Aspects of Contorted Geometry In The Early Universe And The Related Formalism

Panagopoulos Georgios, ge19007

Supervisor: Nikolaos E. Mavromatos

Undergraduate Thesis Defense, 03/10/2024

School of Applied Mathematical and Physical Sciences, Department of Physics

National Technical University of Athens

ge19007@mail.ntua.gr

Motivation

- General Relativity has proven to be an extremely accurate theory, but something is missing: spin.
- Spin is a fundamental quantity in Quantum theories. In fact, the positive energy irreducible representations of the Poincaré group, which are associated with particles, are indexed by *mass* and *spin* (Schwichtenberg 2018).
- In order to get one step closer to Quantum Gravity: Search for a minimal extension of General Relativity that incorporates spin.
- The answer is simple: Assume a non-vanishing torsion!

Outline

- Einstein-Cartan Theory (Formalism)
- QED in Contorted Spacetime
- String-Inspired Inflation Due To Torsion

Einstein-Cartan Theory (Formalism)

Vielbeins

- The theory in which torsion doesn't vanish is called Einstein-Cartan. It is vastly useful, if not necessary, to reformulate using "vielbeins" (Carroll 2019).
- Imagine constructing an orthonormal basis of vectors $\hat{e}_a|_p$ at each point of the manifold. Spacetime is locally flat, and thus:

$$g(\hat{e}_a, \hat{e}_b) = \eta_{ab} \tag{1}$$

  Latin indices $\rightarrow$ flat spacetime (tangent space)!
- Thus, we have passed from a general basis $\hat{e}_\mu \equiv \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ to an orthonormal one, $\hat{e}_a$, called a *vierbein* or *tetrad*. The two bases are connected via the transformation law:

$$\hat{e}_\mu = \hat{e}_\mu{}^a \hat{e}_a \tag{2}$$

  The transformation matrix $\hat{e}_\mu{}^a$ is called a *vierbein field*.

Vielbeins

- The vierbein fields satisfy the orthogonality relations:

$$\hat{e}_{\mu}{}^{a}\hat{e}^{\mu}{}_{b} = \delta^{a}_{b} \tag{3}$$
$$\hat{e}^{\mu}{}_{a}\hat{e}_{\nu}{}^{a} = \delta^{\mu}_{\nu} \tag{4}$$

- We can also identify: $\hat{e}^{\mu} \equiv dx^{\mu}$ and thus:

$$dx^{\mu} = \hat{e}^{\mu}{}_{a}\hat{e}^{a} \tag{5}$$

where $\hat{e}^{\mu}{}_{a}$ is the inverse of $\hat{e}_{\mu}{}^{a}$.

- Through the condition (1) the metric, initially given as $g = g_{\mu\nu}dx^{\mu}dx^{\nu}$, can be expressed in terms of the vierbein fields as:

$$g_{\mu\nu} = \eta_{ab}\hat{e}_{\mu}{}^{a}\hat{e}_{\nu}{}^{b} \tag{6}$$

Coordinate Transformations & Spin Connection

- Greek indices transform with General Coordinate Transformations (GCT), while Latin indices transform with Local Lorentz Transformations (LLT). Example:

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_{a} \frac{\partial x^{\mu'}}{\partial x^{\mu}} \Lambda^{b}{}_{b'} \frac{\partial x^{\nu}}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu} \tag{7}$$

- The covariant derivative in General Relativity is given as:

$$\bar{\bar{\nabla}}_{\mu} v^{\nu} = \partial_{\mu} v^{\nu} + \bar{\bar{\Gamma}}^{\nu}_{\mu\lambda} v^{\lambda} \tag{8}$$

We claim that, for a vector expressed in Latin indices, a similar equation holds:

$$\bar{\bar{\nabla}}_{\mu} v^{a} = \partial_{\mu} v^{a} + \bar{\varpi}_{\mu}{}^{a}{}_{b} v^{b} \tag{9}$$

The quantity $\bar{\varpi}_{\mu}{}^{a}{}_{b}$ defines the so-called *spin connection* (Carroll 2019), which is anti-symmetric in a, b.

Spin Connection

- The spin connection coefficients are given in terms of the Γ symbols:

$$\bar{\omega}_{\mu}{}^{a}{}_{b} = \hat{e}^{\lambda}{}_{b}\,\hat{e}_{\nu}{}^{a}\,\bar{\Gamma}^{\nu}_{\mu\lambda} - \hat{e}^{\lambda}{}_{b}\,\partial_{\mu}\hat{e}_{\lambda}{}^{a} \tag{10}$$

- The covariant derivative of a tetrad field vanishes:

$$\bar{\nabla}_{\mu}\hat{e}_{\nu}{}^{a} = \partial_{\mu}\hat{e}_{\nu}{}^{a} - \hat{e}_{\sigma}{}^{a}\,\bar{\Gamma}^{\sigma}_{\mu\nu} + \bar{\omega}_{\mu}{}^{a}{}_{b}\hat{e}_{\nu}{}^{b} = 0 \tag{11}$$

This is called the *tetrad postulate*.

- The spin connection coefficients transform as (Yepez 2011):

$$\bar{\omega}_{\mu}{}^{a'}{}_{b'} = \bar{\omega}_{\mu}{}^{c}{}_{b}\,\Lambda^{b}{}_{b'}\Lambda^{a'}{}_{c} - \Lambda^{b}{}_{b'}\partial_{\mu}\Lambda^{a'}{}_{b} \tag{12}$$

and thus do not form a tensor, much like the Γ symbols.

Differential Forms Viewpoint

- Differential Forms: Fully anti-symmetric lower index tensors. Examples: X_μ is a 1-form, $A_{\mu\nu}$, where μ, ν anti-symmetric is a 2-form and so on. What about mixed-index objects like $X_\mu{}^a$?
- Fundamental change of viewpoint: View mixed-index tensors as tensor-valued differential forms. Examples: $X_\mu{}^a$ is a vector-valued differential form. It's a differential form for each value of a.
- This viewpoint is useful because we can suppress the Greek indices by writing the objects in differential form notation. For example:

$$\overline{\omega}^a{}_b = \overline{\omega}_\mu{}^a{}_b \, dx^\mu \tag{13}$$

Exterior Covariant Derivative

- The derivative operator for scalar-valued differential forms (no Latin indices) is the exterior derivative:

$$(dX)_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu \tag{14}$$

- This derivative operator is not suitable for tensor-valued differential forms, as the result does not transform properly under LLTs. To amend, we define the *Exterior Covariant Derivative* operator:

$$\begin{aligned}(\bar{D}X)_{\mu\nu}{}^a &= (dX)_{\mu\nu}{}^a + (\bar{\omega} \wedge X)_{\mu\nu}{}^a \\ &= \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a + \bar{\omega}_\mu{}^a{}_b X_\nu{}^b - \bar{\omega}_\nu{}^a{}_b X_\mu{}^b \\ &= \bar{\bar{\nabla}}_\mu X_\nu{}^a - \bar{\bar{\nabla}}_\nu X_\mu{}^a\end{aligned} \tag{15}$$

Exterior Covariant Derivative

- In differential form notation, we can write this as:

$$(\bar{\boldsymbol{D}}\boldsymbol{X})^a = (\bar{\boldsymbol{D}}\boldsymbol{X}^a) = \boldsymbol{d}\boldsymbol{X}^a + \bar{\boldsymbol{\omega}}^a{}_b \wedge \boldsymbol{X}^b \tag{16}$$

- For a general tensor-valued p-form, the exterior covariant derivative is given as (Duncan, Kaloper, and Olive 1992):

$$(\bar{\boldsymbol{D}}\boldsymbol{X})^{a\ldots}{}_{b\ldots} = (\boldsymbol{d}\boldsymbol{X})^{a\ldots}{}_{b\ldots} + (\bar{\boldsymbol{\omega}}^a{}_c \wedge \boldsymbol{X}^{c\ldots}{}_{b\ldots}) + \cdots - (-1)^p(\boldsymbol{X}^{a\ldots}{}_{d\ldots} \wedge \bar{\boldsymbol{\omega}}^d{}_b) - \cdots \tag{17}$$

Cartan's Structure Equations

- From the metric compatibility condition $\bar{\nabla} g = 0$ we get that the spin connection coefficients are anti-symmetric in their Latin indices:

$$\bar{\omega}_{\mu ab} = -\bar{\omega}_{\mu ba} \tag{18}$$

- Expressing the torsion and curvature tensors in terms of the vierbeins and spin connection coefficients leads to Cartan's Structure equations (Duncan, Kaloper, and Olive 1992):

$$\boldsymbol{T}^a = \bar{\boldsymbol{D}}\hat{\boldsymbol{e}}^a = \boldsymbol{d}\hat{\boldsymbol{e}}^a + \bar{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{e}}^b \tag{19}$$

$$\bar{\boldsymbol{R}}^a{}_b = \boldsymbol{d}\bar{\boldsymbol{\omega}}^a{}_b + \bar{\boldsymbol{\omega}}^a{}_c \wedge \bar{\boldsymbol{\omega}}^c{}_b \tag{20}$$

where $\boldsymbol{T}^a = T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu$ is the torsion 2-form and torsion is defined in terms of affine connection as $T^\lambda{}_{\mu\nu} = 2\bar{\Gamma}^\lambda{}_{[\mu\nu]}$ and $\bar{\boldsymbol{R}}^a{}_b = \bar{R}^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$ is the curvature 2-form.

Bianchi Identities

- Taking the exterior covariant derivatives of Cartan's Structure equations leads to the Bianchi identities:

$$\bar{\boldsymbol{D}}\boldsymbol{T}^a = \bar{\boldsymbol{D}}^2\hat{\boldsymbol{e}}^a = \bar{\boldsymbol{R}}^a{}_b \wedge \hat{\boldsymbol{e}}^b \tag{21}$$

$$\bar{\boldsymbol{D}}\bar{\boldsymbol{R}}^a{}_b = 0 \tag{22}$$

- These correspond to the familiar Bianchi identities (Carroll 2019):

$$\bar{R}^\rho{}_{[\sigma\mu\nu]} = 0 \tag{23}$$

$$\bar{\nabla}_{[\lambda|}\bar{R}^\rho{}_{\sigma|\mu\nu]} = 0 \tag{24}$$

Contorsion

- The contorted Γ symbols can be split into parts with and without torsion (Nakahara 2003):

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \frac{1}{2}(T^{\lambda}{}_{\mu\nu} + T_{\mu}{}^{\lambda}{}_{\nu} + T_{\nu}{}^{\lambda}{}_{\mu}) \tag{25}$$

where the $\Gamma^{\lambda}_{\mu\nu}$ are the familiar Christoffel symbols.

- We define the *contorsion* tensor:

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(T^{\lambda}{}_{\mu\nu} + T_{\mu}{}^{\lambda}{}_{\nu} + T_{\nu}{}^{\lambda}{}_{\mu}) \tag{26}$$

such that $\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}$.

- The torsion tensor is anti-symmetric in its lower indices, $T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu}$ and thus the contorsion tensor is anti-symmetric in its first and third indices:

$$K_{\lambda\mu\nu} = -K_{\nu\mu\lambda} \tag{27}$$

Contorsion & Spin Connection

- The contorsion tensor can also be used to split the spin connection into two parts with and without torsion:

$$\bar{\boldsymbol{\omega}}^{a}{}_{b} = \boldsymbol{\omega}^{a}{}_{b} + \boldsymbol{K}^{a}{}_{b} \Leftrightarrow \bar{\omega}_{\mu}{}^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b} + K^{a}{}_{\mu b} \tag{28}$$

- The torsionless part is defined such that:

$$\boldsymbol{D}\hat{\boldsymbol{e}}^{a} = \boldsymbol{d}\hat{\boldsymbol{e}}^{a} + \boldsymbol{\omega}^{a}{}_{b} \wedge \hat{\boldsymbol{e}}^{b} = 0 \tag{29}$$

where D is a torsionless exterior covariant derivative and

$$\bar{\boldsymbol{D}} = \boldsymbol{D} + \boldsymbol{K}^{a}{}_{b} \wedge \tag{30}$$

- Furthermore, the torsionless spin connection coefficients are also anti-symmetric:

$$\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba} \tag{31}$$

Contorsion & Cartan's Structure Equations

- The first of Cartan's Structure equations becomes:

$$\boldsymbol{T}^{a} = \boldsymbol{K}^{a}{}_{b} \wedge \hat{\boldsymbol{e}}^{b} \tag{32}$$

- By defining a torsionless curvature tensor:

$$\boldsymbol{R}^{a}{}_{b} = \boldsymbol{d}\boldsymbol{\omega}^{a}{}_{b} + \boldsymbol{\omega}^{a}{}_{c} \wedge \boldsymbol{\omega}^{c}{}_{b} \tag{33}$$

which obviously satisfies the Bianchi identity $\boldsymbol{D}\boldsymbol{R}^{a}{}_{b} = 0$, the second of Cartan's Structure equations becomes:

$$\bar{\boldsymbol{R}}^{a}{}_{b} = \boldsymbol{R}^{a}{}_{b} + \boldsymbol{D}\boldsymbol{K}^{a}{}_{b} + \boldsymbol{K}^{a}{}_{c} \wedge \boldsymbol{K}^{c}{}_{b} \tag{34}$$

Einstein-Cartan Action

- The Einstein-Cartan Action describing contorted spacetime is simply (Duncan, Kaloper, and Olive 1992):

$$S_G = \frac{1}{16\pi G} \int \bar{R} \sqrt{-g}\, d^4x \tag{35}$$

- It is useful to write it in differential form notation (Gasperini 2017):

$$S_G = \frac{1}{16\pi G} \int \bar{\boldsymbol{R}}_{ab} \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b) \tag{36}$$

- After some computations, we end up with:

$$S_G = \frac{1}{16\pi G} \int (R + \Delta) \sqrt{-g}\, d^4x \tag{37}$$

where Δ is a scalar produced by contractions of the contorsion tensor.

Einstein-Cartan Action

- The contorsion tensor has 24 independent components and can be decomposed into a vector part containing 4 of them and a tensor part containing the other 20 (Cvitanović 2008):

$$K_{abc} = \frac{1}{2}\epsilon_{abcd}S^d + \hat{K}_{abc} \tag{38}$$

- This, in turn, leads to the splitting of the scalar term Δ in two parts:

$$\Delta = \frac{3}{2}S_d S^d + \hat{\Delta} \tag{39}$$

- Therefore, the Einstein-Cartan action can be written as:

$$\begin{aligned} S_G &= \frac{1}{16\pi G}\int \bar{R}\sqrt{-g}\, d^4x \\ &= \frac{1}{16\pi G}\int (R + \hat{\Delta})\sqrt{-g}\, d^4x + \frac{3}{32\pi G}\int \boldsymbol{S} \wedge *\boldsymbol{S} \end{aligned} \tag{40}$$

QED in Contorted Spacetime

Spinor Covariant Derivative & Action Terms

- We define the spinor covariant derivative as (Duncan, Kaloper, and Olive 1992):

$$\bar{\boldsymbol{D}}\psi = \boldsymbol{d}\psi - \frac{i}{4}\overline{\omega}_{ab}\sigma^{ab}\psi \quad \& \quad \bar{\boldsymbol{D}}\overline{\psi} = \boldsymbol{d}\overline{\psi} + \frac{i}{4}\overline{\omega}_{ab}\overline{\psi}\sigma^{ab} \quad (41)$$

where $\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$.

- To get the fermionic action, we take the one used for free fermions in flat spacetime and replace the partial derivative with a covariant derivative $\overline{\mathcal{D}}_\mu = \bar{\boldsymbol{D}}_\mu - ieA_\mu$, thus getting:

$$S_{QED}^{Curved+Torsion} = \frac{1}{2}\int \left(i\overline{\psi}\gamma^\mu(\overline{\mathcal{D}}_\mu\psi) + h.c.\right)\sqrt{-g}\, d^4x \quad (42)$$

Spinor & Electromagnetic Action

- Considering that the fermionic axial current is defined as:

$$j^5_\mu = \bar{\psi}\gamma^d\gamma^5\psi \tag{43}$$

the fermionic action can be expanded and finally written as:

$$\begin{aligned} S_{QED}^{Curved+Torsion} =& \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu \boldsymbol{D}_\mu\psi - i(\boldsymbol{D}_\mu\bar{\psi})\gamma^\mu\psi \right]\sqrt{-g}\, d^4x \\ &+ e\int (\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\, d^4x - \frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j}^5 \end{aligned} \tag{44}$$

- We also add the (minimal) action for the EM field:

$$S_{EM} = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\, d^4x = -\frac{1}{2}\int \boldsymbol{F}\wedge *\boldsymbol{F} \tag{45}$$

The Equations of Motion

- By varying the total action with respect to the fields $A_\mu, S_\mu, \psi, \overline{\psi}$ & $g^{\mu\nu}$ we get the equations of motion.
- Variation with respect to A_μ (term (45)) gives us the Maxwell equations:

$$\boldsymbol{dF} = 0 \quad \& \quad \boldsymbol{d} * \boldsymbol{F} = *\boldsymbol{j} \tag{46}$$

- Next, we vary with respect to S_μ. The relevant action is:

$$S_{Torsion} = \frac{3}{32\pi G}\int \boldsymbol{S} \wedge *\boldsymbol{S} - \frac{3}{4}\int \boldsymbol{S} \wedge *\boldsymbol{j}^5 \tag{47}$$

and the resulting equation of motion is

$$\boldsymbol{S} = 4\pi G \boldsymbol{j}^5 \tag{48}$$

The Equations of Motion

- Variation of the fermionic action (44) with respect to $\bar{\psi}$ results in the modified Dirac equation:

$$i\gamma^{\mu}\mathcal{D}_{\mu}\psi - \frac{3}{4}S_{\mu}\gamma^{\mu}\gamma^{5}\psi = 0 \tag{49}$$

- Furthermore, the axial current j^{5}_{μ} is classically conserved, i.e. $\boldsymbol{d} * \boldsymbol{j}^{5} = 0$. As a consequence (from the Equations of Motion), torsion is also conserved classically:

$$\boxed{\boldsymbol{d} * \boldsymbol{S} = 0} \tag{50}$$

- Finally, variation with respect to the metric gives the Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{51}$$

The Equations of Motion

- There are three distinct contributions to the stress-energy tensor:

$$T_{\mu\nu} = T^{A}_{\mu\nu} + T^{\psi}_{\mu\nu} + T^{S}_{\mu\nu} \tag{52}$$

- These are given as:

$$T^{A}_{\mu\nu} = F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} \tag{53}$$

$$T^{S}_{\mu\nu} = -\frac{3}{16\pi G}\left(S_{\mu}S_{\nu} - \frac{1}{2}g_{\mu\nu}S_{\lambda}S^{\lambda}\right) \tag{54}$$

$$\begin{aligned} T^{\psi}_{\mu\nu} = &-\frac{1}{2}\left[\bar{\psi}\gamma_{(\mu}\mathcal{D}_{\nu)}\psi - (\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right] \\ &+ \frac{3}{4}S_{(\mu}\bar{\psi}\gamma_{\nu)}\gamma^{5}\psi \end{aligned} \tag{55}$$

Anomaly and Axions

- When we pass onto a Quantum theory, the axial current is no longer conserved due to an anomaly in the one-loop level:

$$\boldsymbol{d} * \boldsymbol{j}^5 = -\frac{e^2}{4\pi^2}\boldsymbol{F} \wedge \boldsymbol{F} - \frac{1}{96\pi^2}\mathrm{tr}(\bar{\boldsymbol{R}} \wedge \bar{\boldsymbol{R}}) \tag{56}$$

  Does this mean that the torsion S isn't conserved either?

- It is impossible to know the full quantum properties of torsion. We will hypothesize that these quantum properties are such that a suitable counterterm that maintains $\boldsymbol{d} * \boldsymbol{S} = 0$ at a quantum level exists.

- We can enforce this equation as a constraint in the path integral as a delta functional:

$$Z_S^C = \int \mathcal{D}\boldsymbol{S}\,\delta(\boldsymbol{d} * \boldsymbol{S}) \exp\left[i \int \left(\tfrac{3}{32\pi G} \int \boldsymbol{S} \wedge *\boldsymbol{S} - \tfrac{3}{4} \int \boldsymbol{S} \wedge *\boldsymbol{j}^5\right)\right] \tag{57}$$

Anomaly and Axions

- This delta functional can be written as:

$$\delta(\boldsymbol{d} * \boldsymbol{S}) = \int \mathcal{D}\Phi e^{i \int \Phi \boldsymbol{d} * \boldsymbol{S}} \tag{58}$$

  where Φ is a scalar field.

- The resulting partition function is:

$$Z_S^C = \int \mathcal{D}\phi \exp\left[i \int \left(-\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2f_\phi^2} j^5_\mu (j^5)^\mu - \frac{1}{f_\phi} j^5_\mu (\partial^\mu \phi) \right) \sqrt{-g}\, d^4x \right] \tag{59}$$

  where $\Phi = \sqrt{\frac{3}{16\pi G}}\phi$ and $f_\phi = \frac{1}{\sqrt{3\pi G}}$.

- The static torsion field has thus been replaced by a pseudoscalar axion field which has a dynamical behaviour (Duncan, Kaloper, and Olive 1992).

- Effectively, QED on contorted spacetime is equivalent to QED in torsionless spacetime coupled to an axion.

Outlook on Einstein-Cartan Theories

- There are many different paths to explore from here. For example, we may consider non-minimal models where the EM field couples to torsion classically.
- Current hot topics include Cosmological models with torsion (Mavromatos, Pais, and Iorio 2023). These models predict that instead of a Big Bang, there was a Big Bounce. Similarly, black holes do not form singularities but reach a bounce and new universes are formed inside the event horizon (Popławski 2012). Research is ongoing!

String-Inspired Inflation Due To Torsion

String Theory Essentials

- String theory (Polchinski 1998; Green, Schwarz, and Witten 2012) predicts three massless gravitational fields (gravitational multiplet): the spin-0 Dilaton Φ, the spin-1 antisymmetric Kalb-Ramond field $B_{\mu\nu}$ and the spin-2 graviton $g_{\mu\nu}$.
- The Kalb-Ramond field has a $U(1)$ gauge symmetry:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\theta_{\nu]} \tag{60}$$

- In the low energy regime, the action depends on the strength of the Kalb-Ramond field:

$$H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} \Leftrightarrow \boldsymbol{H} = \boldsymbol{dB} \tag{61}$$

String Theory Essentials

- Due to the presence of anomalies and our desire of their cancellation, the field strength of the Kalb-Ramond field has to be modified as:

$$\boldsymbol{H} = \boldsymbol{dB} + \frac{a'}{8\kappa}\left(\Omega_{3L} - \Omega_{3Y}\right) \tag{62}$$

  where Ω_{3L}, Ω_{3Y} are the Lorentz and gauge Chern-Simons terms respectively.

- The Bianchi identity that results from this equation is:

$$\boldsymbol{dH} = \frac{a'}{8\kappa} Tr\left(\boldsymbol{R} \wedge \boldsymbol{R} - \boldsymbol{F} \wedge \boldsymbol{F}\right) \tag{63}$$

String Theory Essentials

- In index notation, this becomes:

$$\eta_{abc}{}^{\mu}\nabla_{\mu}\mathcal{H}^{abc} = \frac{a'}{16\kappa}\sqrt{-g}\left(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \equiv \sqrt{-g}\,\mathcal{G}(\omega,\boldsymbol{A}) \quad (64)$$

where η is the Levi-Civita *tensor* and $\tilde{R}$, $\tilde{F}$ are the dual quantities defined as:

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\eta_{\mu\nu\lambda\kappa}R^{\lambda\kappa}{}_{\rho\sigma} \quad (65)$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu\rho\sigma}F^{\rho\sigma} \quad (66)$$

- After compactification, the effective string action in four spacetime dimensions is:

$$S_B = \int\left(\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right)\sqrt{-g}\,d^4x \quad (67)$$

where we have assumed the Dilaton to be irrelevant, $\Phi \approx 0$ and $\mathcal{H}_{\lambda\mu\nu} = \kappa^{-1}H_{\lambda\mu\nu}$.

Torsion Induced Axion

- Comparison with the Einstein-Cartan action $S_G = \frac{1}{16\pi G}\int (R+\Delta)\sqrt{-g}\, d^4x$ indicates that the Kalb-Ramond field strength plays the role of torsion in this effective field theory. A contorted connection can thus be defined as:

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \frac{\kappa}{\sqrt{3}}\mathcal{H}^{\lambda}{}_{\mu\nu} \quad (68)$$

- We can enforce the Bianchi identity in the partition function of this action, just like in Einstein-Cartan theory:

$$Z_H = \int \mathcal{D}\mathcal{H}_{\lambda\mu\nu}\delta(\eta^{\mu\nu\rho\sigma}\nabla_{\mu}\mathcal{H}_{\nu\rho\sigma} - \mathcal{G}(\omega, \boldsymbol{A}))e^{-i\int \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\sqrt{-g}\, d^4x} \quad (69)$$

- The resulting action is characterized by the replacement of the Kalb-Ramond field strength with an axion b:

$$S_B = \int \left(\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_{\mu}b\,\partial^{\mu}b - \frac{\alpha'\sqrt{2}}{192\kappa}b\left(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\right)\sqrt{-g}\, d^4x \quad (70)$$

The Cotton & Stress-Energy Tensors

- We study the inflationary period, so the gauge part of the anomaly vanishes.
- The presence of the anomaly term, which couples to the axion field, leads to the modification of the Einstein equations.
- There exists the usual matter (axion) stress-energy tensor:

$$T^{b}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_b}{\delta g^{\mu\nu}} = \partial_\mu b \partial_\nu b - \frac{1}{2} g_{\mu\nu} \partial_\rho b \partial^\rho b \qquad (71)$$

- However, there is also another tensor, called the Cotton tensor, which comes from the anomaly term:

$$\mathcal{C}_{\mu\nu} = -\frac{1}{4\sqrt{-g}}\frac{\delta S_C}{\delta g^{\mu\nu}} \qquad (72)$$

where

$$S_C = \int b R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \sqrt{-g}\, d^4x \qquad (73)$$

The Cotton & Stress-Energy Tensors

- Thus, the Einstein equations take the form:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{a'\kappa\sqrt{2}}{24}\mathcal{C}^{\mu\nu} + \kappa^2 T_b^{\mu\nu} \tag{74}$$

- The matter stress-energy tensor $T_b^{\mu\nu}$ is no longer conserved because:

$$\nabla_\mu \mathcal{C}^{\mu\nu} = -\frac{1}{8}\partial^\nu b R^{\rho\lambda\sigma\kappa}\tilde{R}_{\rho\lambda\sigma\kappa} \tag{75}$$

- This happens because the axion is coupled to the gravitational field and there is an exchange of energy between them.

- Thus, we define a new, generalized stress-energy tensor:

$$\kappa^2 \tilde{T}^{\mu\nu}_{total} = \frac{a'\kappa\sqrt{2}}{24}\mathcal{C}^{\mu\nu} + \kappa^2 T_b^{\mu\nu} \tag{76}$$

such that:

$$\nabla_\mu \tilde{T}^{\mu\nu}_{total} = 0 \tag{77}$$

Running Vacuum Model

- The RVM serves as an alternative to ΛCDM.
- In the ΛCDM model, the vacuum energy density is:

$$\rho_{\Lambda} = \frac{\Lambda}{8\pi G} \tag{78}$$

- In the RVM model it is assumed to be a sum of even powers of the Hubble constant:

$$\rho_{RVM}(H) = \frac{\Lambda(H^2)}{8\pi G} = \frac{3}{8\pi G}\left(c_0 + \nu H^2 + \beta \frac{H^4}{H_I^2} + \cdots\right) \tag{79}$$

where c_0, ν, β are real constants, $H_I \sim 10^{-5} M_{Pl}$ is the inflationary scale and M_{Pl} is the Planck mass.

Gravitational Wave Condensate

- We consider (Dorlis, Mavromatos, and Vlachos 2024) tensor perturbations (gravitational waves) of the FLRW metric:

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{80}$$

- Assume an action of the form:

$$S = \int \left(\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - AbR_{CS} \right) \sqrt{-g}\, d^4x \tag{81}$$

where $R_{CS} = R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ is the gravitational Chern-Simons term and $A = \frac{a'\sqrt{2}}{192\kappa}$.

Gravitational Wave Condensate

- In the linear polarization basis,

$$h_{ij} = h_+ \epsilon_{ij}^{(+)} + h_\times \epsilon_{ij}^{(\times)} \tag{82}$$

and we can write the pertubation tensor as:

$$h = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{83}$$

- We can switch to the chiral basis,

$$h_{L,R} = \frac{1}{\sqrt{2}} \left(h_+ \pm i h_\times \right) \tag{84}$$

Gravitational Wave Condensate

- In that case, the linearized Einstein equations become:

$$\Box h_L = -\frac{4iA\kappa^2}{a^2}\left(2\dot{a}\dot{b} + a\ddot{b}\right)\partial_t\partial_z h_L - \frac{4iA\kappa^2\dot{b}}{a}\partial_t^2\partial_z h_L + \frac{4iA\kappa^2\dot{b}}{a^3}\partial_z^3 h_L \quad (85)$$

$$\Box h_R = +\frac{4iA\kappa^2}{a^2}\left(2\dot{a}\dot{b} - a\ddot{b}\right)\partial_t\partial_z h_R + \frac{4iA\kappa^2\dot{b}}{a}\partial_t^2\partial_z h_R - \frac{4iA\kappa^2\dot{b}}{a^3}\partial_z^3 h_R \quad (86)$$

- Therefore, left and right handed gravitational waves behave differently. As such, the Chern-Simons term does not vanish:

$$R_{CS} = \frac{4i}{a^3}\left[\left(\partial_z^2 h_L \partial_z\partial_t h_R + a^2\partial_t^2 h_L \partial_z\partial_t h_R + a\dot{a}\partial_t h_L \partial_z\partial_t h_R\right) - (L \leftrightarrow R)\right] + \mathcal{O}(h^4) \quad (87)$$

Gravitational Wave Condensate

- We can now quantize these gravitational waves and calculate the condensate $\langle R_{CS} \rangle_I$ during inflation:

$$\langle R_{CS} \rangle_I = -\mathcal{N}_I \frac{A\kappa^4 \mu^4}{\pi^2} \dot{\bar{b}}_I H_I^3 \tag{88}$$

where $\mathcal{N}_I$ is the density of gravitational wave sources during inflation and μ is the UV energy cutoff of the effective field theory we're working with, while $\dot{b}_I$ symbolizes the axion field during the inflation era.

- Approximately, we have that (Basilakos, Mavromatos, and Solà Peracaula 2020a):

$$\dot{\bar{b}}_I \sim \sqrt{2\epsilon} M_{Pl} H_I \tag{89}$$

where ϵ is a phenomenological parameter that we fix as $\epsilon \sim 10^{-2}$. Therefore, $\langle R_{CS} \rangle_I \sim H_I^4$.

Vacuum Energy Density

- The axion stress-energy tensor and cotton tensor each contribute a $\sim H_I^2$ term to the vacuum energy density:

$$\rho_b + \rho_{gCS} \simeq -0.496\epsilon M_{Pl}^2 H_I^2 \tag{90}$$

- Post-quantization, the action can be written as:

$$S = \int \left(\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - \frac{\alpha' \sqrt{2}}{192\kappa} \bar{b}(x) \langle R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \rangle_I - \frac{\alpha' \sqrt{2}}{192\kappa} : b(x) R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} : \right) \sqrt{-g}\, d^4x \tag{91}$$

- This new linear axion potential adds an additional term to the vacuum energy density, of order $\sim H_I^4$, because of the condensate:

$$\rho_\Lambda = 8.6 \times 10^{10} \sqrt{\epsilon} \frac{|\bar{b}(0)|}{M_{Pl}} H^4 \tag{92}$$

- The total vacuum energy density is then:

$$\rho_{vac}(H) = \rho_b + \rho_{gCS} + \rho_\Lambda = -\frac{1}{2}\epsilon M_{Pl}^2 H^2 + 8.6 \times 10^{10} \sqrt{\epsilon} \frac{|\bar{b}(0)|}{M_{Pl}} H^4 \tag{93}$$

Inflation

- We have a RVM-type expression for the vacuum energy density with constants $c_0 = 0, \nu < 0$ and $\beta > 0$.
- The conservation of the total stress-energy tensor of vacuum matter and radiation (Mavromatos 2022) leads to the differential equation:

$$\dot{H} + \frac{3}{2}(1+\omega_m)H^2\left(1 - \nu - \beta\frac{H^2}{H_I^2}\right) = 0 \qquad (94)$$

where $\omega_m = \frac{p_m}{\rho_m}$ and the subscript "m" refers to both matter and radiation.

Inflation

- We can solve this and get:

$$H(a) = \left(\frac{1-v}{\beta}\right)^{\frac{1}{2}} \frac{H_I}{\sqrt{Da^{3(1-v)(1+\omega_m)} + 1}} \tag{95}$$

where $D > 0$ is an integration constant.

- Since $v < 0$ and $\omega_m = 0$ in the vacuum, the power of a in the superscript is positive. In the early universe, $a \ll 1$ and thus $Da^{3(1-v)(1+\omega_m)} \ll 1$, which means that the Hubble parameter is mostly constant:

$$H \simeq H_I \tag{96}$$

- Therefore, inflation appears naturally in this model (no need for inflaton!).

Outlook

- The effects of torsion in the evolution of our universe do not end in the inflationary era (Basilakos, Mavromatos, and Solà Peracaula 2020b).
- The Kalb-Ramond axion field can be used to explain the matter/antimatter asymettry observed in our universe.
- It also breaks CPT and Lorentz symmetry.
- In later stages of the universe, the axion can also acquire a mass, making it a candidate for Dark Matter.
- This means that if torsion exists, then it potentially plays a huge role in the evolution of the universe as we know it today.

Bibliography

- Schwichtenberg, Jakob (2018). *Physics from Symmetry*. Springer International Publishing. ISBN: 9783319666310. DOI: 10.1007/978-3-319-66631-0. URL: http://dx.doi.org/10.1007/978-3-319-66631-0.
- Carroll, Sean M. (2019). *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press.
- Yepez, Jeffrey (2011). “Einstein’s vierbein field theory of curved space”. In: *arXiv: General Relativity and Quantum Cosmology*. URL: https://api.semanticscholar.org/CorpusID:118508699.

Bibliography

Duncan, M.J, N Kaloper, and K.A Olive (1992). "Axion hair and dynamical torsion from anomalies". In: *Nuclear Physics B* 387.1, pp. 215–235. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(92)90052-D. URL: https://www.sciencedirect.com/science/article/pii/055032139290052D.

Nakahara, M. (2003). *Geometry, Topology and Physics, Second Edition*. Taylor & Francis Incorporated. ISBN: 9781315275826. DOI: https://doi.org/10.1201/9781315275826. URL: https://www.taylorfrancis.com/books/mono/10.1201/9781315275826/geometry-topology-physics-mikio-nakahara.

Bibliography

Gasperini, Maurizio (2017). *Theory of Gravitational Interactions*. Springer International Publishing. ISBN: 9783319496825. DOI: 10.1007/978-3-319-49682-5. URL: http://dx.doi.org/10.1007/978-3-319-49682-5.

Cvitanović, Predrag (2008). *Group Theory: Birdtracks, Lie's, and Exceptional Groups*. Princeton University Press. ISBN: 9780691118369. URL: http://www.jstor.org/stable/j.ctt7rnjq (visited on 08/19/2024).

Mavromatos, Nick E., Pablo Pais, and Alfredo Iorio (Dec. 2023). "Torsion at Different Scales: From Materials to the Universe". In: *Universe* 9.12, p. 516. ISSN: 2218-1997. DOI: 10.3390/universe9120516. URL: http://dx.doi.org/10.3390/universe9120516.

Bibliography

- Popławski, Nikodem (May 2012). "Nonsingular, big-bounce cosmology from spinor-torsion coupling". In: *Physical Review D* 85.10. ISSN: 1550-2368. DOI: 10.1103/physrevd.85.107502. URL: http://dx.doi.org/10.1103/physrevd.85.107502.
- Polchinski, Joseph (1998). *String Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press.
- Green, Michael B., John H. Schwarz, and Edward Witten (2012). *Superstring Theory: 25th Anniversary Edition*. Cambridge Monographs on Mathematical Physics. Cambridge University Press.

Bibliography

- Dorlis, Panagiotis, Nick E. Mavromatos, and Sotirios-Neilos Vlachos (2024). "Condensate-Induced Inflation from Primordial Gravitational Waves in String-Inspired Chern-Simons Gravity". In: *arXiv: General Relativity and Quantum Cosmology*. URL: https://arxiv.org/abs/2403.09005.
- Basilakos, Spyros, Nick E. Mavromatos, and Joan Solà Peracaula (2020a). "Quantum anomalies in string-inspired running vacuum universe: Inflation and axion dark matter". In: *Physics Letters B* 803, p. 135342. ISSN: 0370-2693. DOI: https://doi.org/10.1016/j.physletb.2020.135342. URL: https://www.sciencedirect.com/science/article/pii/S0370269320301465.

Bibliography

Mavromatos, Nick E. (2022). *Lorentz Symmetry Violation in String-Inspired Effective Modified Gravity Theories*. DOI: 10.48550/ARXIV.2205.07044. URL: https://arxiv.org/abs/2205.07044.

Basilakos, Spyros, Nick E. Mavromatos, and Joan Solà Peracaula (Feb. 2020b). "Gravitational and chiral anomalies in the running vacuum universe and matter-antimatter asymmetry". In: *Phys. Rev. D* 101 (4), p. 045001. DOI: 10.1103/PhysRevD.101.045001. URL: https://link.aps.org/doi/10.1103/PhysRevD.101.045001.

Thank you for your attention!