International Journal of Bifurcation and Chaos, Vol. 16, No. 6 (2006) 1753–1764

SOLITARY WAVE INTERACTIONS WITH CONTINUOUS WAVES

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Received February 14, 2005; Revised April 25, 2005

Solitary wave propagation under interaction with continuous waves is studied in the context of the Nonlinear Schrödinger Equation. An analytical approach, based on the conserved quantities of the wave evolution, is used to study transverse velocity variations for the case of nonzero transverse wavenumber difference between the solitary and continuous waves. The method is applicable for any number of transverse dimensions and any kind of nonlinearity. Moreover, the presence of a coherent continuous background is shown to be responsible for the creation of solitary rings and spirals, under interaction with solitary structures with nonzero topological charge. Numerical simulations for specific cases were used to confirm the analytical results.

Keywords: Solitons; nonlinear wave propagation.

1. Introduction

The N-dimensional Nonlinear Schrödinger Equation (NLSE)

$$iu_z + \Delta_\perp u + F(|u|^2)u = 0 \tag{1}$$

is one of the most fundamental models for nonlinear wave propagation in a variety of branches of physics, describing the slowly varying envelope of a wave-train in conservative, dispersive systems. In the context of hydrodynamics, it occurs as a model describing gravity waves on deep water for which the modulational instability and envelope soliton formation predicted by the model has also been demonstrated experimentally [Ablowitz & Segur, 1979; Yuen & Lake, 1975]. In plasma physics the NLSE was first derived for nonlinear hydromagnetic waves using the reductive perturbation method [Taniuti & Washimi, 1968], while later on it has been derived for many wave modes of plasmas [Pecseli, 1985]. A quite similar equation to the NLSE with complex nonlinearity describes wave envelopes in nonconservative systems, and appears in the theory of superconductivity as the Ginzburg–Landau equation [Newell & Whitehead, 1969]. In the context of nonlinear optics, the NLSE describes the propagation of light beams in dispersive media with intensity-dependent refractive index [Chiao *et al.*, 1964; Kelley, 1965]. There are also several applications where the NLSE does not describe a wave envelope. The 1-D equation describes the propagation of the Davydov-solitons on an α-helix protein [Davydov, 1979], while the 3-D equation applies to the description of the Bose–Einstein Condensates (BEC) [Ginzburg & Pitaevski, 1958]. Finally, the NLSE occurs as a model in quantum field theories (see Sec. 4.5 in [Ablowitz & Segur, 1981]).

The physical context of this work mostly refers to nonlinear optics, which is a field of continuously increasing research interest because of its applications in optical signal transmission and processing [Kivshar & Agrawal, 2003]. However the application of the results, concerning interactions between solitary and continuous waves, to any of the aforementioned fields is straightforward. In the context of nonlinear optics, NLSE describes the evolution of the complex envelope $u(z, \mathbf{r}_\perp)$ of an electric field within the paraxial model of self-focusing [Kelley, 1965]. The z coordinate measures the propagation distance and the transverse coordinates $(\mathbf{r}_\perp)$ might be spatial and/or temporal, with the corresponding terms of linear operator $\Delta_\perp$ describing diffraction and/or (anomalous) dispersion, respectively. Thus, the 1-D equation describes a light pulse or a self-focusing beam in a planar waveguide, while the 2-D equation describes a self-focusing beam in a bulk medium or a localized wave packet in both spatial and temporal dimensions. Finally, the 3-D equation models the evolution of spatio-temporarily localized structures, namely "light-bullets". In the subsequent analysis we refer to the transverse coordinates as being spatial, although the arguments also hold for the spatiotemporal case.

The nonlinear function $F(I)$, models the intensity dependent refractive index of the medium and has the form $F(I) = I/(1 + s^2 I)$, where $I = |u|^2$ and $0 \leq s \leq 1$ is the saturation parameter of the nonlinear medium. For $s = 0$, we have a cubic (Kerr) nonlinearity and for $s \ll 1$ the function approximates the competing cubic-quintic nonlinearity $(F(I) = I - s^2 I^2)$. The 1-D equation with $s = 0$, is known to be completely integrable in terms of the Inverse Scattering Transform (IST), it admits soliton solutions and has an infinite number of conserved quantities, which are related to symmetries [Ablowitz & Segur, 1981]. Although, the NLSE is not known to be completely integrable in higher dimensions, the balance between diffraction (and/or dispersion) and nonlinearity can result in the formation of localized structures. The stability of these structures is a subject of major interest in the last decade, and it has been shown that for $s = 0$, they can diffract or collapse in a finite distance of propagation, depending on their initial power [Rasmussen & Rypdal, 1986; Berge, 1998]. Collapse arresting mechanisms such as high saturation of the refractive index [Marburger & Dawes, 1968] and the effect of nonparaxiality have been proposed [Fibich, 1996; Sheppard & Haelterman, 1998], while for a variety of applications, including all-optical switching devices, collapsing does not form a drawback, provided that the power of the beam and the length of propagation are chosen within appropriate value ranges [Desaix *et al.*, 1991; Fibich & Gaeta, 2000].

Although a great part of the research interest has been focused on the fundamental self-trapped guided mode (i.e. mode without nodes) [Rasmussen & Rypdal, 1986], it has been also known that self-trapping can also occur for higher-order beams of radially symmetric intensity and azimuthal phase modulation [Afanasjev, 1995; Atai *et al.*, 1994]. The latter are beams with intensity which vanish at the beam center forming a ringlike structure, for the 2-D case. The beam phase has a spiral structure with a singularity at the origin representing a phase dislocation and resembling the structure of an optical vortex. Such phase structures can be associated with a nonzero angular momentum of the corresponding solitary structures. The existence of the aforementioned structures have been demonstrated in many analytical, numerical and experimental studies, in a variety of nonlinear optical media, including Kerr [Kruglov *et al.*, 1992; Afanasjev, 1995], saturable [Skryabin & Firth, 1998; Anastassiou *et al.*, 2001; Tikhonenko *et al.*, 1995] and quadratic [Skryabin & Firth, 1998] types of nonlinearities. Their stability is also a field of interest since these ringlike structures are intrinsically dynamical, undergoing expanding or shrinking, while retaining their solitary character and radial symmetry despite this dynamic evolution [Afanasjev, 1995]. However, both fundamental and ringlike structures are shown to undergo an azimuthal symmetry-breaking instability resulting in beam filamentation and formation of a set of fundamental modes [Skryabin & Firth, 1998].

In this work, we investigate interactions between Solitary Waves (SW) and Continuous Waves (CW) of the NLSE. For the case of nonzero transverse wavenumber difference between the two waves, these interactions are shown capable of affecting certain parameters of the solitary beams, the most important being their transverse velocity. Thus, the intentional injection of an appropriate CW can be used as a control mechanism for changing beam's transverse velocity, resulting in the capability of multidimensional beam steering. This feature of the interactions is very desirable in designing all-optical and dynamically reconfigurable switching devices for potential applications in signal processing and telecommunications. In fact, a variety of beam steering techniques has been investigated for $N = 1, 2$ [Cao *et al.*, 1994; Snyder & Sheppard, 1993; Kang *et al.*, 1996; Christou *et al.*, 1996], and the propagation of one-dimensional solitons lying on a CW background has been studied both analytically and numerically [Akhmediev & Wabnitz, 1992; Kominis & Hizanidis, 2004a, 2004b].

For the case of zero transverse wavenumber difference the CW in the traveling-wave frame of reference of the former, have the form of a constant background. One may interpret the presence of this CW background as either an undesirable residual illumination or, more importantly, as a control signal for determining the evolution of a solitary structure in the context of optical data processing. The initial spatial profiles considered, do not necessarily correspond to exact self-trapped solutions. However, they provide physical insight and reflect key features underlying the evolution of more general initial conditions under the presence of a CW background. It is shown that the latter causes a drastic effect at the propagation of the superimposed solitary structures.

In order to study this kind of interactions in higher dimensions, one may apply standard variational methods to a perturbed NLSE, which is obtained if we treat the CW as an effective external potential. However, this method results in a nonautonomous, multidimensional dynamical system for the amplitude, width, phase, center position and transverse velocity of the beam which possess complex dynamical features, thus, preventing from simple intuitive understanding of the important interaction features. Instead, a much simpler and intuitive analytical approach based on two conserved quantities of the NLSE, namely the "mass" and the "momentum", is utilized. This approach applies for any dimension and nonlinearity function. More importantly, it results in simple formulas for "mass" and "momentum" variation, which provide useful guidelines for optimal parameter selection for efficient beam steering as well as understanding and prediction of the evolution of more complex solitary structures under the interaction. As shown in the following, the presence of a CW affects both the "mass" and the "momentum" of the solitary beam. The "mass" dependent self-focusing instability of the higher-dimensional NLSE makes it necessary to select the interaction parameters so that the resulting "mass" of the beam do not lead to self-focusing. The latter is quite undesirable because, for the Kerr-type nonlinearity leads to beam collapse, while even for a saturable nonlinearity, where there is no collapse, self-focusing and the corresponding beam amplitude increase reduces the steering efficiency of the interaction. Direct numerical simulations are used to confirm the analytically obtained estimations.

The subsequent analysis is organized as follows: In the following section, solitary and continuous wave solutions are defined and general properties of the NLSE are given. The analysis of the subsequent sections investigates interactions between solitary and continouus waves with nonzero and zero transverse wavenumber differences, respectively. Finally, the conclusions are summarized in the last section.

2. Solutions and Conserved Quantities of the NLSE

In order to obtain SW solutions of the NLSE, substitution of a standing wave solution of the form $u_s = U(\mathbf{r}_\perp; \lambda)\exp(ik_z^S z)$ to the NLSE results in the stationary equation

$$-k_z^S U + \Delta_\perp U + F(|U|^2)U = 0 \tag{2}$$

and by assuming an azimuthal dependency of the form $\exp(im\theta)$ and applying the appropriate boundary conditions, the SW can be numerically obtained, using standard shooting methods. It is shown that, for $m = 0$ the SW has nonzero intensity in the center, corresponding to a bright beam, while for $m \neq 0$ the SW has the form of a ring-like structure. The azimuthal number m is directly related to the angular momentum (or the topological charge) of the structure [Kivshar & Agrawal, 2003]. Since the NLSE in invariant to Galilean transformations, any stationary solution is a characteristic member of a family of traveling wave solutions of the form

$$u_s = U(z - \mathbf{k}_\perp^\mathbf{s} \cdot \mathbf{r}_\perp; k_z^S)\exp(-i\mathbf{k}_\perp^\mathbf{s} \cdot \mathbf{r}_\perp + ik_z^S z) \tag{3}$$

On the other hand, the CW are small amplitude solutions of the NLSE for which the nonlinear term is negligible. Thus the CW have the form

$$u_{cw} = \alpha \exp(-i\mathbf{k}_\perp^\mathbf{cw} \cdot \mathbf{r}_\perp + ik_z^{cw} z + i\phi) \tag{4}$$

where $k_z^{cw} = -(1/2)|\mathbf{k}_\perp^\mathbf{cw}|^2$ is the dispersion relation of the linearized NLSE.

The "mass" and "momentum" of a solution of (1) are defined as

$$P = \int |u|^2 dS \tag{5}$$

$$\mathbf{M} = \frac{i}{2}\int (u^\star \nabla_\perp u - u \nabla_\perp u^\star) dS \tag{6}$$

where dS is the area element normal to z, and are conserved when u evolves under (1). Since we are interested in interactions between SW and CW, we consider u as a superposition of a solitary part u_s and a CW part u_{cw}

$$u = u_s + u_{cw} \tag{7}$$

Substituting (7) in (5) we obtain

$$\begin{aligned} P &= P_s + P_{cw} + \Delta P_s, \\ \Delta P_s &= \int (u_s u_{cw}^\star + u_s^\star u_{cw}) dS \end{aligned} \tag{8}$$

where $P_s = \int |u_s|^2 dS$, and $P_{cw} = \int_{S_0} |u_{cw}|^2 dS$ with S_0 being an area much bigger than the characteristic width of the beam, but finite, so that the CW has finite "mass", according to the usual practice of "mass" definition for solitary waves with nonzero background [Kivshar & Luther-Davies, 1998]. Similarly, substitution of (7) in (6) results in

$$\begin{aligned} \mathbf{M} &= \mathbf{M_s} + \mathbf{M_{cw}} + \Delta\mathbf{M_s}, \\ \Delta\mathbf{M_s} &= i \int (u_s^\star \nabla_\perp u_{cw} - u_s \nabla_\perp u_{cw}^\star) dS \end{aligned} \tag{9}$$

where $\mathbf{M_s}$ and $\mathbf{M_{cw}}$ are defined analogously. It can be easily shown that:

$$\begin{aligned} \mathbf{M_s} &= P_s \mathbf{k}_\perp^{\mathbf{s}} \\ \mathbf{M_{cw}} &= P_{cw} \mathbf{k}_\perp^{\mathbf{cw}} \\ \Delta\mathbf{M_s} &= \Delta P_s \mathbf{k}_\perp^{\mathbf{cw}} \end{aligned} \tag{10}$$

Under the assumption of small amplitude CW and efficiently short distance of propagation, the effect of Modulational Instability (MI) is negligible and the background (CW) remains constant. Thus, it is reasonable to assume that the variations ΔP and $\Delta\mathbf{M}$ should be considered as variations of the "mass" and "momentum" of the SW and not the CW. This assumption has also been confirmed by direct numerical simulations. The variation of the transverse wavenumber of the beam can be written in the following form

$$\Delta\mathbf{k}_\perp^{\mathbf{s}} = \frac{\Delta P_s}{P_s} \Delta\mathbf{k}_\perp, \quad \Delta\mathbf{k}_\perp \equiv \mathbf{k}_\perp^{\mathbf{cw}} - \mathbf{k}_\perp^{\mathbf{s}} \tag{11}$$

Thus, the variation of the transverse wavenumber (velocity) of the beam ($\Delta\mathbf{k}_\perp^{\mathbf{s}}$) has the same direction with the transverse wavenumber difference between the beam and the CW ($\Delta\mathbf{k}_\perp$).

3. Nonzero Transverse Wavenumber Difference

In this section we investigate interactions between SW with $m = 0$ and CW with different transverse wavenumbers. As it is shown in (11) the capability of beam steering at any desired direction, is provided, by appropriate choice of $\Delta\mathbf{k}_\perp$, as shown in Fig. 1, for the two-dimensional case. It is remarkable that for the case of a sole transverse dimension and Kerr-type nonlinearity i.e. the completely integrable case, Eqs. (8) and (11), coincide with those obtained by means of perturbation on the associated linear eigenvalue problem of the NLSE, according to the IST method [Hasegawa & Kodama, 1982]. However, our approach extends the capability of estimating the variation of the "mass" and "momentum" (or the transverse wavenumber/velocity $\mathbf{k}_\perp^{\mathbf{s}}$) due to the presence of a CW background, in two directions: higher-dimensional NLSE and more general nonlinearity functions can be studied if u_s can be found numerically or be approximated by a Gaussian (or super-Gaussian) [Karlsson, 1992]. For the two-dimensional NLSE, a circular solitary beam can be written in the following form

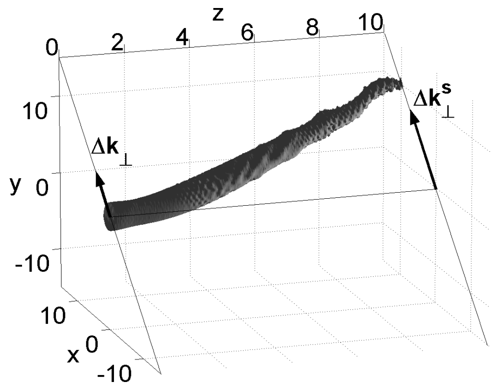


Fig. 1. Steering of a two-dimensional circular Gaussian beam, under interaction with a CW.

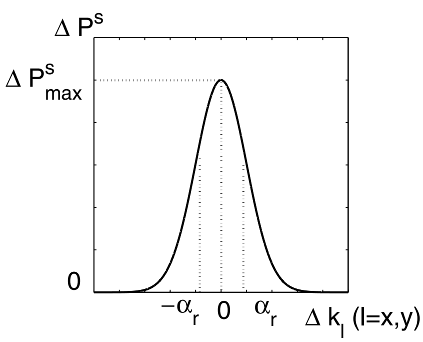
$$\begin{aligned} u_s = &A \exp\left(-\frac{x^2+y^2}{2a_r^2}\right) \\ &\times \exp\left(-ik_x^s x - ik_y^s y + i\sigma\right) \end{aligned} \tag{12}$$

Using (8), the variation of the "mass" is

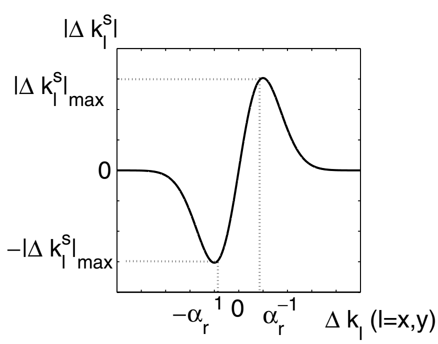
$$\Delta P^s = 4\pi\alpha A a_r^2 \cos(\Delta\phi) \exp\left(\frac{-a_r^2|\Delta\mathbf{k}_\perp|^2}{2}\right) \tag{13}$$

where $\Delta\phi \equiv \phi - \sigma$ is the initial phase difference and a_r is the width of the beam. The variation of the transverse velocity (or wavenumber) of the beam $\Delta\mathbf{k}_\perp^{\mathbf{s}}$ can be obtained directly from (11) and (13).

As it can be seen from (11) and (13), variations of the "mass" and the transverse velocity depend critically on the amplitude of the CW background and the initial phase difference between the beam and the CW. As expected from the perturbative character of our approach, both variations are linearly dependent on the amplitude of the CW. Moreover, the initial phase difference is shown to be crucial for the capability of altering the "mass" or the transverse velocity of a beam, under interaction with a CW. Considering, the dependence of ΔP^s on the transverse wavenumber difference between the beam and the CW ($\Delta\mathbf{k}_\perp$), it is obvious that the maximum ΔP is attained for $\Delta\mathbf{k}_\perp = 0$ [Fig. 2(a)]. Deviations from $\Delta\mathbf{k}_\perp = 0$, have effects which depend strongly on the characteristic size of the beam, since ΔP^s is a Gaussian function of $|\Delta\mathbf{k}_\perp|$, whose width depends on a_r. The maximum velocity along a transverse direction can be achieved for $|\Delta\mathbf{k}_\perp|_{\max} = a_r^{-1}$. For transverse wavenumber differences which are not comparable with the characteristic size of the beam, the CW background cannot affect significantly the transverse velocity of the beam [Fig. 2(b)]. According to formulas (11) and (13), the injection of a CW of appropriate transverse wavenumber difference for beam steering, leads to an increasing beam "mass", which, due to the self-focusing effect, can be



(a)



(b)

Fig. 2. Gaussian beam's variation of (a) "mass" (ΔP^s) and (b) transverse wavenumber ($\Delta k_l^s, l = x, y$) due to the presence of a CW with transverse wavenumber difference Δk_l, ($l = x, y$).

detrimental for the evolution of the beam width and amplitude. The latter determines strongly the propagation length in which the amplitude of the CW is large enough in comparison with the beam amplitude so that the interaction mechanism actually works.

It is well known that solitary wave propagation in nonlinear media governed by the NLSE is not stable in general [Rasmussen & Rypdal, 1986]. For a Kerr type nonlinearity the two-dimensional case is critical for stable propagation. Catastrophic collapse or diffraction can occur depending on the initial "mass" of a beam: namely, there is a critical value for the "mass" ($P_{\rm cr}$) above which the amplitude of the beam increases to infinity and its width decreases to zero after a finite propagation distance. Beams with "mass" values below ($P_{\rm cr}$) continuously diffract and are also destroyed. The critical value of the "mass" as well as the collapse distance, have been calculated analytically for the Gaussian approximation ($P_{\rm cr} = 4\pi$), and numerically for the exact stationary solution of the NLSE ($P'_{\rm cr} = 11.7$) [Desaix *et al.*, 1991]. Beam destruction under the aforementioned instability can be easily avoided in practical applications, by the choice of a medium length shorter than the collapse distance. On the other hand, a saturable nonlinearity can be used as a collapse-arresting mechanism. In this case the beam "mass" needed for self-trapping increases with the saturation parameter [Karlsson, 1992]. Under interaction with a CW, the self-focusing effect implies that the beam characteristics will not remain constant along the propagation distance. However, beam amplitude, width, and transverse wavenumber evolution under propagation can be well understood and predicted by utilizing equations (11) and (13) combined with stability considerations. This provides the capability of an appropriate selection of the interaction parameters for an effective beam steering as shown in numerical simulations.

For the case of a Gaussian beam with $P = P_{\rm cr}$ interacting with a CW having $\Delta\phi = 0$, $\Delta k_x = a_r^{-1}$ and $\alpha = 0.2$, the "mass" increasing, according to (13), results in beam collapse after a short propagation distance for the case of a Kerr-type nonlinearity ($s = 0$). In order to prevent beam collapsing, nonlinearities with nonzero saturation parameter can be used as shown in Figs. 3(a)–3(c). Since the "mass" needed for beam self-trapping increases with the saturation parameter, for a large s, the beam "mass" increasing due to the presence

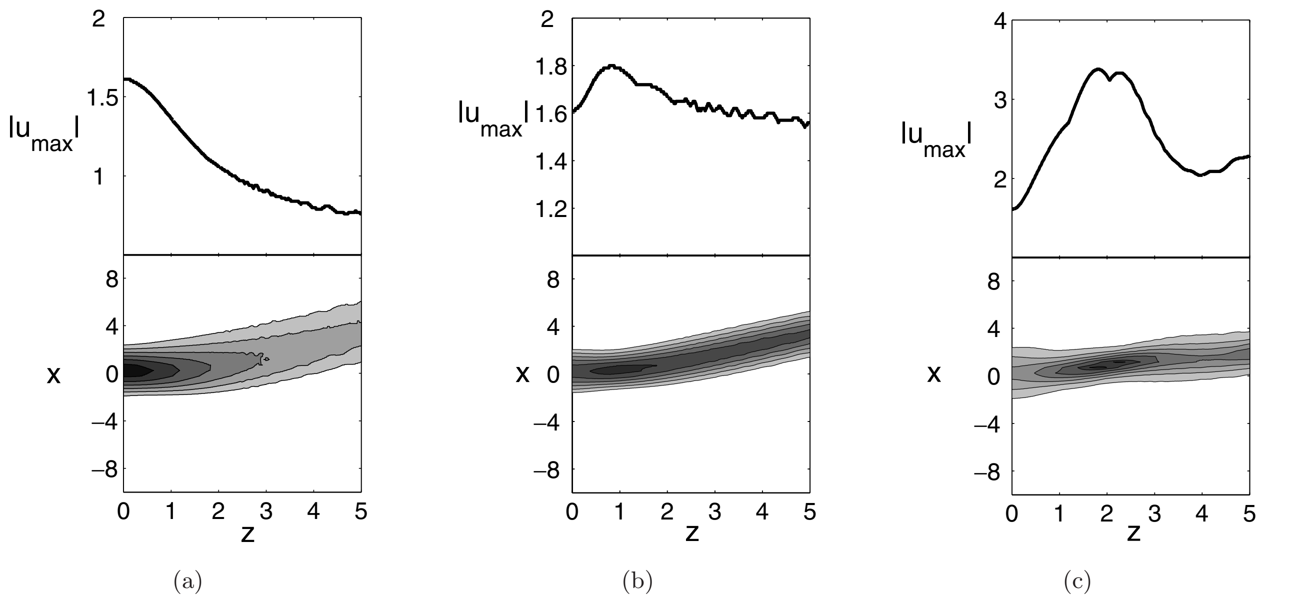


Fig. 3. Evolution of a Gaussian beam with $A = \sqrt{2}$, $a_r^2 = 2$ ($P = P_{\rm cr}$) under the presence of a CW with $\alpha = 0.2$, $\Delta k_y = 0$, $\Delta k_x = a_r^{-1}$ and saturation parameter $s =$ (a) 1, (b) 0.5, (c) 0.25.

of a specific CW may not be sufficiently large for self-trapping so that the beam diffracts [Fig. 3(a)], while a small s can lead to beam self-focusing and amplitude increasing [Fig. 3(c)]. In the first case the amplitude of the beam becomes more comparable with the amplitude of the CW and the transverse velocity of the beam can change significantly in contraposition with the second case, in which the interaction becomes weaker as the beam amplitude increases. However, intermediate values of s can prevent large amplitude variations of the beam and efficient beam steering [Fig. 3(b)]. On the other hand, beam collapse can be avoided, if the initial "mass" of the beam is below $P_{\rm cr}$, even for a Kerr-type nonlinearity, which is the case considered in the following.

The strong dependence of beam evolution on the CW parameters is shown for a Gaussian beam,

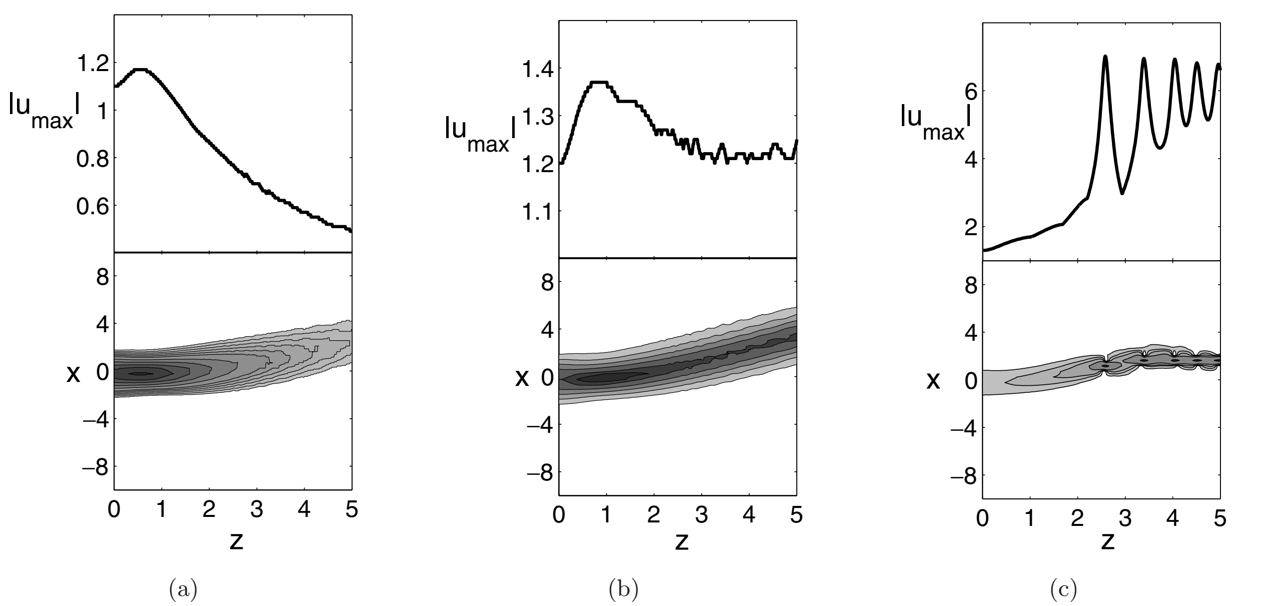
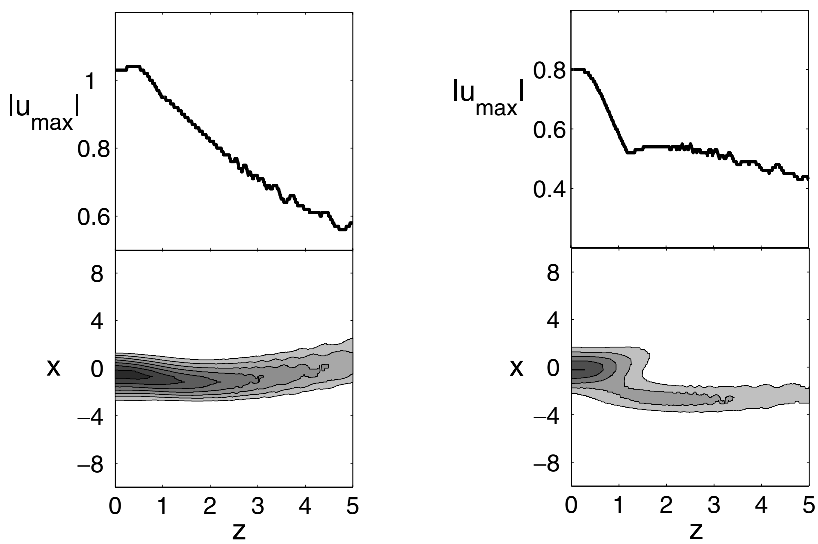


Fig. 4. Evolution of a Gaussian beam with $A = 1$, $a_r^2 = 2$ ($P = P_{\rm cr}/2$) under the presence of a CW with $\Delta\phi = 0$, $\Delta k_y = 0$ and $\Delta k_x = a_r^{-1}$ and $\alpha =$ (a) 0.1, (b) 0.2, (c) 0.3. The saturation parameter is $s = 0$ (Kerr-type nonlinearity).

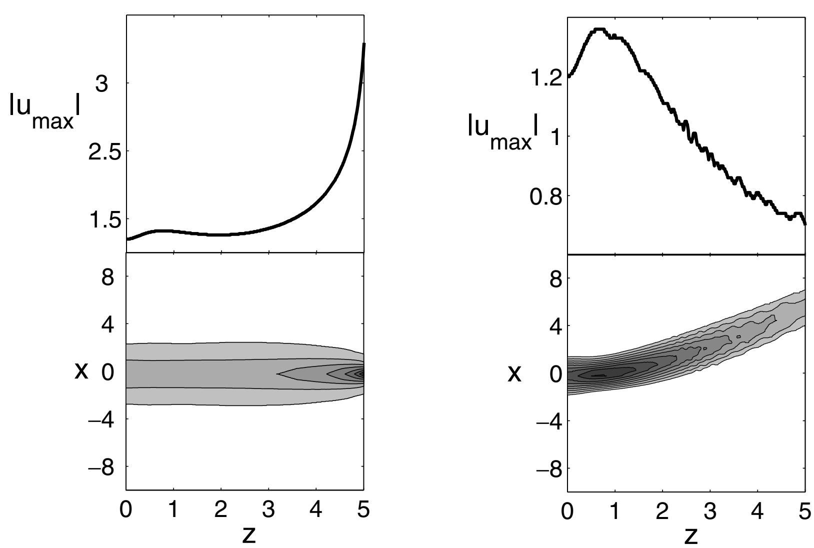
having $P = P_{\mathrm{cr}}/2$, in the following figures obtained by direct numerical simulations of the NLSE. In Figs. 4(a)–4(c) the CW has been choosen so that $\Delta\phi = 0$, $\Delta k_x = a_r^{-1}$ and $\alpha = 0.1, 0.2, 0.3$, respectively. According to (13) the increase of beam's "mass" is proportional to the CW amplitude (α), so that depending on α the resulting beam "mass" can lead to diffraction [Fig. 4(a)], quasistable propagation [Fig. 4(b)], or self-focusing [Fig. 4(c)]. In the latter case it is shown that the increase of the beam's amplitude and the corresponding decrease of beam's width reduces the efficiency of tranversal steering, after a distance of propagation.

The effect of a nonzero initial phase difference is shown in Figs. 5(a) and 5(b) for $\Delta\phi = \pi/2, \pi$ respectively, while the rest of the parameters are the same with the case of Fig. 4(b). For $\Delta\phi = \pi/2$, the mass of the beam does not increase significantly, in



(a) (b)

Fig. 5. Evolution of a Gaussian beam with $A = 1$, $a_r^2 = 2$ ($P = P_{\mathrm{cr}}/2$) under the presence of a CW with $\alpha = 0.2$, $\Delta k_y = 0$, $\Delta k_x = a_r^{-1}$ and $\Delta\phi =$ (a) $\pi/2$, (b) π. The saturation parameter is $s = 0$ (Kerr-type nonlinearity).



(a) (b)

Fig. 6. Evolution of a Gaussian beam with $A = 1$, $a_r^2 = 2$ ($P = P_{\mathrm{cr}}/2$) under the presence of a CW with $\alpha = 0.2$, $\Delta\phi = 0$, $\Delta k_y = 0$ and $\Delta k_x =$ (a) 0, (b) 1. The saturation parameter is $s = 0$ (Kerr-type nonlinearity).

agreement with (13) so that the beam continuously diffracts, while the transverse wavenumber does not change significantly in agreement with (11). A more radical evolution occurs for $\Delta\phi = \pi$, since according to (13) the presence of the CW actually decreases the "mass" of the beam, resulting in drastic beam diffraction and the formation of a secondary beam which is fixed in the transverse dimension.

The dependence of the interaction of the beam with the CW, on the tranverse wavenumber difference Δk_x is shown to be critical for both the stability and the capability of effective beam steering. In Figs. 6(a) and 6(b) the parameters of the beam and the CW are the same with Fig. 4(b) except that $\Delta k_x = 0, 1$, respectively. The presence of a CW with the same transverse wavenumber with the beam is shown to result in a significant increasing of the beam's "mass" and continuous self-focusing, which makes the CW incapable of altering the transverse wavenumber of the beam. On the other hand, a transverse wavenumber difference which is larger than the inverse characteristic initial size of the beam results in beam diffraction, since the presence of the CW does not increase significantly the "mass" of the beam so that it can be self-focused. However, as the beam diffracts the transverse wavenumber can change significantly. Comparing Figs. 6(a) and 6(b) with Fig. 4(b) we can conclude that effective beam steering under necessary stability of beam can be achieved for a transverse wavenumber difference in the vicinity of the maximum shown in Fig. 2(b).

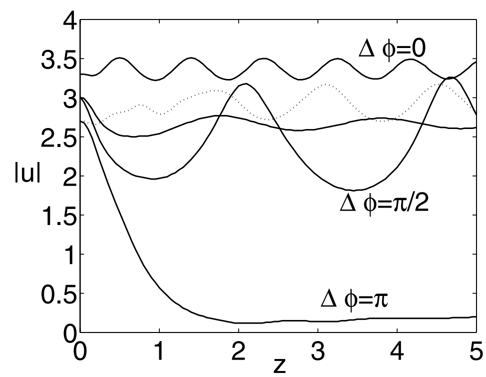


Fig. 7. Amplitude at the central point during propagation of a circular Gaussian beam having $A = 3$, $a = 0.25$ (dashed line). The presence of a CW background with amplitude $\alpha = 0.3$ (solid lines) causes amplitude variation ($\Delta\phi = 0, \pi/2$) and drastic reduction ($\Delta\phi = \pi$), depending on the relative phase. The introduction of beam ellipticity $a_x = 0.25/0.1$, $a_y = 0.25 \cdot 0.1$ stabilizes propagation and prevents diffraction under the presence of a CW having $\alpha = 0.3$, $\Delta\phi = \pi$ (dotted line).

4. Zero Transverse Wavenumber Difference

In this section, interactions between a variety of SW of the general form $u_s = f(\mathbf{r}_\perp)e^{im\theta}$ and CW with same transverse wavenumbers is studied. Numerical simulations have shown that the saturation parameter s of the nonlinearity, apart from determining the energy threshold as well as the exact profile of a guided mode [Karlsson, 1992], do not enter into the qualitative features of the interactions with the CW. Thus, for simplicity reasons, we restrict our analysis to the case $s = 0$.

Firstly, we consider solitary structures in the form of the fundamental guided mode, having $m = 0$ and radial profile $f(r) = A\exp(-r^2/2a^2)$. In Fig. 7, the critical dependence of the amplitude of the beam center on the relative CW phase is shown. A zero phase difference $\Delta\phi = 0$ causes an oscillative beam propagation around an increased mean amplitude, while $\Delta\phi = \pi/2$ results in larger oscillations around a mean value which is not significantly different from the case of absent CW background. The dependency of the mean change of the "mass" of the beam, under propagation, on the relative phase of the CW is in agreement with Eq. (13). However, a relative phase of $\Delta\phi = \pi$ is shown to result in drastic amplitude decrease of the beam center, and diffraction. The corresponding diffraction pattern consists of a, continuously increasing under propagation, set of surrounding rings as shown in Fig. 8 (top). It must be emphasized that this kind of evolution is quite different from the case, in which the beam trivially broadens and decay due to its "mass" reduction bellow the self-focusing threshold. A quite similar dependence of the interaction between a solitary beam and a CW on the relative phase difference has also been shown to occur in the case of one-dimensional NLS equation [Kominis & Hizanidis, 2004b].

On the other hand, the removal of radial symmetry of the beam profile in terms of introduction of a beam ellipticity $f(x, y) = A\exp(-x^2/2a_x^2 - y^2/2a_y^2)$, $a_x \neq a_y$ remarkably prevents beam diffraction as shown in Fig. 7. Moreover, under higher CW amplitude the elliptical beam do not diffract, but evolves into a set of self-focused circular beams as shown in Fig. 8 (bottom), placed in the direction of smaller initial beam width, after emitting part of the initial energy

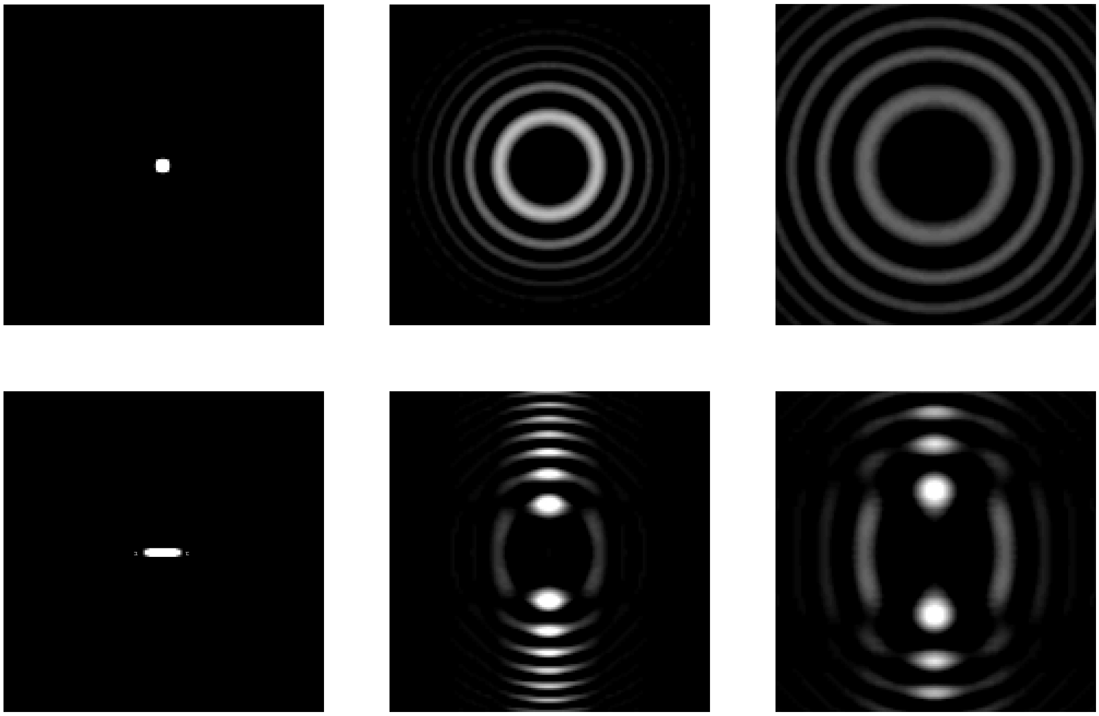


Fig. 8. Diffraction pattern of a circular beam ($A = 3$, $a = 0.25$) (top) and splitting of an elliptical beam ($A = 3$, $a_x = 0.25/0.1$, $a_y = 0.25 \cdot 0.1$) (bottom) under the presence of a π-out-of-phase CW, having $\alpha = 0.3$ (top) and $\alpha = 0.45$ (bottom), at propagation distances $z = 0$, $z = 2.5$, $z = 5$ (left to right). Axes limits are $-20, 20$.

in the form of small amplitude (linear) diffractive waves.

It is well known that a Gaussian beam undergoes instabilities under propagation when the azimuthal symmetry is broken, even under the presence of small random fluctuations [Soto-Crespo *et al.*, 1992]. This break of azimuthal symmetry is expected to be very common in realistic situations where an azimuthal modulation of the beam can be generated by a computer-generated hologram [Tikhonenko *et al.*, 1995] in terms of a factor $e^{im\theta}, m \neq 0$. Under such situation the initial beam breaks into filamentations which can self-focus or diffract, depending on their energy. In Fig. 9 (top) diffraction patterns of azimuthal modulated beams are shown. However, the presence of a CW background drastically changes the diffraction pattern of the beam under propagation to a spiral structure as shown in Fig. 9 (bottom). The number of spiral arms is equal to $|m|$, while their direction is determined by the sign of m. The relative phase of the background determines the angle at which each arm is emerging from the central point. It is remarkable that the phase of the diffraction pattern has also been found to have an almost identical structure to the amplitude.

Finally, the presence of a CW background can also be responsible for a specific breakup scenario of ring beams carrying angular momentum. We consider a ring beam with radial profile $f(r) = A\,\text{sech}(r - r_0)$ and nonzero, integer azimuthal modulation $m \neq 0$. It has been demonstrated in many numerical and analytical studies [Skryabin & Firth, 1998] that such ring-profile vortex beams undergo an azimuthal symmetry-breaking instability, and they usually decay into $2|m|$ fundamental guided modes. This breakup scenario has been numerically observed [Atai *et al.*, 1994], and deviations from the breaking up into something besides $2|m|$ beams is consistent with analytical predictions showing that perturbations with different azimuthal modulation numbers can also grow, but just not as fast [Bigelow *et al.*, 2004]. However, we show that the presence of a CW background determines a robust breakup scenario into exactly $|m|$ beams. This scenario may have the following semi-quantitative interpretation: Considering a ring intensity profile with an azimuthal modulation number m there must be $|m|$ azimuthal values at which the relative phase of the ring and the CW background is zero. Then in analogy with (8) and (13), small areas surrounding these points are expected to increase their energy in

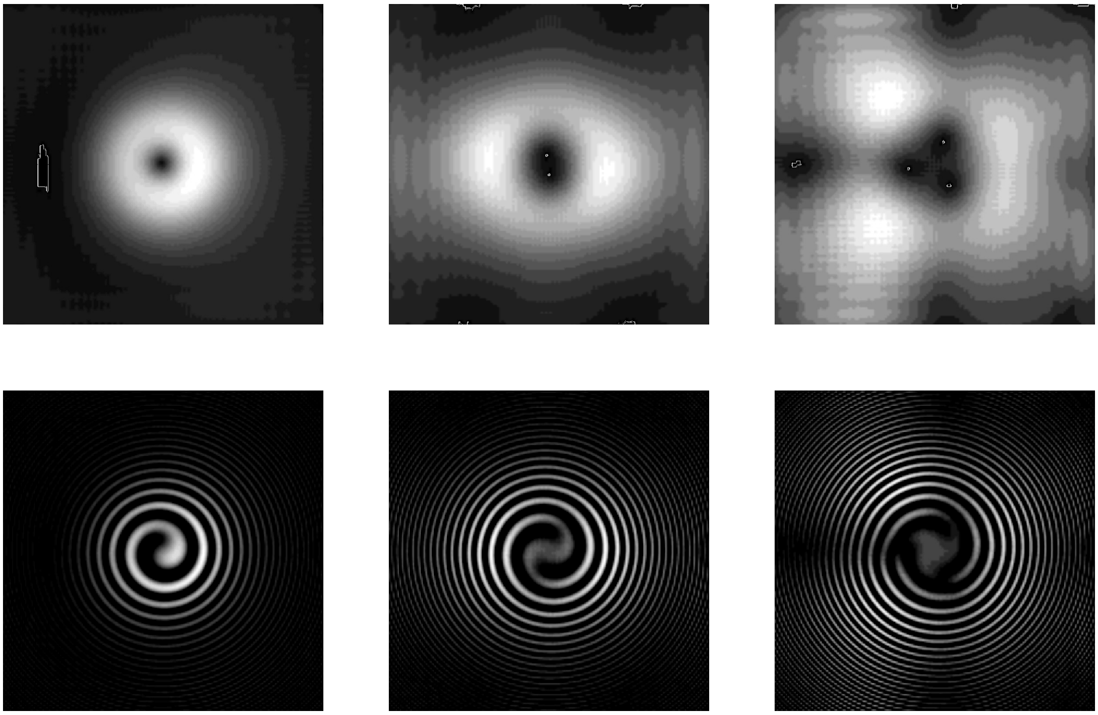


Fig. 9. Diffraction patterns of circular Gaussian beams with nonzero angular momentum, having $A = 3$, $a = 0.25$, $m = 1, 2, 3$ (left to right), under no CW background (top) and under the presence of a CW having $\alpha = 0.3$ and $\Delta\phi = 0$ (bottom). The propagation distance is $z = 1$. Axes limits are $-20, 20$.

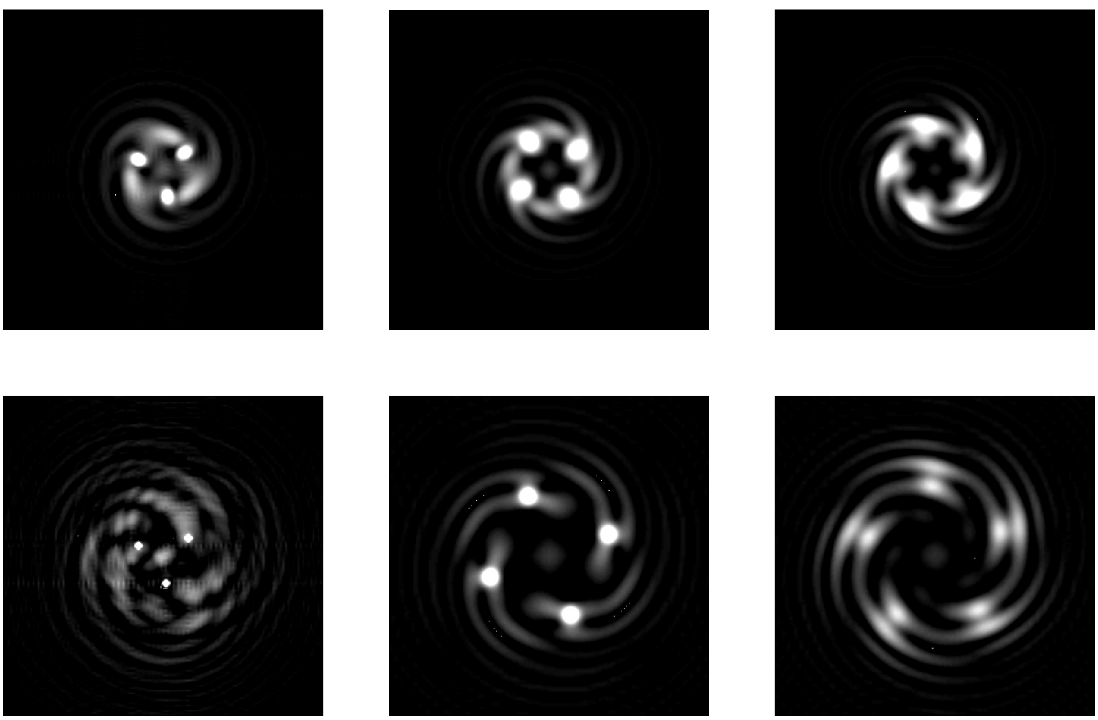


Fig. 10. Breakup of a sech-shaped bright ring, having $A = 1$, $r_0 = 5$ and $m = 3, 4, 5$ (left to right) under the presence of a CW with $\alpha = 0.1$ and $\Delta\phi = 0$, at propagation distances $z = 5$ (top) and $z = 10$ (bottom). Axes limits are $-40, 40$.

comparison with their neighborhoods and evolve as self-focusing beams. Parts of the ring that are π-out-of-phase with the background, lose energy which is "captured" by the self-focusing parts. The relative phase of the background does not change the number of the resulting beams, but only the angles at which they initially emerge from the ring. In Fig. 10 the breakup of an initially ring shaped structure with nonzero angular momentum is shown. In all cases $|m|$ solitary beams fly off the ring tangentially, and the initial spin angular momentum of the ring is transformed to the net orbital angular momentum of the moving beams, so that the total angular momentum is conserved. It is remarkable that as m increases the emerging beams cannot propagate as self-focused beams and diffract along the radial direction (Fig. 10).

5. Conclusions

In conclusion, the interaction of a solitary wave of the multidimensional NLSE with a CW was investigated. An analytical approach, based on two conserved quantities of the wave, namely the "mass" and "momentum" has been applied and it was shown that an initial difference between the transverse wavenumber of the beam and the CW can lead to a variation of the transverse wavenumber ("momentum") of the solitary wave along the same direction with the aforementioned difference, while the "mass" of the wave also changes due to the presence of the CW. Although the approach applies to any number of transverse dimensions and type of nonlinearity, the case of Gaussian beams in a two-dimensional bulk medium with Kerr-type or saturable nonlinearity was studied and the critical dependence of the beam's evolution on the characteristics of the CW was predicted in terms of analytical relations. The latter, combined with widely known stability considerations of the two-dimensional NLSE, are shown to be capable of explaining and predicting the evolution of a radially symmetric beam as obtained by direct simulations. In the context of nonlinear optics, the capability of two-dimensional spatial beam steering under the injection of an appropriate CW, is very promising for potential applications in all-optical signal control. Moreover, considering the two transverse applications as spatial and temporal the aforementioned results can also be applied to the controlled space and time steering (and corresponding frequency conversion) in planar geometries. On the other hand, in the three-dimensional case, a "light bullet" can be controlled both in two-dimensional space and time.

Also, the interaction of a variety of solitary structures with a CW background with the same transverse wavenumber was shown to determine different evolution scenarios of the latter under propagation in a nonlinear medium, including controllable diffraction patterns as well as splitting into a predefined number of beams. Such interactions are very promising for the potential applications of spatial solitary waves to the development of photonic devices that are able to perform data processing operations.

Finally, since the multidimensional NLSE is a universal model for wave propagation under the presence of nonlinearity and dispersion and/or diffraction, interesting applications in other branches of physics are expected. Extensions to elliptic gaussian beams as well as to more general beam profiles can also be considered. This is a subject of current and future investigation.

Acknowledgments

The project is co-funded by the European Social Fund (75%) and National Resources (25%) — Operational Program for Educational and Vocational Training II (EPEAEK II — PYTHAGORAS).

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