - x_0)]\tanh[\sqrt{\beta}(x - x_0)] \right) \tag{5}$$

Equation (5) describes an infinite family of solutions homoclinic to the origin $(q, p) = (0, 0)$, which is parameterized by x_0 corresponding to the location of the maximum for $q_0(x)$. The infinite number of solutions is related to the fact that the unperturbed stationary system (4) is autonomous and the unperturbed NLSE (1) is translationally invariant with respect to x. Under the presence of perturbations only a discrete number of such solutions persist corresponding to values of x_0 given by the zeros of the function [44, 45]

$$M(x_0) = -\varepsilon \int_{-\infty}^{+\infty} p_0(x) \left[n_0(x)q_0(x) + n_2(x)q_0(x)^3 \right] dx \tag{6}$$

corresponding to the Melnikov function. Alternatively, $M(x_0)$ can be written as

$$M(x_0) = \varepsilon \int_{-\infty}^{+\infty} \left[n_0'(x + x_0) \frac{q_0^2(x)}{2} + n_2'(x + x_0) \frac{q_0^4(x)}{2} \right] dx \tag{7}$$

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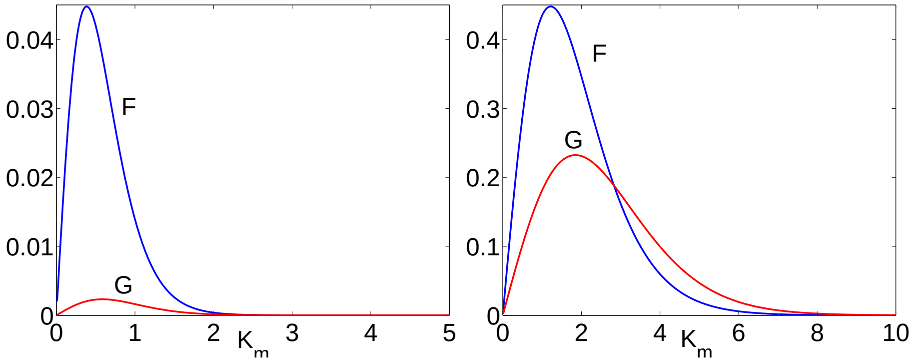
which is sometimes more appropriate for calculations (primes denote differentiation with respect to x).



(a) (b)

Fig. 1. Dependence of the functions $F(K_m^{(0)}, \beta)$ and $G(K_m^{(2)}, \beta)$ on the wavenumber K_m for $\beta = 0.1$ (a) and $\beta = 1$ (b)

In the following we investigate the information provided by the zeros of the function $M(x_0)$ for the existence of solitary stationary waves in a large variety of inhomogeneous media described by different functions $n_0(x)$ and $n_2(x)$. The most generic form for the modulation of the linear and the nonlinear refractive index is given by

$$n_i(x) = \sum_m A_m^{(i)} \cos\left(K_m^{(i)} x + \phi_m^{(i)}\right), \qquad i = 0, 2 \tag{8}$$

corresponding to Fourier series for the case of periodic modulations, generalized Fourier series for quasi-periodic modulations, or finite trigonometric sums. The corresponding Melnikov function is

$$\begin{aligned} M(x_0) &= \varepsilon \frac{\pi\sqrt{\beta}}{2} \sum_m A_m^{(0)} F(K_m^{(0)}, \beta) \sin\left(K_m^{(0)} x_0 + \phi_m^{(0)}\right) \\ &\quad + \varepsilon \frac{\pi\sqrt{\beta}}{2} \sum_m A_m^{(2)} G(K_m^{(2)}, \beta) \sin\left(K_m^{(2)} x_0 + \phi_m^{(2)}\right) \end{aligned} \tag{9}$$

where

$$F(K_m^{(0)}, \beta) = \frac{(K_m^{(0)})^2}{\sinh\left(\frac{\pi K_m^{(0)}}{2\sqrt{\beta}}\right)} \tag{10}$$

$$G(K_m^{(2)}, \beta) = \frac{(K_m^{(2)})^2[(K_m^{(2)})^2 + 4\beta]}{12 \sinh\left(\frac{\pi K_m^{(2)}}{2\sqrt{\beta}}\right)} \tag{11}$$

The functions $F(K_m^{(0)}, \beta)$ and $G(K_m^{(2)}, \beta)$ contain all the essential information related to the effect of the inhomogeneity on the homoclinic solution. Both functions are strongly localized with respect to the wavenumber $K_m^{(i)}$ $(i = 0, 2)$ with their maximum and width depending strongly on

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the propagation constant β, as shown in Figs. 1(a) and (b). The latter is quite important since, as we show in the following, the effect of each Fourier component of the inhomogeneity on a stationary solution is not uniform with respect to its propagation constant β. Since β is the longitudinal wavenumber of a stationary solution, this fact reflects the interplay between the transverse and the longitudinal periodicity. On the other hand, noting that the power of the unperturbed solution (5) is $P = \int |u|^2 dx = 2\sqrt{\beta}$, the strong dependence of the functions F and G results in power dependent bifurcations of the stationary solutions, as we show in the following.

Due to the fact that the Melnikov function is linear in the Fourier components, without loss of generality, we can consider one or a few Fourier components of the expansion, while the results can be directly extended for the case where all terms of the Fourier expansion are taken into account. A wide variety of configurations can be investigated with the utilization of Eq. (9). In the following we focus on specific cases having essential qualitatively different properties. We consider the case where $\varepsilon = 0.1$ so that the perturbative character of our approach is relevant; however, most of the results also hold even for stronger perturbations.

3. Lattice solitons in media with periodically and quasiperiodically modulated linear refractive index

Firstly, let us consider the case where only the linear refractive index is spatially modulated, so that $A_m^{(2)} = 0$. More specifically, we start from the commonly studied case of a simple harmonic lattice where only one Fourier component ($A_1^{(0)}$) of $n_0(x)$ is nonzero. The respective Melnikov function (9)

$$M(x_0) = A_1^{(0)} F(K_1^{(0)}, \beta) \sin\left(K_1^{(0)} x_0 + \phi_1^{(0)}\right) \tag{12}$$

has two zeros within the period of the spatial modulation. The corresponding soliton solutions have been considered in a large number of previous works, where they have been identified as on-site (stable) and off-site (unstable) solutions, located at the maxima and the minima of the periodic linear refractive index, respectively. The stability of the solution depends on the sign of the derivative of the Melnikov function with respect to x_0, $M'(x_0)$ [46]. Note that the number and the location (x_0) as well as the stability type of the solutions does not depend on the propagation constant β.

Let us now consider the more general case where the linear refractive index is periodic but has two Fourier components $K_1^{(0)} = 1, K_2^{(0)} = 2$, with $A_1^{(0)} = 1$ and $\phi_1^{(0)} = 0$. In the following we investigate the effect of the second Fourier component on soliton formation and propagation for solitons having different β. For an analogous case of gap solitons, it has been shown in a more general setting that only two solutions bifurcate in periodic potentials on a single period if β is small [43]. According to Eq. (9), the stationary solutions correspond to the zeros of the Melnikov function given by the equation

$$M(x_0) = F(1,\beta)\sin(x_0) + A_2^{(0)} F(2,\beta)\sin\left(2x_0 + \phi_2^{(0)}\right) = 0 \tag{13}$$

The spatial profile of the linear refractive index along with the positions of the corresponding zeros of the Melnikov function are shown in Fig. 2 for the case $A_2^{(0)} = 1$, $\phi_2^{(0)} = 0$ and for propagation constants $\beta = 0.1$ (circles) and $\beta = 1$ (asterisks). It is shown that for $\beta = 0.1$ the Melnikov function has only two zeros (circles) located at $x = 0, \pi$, while for $\beta = 1$ two additional zeros (asterisks) appear at $x \simeq 2.25, 4.05$ (in symmetric positions with respect to $x = \pi$). The latter correspond to asymmetric solution profiles. The effect of the second Fourier component on the solutions corresponding to different propagation constants, differs for the two cases due to the strong dependence of the Melnikov function (9) on β through the function F as shown in Fig. 1. Therefore, there is no significant effect of the second Fourier component

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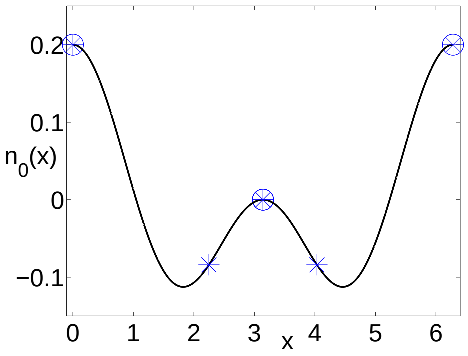
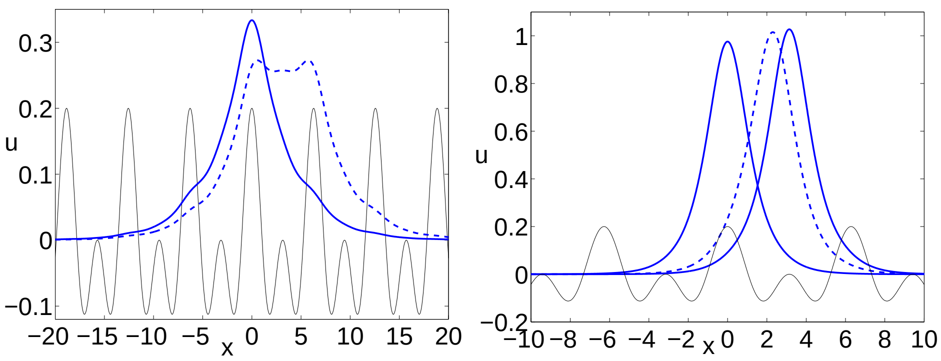


Fig. 2. Linear refractive index profile along with the corresponding position of the zeros of the Melnikov function for the case where the linear refractive index is modulated by two commensurate wavenumbers ($n_0(x) = \cos(x) + \cos(2x)$, $n_2(x) = 0$). Circles correspond to $\beta = 0.1$ and asterisks to $\beta = 1$.



(a) (b)

Fig. 3. Profiles of stable (solid line) and unstable (dashed line) stationary solutions for the case where the linear refractive index is modulated by two commensurate wavenumbers ($n_0(x) = \cos(x) + \cos(2x)$, $n_2(x) = 0$). The propagation constant is $\beta = 0.1$ (a) and $\beta = 1$ (b).

for solutions with propagation constant $\beta = 0.1$. This is a case of a power (or β) dependent bifurcation: for small β only two stationary solutions located at $x_0 = 0, \pi$ exist with the former being stable and the latter being unstable, while for higher β two additional stationary solutions appear, resulting also to a change of the stability of the solution located at $x_0 = \pi$. The corresponding bifurcation (transition) value for β is given from the equation $M'(x_0 = \pi) = 0$. It is worth mentioning that, in contrast to the case of a lattice modulated by a single wavenumber, the appearance of more than one wavenumbers results in nonuniform existence and stability properties with respect to β. This interesting feature suggests a power selectivity property of a polychromatic lattice which is promising for applications.

The stationary solutions for each case are shown in Fig. 3 (due to the symmetry of the linear refractive index profile, only solutions in $[0, \pi]$ are shown). Their evolution under propagation is depicted in Fig. 4, where it is confirmed that stable and unstable solutions are alternate. (In all cases a random noise 1% of the maximum of solution amplitude has been superimposed on the

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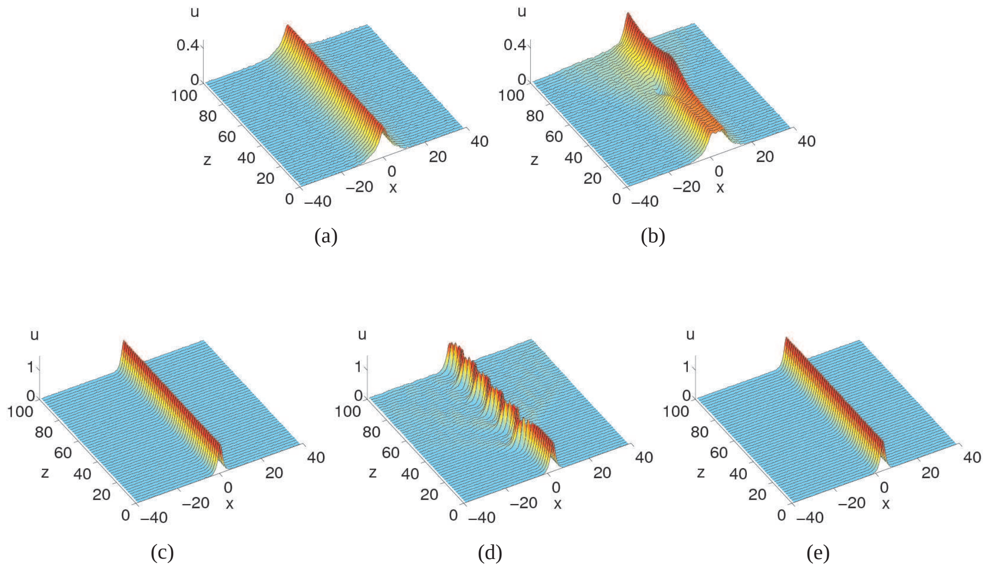

Fig. 4. Propagation of the stationary solutions shown in Figs. 3. (a) $\beta = 0.1$, $x_0 = 0$, (b) $\beta = 0.1$, $x_0 = \pi$, (c) $\beta = 1$, $x_0 = 0$, (d) $\beta = 1$, $x_0 = 2.25$, (e) $\beta = 1$, $x_0 = \pi$.

amplitude and the phase of the stationary profiles for the numerical study of their stability.) It is also confirmed that the stationary solution located at the local maximum at $x = \pi$ is unstable for $\beta = 0.1$, while it is stable for $\beta = 1$. Qualitatively similar conclusions can be made for different selections of the amplitude $A_2^{(0)}$, the wavenumber $K_2^{(0)}$ and the phase $\phi_2^{(0)}$ of the second Fourier component, depending on the number and location of the zeros of the equation (13). Also, cases with multiple Fourier components can be investigated on the same basis.

The lattices considered in the previous cases were periodic as it is the case when all wavenumbers modulating the linear or the nonlinear refractive index of the medium are commensurate. As a result of this periodicity we were allowed to restrict the study of the soliton position and stability in one period. However, a more complicated case occurs when the modulating wavenumbers are incommensurate, where the corresponding linear (or nonlinear) refractive index profile is quasiperiodic. The refractive index form as well as the respective soliton position are not repeated in the transverse dimension, and an infinite number of different localized stationary solution, having the same propagation constant β, exist in irregularly distributed positions along the lattice. In accordance to this feature, the corresponding Melnikov function $M(x_0)$ (9) is also quasiperiodic and its irregularly distributed zeros predict the location of the stationary solutions in the lattice. Additionally, the stability of a solution corresponding to x_0 is determined by the sign of $M'(x_0)$. It is obvious that the information provided by the Melnikov function is even more important in this complex case. As an example we consider the case where the linear refractive index is modulated by two incommensurate wavenumbers, so that the respective parameters are $K_1^{(0)} = 1$, $K_2^{(0)} = \pi/2$, $A_1^{(0)} = A_2^{(0)} = 1$ and $\phi_1^{(0)} = \phi_2^{(0)} = 0$. The form of the nonperiodic linear refractive index profile along with the irregularly distributed corresponding zeros of the Melnikov function, for a finite part of the lattice, are shown in Fig. 5 for propagation constants $\beta = 0.1$ (circles) and $\beta = 1$ (asterisks). In Fig. 6, the profiles of the first three solutions (on the right of the origin) are shown, while their respective evolution

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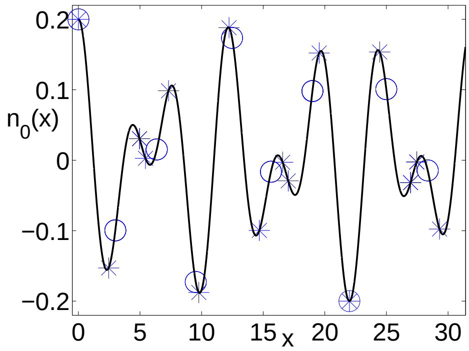

Fig. 5. Linear refractive index profile along with the corresponding position of the zeros of the Melnikov function for the case where the linear refractive index is modulated by two incommensurate wavenumbers ($n_0(x) = \cos(x) + \cos(\pi x/2)$, $n_2(x) = 0$). Circles correspond to $\beta = 0.1$ and asterisks to $\beta = 1$.

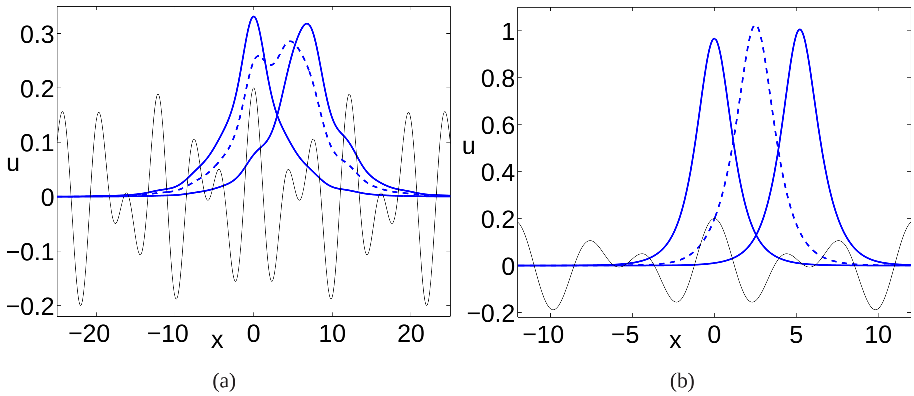

Fig. 6. Profiles of stable (solid line) and unstable (dashed line) stationary solutions for the case where the linear refractive index is modulated two incommensurate wavenumbers ($n_0(x) = \cos(x) + \cos(\pi x/2)$, $n_2(x) = 0$). The propagation constant is $\beta = 0.1$ (a) and $\beta = 1$ (b).

under propagation is depicted in Fig. 7, where the alternate character of the stability type of each solution, as predicted by the sign of $M'(x_0)$, is confirmed.

4. Lattice solitons in media with modulated linear and nonlinear refractive indices

In the previous section we have investigated the location, the profile shape and the stability of solitons in several cases where the linear refractive index is modulated by a single or multiple wavenumbers being either commensurate or incommensurate. For the case where only the nonlinear refractive index is modulated the analysis of previous sections applies directly and the results are qualitatively similar with the difference that the function G (11) appears in the corresponding Melnikov function (9) instead of the function F (10). A more interesting case occurs when both the linear and the nonlinear refractive indices are spatially modulated. As an example we firstly investigate the case where both refractive indices are modulated by the same wavenumber and we consider the following parameter values $K_1^{(0)} = K_1^{(2)} = 1$, $A_1^{(0)} = 1$ and

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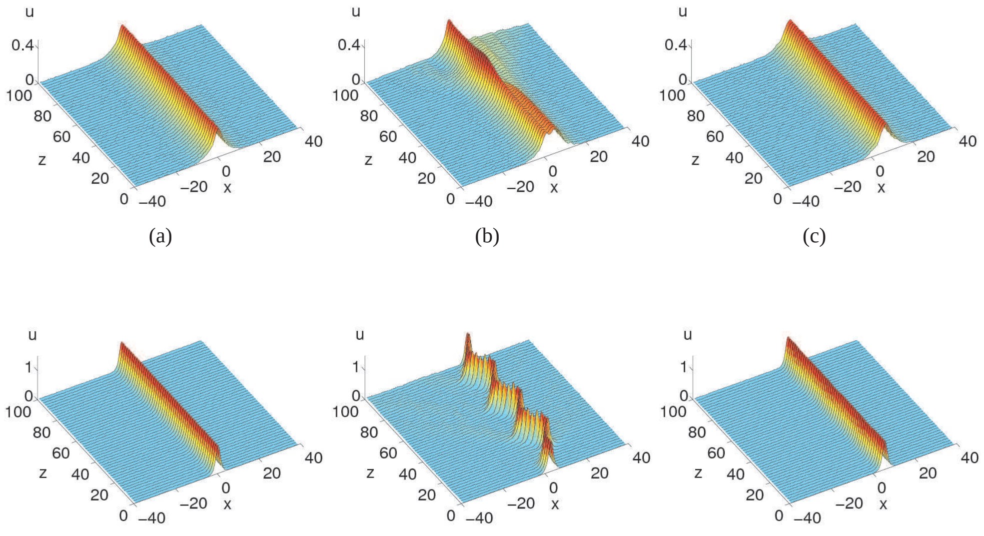

(a) (b) (c)

(d) (e) (f)

Fig. 7. Propagation of the stationary solutions shown in Figs. 6. (a) $\beta = 0.1$, $x_0 = 0$, (b) $\beta = 0.1$, $x_0 = 3$, (c) $\beta = 0.1$, $x_0 = 6.36$, (d) $\beta = 1$, $x_0 = 0$, (e) $\beta = 1$, $x_0 = 2.45$, (f) $\beta = 1$, $x_0 = 4.96$.

$\phi_1^{(0)} = 0$. The location of the stationary solutions are given from the zeros of the corresponding Melnikov function, which, according to (9), is

$$M(x_0) = F(1,\beta)\sin(x_0) + A_1^{(2)} G(1,\beta)\sin\left(x_0 + \phi_1^{(2)}\right) = 0 \tag{14}$$

More specifically, if $\phi_1^{(2)} = 0$, the Melnikov function is

$$M(x_0) = \left[F(1,\beta) + A_1^{(2)} G(1,\beta)\right]\sin(x_0) \tag{15}$$

In comparison to the case where only the linear refractive index is modulated by the same wavenumber (investigated in the previous section) we have the following features: When the Melnikov function (15) is not identically zero, i.e. when

$$F(1,\beta) + A_1^{(2)} G(1,\beta) \tag{16}$$

is not zero, (i) The number of the stationary solutions within a period of the lattice as well as and their positions ($x_0 = 0, \pi$) are the same in both cases, (ii) Their stability type can be either the same or interchanged depending on the sign of the quantity (16). For example, let us consider the case of a nonlinear refractive index profile having $A_1^{(2)} = -4.8$ and a set of stationary solutions corresponding to $\beta = 1$ and $\beta = 0.1$. The profiles of the solutions (corresponding to $x_0 = 0, \pi$) are shown in Fig. 8, while their respective evolution under propagation is depicted in Fig. 9.

It is obvious that, in comparison with the case where only the linear refractive index is modulated, the stability type of the two solutions can be interchanged depending on β i.e. for $\beta = 1$

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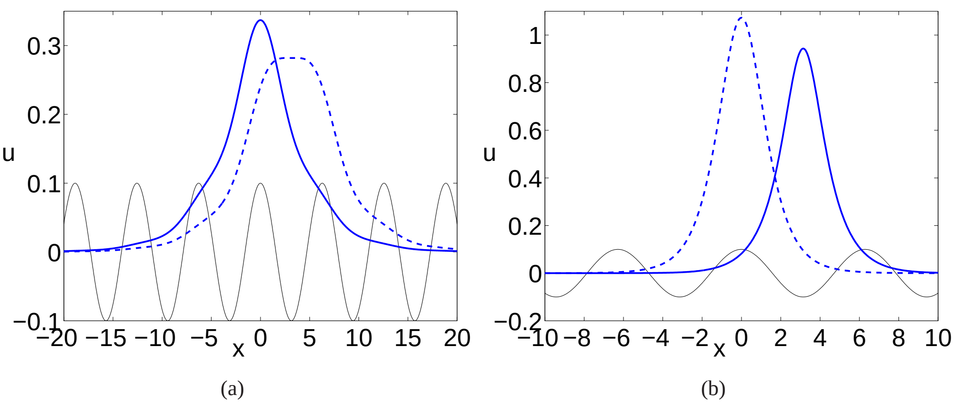

Fig. 8. Profiles of stable (solid line) and unstable (dashed line) stationary solutions for the case where $n_0(x) = \cos(x)$ and $n_2(x) = -4.8\cos(x)$. The propagation constant is $\beta = 0.1$ (a) and $\beta = 1$ (b).

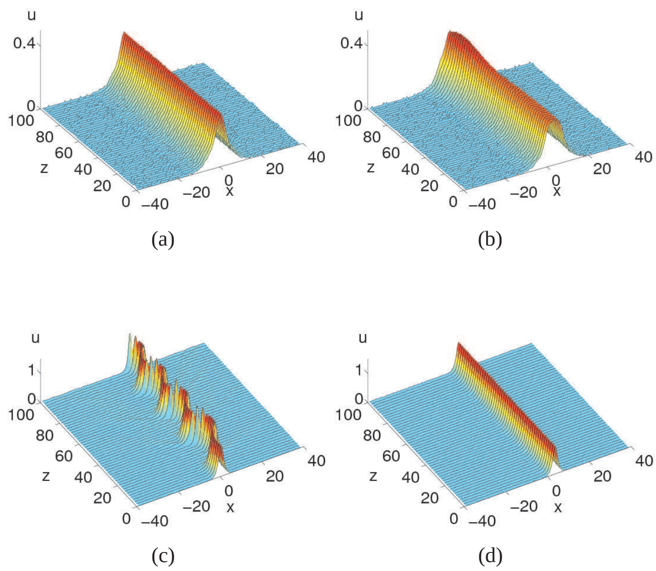

Fig. 9. Propagation of the stationary solutions shown in Figs. 8. (a) $\beta = 0.1$, $x_0 = 0$, (b) $\beta = 0.1$, $x_0 = \pi$, (c) $\beta = 1$, $x_0 = 0$, (d) $\beta = 1$, $x_0 = \pi$.

the soliton located at $x_0 = 0$ becomes now unstable, while the soliton at $x_0 = \pi$ becomes stable. It is worth emphasizing that, since the quantity (16) depends on the propagation constant β (and therefore on the power P), the stability type of the solutions is not uniform for all β, so that for different β we can have different stability type of the two solutions. As in the case where only the linear refractive index is modulated and we have more than one wavenumbers (Figs. 2, 3, 4), the stability type of the solution depends on the propagation constant (or the power); the difference here is that we have an exchange of stability type between the two solutions, while no additional solution appears.

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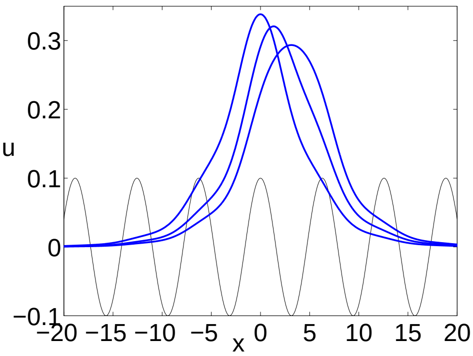

Fig. 10. Profiles of stationary solutions for the case where the quantity (16) is zero for $\beta = 0.1$, ($n_0(x) = \cos(x)$, $n_2(x) = -8.57\cos(x)$).

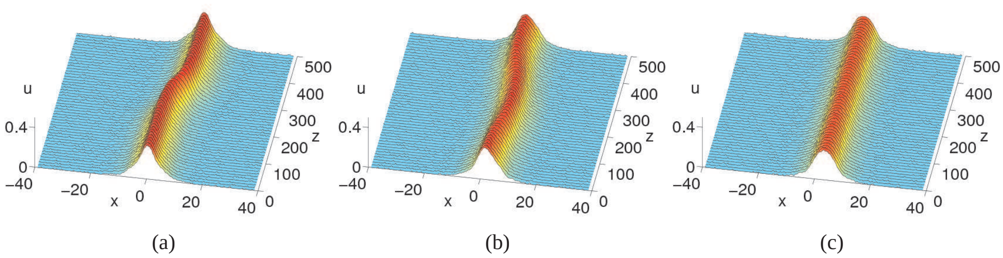

Fig. 11. Propagation of the stationary solutions shown in Fig. 10.

An interesting case occurs when the Melnikov function vanishes identically, i.e. when the quantity (16) is zero. Note that for any case of a negative $A_1^{(2)}$, there exist a propagation constant β, for which the Melnikov function vanishes identically. Such case corresponds to a bifurcation point in the parameter space, and the investigation based on the Melnikov function becomes inconclusive. Keeping in mind that the previously presented Melnikov method, is actually a first-order perturbation theory [44, 45], in order to conclude on the existence and the stability of stationary solutions, higher-order theory should be used. However, since higher-order calculations may become quite complicated, in the following we investigate numerically this case. As an example, we consider the case where the quantity (16) is zero for $\beta = 0.1$, corresponding to $A_1^{(2)} = -8.57$. As shown in Fig. 10, additional stationary solutions appear in this case. Therefore, we have found 3 stationary solutions corresponding to x_0 in the interval $[0, \pi]$, while a fourth solution exist in the interval $(\pi, 2\pi)$ symmetrically with respect to π. In an analogous case for gap solitons it has been shown that four branches of solutions bifurcate in the limit of small beta [43]. However, in the context of our analysis, due to the identical vanishing of the Melnikov function, we cannot exclude the possibility of the existence of more solutions than those shown in Fig. 10 within the interval $[0, \pi]$ for any β. The respective evolution under propagation of these solutions is depicted in Fig. 11, where it is shown that the instability of the first two solutions is significantly slower than all the previous cases corresponding to a nonvanishing Melnikov function. The latter is important from the point of view of practical applications, where the propagation distances of interest can be too small for the appearance

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of the instability. For the case where the linear refractive index is modulated by a large number of wavenumbers, we can always find a nonlinear refractive index profile with parameters $(A_m^{(2)}, K_m^{(2)}, \phi_m^{(2)})$ for which the Melnikov function vanishes for all x_0, for a specific propagation constant β. However, it is worth emphasizing that such a case corresponds to a bifurcation point and is structurally unstable in the parameter space, therefore any deviation of the parameters from their bifurcation values results in drastic and qualitatively different features. This case has been previously studied with respect to the enhancement of the mobility of a solitary wave [26].

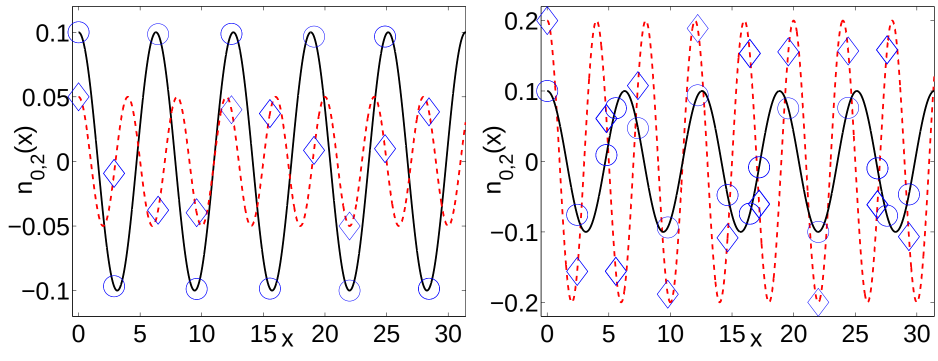

Fig. 12. Linear (solid line) and nonlinear (dashed line) refractive index profiles along with the corresponding relative position of the zeros of the Melnikov function (circles: position relative to the linear refractive index profile, rhombs: position relative to the nonlinear refractive index profile) for the case where the linear and the nonlinear refractive indices are modulated by two incommensurate wave ($n_0(x) = \cos(x)$, $n_2(x) = A_1^{(2)} \cos(\pi x/2)$). (a) $A_1^{(2)} = 0.5$, (b) $A_1^{(2)} = 2$.

Finally, it is interesting to consider the case where the wavenumbers modulating the linear and nonlinear refractive indices are incommensurate. In this case, even though each index profile is periodic, the position of the stationary solitary solutions within the lattice appear irregularly distributed with respect to either the linear or the nonlinear refractive index profile. For example in the case where $A_1^{(0)} = 1, K_1^{(0)} = 1, A_1^{(2)} = 0.5, 2, K_1^{(2)} = \pi/2$ and $\phi_1^{(0)} = \phi_1^{(2)} = 0$ the positions of the stationary solutions with respect to the linear and the nonlinear refractive index profiles are shown in Fig. 12, for $\beta = 1$. It is shown that for $A_1^{(2)} = 0.5$ the positions with respect to the linear refractive index profile are close to its extrema, while the positions with respect to the nonlinear refractive index profile appear irregularly distributed (Fig. 12(a)). For stronger modulation ($A_1^{(2)} = 2$) of the nonlinear refractive index the positions of the stationary solutions are irregularly distributed with respect to both the linear and the nonlinear refractive index profiles (Fig. 12(b)). In general, it can be easily shown that depending on the ratio

$$a = \frac{A_1^{(0)} F(K_1^{(0)}, \beta)}{A_1^{(2)} G(K_1^{(2)}, \beta)} \tag{17}$$

the infinite number of zeros of the respective Melnikov function in the $x-$ axis, if reduced within the same interval T (x_0 mod$T$), can densely fill the interval of one period (T) of either the linear or the nonlinear refractive index profiles. This means that for an infinite lattice there is always a part of the lattice where a stationary solution can be found with any location relative to the underlying linear or nonlinear refractive index profiles. Note that the absolute position of this location can be controlled by the appropriate choice of the phases $(\phi_1^{(0)}, \phi_1^{(2)})$ so that the position can always be located within the first period (with respect to the origin) of the linear or the nonlinear refractive index profile. This mechanism of controlling the position of the soliton can be proved useful in applications where one of the index profiles $n_0(x)$ or $n_2(x)$ is determined by the geometrical structure of the configuration, while the other is dynamically induced by an optical control wave.

5. Summary and conclusions

The location and the stability of localized states in general configurations where the linear and/or the nonlinear refractive index are transversely modulated, have been studied. The inhomogeneity of the medium and the resulting breaking of the translational invariance have been related to homoclinic bifurcations of the underlying dynamical system describing stationary solutions. The latter has been studied with the utilization of the Melnikov's method allowing for the derivation of an analytical expression from which the position of the localized state within the photonic structure as well as its stability can be determined.

The results of the method are shown to be applicable to a large variety of different configurations. Characteristic cases having qualitatively different features were investigated, and the predictions of the theory were successfully tested by numerical simulations. Among these interesting features we can refer to the power (or propagation constant) dependence of the position and the stability type of the respective localized states, for the cases where either the linear refractive index is spatially modulated by more than one wavenumbers or both linear and nonlinear refractive are spatially modulated. It is shown that, in contrast to cases of monochromatic modulation of solely the linear or the nonlinear refractive index, more complex spatial modulations result in the capability of the photonic structure for discriminating solitons having different characteristics. Moreover, it is worth mentioning the case where incommensurate wavenumbers modulate the linear and/or nonlinear refractive index which result a plethora of asymmetric localized states located in irregularly distributed positions within the photonic structures. The aforementioned features along with the capability of controlling the spatial profiles of the linear and the nonlinear refractive indices, either in the fabrication process or dynamically by using optical control signals, opens the possibility of potential applications having technological interest. The method and results presented in this work are expected to be proved useful for the designing of appropriately engineered photonic structures having the desirable properties.

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