Lattice solitons in self-defocusing optical media: analytical solutions of the nonlinear Kronig–Penney model

Y. Kominis and K. Hizanidis

School of Electrical and Computer Engineering, National Technical University of Athens, Zographou GR-15773, Greece

Received May 23, 2006; accepted June 19, 2006; posted July 10, 2006 (Doc. ID 71254); published September 11, 2006

A novel method for obtaining analytical solitary wave solutions of the nonlinear Kronig–Penney model in periodic photonic structures with self-defocusing nonlinearity is applied for providing generic families of solutions corresponding to the gaps of the linear band structure. Characteristic cases are shown to be quite robust under propagation. © 2006 Optical Society of America

OCIS codes: 190.4420, 130.2790, 190.5530.

Periodic photonic structures fabricated in nonlinear dielectric media recently became a subject of intense theoretical and experimental research. The formation of self-trapped localized modes, among others, is of major importance.1,2 These modes have the form of gap solitons inside the photonic gaps of the periodic structure and result from the dynamical balancing between the nonlinearity and the diffraction. The robustness of these waves under propagation facilitates their experimental observation and is very promising for applications in integrated photonic devices and waveguide arrays, such as multiport beam coupling, steering, and switching.3–6 On the other hand, the related field of Bose–Einstein condensates loaded in optical lattices7–11 is of increasing interest, and the theoretical studies in both fields progress in parallel.

The formation and propagation of localized modes in photonic structures have been theoretically studied mostly on the basis of either the tight-binding approximation or the coupled-mode theory rendering simplified discrete models.12–16 These approximations provide accurate modeling only under the corresponding assumptions, while a more general model is the nonlinear Schrödinger equation (NLS) with spatially periodic coefficients

\[
\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial x^2} + \epsilon(x) \psi + g(x, |\psi|^2) \psi = 0,
\]

where \(z\), \(x\), and \(\psi\) are the normalized propagation distance, transverse dimension, and electric field, respectively. The periodic transverse variation of the linear refractive index is given by \(\epsilon(x)\), while the spatial and intensity dependence of the nonlinear refractive index is provided through \(g(x, |\psi|^2)\). This model has been previously studied for the case of the periodic function in the form of the periodic sequence of Dirac functions.1,17 In recent work,18 the case of a more realistic model with piecewise-constant coefficients, namely, a nonlinear Kronig–Penney model, has been considered, and a new method was utilized to provide analytical solutions for the case of a self-focusing nonlinearity. In this Letter, the same approach is applied for the case of self-defocusing nonlinearity (or repulsive interatomic interactions for the case of Bose–Einstein condensates). The analytical solutions thus obtained correspond to localized excitations on a finite periodic background in the form of dark and antidark solitons.

The stationary solutions of Eq. (1) have the form \(\psi(x, z) = u(x; \beta)e^{i\beta z}\) and satisfy the nonlinear ordinary differential equation

\[
\frac{d^2 u}{dx^2} + [\epsilon(x) - \beta]u + g(x, u^2)u = 0,
\]

where \(\beta\) is the propagation constant and \(u(x; \beta)\) is the real transverse wave profile. The periodic structure (Fig. 1) under consideration consists of linear and nonlinear (Kerr type) layers with the linear and the nonlinear refractive index given by \([\epsilon(x), g(x, u^2)]\) = \((\epsilon_N, -2u^2)\), for \(x \in U_N\) and \([\epsilon(x), g(x, u^2)]\) = \((\epsilon_L, 0)\) for \(x \in U_L\), where \(U_N = \cup_k [kT - N/2, kT + N/2]\); \(U_L = \cup_k [kT + N/2, (k + 1)T - N/2]\); \(L\) and \(N\) are the lengths of the linear and the nonlinear layers, respectively; and \(T = L + N\) is the spatial period of the structure. In each part of the photonic structure the wave profile is described by the following equations:

\[
\frac{d^2 u}{dx^2} + (\epsilon_N - \beta)u - 2u^3 = 0, \quad x \in U_N,
\]

\[
\frac{d^2 u}{dx^2} + (\epsilon_L - \beta)u = 0, \quad x \in U_L.
\]

The stationary solutions of Eq. (2) can be provided by composing solutions of these two dynamical systems that have matched conditions for \(u\) and its de-
Fig. 3. (Color online) Field profiles of the analytically obtained stationary solutions for a periodic structure with \( L = \frac{3\pi}{2}, N = \pi \), and \( \epsilon_N = 0 \). (left column) \( \epsilon_L = 0.1, n = 2 \); (middle column) \( \epsilon_L = -0.5, n = 1 \); and (right column) \( \epsilon_L = -0.5, n = 2 \). Top and bottom rows, respectively, correspond to \( x_0 = 0, N/2 \).

For each \( \beta \), a family consisting of an infinite number of solutions parameterized by \( x_0 \in [-N/2, N/2] \) is obtained. Owing to the symmetry of the periodic structure, the analysis is restricted to solutions with \( x_0 \in [0, N/2] \). The solutions are symmetric or antisymmetric with respect to the center of the nonlinear or linear layer for \( x_0 = 0, N/2 \), respectively, while they are in general asymmetric for \( x_0 \neq 0, N/2 \). In Fig. 3 several spatial profiles are shown for \( x_0 = 0, N/2 \) for the case of a periodic structure having \( L = 4\pi, N = \pi, \) and \( \epsilon_N = 0 \). For a \( \Delta \epsilon = \epsilon_L - \epsilon_N > 0 \) [Fig. 3 (left column)], one obtains dark solitary wave profiles formed as localized dips on a finite periodic background for \( n = 2 \). For a negative \( \Delta \epsilon \), antidark solitary wave profiles are obtained for \( n = 1 \) [Fig. 3 (middle column)], while for \( n = 2 \) the profile changes from (slightly) dark to antidark when \( x_0 \) increases from zero to \( N/2 \) [Fig. 3 (right column)].
show that the stability issue and the associated growth rate depend on the energy and the shape of the initial profile.

In conclusion, a novel method is applied for providing analytical solitary wave solutions in periodic photonic structures. These solutions form a generic family, corresponding to the gaps of the linear band structure, shown to be quite robust under propagation in several cases.

The project is funded by the joint Greek–European Union grant EPAEK II–PYTHAGORAS. K. Hizanidis’s e-mail address is kyriakos@central.ntua.gr.

**References**