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ANALYTICAL SOLUTIONS OF SYSTEMS WITH PIECEWISE LINEAR DYNAMICS

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A class of nonautonomous dynamical systems, consisting of an autonomous nonlinear system and an autonomous linear periodic system, each acting by itself at different time intervals, is studied. It is shown that under certain conditions for the durations of the linear and the nonlinear time intervals, the dynamics of the nonautonomous piecewise linear system is closely related to that of its nonlinear autonomous component. As a result, families of explicit periodic, nonperiodic and localized breather-like solutions are analytically obtained for a variety of interesting physical phenomena.

Keywords: Piecewise linear systems; analytical solutions; nonautonomous systems.

1. Introduction

Nonautonomous dynamical systems describe a variety of physical phenomena and processes where the evolution of a system depends explicitly on the specific time interval and are encountered in many areas of physical interest. In addition, when the independent variable is not time but corresponds to a spatial dimension, such systems describe pattern formation in spatially inhomogeneous media. Interesting dynamics are related to this temporal and spatial inhomogeneity, corresponding to periodic, nonperiodic, localized and irregular (chaotic) solutions of the underlying dynamical systems.

The explicit dependence of a dynamical system on the evolution variable (time or space) results in an additional dimension in the phase space of the system (extended phase space). In most cases this explicit dependence leads to chaotic dynamics. Indeed, even within the class of integrable Hamiltonian systems, it is well known that small nonautonomous perturbations lead (in general) to nonintegrability and irregular dynamics. Perturbative approaches, on the other hand, allow for the investigation of the relation between the dynamics of the autonomous integrable system and the nonautonomous perturbed system. The Poincare–Birkhoff theorem as well as Melnikov's theory for periodic orbits [Guckenheimer & Holmes, 1990; Wiggins, 1990] predict that when certain resonance conditions between the unperturbed system and the time-dependent perturbation are satisfied, a finite discrete family of periodic solutions persist under perturbation. Additionally, the Melnikov theory for homoclinic orbits [Guckenheimer & Holmes, 1990;

Wiggins, 1990] relates the existence of an unperturbed orbit homoclinic to a fixed point with the persistence of a discrete number of orbits homoclinic to a periodic orbit.

In this work, we study a class of nonautonomous dynamical systems consisting of an autonomous nonlinear system and an autonomous linear periodic system, each acting alone at different time intervals. We show that under certain conditions for the durations of the linear and nonlinear time intervals, the behavior of the nonautonomous piecewise linear system is closely connected with the dynamics of its nonlinear autonomous component. In particular, we prove that a Poincare surface of section of the nonautonomous system is identical to the phase space of the autonomous nonlinear system. This result applies to a general class of dynamical systems including dissipative, integrable and nonintegrable Hamiltonian systems.

It is also demonstrated that the existence of periodic solutions or fixed points of the autonomous nonlinear system results in the existence of continuous families of periodic solutions for the nonautonomous system. Among all the systems for which the above results can be applied, we have chosen an example where the autonomous nonlinear system is a Hamiltonian integrable system. Our approach differs crucially from the perturbative approaches mentioned above for the following reasons: (a) The nonautonomous system under consideration cannot be considered as a perturbation of an integrable one since its time-dependent terms are not small. (b) The resulting families of solutions are not discrete but form continuous families parameterized by the initial time t_0, as shown in Sec. 2. Thus, we are able to obtain in Sec. 3 explicitly periodic and nonperiodic solutions of arbitrary complexity as we show in a simple example where the nonlinear system is a one-degree of freedom Hamiltonian oscillator. In Sec. 4, we relate these solutions to recently studied phenomena of nonlinear wave propagation and localization in inhomogeneous media occurring in a variety of physical problems as well as to particle beam dynamics in storage rings of high-energy accelerators. Finally, in Sec. 5 we present our conclusions and discuss possible future directions of this research.

2. Piecewise Linear Dynamical Systems

Let us consider two autonomous systems in the N-dimensional phase space ($\mathbf{x} \in \mathbb{R}^N$)

$$\dot{\mathbf{x}} = F_{NL}(\mathbf{x}) \tag{1}$$

and

$$\dot{\mathbf{x}} = F_L(\mathbf{x}) \tag{2}$$

with the former being any nonlinear system and the latter being a linear system with periodic solutions,

$$\mathbf{x}(t) = S_L(t - t_0; T_i)\mathbf{x}(t_0) \tag{3}$$

where $S_L(t - t_0; T_i)$ is the evolution operator and $\{T_i\}$ the possible periods of the system. Suppose now that we have a nonautonomous system consisting of both the above autonomous systems in the sense that, within finite time intervals, the dynamics of the system is determined exclusively by one of these systems. Such a composite system can be written in the following form

$$\dot{\mathbf{x}} = u(t)F_{NL}(\mathbf{x}) + [1 - u(t)]F_L(\mathbf{x}) \tag{4}$$

where $u(t)$ is a piecewise constant function taking only the values 0 and 1 in the time intervals T_{NL} and T_L, respectively,

$$u(t) = \begin{cases} 1, & \text{if } t \in T_{NL} \\ 0, & \text{if } t \in T_L \end{cases} \tag{5}$$

where $T_{NL} = \cup_i(t_i, t_{i+1})$ and $T_L = \cup_j(t_j, t_{j+1})$ are unions of finite time intervals of durations $\Delta t_{NL,i}$ and $\Delta t_{L,j}$, respectively (Fig. 1). The function $u(t)$ can be either periodic or nonperiodic. Note that an interesting case occurs when the linear system (2) coincides with the linear part of the nonlinear system (1). Then we have a nonautonomous system (4) corresponding to a situation where the nonlinearity is being switched on and off in the respective time intervals. The latter is of particular interest for systems describing particle beam dynamics in storage rings of accelarators, as discussed in Sec. 4.

In order to study this composite dynamical system, we shall consider it in the $(N+1)$-dimensional

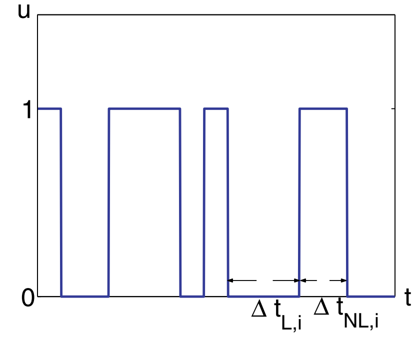


Fig. 1. Form of the function $u(t)$. Δt_{NL} and Δt_L are the durations of the nonlinear and linear time intervals, respectively.

extended phase space and utilize Poincare surfaces of section $P^{(t_s)}$ at time t_s. We are thus able to prove the following proposition:

Proposition 1. *If T_L consists of time intervals which are integer multiples of the least common multiplier* (LCM) *of the periods of the solution of the linear system (2) $T_{\rm LCM} = {\rm LCM}\{T_i\}$, then the Poincare surfaces of section P^{t_s} of the nonautonomous system (4) are identical to the phase space of the nonlinear system (1) when $t_s \in T_{NL}$. When $t_s \in T_L$ the respective Poincare surface of section P^{t_s} is related to the phase space of the nonlinear system (1) by the simple transformation provided by (3).*

Proof. The Poincare mapping from a time t_0 to a time t_s is given by P^{t_s} : $\mathbf{x}(t_s) = S_{NL}(\Delta t_{NL,i})$ $S_L(\Delta t_{L,i-1})\, S_{NL}(\Delta t_{NL,i-1}) \cdots S_L(\Delta t_{L,i-\ell+1})\, S_{NL}$ $(\Delta t_{NL,i-\ell})\mathbf{x}(t_0)$, where $S_{NL}(\Delta t_{NL,i})$ and $S_L(\Delta t_{L,i})$ are the evolution operators for the respective linear and nonlinear time intervals between t_0 and t_s. When a linear time interval is equal to an integer multiple of $T_{\rm LCM}$ then the linear evolution operator is the identity operator $S_L(\Delta t_{L,i}) = I$. Therefore, the Poincare mapping can be written as P^{t_s} : $\mathbf{x}(t_s) = S_{NL}(\Delta t_{NL,i})I \cdots IS_{NL}(\Delta t_{NL,i-\ell})\mathbf{x}(t_0)$, or $\mathbf{x}(t_s) = S_{NL}(\sum_i \Delta t_{NL,i})\mathbf{x}(t_0)$ so that an initial point of the phase space $\mathbf{x}(t_0)$ actually evolves as if only the nonlinear time intervals are taken into account. Thus, when we consider $t_s \in T_{NL}$, the Poincare surface of section P^{t_s} is identical to the phase space of the nonlinear system (1). When $t_s \in T_L$, the Poincare surface of section is related to the phase space of the nonlinear system (1), by the simple transformation $\mathbf{x}(t_s) = S_L(t_s - t_0)\mathbf{x}(t_0)$. ■

This proposition states that the dynamics of the nonautonomous system (4) is in fact identical to the dynamics of the autonomous nonlinear system (1): The evolution of $\mathbf{x}$, under the nonautonomous system is only interrupted by a periodic evolution in the time linear intervals (where $t \in T_L$), and returns exactly to the previous state after evolving according to the linear system when $\Delta t_{L,j}$ have the appropriate length (Fig. 1). Since the autonomous nonlinear system (1) is arbitrary, the above statement holds for any kind of dynamical system, be it an integrable or nonintegrable Hamiltonian system or a dissipative system. Any number of degrees of freedom can be considered and any kind of dynamical behavior for the autonomous nonlinear system (1), regular or chaotic. Therefore, knowledge of the dynamics of the nonlinear autonomous system (1) suggests also knowledge for the class of nonautonomous systems (4), for any piecewise constant function $u(t)$ satisfying the assumptions of Proposition 1.

For the case where the nonlinear autonomous system (1) possesses a symmetry, i.e. if it is invariant under a transformation C

$$S_{NL}(t - t_0)C\mathbf{x} = CS_{NL}(t - t_0)\mathbf{x}, \quad \forall\ t, t_0 \quad (6)$$

the linear time intervals need not only be integer multiples of $T_{\rm LCM}$ as required in Proposition 1. Thus, a larger class of functions $u(t)$, fulfills the conditions of Proposition 1 to hold. In such cases, we have the following:

Corollary 1. *If there exists a time interval $\Delta t_{L,C}$ so that $S_L(\Delta t_{L,C}) = C$, then the results of Proposition 1 also hold for functions $u(t)$ for which the linear time intervals are integer multiples of $\Delta t_{L,C}$.*

Proof. We use the property $S_{NL}[S_L(n\Delta t_{L,C})\mathbf{x}] = S_L(n\Delta t_{L,C})[S_{NL}\mathbf{x}]$, for n integer. The proof is quite similar to the one of Proposition 1. ■

A case of special interest is when the function $u(t)$ is periodic, with the linear and nonlinear time intervals being Δt_{NL} and Δt_L, respectively. In this case we can prove that, under certain conditions, the existence of a periodic orbit of the autonomous nonlinear system (1) results in the existence of a periodic orbit for the nonautonomous system (4), as follows:

Proposition 2. *If the autonomous nonlinear system (1) has a periodic orbit of period T, and if the time intervals Δt_L and Δt_{NL} are such that $\Delta t_L = nT_{\rm LCM}$ and $\ell\Delta t_{NL} = mT$, with ℓ, m, n integers, then the nonautonomous system (4) has a periodic orbit, which coincides with the one of system (1) in the nonlinear time intervals. Also, if the autonomous nonlinear system (1) has a fixed point, then if $\Delta t_L = nT_{\rm LCM}$, with n integer, the nonautonomous system (4) has a periodic orbit, which has a constant value in the nonlinear time intervals, determined by the fixed point of the system (1).*

Proof. For the evolution of a phase space point $\mathbf{x}(t_0)$, belonging to a periodic orbit of the system (1), under the system (4) we have: $\mathbf{x}(t_s) = S_{NL}(\Delta t_{NL,i})\quad S_L(\Delta t_{L,i-1}) \cdots S_L(\Delta t_{L,i-n+1})\, S_{NL}$ $(\Delta t_{NL,i-n})\mathbf{x}(t_0)$. When $\Delta t_L = nT_{\rm LCM}$, $\mathbf{x}(t_s) = S_{NL}(\sum_i \Delta t_{NL,i}))\mathbf{x}(t_0)$, as in Proposition 1. When the additional condition $\ell\Delta t_{NL} = mT$ holds, we can write $\mathbf{x}(t_s) = S_{NL}(\ell T)\mathbf{x}(t_0) = S_{NL}(T)\cdots S_{NL}(T)\,\mathbf{x}(t_0)$ (ℓ times) and finally $\mathbf{x}(t_s) = \mathbf{x}(t_0)$. For the case where $\mathbf{x}(t_0)$ is a fixed point the proof is very similar and utilizes the property $\mathbf{x}(t_s) = S_{NL}(\Delta t)\mathbf{x}(t_0)$ for any Δt. ■

For the case where the system (1) has more than one periodic orbits related with a symmetry property, in accordance to Corollary 1, we can choose the duration of the linear time interval Δt_L appropriately in order to have additional periodic orbits. In the following section, we clarify these general results with a simple example.

3. Nonautonomous Hamiltonian Systems with a Piecewise Linear Part

In the previous section, we have related the solutions of a single autonomous nonlinear system to the dynamics of a large class of nonautonomous systems. Among all the autonomous nonlinear systems for which the results of the respective propositions hold, in the following, we focus specifically on the class of integrable Hamiltonian systems. Integrable systems are very rare but are of great importance since they serve as starting points for studying larger classes of "nearby" perturbed systems. Perturbations, in general, result in symmetry breaking and loss of integrability. On the other hand, extensions by additional degrees of freedom also result in loss of integrability.

The results of the previous section suggest that an integrable Hamiltonian system of N-degrees of freedom can be used to provide knowledge about the dynamics of an extended system belonging to a large class of nonautonomous systems having an additional 1/2 degree of freedom (due to time dependence). Different nonautonomous systems can be obtained for different forms of the function $u(t)$.

In what follows, in order to exhibit the dynamical features of the respective class of nonautonomous Hamiltonian systems and clarify the results of the previous section, we shall consider an example where the nonlinear autonomous system (1) is a one degree of freedom Hamiltonian system. More specifically, we take a nonlinear (Duffing) oscillator with a Hamiltonian:

$$H_{NL}(q,p) = \frac{p^2}{2} + a\frac{q^2}{2} + \frac{q^4}{4} = h \tag{7}$$

As shown in Fig. 2, the phase space of the system consists of curves, labeled by the value of the

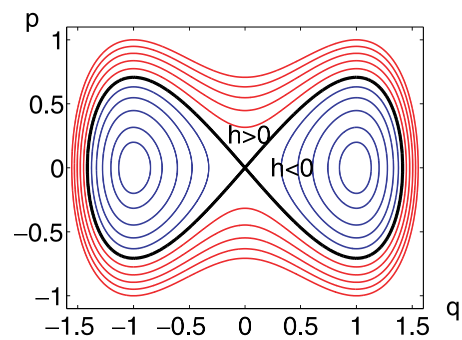


Fig. 2. Phase space of the Hamiltonian system (7), for $a = -1$.

Hamiltonian h which is a constant of the motion. For $a < 0$ we have two families of nonharmonic periodic orbits, corresponding to libration ($h < 0$) and rotation ($h > 0$) type of oscillations, which are separated by an orbit homoclinic to the origin ($h = 0$) having an infinite period. For $a = -1$, the period of the oscillation is

$$T(k) = \begin{cases} 2K(k)\sqrt{2-k^2}, & \text{if } k \in (0,1) \\ & \text{(libration)} \\ 4K(k)\sqrt{2k^2-1}, & \text{if } k \in (1, 1/\sqrt{2}) \\ & \text{(rotation)} \end{cases} \tag{8}$$

where $k(K)$ is the complete elliptic integral of the first kind and k is the elliptic modulus related to the Hamiltonian h as follows

$$h(k) = \frac{k^2-1}{(2-k^2)^2} \tag{9}$$

The phase space of the system possesses a symmetry property, since it is symmetric with respect to the origin. Therefore, Corollary 1 applies in this case, with the transformation C being

$$C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

As a linear system (2) let us use a system with Hamiltonian:

$$H_L(q,p) = \frac{p^2}{2} + \omega_0^2\frac{q^2}{2} \tag{11}$$

where $\omega_0 = 2\pi/T_0$ is the period of the oscillation. In the example, that follows we take $\omega_0 = \pi$. Also, we consider the case of a function $u(t)$ of the form (5) which is time-periodic.

In accordance to Proposition 2, the existence of two fixed points along with the symmetry property of the phase space of system (7) results in the existence of families of periodic orbits when the linear time interval is an integer multiple of the half-period of the linear system (11), i.e. $\Delta t_L = n\pi/\omega_0$ with $n = 1, 2, \ldots$. The periodic solutions consist of constant parts (corresponding to the fixed points) in the nonlinear time intervals and sinusoidal parts in the linear time intervals, while the number of zeros in the linear time intervals is equal to n, as shown in Figs. 3(a), 3(c) and 3(e). The value of the Hamiltonian h (7) is the same for all nonlinear time intervals and equal to the one corresponding to the fixed points. In Figs. 3(b), 3(d) and 3(f), it is shown that solutions starting from the fixed points in the nonlinear time intervals are not related to periodic solutions of the nonautonomous system when $\Delta t_L \neq n\pi/\omega_0$. In this case, the nonautonomous system, after evolving in a linear time interval, does not return to one of the fixed points of (7). The value

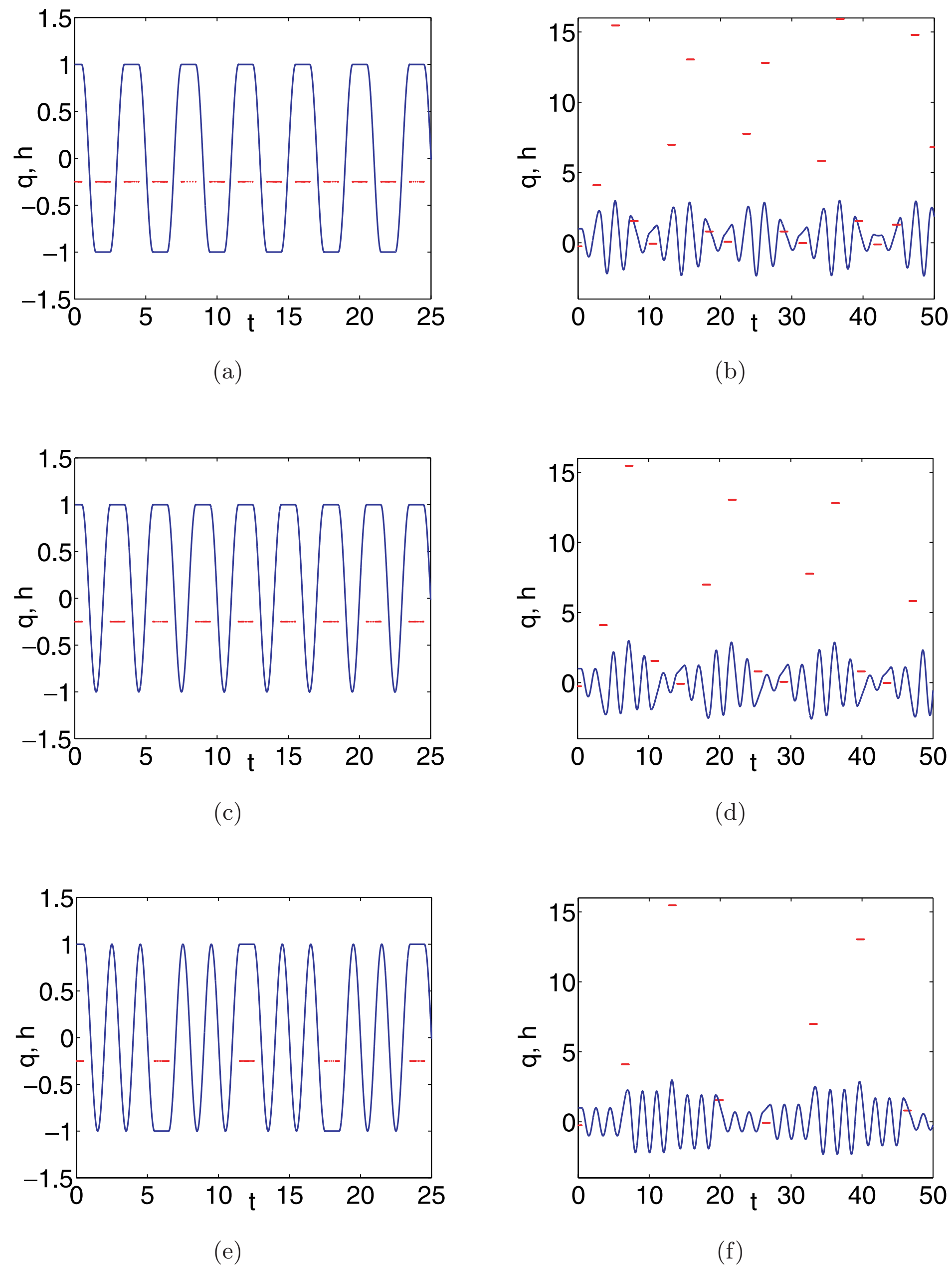


Fig. 3. (a), (c), (e): Periodic solutions of the nonautonomous system, related to the fixed points of the system (7), according to Proposition 2. The duration of the linear time intervals is $\Delta t_L = n\pi/\omega_0$ with $n = 1, 2, 5$, respectively. (b), (d), (f): Nonperiodic solutions are obtained for the same initial conditions when the linear time intervals are not integer multiples of the half-period of the linear system (11), $\Delta t_L = n\pi/\omega_0 + \pi/5$ for $n = 1, 2, 5$, respectively. The piecewise constant lines (red) depict the value of h in the nonlinear time intervals. The duration of the nonlinear time intervals is $\Delta t_{NL} = 1$, in all cases.

of h is different in each nonlinear time interval and a *nonperiodic* solution is obtained. In this case, due to the "nonresonance" between Δt_L and the half-period of the linear part π/ω_0, we expect that these nonperiodic solutions are actually quasiperiodic and correspond to orbits lying on invariant tori of the composite Hamiltonian system, guaranteeing thus the regularity of the motion in their vicinity.

Similarly, the periodic solutions of (7) result in periodic solutions of the nonautonomous system when in addition to the condition $\Delta t_L = n\pi/\omega_0$, $n = 1, 2, \ldots$, we also have $\ell \Delta t_{NL} = mT$, with ℓ, m integers, as stated in Proposition 2. In Figs. 4(a) and 4(b) such periodic solutions are shown. These solutions are constructed using initial conditions (taken in the nonlinear time interval)

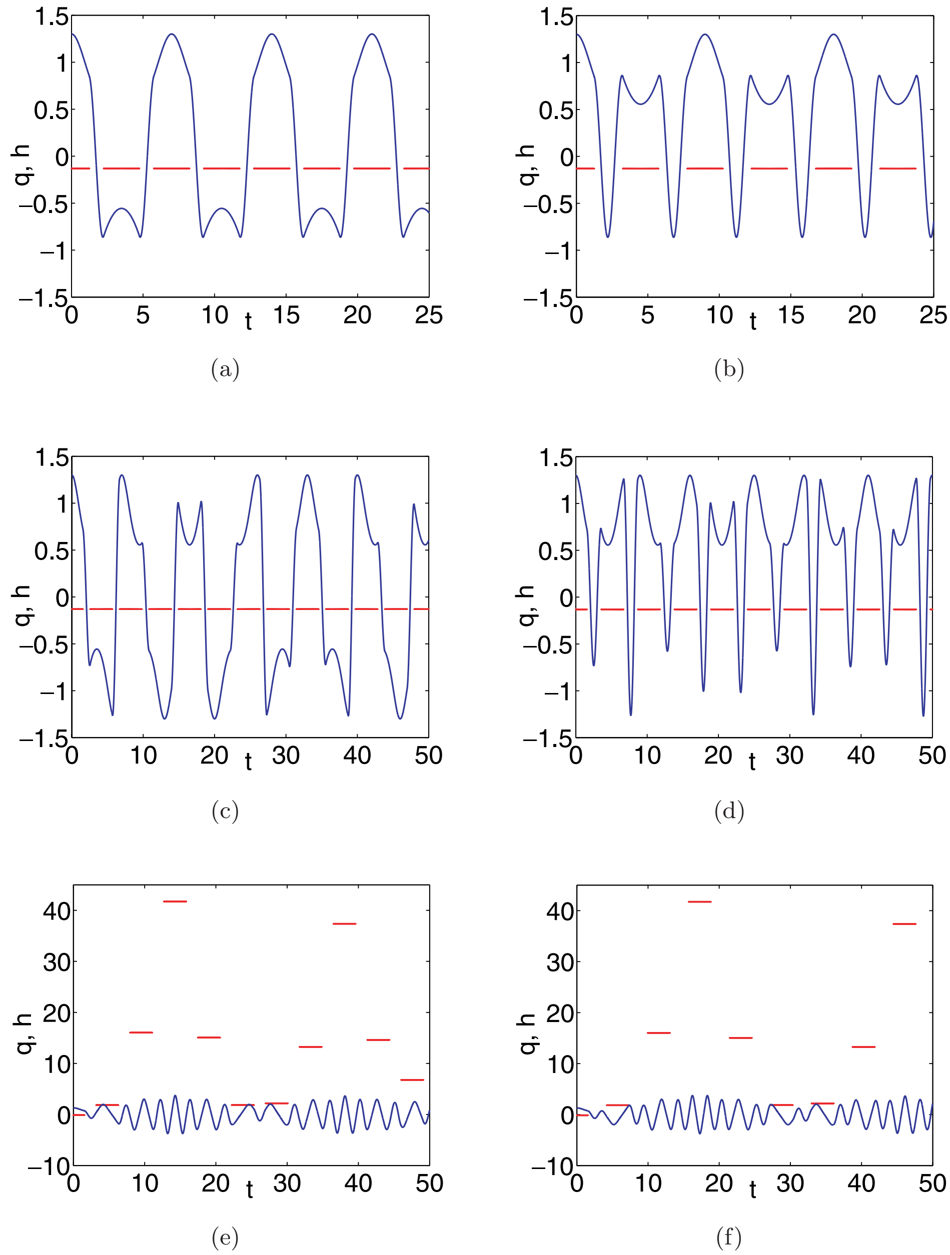


Fig. 4. (a), (b): Periodic solutions of the nonautonomous system, related to a periodic solution of the system (7) having period $T = 5$, according to Proposition 2. The respective initial conditions (in the nonlinear time interval) are $(p_0, q_0) = (0, 1.3005)$. The durations of the nonlinear and the linear time intervals are $\Delta t_{NL} = T/2 = 2.5$ and $\Delta t_L = n\pi/\omega_0$ with $n = 1, 2$, respectively. (c), (d): Nonperiodic solutions obtained for the same initial conditions when $\Delta t_{NL} = T/2 + \pi/5$ and $\Delta t_L = n\pi/\omega_0$ with $n = 1, 2$, respectively. (e), (f): Nonperiodic solutions obtained for the same initial conditions when $\Delta t_{NL} = T/2$ and $\Delta t_L = n\pi/\omega_0 + \pi/5$ with $n = 1, 2$, respectively. The piecewise constant lines (red) depict the value of h in the nonlinear time intervals.

$(p_0, q_0) = (0, 1.3005)$. These correspond to a periodic solution of (7) with period $T = 5$. The durations of the nonlinear and the linear time intervals are $\Delta t_{NL} = T/2 = 2.5$ and $\Delta t_L = n\pi/\omega_0$ with $n = 1, 2$, respectively. As shown in Figs. 4(c) and 4(d), for the case where the condition for the nonlinear time interval does not hold, the resulting orbits are not periodic. However, the condition for the linear time interval ensures that, after evolving in a linear time interval, the solution returns to a periodic solution of (7), so that the value of h is the same for all nonlinear time intervals. The amplitude of the solution remains constant, but the periodicity is not ensured due to the fact that the "phase" of the nonlinear oscillation is not "appropriate" after evolution in the linear time interval. The respective case where the condition for the linear time interval is violated is shown in Figs. 4(e) and 4(f).

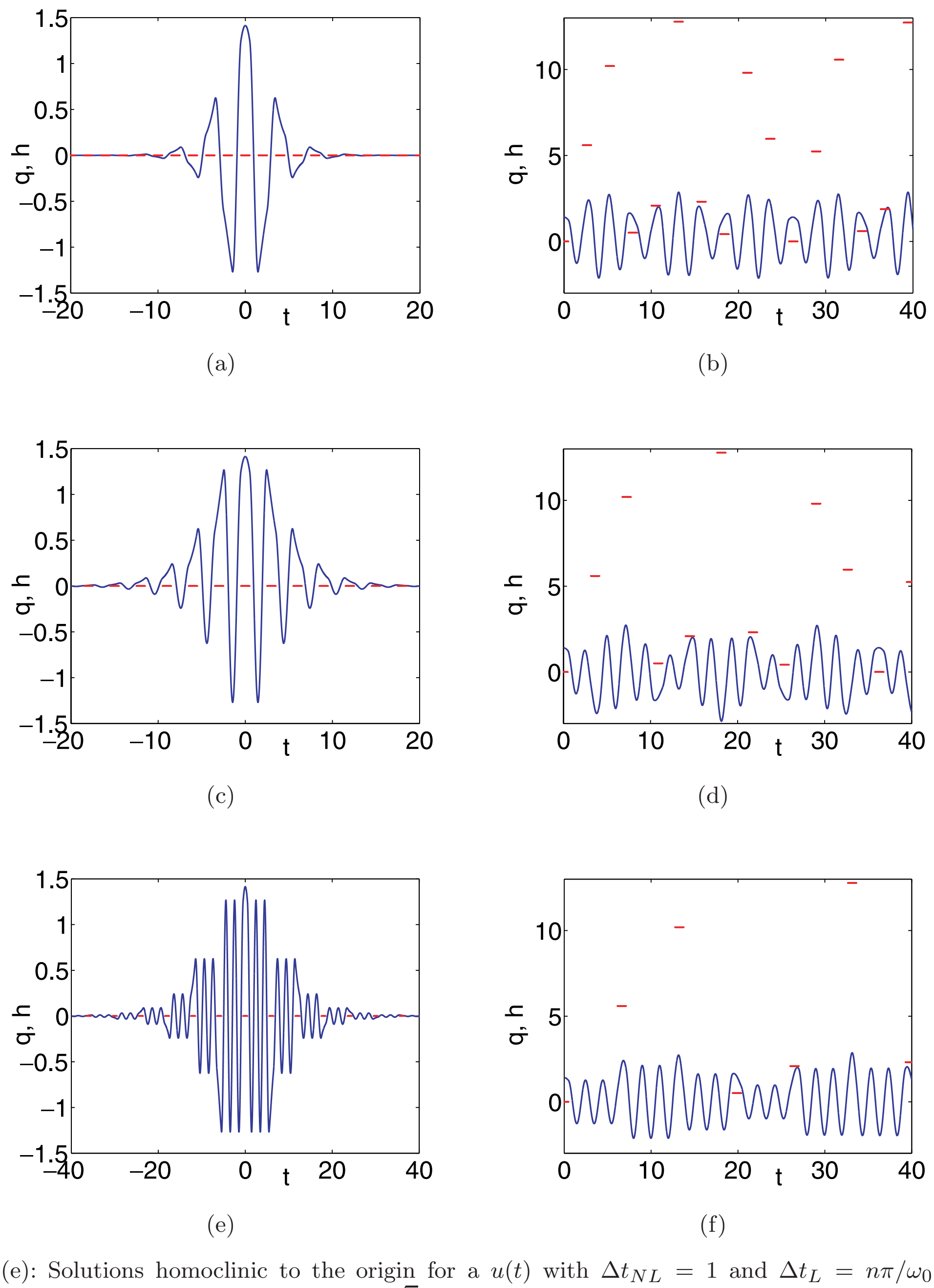
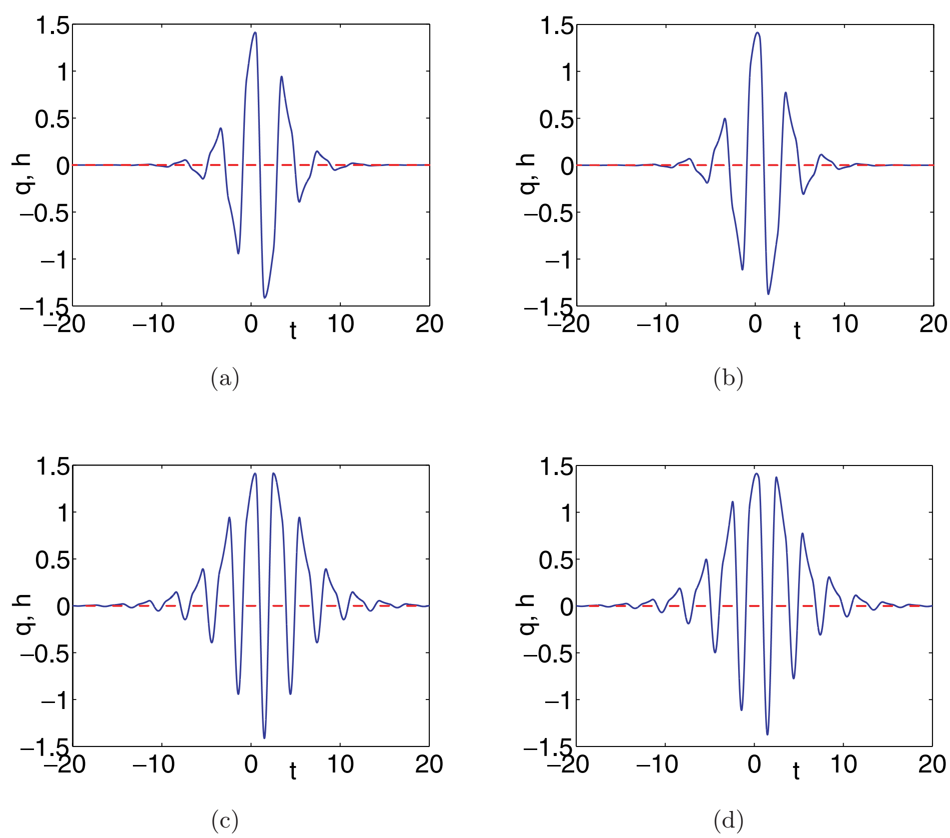


Fig. 5. (a), (c), (e): Solutions homoclinic to the origin for a $u(t)$ with $\Delta t_{NL} = 1$ and $\Delta t_L = n\pi/\omega_0$ with $n = 1, 2, 5$, respectively. The initial conditions are $(p_0, q_0) = (0, \sqrt{2})$ at $t_0 = 0$ (center of the nonlinear time interval). (b), (d), (f): Solutions corresponding to the same initial conditions but with $\Delta t_L = n\pi/\omega_0 + \pi/5$. The piecewise constant lines (red) depict the value of h in the nonlinear time intervals.

According to Proposition 1, when the duration of the linear time intervals (with $u(t) = 0$) is appropriate, the dynamics of the nonautonomous system is directly related to that of the autonomous nonlinear system (7). In addition to the families of periodic orbits, investigated in the previous paragraphs, it is particularly interesting to investigate the solutions of the nonautonomous system related to the *homoclinic* solution of (7). Starting from an initial condition located in the homoclinic orbit of the nonlinear system (7), the solution coincides with the homoclinic solution within the nonlinear time interval. After evolving in the linear time interval of duration $\Delta t_L = n\pi/\omega_0$, the solution of the nonautonomous system returns to the homoclinic solution either in the same (n: even) or in the symmetric (n: odd) branch. These asymptotic solutions are homoclinic to the origin and are reminiscent of the so-called *breather* type, which have been extensively studied in the literature as spatially localized periodic solutions of nonlinear lattices (for a recent review see [Flach & Gorbach, 2008]). In Figs. 5(a), 5(c) and 5(e) such solutions are shown for a $u(t)$ with $\Delta t_{NL} = 1$ and $\Delta t_L = n\pi/\omega_0$ with $n = 1, 2, 5$, respectively. The initial conditions are $(p_0, q_0) = (0, \sqrt{2})$ at $t_0 = 0$ which corresponds to the center of the nonlinear time interval. The violation of the condition $\Delta t_L = n\pi/\omega_0$ results in delocalized nonperiodic solutions, as shown in Figs. 5(b), 5(d) and 5(f).

It is quite interesting that additional localized solutions can thus be obtained from the same initial conditions by changing the initial time t_0. For the case where $t_0 = 0$ the resulting solutions are symmetric with respect to the center of the nonlinear time interval as shown in Figs. 5(a), 5(c) and 5(e). When t_0 corresponds to the edge of the nonlinear time interval the resulting solutions are either antisymmetric (n: odd) or symmetric (n: even) with respect to the center of the linear time interval, as shown in Figs. 6(a) and 6(c). For an intermediate value of t_0 the solutions are in general nonsymmetric as shown in Figs. 6(b) and 6(d). Note that the same arguments for the selection of the initial time t_0 apply also for the case of periodic orbits, so that each one of the periodic solutions shown previously is a member of family of solutions parameterized by t_0. Homoclinic solutions constructed by



(a) (b)

(c) (d)

Fig. 6. (a), (b): Homoclinic solutions corresponding to initial conditions $(p_0, q_0) = (0, \sqrt{2})$ at $t_0 = 0.5$ (edge of the nonlinear interval) and $t_0 = 0.25$ (inside the nonlinear interval) for $\Delta t_L = n\pi/\omega_0$, $n = 1$. (c), (d): Same as in (a) and (b) for $n = 2$.

this method have been shown to correspond to stationary solitary wave solutions propagating in optical lattices of a variety of different configurations [Kominis, 2006; Kominis & Hizanidis, 2006; Kominis *et al.*, 2007].

4. Application to Physical Problems

Nonautonomous systems of the class discussed in the previous sections are met in a variety of problems of physical interest. Among them, in what follows, we refer to: (a) the formation of localized and periodic waves in nonlinear inhomogeneous media of interest to nonlinear optics or Bose–Einstein Condensates, and (b) particle beam dynamics in storage rings of high-energy accelerators.

Nonlinear wave propagation in a transversely inhomogeneous nonlinear (Kerr-type) optical medium is described by a NonLinear Scrödinger Equation (NLSE), with periodically varying coefficients:

$$i\frac{\partial\Psi}{\partial Z} + \frac{\partial^2\Psi}{\partial T^2} + \epsilon(T)\Psi + g(T, |\Psi|^2)\Psi = 0 \qquad (12)$$

where Z, T and Ψ are the normalized propagation distance, transverse dimension and electric field, respectively. The periodic transverse variation of the linear refractive index is given by $\epsilon(T)$, while the spatial and intensity dependence of the nonlinear refractive index is provided through $g(T, |\psi|^2)$. The stationary solutions of (12) have the form $\Psi(T, Z) = X(T; \beta)e^{i\beta Z}$, and satisfy the nonlinear ordinary differential equation

$$\frac{d^2X}{dT^2} + [\epsilon(T) - \beta]X + g(T, X^2)X = 0 \qquad (13)$$

where β is the propagation constant and $X(T; \beta)$ is the real transverse wave profile. For the case where the photonic structure is a waveguide array consisting of alternating linear and nonlinear layers of constant refractive indices, the dynamical system (13) belongs to the class of nonautonomous systems discussed in the previous sections. It has been shown [Kominis, 2006; Kominis & Hizanidis, 2006; Kominis *et al.*, 2007] that stationary solitary wave solutions can be analytically obtained under conditions similar to those of Proposition 1, involving the propagation constant β, the value of the linear refractive index and the width of the linear layer. Utilizing the results of this work, we demonstrate that there is a more fundamental relation between the dynamics of the system (13) and that of the respective autonomous nonlinear system describing stationary solutions in a homogeneous nonlinear medium, allowing for obtaining not only periodic but also quasiperiodic and homoclinic wave solutions.

On the other hand, the evolution of the mean-field wave function of a Bose–Einstein condensate in an optical trap obeys an equation identical to (12), commonly referred as the Gross–Pitaevskii equation, in the respective literature. For the case of a periodic, piecewise-constant scattering length [Rodrigues *et al.*, 2008] the stationary wave solutions are described by a system for which the results of this work apply directly.

Another area of physical problems where systems of the class discussed in this work appear is the study of particle beam dynamics in storage rings of high-energy accelerators (see e.g. [Month & Herrera, 1980; Carrigan *et al.*, 1982; Turchetti & Scandale, 1991]). In such devices, one often considers experiments in which particle motion is determined by a linear (harmonic) Hamiltonian system interrupted periodically by short nonlinear "kicks," due to magnetic focusing elements spaced at equal distances l along the ring. The time intervals corresponding to these "kicks" are very short and are commonly modeled by δ functions. The respective Hamiltonian has the general form:

$$H(\mathbf{q}, \mathbf{p}, t) = H_L(\mathbf{q}, \mathbf{p}) + H_{NL}(\mathbf{q}, \mathbf{p})\sum_n \delta(t - nl) \qquad (14)$$

where

$$H_L = \sum_k \left(\frac{p_k^2}{2} + \omega_{0(k)}\frac{q_k^2}{2}\right) \qquad (15)$$

and H_{NL} corresponds to the nonlinear particle motion due to passing through focusing elements. If we consider δ functions as a limit of a piecewise constant function, which is actually more realistic, Proposition 1 applies, when the distance l takes a value which is a multiple of the period of the linear harmonic motion, and the dynamics of the composite system is thus related to that of the respective autonomous nonlinear system. Under this condition, the beam returns to its initial state after evolving between two focusing elements. As a result, when the autonomous nonlinear system is integrable, as in the case of "flat" (one-dimensional) beams represented by a one-degree of freedom Hamiltonian, one can readily obtain periodic solutions according to Proposition 2. For the case of a nonintegrable autonomous nonlinear system, the condition for the distance l ensures that the additional degree of freedom due to the "time" dependence of the Hamiltonian does not change qualitatively the properties of the orbits: Particle motion is only interrupted periodically by its passage through a nonlinear focusing element and returns to its initial state after evolving linearly. Therefore, periodic or quasiperiodic orbits remain qualitatively intact and the system's dynamics is essentially determined by the respective autonomous nonlinear system describing particle motion due to the focusing elements. The above considerations are directly related to the determination of the beam's *dynamical aperture*, i.e. the maximal domain containing the particles close to their ideal circular path for the longest possible time (see e.g. [Bountis & Skokos, 2006]), which crucially depends on the spacing of the focusing element and the time duration of their effect.

5. Summary and Conclusions

In this work we have investigated the dynamics of a nonautonomous dynamical system consisting of time intervals where the evolution is determined by an autonomous linear periodic system and intervals where the evolution is determined by an autonomous nonlinear system. By utilizing a Poincare surface of section approach, we have first established, by our Proposition 1, a relation that connects directly the dynamics of the autonomous nonlinear system to that of the nonautonomous system in the extended phase space, when the durations of the respective time intervals fulfill certain conditions. Then, through our Proposition 2, we proved that continuous families of periodic solutions of the nonautonomous system can be obtained using the periodic solutions and the fixed points of the respective autonomous nonlinear system, while a violation of one of our periodicity conditions leads to nonperiodic solutions of the quasiperiodic type. We have also demonstrated that the existence of symmetry properties results in additional classes of solutions, while families of homoclinic oscillations localized in time were obtained as well. Such homoclinic solutions can also be obtained by our method in nonlinear lattices, where they are localized in space and are truly of the breather type [Kominis *et al.*, preprint]. Finally, we have discussed a variety of physical problems of technological interest, where our results are directly applicable and which we plan to investigate thoroughly in a future publication.

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