NATIONAL TECHNICAL UNIVERSITY OF ATHENS

SCHOOL OF APPLIED MATHEMATICAL AND PHYSICAL SCIENCES

DEPARTMENT OF PHYSICS

Aspects of Contorted Geometry In The Early Universe And The Related Formalism

Undergraduate Thesis of Georgios Panagopoulos

Academic Year 2023-2024

Supervisor

Nikolaos E. Mavromatos
Professor, School of Applied Mathematical and Physical Sciences, Department of Physics

Thesis Committee

Nikolaos E. Mavromatos
Professor, School of Applied Mathematical and Physical Sciences, Department of Physics

Konstantinos N. Anagnostopoulos
Professor, School of Applied Mathematical and Physical Sciences, Department of Physics

Christoforos Kouvaris
Professor, School of Applied Mathematical and Physical Sciences, Department of Physics

Abstract

This thesis explores gravitational theories with the presence of torsion. In the first chapter, after a brief introduction to the tetrad and differential forms formalisms, the Einstein-Cartan theory is developed. The Einstein-Cartan theory is a minimal extension of General Relativity that assumes a non-vanishing torsion. In the second chapter, the theory of Quantum Electrodynamics is considered in contorted spacetimes, with a dual purpose. The first is to show that in such a theory, spin is connected to torsion, much like mass is connected to curvature in General Relativity. The second is to show that, by making a reasonable assumption, torsion can be physically realized as an axion due to the anomaly present in the axial current in Quantum Electrodynamics. Finally, in the third chapter we consider a string-inspired model, in which the field strength of the Kalb-Ramond field takes on the role of torsion and thus behaves analogously with the torsion in Einstein-Cartan theory, producing an axionic degree of freedom. This theory is explored further, showing how, due to the anomalies present in the string theory, gravitational wave condensates in the early universe can explain inflation. This is done through an alternative cosmological model to Λ-Cold Dark Matter (ΛCDM), called the Running Vacuum Model.

Keywords

Torsion, contorsion, tetrads, spin connection, differential forms, Cartan equations, Einstein-Cartan theory, Quantum Electrodynamics (QED) in contorted spacetime, anomaly, axions, strings, Kalb-Ramond field, Hirzebruch-Pontryagin topological density, Cotton tensor, gravitational wave condensate, running vacuum model, inflation.

Acknowledgments

I would like to thank everyone that has contributed to my efforts to complete this thesis and therefore my undergraduate degree. First and foremost, my parents, Rena and Yiannis, who satisfied my curiosity about the world as a child, supported my interest in science unconditionally and enabled me to fully focus on my undergraduate studies. Secondly, my supervisor Professor Nikolaos Mavromatos, for introducing me to this fascinating topic for a thesis and his invaluable guidance through all the unfamiliar knowledge in this field. Lastly, I would like to thank my friends and fellow students, with whom I shared this five-year journey of learning and growth. Our discussions, collaborations, and their constructive feedback have been instrumental in shaping my development as a physicist.

Contents

Chapter 1

Curved Spacetime with Torsion

The theory of General Relativity has proven itself to be one of the most successful physical theories we have. Today, more than a hundred years after its initial formulation, the predictions of General Relativity are still being confirmed experimentally to stunning accuracy. However, there is still a big drawback to this theory: it is not Quantum. General Relativity is a purely classical theory. It is widely believed that there is an underlying theory in which even gravity is quantized, and General Relativity is nothing but the macroscopic limit of that theory. There are many attempts to construct a quantum theory of gravity, such as String Theory, Loop Quantum Gravity etc. These are great efforts with a lot of depth and will someday be experimentally tested. A more minimalistic approach, or rather, a small step in reconciling General Relativity with Quantum Field Theory is to generalize Einstein's theory. How? Well, in our standard theory of curved spacetime, only the mass/energy of the objects considered plays a role. However, we know that there are more intrinsic properties to matter. The most important and obvious one is the *spin* of the particles. In fact, spin arises as a quantum number from the representations of the Poincaré group which describes the symmetries of Minkowski (flat) spacetime [1]. It is logical, then, to try to minimally extend the General Theory of Relativity to include spin. This chapter will be an introduction to such a theory, called Einstein-Cartan theory, where spin is minimally included by assuming that the torsion, unlike General Relativity, is non-zero.

1.1 Tetrad Formalism

In order to incorporate torsion in our existing theory of General Relativity, it is useful to express it in a different, yet equivalent formalism, called the *tetrad* or *vielbein*[1] formalism, which involves constructing an orthonormal four-vector basis at each point on the spacetime manifold. Let (M, g) be the spacetime Riemannian manifold with metric g. The standard expression for the metric in a chart $U \subseteq M$ (a local "patch" of spacetime) with local coordinates $\{x^\mu\}$ is

$$\boxed{g = g_{\mu\nu} dx^\mu dx^\nu} \tag{1.1.1}$$

where $g_{\mu\nu}$ are the *component functions* of the metric g and $dx^\mu dx^\nu$ is the *symmetric product*[2] of dx^μ and dx^ν, which are the one-forms of the coordinate coframe (dx^μ). In general, the component functions of the metric are dependent on the local coordinates x^μ of spacetime, i.e. $g_{\mu\nu} = g_{\mu\nu}(x)$. Knowing that spacetime is locally flat, however, makes it possible to find an alternative formalism in which the expression of the metric is dependent on the Minkowski metric[3] η_{ab} of the tangent space of the manifold at each point. This is the so-called tetrad formalism.

[1] Vielbein comes from the German words "viele" ("many") and "beine" ("legs"). In four spacetime dimensions, vierbein is sometimes used instead, where vier ("four") replaces viele.

[2] The symmetric product is defined to be

$$dx^\mu dx^\nu = \frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu)$$

where $\otimes$ is the tensor product of two tensors. We can then do a quick manipulation and renaming of indices to show that

$$g = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2}(g_{\mu\nu} dx^\mu \otimes dx^\nu + g_{\nu\mu} dx^\nu \otimes dx^\mu) = g_{\mu\nu} dx^\mu \otimes dx^\nu$$

[3] The convention $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$ is used, like in most texts dealing with General Relativity.

1.1.1 Tetrads

The tetrad formalism is essentially a change of basis from the coordinate frame to a general orthonormal frame on our manifold. Following [2], we can imagine setting up an orthonormal basis of vectors $\hat{e}_a|_p$ at each point $p \in M$ on the spacetime manifold. Since the basis is orthonormal and spacetime is locally flat, at each point we have that the component functions in this basis are nothing more than the Minkowski metric components:

$$g(\hat{e}_a|_p, \hat{e}_b|_p) = \eta_{ab}|_p$$

By repeating this process at all points of the chart we're on we get an orthonormal frame $\hat{e}_a$ and the above condition becomes

$$\boxed{g(\hat{e}_a, \hat{e}_b) = \eta_{ab}} \tag{1.1.2}$$

We have followed standard notational conventions and use Latin indices $a, b, c, d, \ldots$ etc when flat spacetime is considered. The coordinate frame is labeled as $\hat{e}_\mu \equiv \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and is connected to this new frame $\hat{e}_a$ via a basis transformation:

$$\boxed{\hat{e}_\mu = \hat{e}_\mu{}^a \, \hat{e}_a} \tag{1.1.3}$$

The new basis (frame) $\hat{e}_a$ is called a *tetrad or vierbein* basis, while the transformation matrix $\hat{e}_\mu{}^a = \hat{e}_\mu{}^a(x)$ (which is dependent on spacetime points x) is called a *tetrad field* or *vierbein field*. The inverse of the tetrad field is denoted by $(\hat{e}_\mu{}^a)^{-1} = \hat{e}^\mu{}_a$ and satisfies the orthogonality conditions

$$\boxed{\hat{e}_\mu{}^a \, \hat{e}^\mu{}_b = \delta^a_b} \tag{1.1.4}$$

$$\boxed{\hat{e}^\mu{}_a \, \hat{e}_\nu{}^a = \delta^\mu_\nu} \tag{1.1.5}$$

where δ denotes the Kronecker delta. By the vector-covector duality the dual of (1.1.3) is

$$\boxed{\hat{e}^\mu = \hat{e}^\mu{}_a \, \hat{e}^a} \tag{1.1.6}$$

where $\hat{e}^\mu \equiv dx^\mu$ are the coordinate coframe one-forms, i.e.

$$\boxed{dx^\mu = \hat{e}^\mu{}_a \, \hat{e}^a} \tag{1.1.7}$$

Therefore, the metric can be written as (eq. (1.1.1))

$$g = g_{\mu\nu}(\hat{e}^\mu{}_a \, \hat{e}^a)\,(\hat{e}^\nu{}_b \, \hat{e}^b) = g_{\mu\nu}(\hat{e}^\mu{}_a \, \hat{e}^\nu{}_b)\,(\hat{e}^a \, \hat{e}^b)$$

By applying condition (1.1.2), we get that

$$g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu}\hat{e}^\mu{}_a \, \hat{e}^\nu{}_b \, \hat{e}^a \, \hat{e}^b \, \hat{e}_a \, \hat{e}_b = \eta_{ab} \Rightarrow$$
$$\Rightarrow g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu}\hat{e}^\mu{}_a \, \hat{e}^\nu{}_b \, \delta^a_a \, \delta^b_b = \eta_{ab} \Rightarrow g_{\mu\nu}\hat{e}^\mu{}_a \, \hat{e}^\nu{}_b = \eta_{ab}$$

By inverting this last relation we get that the components of the metric can be expressed as

$$\boxed{g_{\mu\nu} = \eta_{ab}\hat{e}_\mu{}^a \, \hat{e}_\nu{}^b} \tag{1.1.8}$$

The tetrad fields $\hat{e}_\mu{}^a(x)$ can be considered as components of a $(1,1)$ tensor e on a "mixed" frame with both curved and flat spacetime components:

$$\boxed{e = \hat{e}_\mu{}^a dx^\mu \otimes \hat{e}_a} \tag{1.1.9}$$

If we consider a vector v we can express it into either the curved or the flat basis:

$$\boxed{v = v^\mu \hat{e}_\mu} \qquad \boxed{v = v^a \hat{e}_a} \tag{1.1.10}$$

By acting into either of these with the tetrad tensor e, we get the same vector but in the different basis. Thus, the tetrad tensor is actually the identity map. The vector transformation law (and its inverse) that occur are

$$\boxed{v^a = \hat{e}_\mu{}^a v^\mu} \qquad \boxed{v^\mu = \hat{e}^\mu{}_a v^a} \tag{1.1.11}$$

1.1.2 Local Lorentz Transformations

We can also consider coordinate transformations between different tetrad bases. The only requirement that needs to be fulfilled is that the transformation must preserve orthogonality. Since tetrads are considered locally in patches of flat spacetime with the Minkowski metric, Lorentz transformations are exactly the type of transformations we're looking for. As such, we have that a change of tetrad basis can be expressed as

$$\boxed{\hat{e}_{a'} = \Lambda^a{}_{a'}(x)\hat{e}_a} \tag{1.1.12}$$

where $\Lambda^a{}_{a'}(x)$ denote the component functions of the Lorentz transformation tensor. Of course, the Minkowski metric remains invariant under Lorentz transformations:

$$\Lambda^a{}_{a'}\Lambda^b{}_{b'}\eta_{ab} = \eta_{a'b'}$$

Due to the local nature of these Lorentz transformations, we call these *Local Lorentz Transformations*. Essentially, we now have a "rule" to transform both (upper and lower) Latin and Greek indices. Therefore, assuming we have a tensor $T^{a\mu}{}_{b\nu}$ with all 4 types of indices, it transforms as

$$\boxed{T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_a \frac{\partial x^{\mu'}}{\partial x^{\mu}} \Lambda^b{}_{b'} \frac{\partial x^{\nu}}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu}} \tag{1.1.13}$$

The curved spacetime (Greek indices) coordinates changes are called *General Coordinate Transformations*.

1.1.3 The Spin Connection

We know that the covariant derivative of a vector expressed in the curved basis is

$$\boxed{\overline{\nabla}_\mu v^\nu = \partial_\mu v^\nu + \bar{\Gamma}^\nu_{\mu\lambda}\, v^\lambda} \tag{1.1.14}$$

where the bar on top of the symbol $\bar{\Gamma}^\nu_{\mu\lambda}$ is used to indicate that this is not the usual Christoffel symbol of the Levi-Civita connection as it is not symmetric, i.e.

$$\bar{\Gamma}^\nu_{\mu\lambda} - \bar{\Gamma}^\mu_{\nu\lambda} \neq 0$$

which means that there is a non-vanishing torsion and we consider a general affine connection. Similarly, the bar over the derivative symbol $\overline{\nabla}$ indicates that the torsion-full (from here on, contorted) connection is being considered. Now, suppose that the vector v is expressed in the flat basis. We can assume that there is a different connection such that the covariant derivative is expressed in the same way as before, i.e. as

$$\boxed{\overline{\nabla}_\mu v^a = \partial_\mu v^a + \bar{\omega}_\mu{}^a{}_b\, v^b} \tag{1.1.15}$$

This new connection coefficient $\bar{\omega}_\mu{}^a{}_b$ that is related to the Latin indices is called the *spin connection*[4]. The bar indicates the presence of torsion as before. We can express the spin connection coefficients in terms of the $\bar{\Gamma}$ symbols by expressing the tensor $\overline{\nabla}v$ in the two different bases. For the curved basis, we have that

$$\overline{\nabla}v = (\overline{\nabla}_\mu v^\nu)dx^\mu \otimes \partial_\nu = (\partial_\mu v^\nu + \bar{\Gamma}^\nu_{\mu\lambda}\, v^\lambda)dx^\mu \otimes \partial_\nu \tag{1.1.16}$$

For the flat basis, we have that

$$\overline{\nabla}v = (\overline{\nabla}_\mu v^a)dx^\mu \otimes \hat{e}_a = (\partial_\mu v^a + \bar{\omega}_\mu{}^a{}_b\, v^b)dx^\mu \otimes \hat{e}_a \tag{1.1.17}$$

We know that $\hat{e}_a = \hat{e}^\rho{}_a\, \hat{e}_\rho = \hat{e}^\rho{}_a\, \partial_\rho$ and $v^a = \hat{e}_\nu{}^a v^\nu$ and thus Equation (1.1.17) becomes

$$\overline{\nabla}v = [\partial_\mu(\hat{e}_\nu{}^a v^\nu) + \bar{\omega}_\mu{}^a{}_b\,(\hat{e}_\nu{}^b v^\nu)]dx^\mu \otimes (\hat{e}^\rho{}_a\, \partial_\rho)$$

which gives us

$$\overline{\nabla}v = (v^\nu \hat{e}^\rho{}_a \partial_\mu \hat{e}_\nu{}^a + \partial_\mu v^\rho + \bar{\omega}_\mu{}^a{}_b\, \hat{e}_\nu{}^b v^\nu \hat{e}^\rho{}_a)dx^\mu \otimes \partial_\rho$$

[4] The placement of the index μ is purely conventional. A different notational convention would be to place it on the right, i.e. to have $\bar{\omega}^a{}_{b\mu}$ instead.

where we have done the contraction $\hat{e}_\nu{}^a \hat{e}_a{}^\rho = \delta^\rho_\nu$. By relabeling the indices $\rho \to \nu \to \lambda$ we get that

$$\bar{\nabla} v = (\partial_\mu v^\nu + v^\lambda \hat{e}^\nu{}_a \partial_\mu \hat{e}_\lambda{}^a + \bar{\omega}_\mu{}^a{}_b\, \hat{e}_\lambda{}^b v^\lambda \hat{e}^\nu{}_a) dx^\mu \otimes \partial_\nu \tag{1.1.18}$$

Then, by comparing equations (1.1.16) and (1.1.18) we find that

$$\bar{\Gamma}^\nu_{\mu\lambda}\, v^\lambda = v^\lambda \hat{e}^\nu{}_a \partial_\mu \hat{e}_\lambda{}^a + \bar{\omega}_\mu{}^a{}_b\, \hat{e}_\lambda{}^b v^\lambda \hat{e}^\nu{}_a$$

and since this must hold for every v^λ, we have that the relation between the affine and spin connection coefficients is:

$$\boxed{\bar{\Gamma}^\nu_{\mu\lambda} = \hat{e}^\nu{}_a \partial_\mu \hat{e}_\lambda{}^a + \bar{\omega}_\mu{}^a{}_b\, \hat{e}_\lambda{}^b \hat{e}^\nu{}_a} \tag{1.1.19}$$

We can easily invert this and solve for the spin connection coefficients:

$$\boxed{\bar{\omega}_\mu{}^a{}_b = \hat{e}^\lambda{}_b\, \hat{e}_\nu{}^a\, \bar{\Gamma}^\nu_{\mu\lambda} - \hat{e}^\lambda{}_b\, \partial_\mu \hat{e}_\lambda{}^a} \tag{1.1.20}$$

Now, if we multiply both sides with $\hat{e}_\sigma{}^b$ we get that

$$\bar{\omega}_\mu{}^a{}_b \hat{e}_\sigma{}^b = \hat{e}^\lambda{}_b\, \hat{e}_\nu{}^a\, \hat{e}_\sigma{}^b\, \bar{\Gamma}^\nu_{\mu\lambda} - \hat{e}^\lambda{}_b\, \hat{e}_\sigma{}^b\, \partial_\mu \hat{e}_\lambda{}^a = \hat{e}_\nu{}^a\, \bar{\Gamma}^\nu_{\mu\sigma} - \partial_\mu \hat{e}_\sigma{}^a$$

By rearranging the terms in one side we get

$$\partial_\mu \hat{e}_\sigma{}^a - \hat{e}_\nu{}^a\, \bar{\Gamma}^\nu_{\mu\sigma} + \bar{\omega}_\mu{}^a{}_b \hat{e}_\sigma{}^b = 0$$

We may recognize this relation as a covariant derivative of the tetrad field $\hat{e}_\sigma{}^a$, i.e.

$$\bar{\nabla}_\mu \hat{e}_\nu{}^a = \partial_\mu \hat{e}_\nu{}^a - \hat{e}_\sigma{}^a\, \bar{\Gamma}^\sigma_{\mu\nu} + \bar{\omega}_\mu{}^a{}_b \hat{e}_\nu{}^b$$

where we have relabeled the indices $\nu \leftrightarrow \sigma$. The term "covariant derivative" is used not to refer the usual covariant derivative encountered in General Relativity, which only considers the affine connection $\bar{\Gamma}$ in relation to Greek indices, but in a broader/generalized sense where Latin indices are considered with regards to the spin connection $\bar{\omega}$. Therefore, we have that the covariant derivative of the tetrad field vanishes:

$$\boxed{\bar{\nabla}_\mu \hat{e}_\nu{}^a = \partial_\mu \hat{e}_\nu{}^a - \hat{e}_\sigma{}^a\, \bar{\Gamma}^\sigma_{\mu\nu} + \bar{\omega}_\mu{}^a{}_b \hat{e}_\nu{}^b = 0} \tag{1.1.21}$$

The above relation can be used as an initial assumption which ultimately acts as a defining relation for the spin connection. That's why it is called the *tetrad postulate*. As our starting point was different, however, this result is somehow expected, since $\hat{e}_\nu{}^a$ are the components of the tensor e, which as we saw is the identity map. Another important question regarding the spin connection coefficients is the way they transform. Just like the $\bar{\Gamma}$ symbols, which arise from the affine connection, we do not expect the spin connection coefficients to be the components of a tensor. Our starting point to find this transformation law is the fact that the covariant derivative of a vector v in the flat basis must be Lorentz invariant [3]. Therefore, if

$$\bar{\nabla}_\mu v^a = \partial_\mu v^a + \bar{\omega}_\mu{}^a{}_b v^b$$

then

$$\boxed{\bar{\nabla}_\mu v^{a'} = \Lambda^{a'}{}_a \bar{\nabla}_\mu v^a} \tag{1.1.22}$$

must hold. We have that

$$\bar{\nabla}_\mu v^{a'} = \bar{\nabla}_\mu (\Lambda^{a'}{}_a\, v^a) = (\bar{\nabla}_\mu \Lambda^{a'}{}_a) v^a + \Lambda^{a'}{}_a \bar{\nabla}_\mu v^{a'}$$

and thus the only way for our Lorentz invariance condition to hold is if

$$\boxed{\bar{\nabla}_\mu \Lambda^{a'}{}_a = 0} \tag{1.1.23}$$

We know how to take covariant derivatives of $(1,1)$-tensors (and, in general, (k,l)-tensors) and so we calculate the covariant derivative of $\Lambda^{a'}{}_b$, which we know should be zero:

$$\boxed{\bar{\nabla}_\mu \Lambda^{a'}{}_b = \partial_\mu \Lambda^{a'}{}_b + \bar{\omega}_\mu{}^{a'}{}_c \Lambda^c{}_b - \bar{\omega}_\mu{}^c{}_b \Lambda^{a'}{}_c = 0} \tag{1.1.24}$$

where the upper index has a positive correction term and the lower index has a negative one. We multiply with $\Lambda^b{}_{b'}$ and get

$$\Lambda^b{}_{b'}\partial_\mu\Lambda^{a'}{}_b + \bar\omega_\mu{}^{a'}{}_c\,\Lambda^b{}_{b'}\Lambda^c{}_b - \bar\omega_\mu{}^c{}_b\,\Lambda^b{}_{b'}\Lambda^{a'}{}_c = 0 \Rightarrow$$
$$\Lambda^b{}_{b'}\partial_\mu\Lambda^{a'}{}_b + \bar\omega_\mu{}^{a'}{}_{b'} - \bar\omega_\mu{}^c{}_b\,\Lambda^b{}_{b'}\Lambda^{a'}{}_c = 0 \tag{1.1.25}$$

and thus

$$\boxed{\bar\omega_\mu{}^{a'}{}_{b'} = \bar\omega_\mu{}^c{}_b\,\Lambda^b{}_{b'}\Lambda^{a'}{}_c - \Lambda^b{}_{b'}\partial_\mu\Lambda^{a'}{}_b} \tag{1.1.26}$$

It can be seen clearly that the spin connection coefficients must not transform like a tensor in order for the covariant derivative of a vector to transform properly.

1.2 Differential Forms Viewpoint

Our new formulation, involving Latin (flat) indices, allows us to employ a very powerful computational tool: the exterior algebra of differential forms. We know that an object like X_μ is a one-form, an anti-symmetric object like $A_{\mu\nu}$ is a 2-form and so on. But what about an object like $X_\mu{}^a$? At first glance, one would say that this is not a differential form. However, the fact that a is a Latin (flat) index allows us to make a very important change in viewpoint: We can see $X_\mu{}^a$ as a *vector-valued one-form*! For each value of the lower index μ, $X_\mu{}^a$ is a four-vector by its second, upper flat index a. In this spirit, any tensor that has any number of lower Greek indices that are completely anti-symmetric and any number of Latin indices can be viewed as a tensor-valued differential form. For example, the tensor $A_{\mu\nu}{}^a{}_b$, where μ,ν are antisymmetric, can be thought of as a $(1,1)$-tensor valued 2-form. As such, "ordinary" differential forms like X_μ are simply scalar-valued differential forms. By application of this "rule", the spin connection coefficients $\bar\omega_\mu{}^a{}_b$ are $(1,1)$-tensor-valued one-forms. This means that we can "suppress" the Greek index by writing the spin connection coefficient as a one-form[5]

$$\boxed{\bar{\boldsymbol\omega}^a{}_b = \bar\omega_\mu{}^a{}_b\,dx^\mu} \tag{1.2.1}$$

where the bold symbol has been used to indicate the one-form as a geometric object and not as simply its component functions. We know that the exterior derivative of a one-form X_μ is a 2-form given by:

$$(dX)_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu$$

and so for a four-vector valued one-form $X_\mu{}^a$ we have that

$$\boxed{(dX)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a} \tag{1.2.2}$$

However, this derivative does not suit our needs. That's because, while it transforms like a 2-form under General Coordinate Transformations (Greek indices), it does *not* transform like a four-vector under Local Lorentz Transformations (Latin indices). To amend this, we must introduce a new derivative operator, the *exterior covariant derivative*, which is defined to be

$$\boxed{(\bar DX)_{\mu\nu}{}^a = (dX)_{\mu\nu}{}^a + (\bar\omega\wedge X)_{\mu\nu}{}^a = \partial_\mu X_\nu{}^a - \partial_\nu X_\mu{}^a + \bar\omega_\mu{}^a{}_b X_\nu{}^b - \bar\omega_\nu{}^a{}_b X_\mu{}^b} \tag{1.2.3}$$

where $\wedge$ symbolizes the wedge product of differential forms and the bar indicates a connection for a non-vanishing torsion is used. We can show that this is equal to

$$\boxed{(\bar DX)_{\mu\nu}{}^a = \bar\nabla_\mu X_\nu{}^a - \bar\nabla_\nu X_\mu{}^a} \tag{1.2.4}$$

In bold notation, suppressing the Greek indices, the exterior covariant derivative of a vector-valued p-form is expressed as:

$$\boxed{(\bar{\boldsymbol D}\boldsymbol X)^a = (\bar{\boldsymbol D}\boldsymbol X^a) = \boldsymbol{dX}^a + \bar{\boldsymbol\omega}^a{}_b\wedge\boldsymbol X^b} \tag{1.2.5}$$

This definition of the exterior covariant derivative can be extended to any tensor-valued p-form $\boldsymbol X^{a\dots}{}_{b\dots}$, as given in [4]:

$$\boxed{(\bar{\boldsymbol D}\boldsymbol X)^{a\dots}{}_{b\dots} = (\boldsymbol{dX})^{a\dots}{}_{b\dots} + (\bar{\boldsymbol\omega}^a{}_c\wedge\boldsymbol X^{c\dots}{}_{b\dots}) + \cdots - (-1)^p(\boldsymbol X^{a\dots}{}_{d\dots}\wedge\bar{\boldsymbol\omega}^d{}_b) - \cdots} \tag{1.2.6}$$

To sum up, we have done two things:

[5]Since the index μ gets contracted anyway, it becomes clearer why there are different notations for its placement. In most cases, we'll be using the contracted forms of the connection coefficients.

1. We have changed our viewpoint, by viewing tensors of mixed Greek and Latin indices as tensor-valued differential forms of Greek indices. This allows us to unclutter our notation and keep only the Latin (flat) indices by representing these differential forms as geometric objects and not as their component functions (thus suppressing all Greek indices).
2. We have defined a suitable derivative operator (the exterior covariant derivative) for tensor-valued differential forms that incorporates the spin connection.

We will thoroughly use the notation and tools introduced above in the rest of the text.

1.3 The Cartan Structure Equations

Now that we have established our formalism in terms of vierbeins and differential forms, we can go ahead and express many notions of General Relativity we're familiar with in these new terms.

1.3.1 Metric Compatibility

The metric compatibility condition of General Relativity is expressed as the vanishing of the covariant derivative of the metric

$$\boxed{\bar{\nabla} g = 0} \tag{1.3.1}$$

In vielbein notation, this is expressed as

$$\bar{\boldsymbol{D}}\boldsymbol{\eta}_{ab} = 0 \Rightarrow \bar{\nabla}_\mu \eta_{ab} = \underbrace{\partial_\mu \eta_{ab}}_{0} - \bar{\omega}_\mu{}^c{}_a\,\eta_{cb} - \bar{\omega}_\mu{}^c{}_b\,\eta_{ac} = -\bar{\omega}_{\mu ab} - \bar{\omega}_{\mu ba} = 0$$

which translates to the two Latin indices being antisymmetric:

$$\boxed{\bar{\omega}_{\mu ab} = -\bar{\omega}_{\mu ba}} \tag{1.3.2}$$

or, equivalently

$$\boxed{\bar{\boldsymbol{\omega}}_{ab} = -\bar{\boldsymbol{\omega}}_{ba}} \tag{1.3.3}$$

1.3.2 Torsion

We know that torsion in General Relativity is a tensor with components[6]

$$\boxed{T^\lambda{}_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\nu\mu}} \tag{1.3.4}$$

We want to express this in terms of Latin indices. We have[7]:

$$\boxed{T^\lambda{}_{\mu\nu} = \hat{e}^\lambda{}_a T^a{}_{\mu\nu}} \tag{1.3.5}$$

From equation (1.1.19) we have that

$$\bar{\Gamma}^\lambda_{\mu\nu} = \hat{e}^\lambda{}_a \partial_\mu \hat{e}_\nu{}^a + \bar{\omega}_\mu{}^a{}_c\,\hat{e}_\nu{}^c \hat{e}^\lambda{}_a$$

and thus

$$\begin{gathered} T^\lambda{}_{\mu\nu} = \hat{e}^\lambda{}_a \partial_\mu \hat{e}_\nu{}^a + \bar{\omega}_\mu{}^a{}_c\,\hat{e}_\nu{}^c \hat{e}^\lambda{}_a - \hat{e}^\lambda{}_a \partial_\nu \hat{e}_\mu{}^a - \bar{\omega}_\nu{}^a{}_c\,\hat{e}_\mu{}^c \hat{e}^\lambda{}_a \Rightarrow \\ T^\lambda{}_{\mu\nu} = \hat{e}^\lambda{}_a (\partial_\mu \hat{e}_\nu{}^a - \partial_\nu \hat{e}_\mu{}^a + \bar{\omega}_\mu{}^a{}_c\,\hat{e}_\nu{}^c - \bar{\omega}_\nu{}^a{}_c\,\hat{e}_\mu{}^c) \end{gathered} \tag{1.3.6}$$

[6] Since the connection coefficients $\bar{\Gamma}$ aren't tensors, we typically do not adhere strictly to index notation and write the top index directly above the first lower index. However, as the difference of these two $\bar{\Gamma}$ symbols is a tensor, we are obliged to *choose* a placement for the upper index. Here, we follow the convention of placing it in the left side, rather than the right side.

[7] In other sources, $T^\lambda{}_{\mu\nu} = -\hat{e}^\lambda{}_a T^a{}_{\mu\nu}$ is used instead. This conventional minus can be traced back to the convention used by the author in the covariant derivative definition. More specifically, while here we use $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda$ other sources use $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\nu\mu}\omega_\lambda$ instead. To preserve the final result (Equation (1.3.8)), the introduction of this minus is necessary.

By comparing equations (1.3.5) and (1.3.6) we find that

$$\boxed{T^{a}{}_{\mu\nu} = \partial_\mu \hat{e}_\nu{}^a - \partial_\nu \hat{e}_\mu{}^a + \bar{\omega}_\mu{}^a{}_c\, \hat{e}_\nu{}^c - \bar{\omega}_\nu{}^a{}_c\, \hat{e}_\mu{}^c} \tag{1.3.7}$$

We recognize this as the exterior covariant derivative of the vector-valued one-form $\hat{\boldsymbol{e}}^a$ (see Equation (1.2.3)). Thus, we can write

$$\boxed{\boldsymbol{T}^a = \overline{\boldsymbol{D}}\hat{\boldsymbol{e}}^a = \boldsymbol{d}\hat{\boldsymbol{e}}^a + \bar{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{e}}^b} \tag{1.3.8}$$

This is called *Cartan's first structure equation*.

1.3.3 Curvature

We can work similarly with the curvature tensor. In General Relativity, the curvature tensor is given by

$$\boxed{\overline{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \bar{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \bar{\Gamma}^\rho_{\mu\sigma} + \bar{\Gamma}^\rho_{\mu\lambda}\bar{\Gamma}^\lambda_{\nu\sigma} - \bar{\Gamma}^\rho_{\nu\lambda}\bar{\Gamma}^\lambda_{\mu\sigma}} \tag{1.3.9}$$

We can change two of the indices to flat and express the curvature in terms of the spin connection as

$$\boxed{\overline{R}^{a}{}_{b\mu\nu} = \partial_\mu \bar{\omega}_\nu{}^a{}_b - \partial_\nu \bar{\omega}_\mu{}^a{}_b + \bar{\omega}_\mu{}^a{}_c \bar{\omega}_\nu{}^c{}_b - \bar{\omega}_\nu{}^a{}_c \bar{\omega}_\mu{}^c{}_b} \tag{1.3.10}$$

Proof

We know that we can pass onto flat indices by using the vielbeins:

$$\boxed{\overline{R}^{\rho}{}_{\sigma\mu\nu} = \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^b\, \overline{R}^{a}{}_{b\mu\nu}} \tag{1.3.11}$$

Now, by using Equation (1.1.19) we get that:

$$\begin{aligned}\partial_\mu \bar{\Gamma}^\rho_{\nu\sigma} &= \partial_\mu(\hat{e}^\rho{}_a \partial_\nu \hat{e}_\sigma{}^a + \bar{\omega}_\nu{}^a{}_c\, \hat{e}_\sigma{}^c \hat{e}^\rho{}_a) = \\ &= (\partial_\mu \hat{e}^\rho{}_a)(\partial_\nu \hat{e}_\sigma{}^a) + \hat{e}^\rho{}_a(\partial_\mu\partial_\nu \hat{e}_\sigma{}^a) + (\partial_\mu \bar{\omega}_\nu{}^a{}_c)\, \hat{e}_\sigma{}^c \hat{e}^\rho{}_a + \bar{\omega}_\nu{}^a{}_c\, (\partial_\mu \hat{e}_\sigma{}^c)\hat{e}^\rho{}_a + \bar{\omega}_\nu{}^a{}_c\, \hat{e}_\sigma{}^c(\partial_\mu \hat{e}^\rho{}_a)\end{aligned}$$

and

$$\begin{aligned}\partial_\nu \bar{\Gamma}^\rho_{\mu\sigma} &= \partial_\nu(\hat{e}^\rho{}_a \partial_\mu \hat{e}_\sigma{}^a + \bar{\omega}_\mu{}^a{}_c\, \hat{e}_\sigma{}^c \hat{e}^\rho{}_a) = \\ &= (\partial_\nu \hat{e}^\rho{}_a)(\partial_\mu \hat{e}_\sigma{}^a) + \hat{e}^\rho{}_a(\partial_\nu\partial_\mu \hat{e}_\sigma{}^a) + (\partial_\nu \bar{\omega}_\mu{}^a{}_c)\, \hat{e}_\sigma{}^c \hat{e}^\rho{}_a + \bar{\omega}_\mu{}^a{}_c\, (\partial_\nu \hat{e}_\sigma{}^c)\hat{e}^\rho{}_a + \bar{\omega}_\mu{}^a{}_c\, \hat{e}_\sigma{}^c(\partial_\nu \hat{e}^\rho{}_a)\end{aligned}$$

as well as

$$\begin{aligned}\bar{\Gamma}^\rho_{\mu\lambda}\bar{\Gamma}^\lambda_{\nu\sigma} &= (\hat{e}^\rho{}_a \partial_\mu \hat{e}_\lambda{}^a + \bar{\omega}_\mu{}^a{}_c\, \hat{e}_\lambda{}^c \hat{e}^\rho{}_a)\left(\hat{e}^\lambda{}_b \partial_\nu \hat{e}_\sigma{}^b + \bar{\omega}_\nu{}^b{}_d\, \hat{e}_\sigma{}^d \hat{e}^\lambda{}_b\right) \\ &= \hat{e}^\rho{}_a \hat{e}^\lambda{}_b \partial_\mu \hat{e}_\lambda{}^a \partial_\nu \hat{e}_\sigma{}^b + \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \hat{e}^\lambda{}_b\, \bar{\omega}_\nu{}^b{}_d\, \partial_\mu \hat{e}_\lambda{}^a + \\ &+ \underbrace{\hat{e}_\lambda{}^c \hat{e}^\lambda{}_b}_{=\delta^c_b} \hat{e}^\rho{}_a\, \bar{\omega}_\mu{}^a{}_c \partial_\nu \hat{e}_\sigma{}^b + \underbrace{\hat{e}_\lambda{}^c\, \hat{e}^\lambda{}_b}_{=\delta^c_b}\, \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \bar{\omega}_\mu{}^a{}_c\, \bar{\omega}_\nu{}^b{}_d = \\ &= \hat{e}^\rho{}_a \hat{e}^\lambda{}_b \partial_\mu \hat{e}_\lambda{}^a \partial_\nu \hat{e}_\sigma{}^b + \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \hat{e}^\lambda{}_b\, \bar{\omega}_\nu{}^b{}_d\, \partial_\mu \hat{e}_\lambda{}^a + \hat{e}^\rho{}_a\, \bar{\omega}_\mu{}^a{}_b \partial_\nu \hat{e}_\sigma{}^b + \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \bar{\omega}_\mu{}^a{}_b\, \bar{\omega}_\nu{}^b{}_d\end{aligned}$$

and

$$\begin{aligned}\bar{\Gamma}^\rho_{\nu\lambda}\bar{\Gamma}^\lambda_{\mu\sigma} &= (\hat{e}^\rho{}_a \partial_\nu \hat{e}_\lambda{}^a + \bar{\omega}_\nu{}^a{}_c\, \hat{e}_\lambda{}^c \hat{e}^\rho{}_a)\left(\hat{e}^\lambda{}_b \partial_\mu \hat{e}_\sigma{}^b + \bar{\omega}_\mu{}^b{}_d\, \hat{e}_\sigma{}^d \hat{e}^\lambda{}_b\right) \\ &= \hat{e}^\rho{}_a \hat{e}^\lambda{}_b \partial_\nu \hat{e}_\lambda{}^a \partial_\mu \hat{e}_\sigma{}^b + \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \hat{e}^\lambda{}_b\, \bar{\omega}_\mu{}^b{}_d\, \partial_\nu \hat{e}_\lambda{}^a + \hat{e}^\rho{}_a\, \bar{\omega}_\nu{}^a{}_b \partial_\mu \hat{e}_\sigma{}^b + \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \bar{\omega}_\nu{}^a{}_b\, \bar{\omega}_\mu{}^b{}_d\end{aligned}$$

and so we have that

$$\begin{aligned}\overline{R}^{\rho}{}_{\sigma\mu\nu} &= (\partial_\mu \hat{e}^\rho{}_a)(\partial_\nu \hat{e}_\sigma{}^a) + \cancel{\hat{e}^\rho{}_a(\partial_\mu\partial_\nu \hat{e}_\sigma{}^a)} + (\partial_\mu \bar{\omega}_\nu{}^a{}_c)\, \hat{e}_\sigma{}^c \hat{e}^\rho{}_a + \bar{\omega}_\nu{}^a{}_c\, (\partial_\mu \hat{e}_\sigma{}^c)\hat{e}^\rho{}_a + \bar{\omega}_\nu{}^a{}_c\, \hat{e}_\sigma{}^c(\partial_\mu \hat{e}^\rho{}_a) \\ &- (\partial_\nu \hat{e}^\rho{}_a)(\partial_\mu \hat{e}_\sigma{}^a) - \cancel{\hat{e}^\rho{}_a(\partial_\nu\partial_\mu \hat{e}_\sigma{}^a)} - (\partial_\nu \bar{\omega}_\mu{}^a{}_c)\, \hat{e}_\sigma{}^c \hat{e}^\rho{}_a - \bar{\omega}_\mu{}^a{}_c\, (\partial_\nu \hat{e}_\sigma{}^c)\hat{e}^\rho{}_a - \bar{\omega}_\mu{}^a{}_c\, \hat{e}_\sigma{}^c(\partial_\nu \hat{e}^\rho{}_a) \\ &+ \hat{e}^\rho{}_a \hat{e}^\lambda{}_b \partial_\mu \hat{e}_\lambda{}^a \partial_\nu \hat{e}_\sigma{}^b + \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \hat{e}^\lambda{}_b\, \bar{\omega}_\nu{}^b{}_d\, \partial_\mu \hat{e}_\lambda{}^a + \hat{e}^\rho{}_a\, \bar{\omega}_\mu{}^a{}_b \partial_\nu \hat{e}_\sigma{}^b + \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \bar{\omega}_\mu{}^a{}_b\, \bar{\omega}_\nu{}^b{}_d \\ &- \hat{e}^\rho{}_a \hat{e}^\lambda{}_b \partial_\nu \hat{e}_\lambda{}^a \partial_\mu \hat{e}_\sigma{}^b - \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \hat{e}^\lambda{}_b\, \bar{\omega}_\mu{}^b{}_d\, \partial_\nu \hat{e}_\lambda{}^a - \hat{e}^\rho{}_a\, \bar{\omega}_\nu{}^a{}_b \partial_\mu \hat{e}_\sigma{}^b - \hat{e}^\rho{}_a\, \hat{e}_\sigma{}^d\, \bar{\omega}_\nu{}^a{}_b\, \bar{\omega}_\mu{}^b{}_d\end{aligned}$$

After rearranging some terms and renaming some indices, we get

$$\overline{R}^{\rho}{}_{\sigma\mu\nu} = \hat{e}^{\rho}{}_{a}\,\hat{e}_{\sigma}{}^{b}\Big[\partial_\mu\bar{\omega}_\nu{}^a{}_b - \partial_\nu\bar{\omega}_\mu{}^a{}_b + \bar{\omega}_\mu{}^a{}_c\,\bar{\omega}_\nu{}^c{}_b - \bar{\omega}_\nu{}^a{}_c\,\bar{\omega}_\mu{}^c{}_b\Big] + \Big[(\partial_\mu\hat{e}^{\rho}{}_a)(\partial_\nu\hat{e}_\sigma{}^a) - (\partial_\nu\hat{e}^{\rho}{}_a)(\partial_\mu\hat{e}_\sigma{}^a)\Big]$$
$$+\hat{e}_\sigma{}^c\Big[\bar{\omega}_\nu{}^a{}_c\,(\partial_\mu\hat{e}^\rho{}_a) - \bar{\omega}_\mu{}^a{}_c\,(\partial_\nu\hat{e}^\rho{}_a)\Big] + \hat{e}^\rho{}_a\Big[\bar{\omega}_\nu{}^a{}_c\,(\partial_\mu\hat{e}_\sigma{}^c) - \bar{\omega}_\mu{}^a{}_c\,(\partial_\nu\hat{e}_\sigma{}^c)\Big] + \hat{e}^\rho{}_a\Big[\bar{\omega}_\mu{}^a{}_b(\partial_\nu\hat{e}_\sigma{}^b) - \bar{\omega}_\nu{}^a{}_b(\partial_\mu\hat{e}_\sigma{}^b)\Big]$$
$$+\hat{e}^\rho{}_a\hat{e}^\lambda{}_b\Big[(\partial_\mu\hat{e}_\lambda{}^a)(\partial_\nu\hat{e}_\sigma{}^b) - (\partial_\nu\hat{e}_\lambda{}^a)(\partial_\mu\hat{e}_\sigma{}^b)\Big] + \hat{e}^\rho{}_a\,\hat{e}_\sigma{}^d\,\hat{e}^\lambda{}_b\Big[\bar{\omega}_\nu{}^b{}_d\,(\partial_\mu\hat{e}_\lambda{}^a) - \bar{\omega}_\mu{}^b{}_d\,(\partial_\nu\hat{e}_\lambda{}^a)\Big]$$

To proceed with the computations, we require a useful identity. We have that

$$\partial_\mu(\hat{e}^\lambda{}_a\hat{e}_\tau{}^a) = (\partial_\mu\hat{e}_\tau{}^a)\hat{e}^\lambda{}_a + \hat{e}_\tau{}^a(\partial_\mu\hat{e}^\lambda{}_a)$$

but also

$$\partial_\mu(\hat{e}^\lambda{}_a\hat{e}_\tau{}^a) = \partial_\mu\delta^\lambda_\tau = 0$$

and thus

$$\boxed{(\partial_\mu\hat{e}_\tau{}^a)\hat{e}^\lambda{}_a = -\hat{e}_\tau{}^a(\partial_\mu\hat{e}^\lambda{}_a)} \tag{1.3.12}$$

which means that we can exchange vielbeins in the derivative while getting a minus sign. Having this in mind, we have that

$$\hat{e}^\rho{}_a\Big[\hat{e}^\lambda{}_b(\partial_\mu\hat{e}_\lambda{}^a)(\partial_\nu\hat{e}_\sigma{}^b) - \hat{e}^\lambda{}_b(\partial_\nu\hat{e}_\lambda{}^a)(\partial_\mu\hat{e}_\sigma{}^b)\Big] = \hat{e}^\rho{}_a\Big[-\hat{e}_\lambda{}^a(\partial_\mu\hat{e}^\lambda{}_b)(\partial_\nu\hat{e}_\sigma{}^b) + \hat{e}_\lambda{}^a(\partial_\nu\hat{e}^\lambda{}_b)(\partial_\mu\hat{e}_\sigma{}^b)\Big]$$
$$= \delta^\rho_\lambda\Big[-(\partial_\mu\hat{e}^\lambda{}_b)(\partial_\nu\hat{e}_\sigma{}^b) + (\partial_\nu\hat{e}^\lambda{}_b)(\partial_\mu\hat{e}_\sigma{}^b)\Big] = \Big[-(\partial_\mu\hat{e}^\rho{}_b)(\partial_\nu\hat{e}_\sigma{}^b) + (\partial_\nu\hat{e}^\rho{}_b)(\partial_\mu\hat{e}_\sigma{}^b)\Big]$$

Similarly, we have that

$$\hat{e}^\rho{}_a\,\hat{e}_\sigma{}^d\Big[\hat{e}^\lambda{}_b\,\bar{\omega}_\nu{}^b{}_d\,(\partial_\mu\hat{e}_\lambda{}^a) - \hat{e}^\lambda{}_b\,\bar{\omega}_\mu{}^b{}_d\,(\partial_\nu\hat{e}_\lambda{}^a)\Big] = \hat{e}^\rho{}_a\,\hat{e}_\sigma{}^d\Big[-\hat{e}_\lambda{}^a\,\bar{\omega}_\nu{}^b{}_d\,(\partial_\mu\hat{e}^\lambda{}_b) + \hat{e}_\lambda{}^a\,\bar{\omega}_\mu{}^b{}_d\,(\partial_\nu\hat{e}^\lambda{}_b)\Big]$$
$$= \delta^\rho_\lambda\,\hat{e}_\sigma{}^d\Big[-\bar{\omega}_\nu{}^b{}_d\,(\partial_\mu\hat{e}^\lambda{}_b) + \bar{\omega}_\mu{}^b{}_d\,(\partial_\nu\hat{e}^\lambda{}_b)\Big] = \hat{e}_\sigma{}^d\Big[-\bar{\omega}_\nu{}^b{}_d\,(\partial_\mu\hat{e}^\rho{}_b) + \bar{\omega}_\mu{}^b{}_d\,(\partial_\nu\hat{e}^\rho{}_b)\Big]$$

and so if we replace back in the expression for the curvature tensor we get

$$\begin{aligned}\overline{R}^{\rho}{}_{\sigma\mu\nu} = \hat{e}^{\rho}{}_{a}\,\hat{e}_{\sigma}{}^{b}&\Big[\partial_\mu\bar{\omega}_\nu{}^a{}_b - \partial_\nu\bar{\omega}_\mu{}^a{}_b + \bar{\omega}_\mu{}^a{}_c\,\bar{\omega}_\nu{}^c{}_b - \bar{\omega}_\nu{}^a{}_c\,\bar{\omega}_\mu{}^c{}_b\Big]\\
&+\Big[(\partial_\mu\hat{e}^\rho{}_a)(\partial_\nu\hat{e}_\sigma{}^a) - (\partial_\nu\hat{e}^\rho{}_a)(\partial_\mu\hat{e}_\sigma{}^a)\Big]\\
&+\hat{e}_\sigma{}^c\Big[\bar{\omega}_\nu{}^a{}_c\,(\partial_\mu\hat{e}^\rho{}_a) - \bar{\omega}_\mu{}^a{}_c\,(\partial_\nu\hat{e}^\rho{}_a)\Big]\\
&+\Big[-(\partial_\mu\hat{e}^\rho{}_b)(\partial_\nu\hat{e}_\sigma{}^b) + (\partial_\nu\hat{e}^\rho{}_b)(\partial_\mu\hat{e}_\sigma{}^b)\Big]\\
&+\hat{e}_\sigma{}^d\Big[-\bar{\omega}_\nu{}^b{}_d\,(\partial_\mu\hat{e}^\rho{}_b) + \bar{\omega}_\mu{}^b{}_d\,(\partial_\nu\hat{e}^\rho{}_b)\Big]\end{aligned}$$

We observe that all terms besides the first equal zero and thus

$$\overline{R}^{\rho}{}_{\sigma\mu\nu} = \hat{e}^\rho{}_a\hat{e}_\sigma{}^b(\partial_\mu\bar{\omega}_\nu{}^a{}_b - \partial_\nu\bar{\omega}_\mu{}^a{}_b + \bar{\omega}_\mu{}^a{}_c\bar{\omega}_\nu{}^c{}_b - \bar{\omega}_\nu{}^a{}_c\bar{\omega}_\mu{}^c{}_b) \tag{1.3.13}$$

Therefore, by direct comparison of Equations (1.3.11) and (1.3.13) we get that

$$\overline{R}^{a}{}_{b\mu\nu} = \partial_\mu\bar{\omega}_\nu{}^a{}_b - \partial_\nu\bar{\omega}_\mu{}^a{}_b + \bar{\omega}_\mu{}^a{}_c\bar{\omega}_\nu{}^c{}_b - \bar{\omega}_\nu{}^a{}_c\bar{\omega}_\mu{}^c{}_b$$

This is the desired result.

We recognize the first two terms of the curvature relation as the exterior derivative of $\bar{\omega}$ and the last two terms as the wedge product of $\bar{\omega}$ with itself. Thus, we have an equivalent expression of

$$\boxed{\overline{\boldsymbol{R}}^a{}_b = \boldsymbol{d}\bar{\boldsymbol{\omega}}^a{}_b + \bar{\boldsymbol{\omega}}^a{}_c \wedge \bar{\boldsymbol{\omega}}^c{}_b} \tag{1.3.14}$$

This is called *Cartam's second structure equation.*

1.3.4 The Bianchi Identities

By taking multiple exterior covariant derivatives of the tetrad one-form, we get the Bianchi identities, known from General Relativity, in our new formalism. The exterior covariant derivative of the torsion form is

$$\overline{\boldsymbol{D}}\boldsymbol{T}^a = \overline{\boldsymbol{D}}^2\hat{\boldsymbol{e}}^a = \overline{\boldsymbol{D}}\boldsymbol{d}\hat{\boldsymbol{e}}^a + \overline{\boldsymbol{D}}(\bar{\boldsymbol{\omega}}^a_b \wedge \hat{\boldsymbol{e}}^b)$$

We have that

$$\overline{\boldsymbol{D}}\boldsymbol{d}\hat{\boldsymbol{e}}^a = \cancelto{0}{\boldsymbol{d}^2\hat{\boldsymbol{e}}^a} + \bar{\boldsymbol{\omega}}^a{}_b \wedge \boldsymbol{d}\hat{\boldsymbol{e}}^b = \bar{\boldsymbol{\omega}}^a{}_b \wedge \boldsymbol{d}\hat{\boldsymbol{e}}^b$$

and

$$\overline{\boldsymbol{D}}(\bar{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{e}}^b) = \boldsymbol{d}(\bar{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{e}}^b) + \bar{\boldsymbol{\omega}}^a{}_b \wedge \bar{\boldsymbol{\omega}}^b{}_c \wedge \hat{\boldsymbol{e}}^c = \boldsymbol{d}\bar{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{e}}^b - \bar{\boldsymbol{\omega}}^a{}_b \wedge \boldsymbol{d}\hat{\boldsymbol{e}}^b + \bar{\boldsymbol{\omega}}^a{}_b \wedge \bar{\boldsymbol{\omega}}^b{}_c \wedge \hat{\boldsymbol{e}}^c$$

and so

$$\begin{aligned}\overline{\boldsymbol{D}}\boldsymbol{T}^a &= \cancel{\bar{\boldsymbol{\omega}}^a{}_b \wedge \boldsymbol{d}\hat{\boldsymbol{e}}^b} + \boldsymbol{d}\bar{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{e}}^b - \cancel{\bar{\boldsymbol{\omega}}^a{}_b \wedge \boldsymbol{d}\hat{\boldsymbol{e}}^b} + \bar{\boldsymbol{\omega}}^a{}_b \wedge \bar{\boldsymbol{\omega}}^b{}_c \wedge \hat{\boldsymbol{e}}^c \\ &= \boldsymbol{d}\bar{\boldsymbol{\omega}}^a{}_b \wedge \hat{\boldsymbol{e}}^b + \bar{\boldsymbol{\omega}}^a{}_b \wedge \bar{\boldsymbol{\omega}}^b{}_c \wedge \hat{\boldsymbol{e}}^c = (\boldsymbol{d}\bar{\boldsymbol{\omega}}^a{}_b + \bar{\boldsymbol{\omega}}^a{}_b \wedge \bar{\boldsymbol{\omega}}^b{}_c) \wedge \hat{\boldsymbol{e}}^c\end{aligned}$$

By relabeling the indices $b \leftrightarrow c$ we arrive at the relation

$$\boxed{\overline{\boldsymbol{D}}\boldsymbol{T}^a = \overline{\boldsymbol{D}}^2\hat{\boldsymbol{e}}^a = \overline{\boldsymbol{R}}^a{}_b \wedge \hat{\boldsymbol{e}}^b} \tag{1.3.15}$$

This is in fact *Bianchi's first identity* of General Relativity [2]:

$$\boxed{\overline{R}^\rho{}_{[\sigma\mu\nu]} = 0} \tag{1.3.16}$$

Note that the proof of the first Bianchi identity presented above made no assumptions about $\hat{e}^a$ besides it being a vector-valued differential form. Therefore, the formula is valid for any vector valued p-form $\boldsymbol{\lambda}^a$:

$$\boxed{\overline{\boldsymbol{D}}^2\boldsymbol{\lambda}^a = \overline{\boldsymbol{R}}^a{}_b \wedge \boldsymbol{\lambda}^b} \tag{1.3.17}$$

Now, consider taking an additional exterior covariant derivative, i.e. calculating $\overline{\boldsymbol{D}}^3\boldsymbol{\lambda}^a$. We can calculate this in two different ways. First,

$$\overline{\boldsymbol{D}}^3\boldsymbol{\lambda}^a = \overline{\boldsymbol{D}}(\overline{\boldsymbol{D}}^2\boldsymbol{\lambda}^a) = \overline{\boldsymbol{D}}(\overline{\boldsymbol{R}}^a{}_b \wedge \boldsymbol{\lambda}^b) = (\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b \wedge \boldsymbol{\lambda}^b) + (\overline{\boldsymbol{R}}^a{}_b \wedge \overline{\boldsymbol{D}}\boldsymbol{\lambda}^b) \tag{1.3.18}$$

However, we also have

$$\overline{\boldsymbol{D}}^3\boldsymbol{\lambda}^a = \overline{\boldsymbol{D}}^2(\overline{\boldsymbol{D}}\boldsymbol{\lambda}^a) = \overline{\boldsymbol{R}}^a{}_b \wedge \overline{\boldsymbol{D}}\boldsymbol{\lambda}^b \tag{1.3.19}$$

where we used $\overline{\boldsymbol{D}}\boldsymbol{\lambda}^a$ instead of $\boldsymbol{\lambda}^a$ on Equation (1.3.17). By comparing Equations (1.3.18) and (1.3.19) we conclude that

$$\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b \wedge \boldsymbol{\lambda}^b = 0$$

Since $\boldsymbol{\lambda}^a$ can be an arbitrary vector value p-form, we choose it to be a vector valued 0-form, i.e. a vector field $\boldsymbol{\lambda}^a = X^a$. By doing this, the wedge product simplifies to an ordinary product:

$$(\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b)X^b = 0$$

Since this equality must hold for an arbitrary such vector field, we conclude that

$$\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b = 0$$

must hold. By using the exterior covariant derivative formula, we get that this relation is:

$$\boxed{\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b = \boldsymbol{d}\overline{\boldsymbol{R}}^a{}_b + \bar{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{R}}^c{}_b - \overline{\boldsymbol{R}}^a{}_c \wedge \bar{\boldsymbol{\omega}}^c{}_b = 0} \tag{1.3.20}$$

which is in fact *Bianchi's second (differential) identity* of General Relativity [2]:

$$\boxed{\overline{\nabla}_{[\lambda|}\overline{R}^{\rho}{}_{\sigma|\mu\nu]} = 0} \tag{1.3.21}$$

We can calculate $\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b$:

$$\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b = \overline{\boldsymbol{D}}(\boldsymbol{d}\overline{\boldsymbol{\omega}}^a{}_b + \overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) = \overline{\boldsymbol{D}}(\boldsymbol{d}\overline{\boldsymbol{\omega}}^a{}_b) + \overline{\boldsymbol{D}}(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b)$$

We have that

$$\overline{\boldsymbol{D}}(\boldsymbol{d}\overline{\boldsymbol{\omega}}^a{}_b) = \cancelto{0}{(\boldsymbol{d}^2\overline{\boldsymbol{\omega}}^a{}_b)} + (\overline{\boldsymbol{\omega}}^a{}_c \wedge \boldsymbol{d}\overline{\boldsymbol{\omega}}^c{}_b) - (\boldsymbol{d}\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) = (\overline{\boldsymbol{\omega}}^a{}_c \wedge \boldsymbol{d}\overline{\boldsymbol{\omega}}^c{}_b) - (\boldsymbol{d}\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) = \boldsymbol{d}(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b)$$

and

$$\begin{aligned}\overline{\boldsymbol{D}}(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) &= \boldsymbol{d}(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) + (\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_d \wedge \overline{\boldsymbol{\omega}}^d{}_b) + (\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_d \wedge \overline{\boldsymbol{\omega}}^d{}_b)\\ &= \boldsymbol{d}(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) + 2(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_d \wedge \overline{\boldsymbol{\omega}}^d{}_b)\end{aligned}$$

and thus we get

$$\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b = 2\boldsymbol{d}(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) + 2(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_d \wedge \overline{\boldsymbol{\omega}}^d{}_b)$$

Bianchi's second identity can therefore be written as

$$\boxed{\overline{\boldsymbol{D}}\overline{\boldsymbol{R}}^a{}_b = \boldsymbol{d}(\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_b) + (\overline{\boldsymbol{\omega}}^a{}_c \wedge \overline{\boldsymbol{\omega}}^c{}_d \wedge \overline{\boldsymbol{\omega}}^d{}_b) = 0} \tag{1.3.22}$$

1.4 The Contorsion Tensor

Assuming a contorted connection $\bar{\Gamma}^\lambda_{\mu\nu}$ it is possible to show that [5]

$$\boxed{\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2}(T^\lambda{}_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu)} \tag{1.4.1}$$

where the $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of conventional General Relativity which do not include torsion:

$$\boxed{\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}} \tag{1.4.2}$$

and $T^\lambda{}_{\mu\nu}$ are the components of the torsion tensor. Therefore, it is is possible to split the affine connection in two parts, a torsion free part $\Gamma^\lambda_{\mu\nu}$ and a contorted part $K^\lambda{}_{\mu\nu}$:

$$\boxed{\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + K^\lambda{}_{\mu\nu}} \tag{1.4.3}$$

where the contorted part $K^\lambda{}_{\mu\nu}$ is a tensor (as it is defined in terms of the torsion tensor), which we call the *contorsion* tensor[8]

$$\boxed{K^\lambda{}_{\mu\nu} = \frac{1}{2}(T^\lambda{}_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu)} \tag{1.4.4}$$

[8]It turns out that there are *four* different expressions for the contorsion tensor, which are based into two conventional choices:

i The choice of the definition of the covariant derivative: Either $\nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda$ (Choice (1a)) or $\nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \Gamma^\lambda_{\nu\mu}\omega_\lambda$ (Choice (1b))

ii The choice of the definition of the torsion tensor: Either $T^\lambda{}_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\nu\mu}$ (Choice (2a)) or $T^\lambda{}_{\mu\nu} = \bar{\Gamma}^\lambda_{\nu\mu} - \bar{\Gamma}^\lambda_{\mu\nu}$ (Choice (2b))

Then, according to our combination of conventional choices, we get different expressions for the contorsion tensor components:

i (1a) & (2a): $K_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu} + T_{\nu\lambda\mu})$ & anti-symmetric in 1st and 3rd indices: $K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}$ (we're using this one)

ii (1a) & (2b): $K_{\lambda\mu\nu} = -\frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu} + T_{\nu\lambda\mu})$ & anti-symmetric in 1st and 3rd indices: $K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}$

iii (1b) & (2a): $K_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} - T_{\mu\lambda\nu} - T_{\nu\lambda\mu})$ & anti-symmetric in 1st and 2nd indices: $K_{\lambda\mu\nu} = -K_{\mu\lambda\nu}$

iv (1b) & (2b): $K_{\lambda\mu\nu} = \frac{1}{2}(T_{\mu\lambda\nu} + T_{\nu\lambda\mu} - T_{\lambda\mu\nu})$ & anti-symmetric in 2nd and 3rd indices: $K_{\lambda\mu\nu} = -K_{\lambda\nu\mu}$

The torsion tensor is antisymmetric in its lower indices:

$$\boxed{T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu}} \tag{1.4.5}$$

and based on that we can easily show that the contorsion tensor is anti-symmetric in its first and third indices:

$$\boxed{K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}} \tag{1.4.6}$$

Analogously, the contorted spin connection $\bar{\boldsymbol{\omega}}^{a}{}_{b}$ can be split it into a torsionless part $\boldsymbol{\omega}^{a}{}_{b}$ and a part $\boldsymbol{K}^{a}{}_{b}$ containing all the information about the torsion:

$$\boxed{\bar{\boldsymbol{\omega}}^{a}{}_{b} = \boldsymbol{\omega}^{a}{}_{b} + \boldsymbol{K}^{a}{}_{b} \Rightarrow \bar{\omega}_{\mu}{}^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b} + K^{a}{}_{\mu b}} \tag{1.4.7}$$

where $\boldsymbol{K}^{a}{}_{b}$ is again the *contorsion*, this time expressed using Latin indices. The torsionless part is defined to be such that

$$\boxed{\boldsymbol{D}\hat{\boldsymbol{e}}^{a} = \boldsymbol{d}\hat{\boldsymbol{e}}^{a} + \boldsymbol{\omega}^{a}{}_{b} \wedge \hat{\boldsymbol{e}}^{b} = 0} \tag{1.4.8}$$

i.e. the torsion contribution from $\boldsymbol{\omega}^{a}{}_{b}$ is zero. It is easy then to show that

$$\boxed{\overline{\boldsymbol{D}} = \boldsymbol{D} + \boldsymbol{K}^{a}{}_{b}\wedge} \tag{1.4.9}$$

Note that we have defined a bar-less exterior covariant derivative $\boldsymbol{D}$, which is defined like $\overline{\boldsymbol{D}}$, but with $\boldsymbol{\omega}^{a}{}_{b}$ instead of $\bar{\boldsymbol{\omega}}^{a}{}_{b}$. This, in turn, gives us that $\boldsymbol{\omega}_{ab}$ is anti-symmetric:

$$\boxed{\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}} \tag{1.4.10}$$

by the metric compatibility condition, as before. The torsion one-form is defined by the contraction[9]

$$\boxed{\boldsymbol{K}^{a}{}_{c} = K^{a}{}_{bc}\,\hat{e}^{b} = K^{a}{}_{\mu c}\,dx^{\mu}} \tag{1.4.11}$$

where the components $K^{a}{}_{bc}$ are given by:

$$\boxed{K_{abc} = \frac{1}{2}(T_{abc} + T_{bac} + T_{cab})} \tag{1.4.12}$$

Since the first and third indices are antisymmetric,

$$\boxed{K_{abc} = -K_{cba}} \tag{1.4.13}$$

it follows that the components of the contorsion one-form are anti-symmetric:

$$\boxed{\boldsymbol{K}_{ab} = -\boldsymbol{K}_{ba}} \tag{1.4.14}$$

Furthermore, by the defining Equation (1.3.8) for torsion and Equation (1.4.8) it is trivial to show that

$$\boxed{\boldsymbol{T}^{a} = \boldsymbol{K}^{a}{}_{b} \wedge \hat{\boldsymbol{e}}^{b}} \tag{1.4.15}$$

thus confirming that the information about the torsion is contained entirely in the contortion. Furthermore, we have that

$$2(K_{abc} - K_{acb}) = T_{abc} + T_{bac} + T_{cab} - T_{acb} - T_{cab} - T_{bac} = 2T_{abc}$$

and thus we get the property

$$\boxed{T^{a}{}_{bc} = 2K^{a}{}_{[bc]} = (K^{a}{}_{bc} - K^{a}{}_{cb})} \tag{1.4.16}$$

Having all of the above in mind, we can define a torsionless curvature 2-form as [4]

$$\boxed{\boldsymbol{R}^{a}{}_{b} = \boldsymbol{d}\boldsymbol{\omega}^{a}{}_{b} + \boldsymbol{\omega}^{a}{}_{c} \wedge \boldsymbol{\omega}^{c}{}_{b}} \tag{1.4.17}$$

[9] The index contracted is the one that doesn't possess any symmetry. Similarly, in the other conventions we described, the contorsion one-form is defined by contracting the lower index that doesn't have any symmetries.

which (obviously) satisfies the Bianchi identity

$$\boxed{\boldsymbol{D}\boldsymbol{R}^a{}_b = 0} \tag{1.4.18}$$

By expanding Equation (1.3.14) we get that

$$\begin{aligned}\overline{\boldsymbol{R}}^a{}_b &= \boldsymbol{d}(\boldsymbol{\omega}^a{}_b + \boldsymbol{K}^a{}_b) + (\boldsymbol{\omega}^a{}_c + \boldsymbol{K}^a{}_c) \wedge (\boldsymbol{\omega}^c{}_b + \boldsymbol{K}^c{}_b) \\ &= \underbrace{\boldsymbol{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b}_{=\boldsymbol{R}^a{}_b} + \underbrace{\boldsymbol{d}\boldsymbol{K}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{K}^c{}_b + \boldsymbol{K}^a{}_c \wedge \boldsymbol{\omega}^c{}_b}_{\boldsymbol{D}\boldsymbol{K}^a{}_b} + \boldsymbol{K}^a{}_c \wedge \boldsymbol{K}^c{}_b\end{aligned}$$

and thus the contorted curvature 2-form can be expressed in terms of the torsionless curvature 2-form and the contorsion one-form:

$$\boxed{\overline{\boldsymbol{R}}^a{}_b = \boldsymbol{R}^a{}_b + \boldsymbol{D}\boldsymbol{K}^a{}_b + \boldsymbol{K}^a{}_c \wedge \boldsymbol{K}^c{}_b} \tag{1.4.19}$$

1.5 Lagrangian Formulation of Einstein-Cartan Theory

In General Relativity, the Einstein Equations in the absence of matter are derived from the so-called Einstein-Hilbert action:

$$\boxed{S_{E-H} = \frac{1}{16\pi G}\int R\sqrt{-g}\, d^4x} \tag{1.5.1}$$

where R is the (torsionless) Ricci scalar and G is Newton's constant of gravitation. In our contorted theory, it makes sense to simply replace the torsionless Ricci scalar with the contorted Ricci scalar $\bar{R}$. Therefore, the starting point for the action in a contorted gravity theory is

$$\boxed{S_G = \frac{1}{16\pi G}\int \bar{R}\sqrt{-g}\, d^4x} \tag{1.5.2}$$

We will show (by following [6]) that this term can be rewritten in differential form notation as

$$\boxed{S_G = \frac{1}{16\pi G}\int \overline{\boldsymbol{R}}_{ab} \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b)} \tag{1.5.3}$$

Proof

First of all, we have that the curvature form can be written as the contraction of the Riemann curvature tensor:

$$\overline{\boldsymbol{R}}_{ab} = \frac{1}{2}\bar{R}_{ab\mu\nu}dx^\mu \wedge dx^\nu$$

Furthermore, the asterisk denotes the Hodge dual, which for a p-form A in D-dimensional space is defined to be

$$\boxed{*A = \frac{1}{(D-p)!}A^{\mu_1\ldots\mu_p}\,\eta_{\mu_1\ldots\mu_p\ldots\mu_D}\,dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_D}} \tag{1.5.4}$$

where η is used to denote the Levi-Civita *tensor*

$$\boxed{\eta_{\mu_1\ldots\mu_D} = \sqrt{-g}\,\epsilon_{\mu_1\ldots\mu_D}} \tag{1.5.5}$$

Therefore, we can see that

$$*(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b) = \frac{1}{2}\hat{e}^{a\rho}\hat{e}^{b\sigma}\eta_{\rho\sigma\kappa\lambda}\, dx^\kappa \wedge dx^\lambda$$

and thus we get

$$\begin{aligned}S_G &= \frac{1}{16\pi G}\int \left[\frac{1}{2}\bar{R}_{ab\mu\nu}dx^\mu \wedge dx^\nu\right] \wedge \left[\frac{1}{2}\hat{e}^{a\rho}\hat{e}^{b\sigma}\eta_{\rho\sigma\kappa\lambda}\, dx^\kappa \wedge dx^\lambda\right] \\ &= \frac{1}{16\pi G}\int \frac{1}{4}\bar{R}_{ab\mu\nu}\hat{e}^{a\rho}\hat{e}^{b\sigma}\eta_{\rho\sigma\kappa\lambda}dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda\end{aligned}$$

Then, by using the relation

$$\boxed{dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} = \sqrt{-g}\,\eta^{\mu_1\cdots\mu_D}\,d^D x} \tag{1.5.6}$$

we get that

$$S_G = \frac{1}{16\pi G}\int \frac{1}{4}\overline{R}^{ab}{}_{\mu\nu}\hat{e}_a{}^{\rho}\hat{e}_b{}^{\sigma}\eta_{\rho\sigma\kappa\lambda}\eta^{\mu\nu\kappa\lambda}\sqrt{-g}\,d^4x$$

For the contraction of the Levi-Civita tensors, we have the identity

$$\boxed{\eta_{\nu_1\ldots\nu_D}\eta^{\mu_1\cdots\mu_D} = (D-p)!\,\delta^{\mu_1\cdots\mu_D}_{\nu_1\ldots\nu_D}} \tag{1.5.7}$$

where we have adopted the convention $\epsilon_{0123} = +1$ and $\delta^{\mu_1\cdots\mu_D}_{\nu_1\ldots\nu_D}$ is the *generalized Kronecker delta* symbol defined as the determinant:

$$\boxed{\delta^{\mu_1\cdots\mu_D}_{\nu_1\ldots\nu_D} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_D} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_D}_{\nu_1} & \cdots & \delta^{\mu_D}_{\nu_D} \end{vmatrix}} \tag{1.5.8}$$

Having this in mind, we get that

$$\eta_{\rho\sigma\kappa\lambda}\eta^{\mu\nu\kappa\lambda} = 2(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho})$$

and thus the action can be written as

$$\begin{aligned} S_G = \frac{1}{16\pi G}\int \frac{1}{2}\overline{R}^{ab}{}_{\mu\nu}\hat{e}_a{}^{\rho}\hat{e}_b{}^{\sigma}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho})\sqrt{-g}\,d^4x &= \frac{1}{16\pi G}\int \frac{1}{2}\overline{R}^{ab}{}_{\mu\nu}(\hat{e}_a{}^{\mu}\hat{e}_b{}^{\nu} - \hat{e}_a{}^{b\nu}\hat{e}_b{}^{\mu})\sqrt{-g}\,d^4x \\ = \frac{1}{16\pi G}\int \frac{1}{2}(\overline{R}^{ab}{}_{ab} - \overline{R}^{ab}{}_{ba})\sqrt{-g}\,d^4x &= \frac{1}{16\pi G}\int \overline{R}^{ab}{}_{ab}\sqrt{-g}\,d^4x \end{aligned}$$

We can recognize $\overline{R}^{ab}{}_{ab}$ as the Ricci curvature scalar:

$$\overline{R} = \overline{R}_{ab}\eta^{ab} = \overline{R}^{c}{}_{acb}\eta^{ab} = \overline{R}^{cb}{}_{cb}$$

and thus we have shown that

$$S_G = \frac{1}{16\pi G}\int \overline{R}\sqrt{-g}\,d^4x \tag{1.5.9}$$

as desired.

By expanding the curvature 2-form in Equation (1.5.3), we get three terms:

$$S_G = \frac{1}{16\pi G}\int (\boldsymbol{R}^a{}_b + \boldsymbol{D}\boldsymbol{K}^a{}_b + \boldsymbol{K}^a{}_c \wedge \boldsymbol{K}^c{}_b) \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b) \tag{1.5.10}$$

The first term is the familiar Einstein-Hilbert action from General Relativity:

$$\boxed{S_{G_1} = S_{E-H} = \frac{1}{16\pi G}\int \boldsymbol{R}^a{}_b \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b)} \tag{1.5.11}$$

The second term turns out to be a surface term

$$S_{G_2} = \frac{1}{16\pi G}\int \boldsymbol{D}\boldsymbol{K}^a{}_b \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b) = \int \boldsymbol{d}(\boldsymbol{K}^a{}_b \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b)) = \int_{\partial M} (\boldsymbol{K}^a{}_b \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b)) \tag{1.5.12}$$

and thus doesn't contribute to the equations of motion and can be ignored. The last term contains all torsion-related information

$$\boxed{S_{G_3} = \frac{1}{16\pi G}\int (\boldsymbol{K}^a{}_c \wedge \boldsymbol{K}^c{}_b) \wedge *(\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b)} \tag{1.5.13}$$

If we return to our previous notation, this term can be written as

$$\boxed{S_{G_3} = \frac{1}{16\pi G}\int \Delta\sqrt{-g}\,d^4x} \tag{1.5.14}$$

where Δ is a factor that comes from contractions of the contorsion tensor[10]. Therefore, the full gravitational action is

$$\boxed{S_G = \frac{1}{16\pi G}\int (R+\Delta)\sqrt{-g}\,d^4x} \tag{1.5.15}$$

In the next chapter, we'll see that this Δ factor can be further split up and only part of it is important for our theory.

[10]Using the convention *iii* as presented in footnote 7, this term is

$$\Delta = K^{\lambda}{}_{\mu\nu}K^{\nu\mu}{}_{\lambda} - K^{\mu\nu}{}_{\nu}K_{\mu\lambda}{}^{\lambda} = T^{\nu}{}_{\nu\mu}T^{\lambda}{}_{\lambda}{}^{\mu} - \frac{1}{2}T^{\mu}{}_{\nu\lambda}T^{\nu\lambda}{}_{\mu} + \frac{1}{4}T_{\mu\nu\lambda}T^{\mu\nu\lambda}$$

Chapter 2

Quantum Electrodynamics in Contorted Curved Spacetime

In this chapter, we will study Quantum Electrodynamics in curved spacetime with non-vanishing torsion. We already have a description (the action) for an Einstein-Cartan spacetime with torsion. In this chapter, we'll add an electromagnetic field alongside a fermion field coupled to it while it's in a contorted and curved background spacetime. We'll then proceed to find the equations of motion and see that the presence of torsion modifies the Dirac equation. Finally, by making some additional assumptions we'll see how torsion can give rise to an axion with which the fermion interacts.

2.1 The Contorted QED Action

In the previous chapter, we presented the gravitational action that describes a contorted curved spacetime. To describe Quantum Electrodynamics (QED) in such a background spacetime, we need to add two more action terms: A term that couples the fermions to the electromagnetic field in a contorted and curved spacetime, and a term describing the electromagnetic field itself. To do this, however, we must define a suitable covariant derivative that acts on spinors.

2.1.1 The Gravitational Covariant Derivative of Spinors

The proper covariant derivative for spinors is called the gravitational covariant derivative and is defined as [4]:

$$\boxed{\overline{\boldsymbol{D}}\psi = \boldsymbol{d}\psi - \frac{i}{4}\overline{\boldsymbol{\omega}}_{ab}\sigma^{ab}\psi \Leftrightarrow \overline{D}_\mu\psi = \partial_\mu\psi - \frac{i}{4}\overline{\omega}_{\mu ab}\sigma^{ab}\psi} \tag{2.1.1}$$

$$\boxed{\overline{\boldsymbol{D}}\bar{\psi} = \boldsymbol{d}\bar{\psi} + \frac{i}{4}\overline{\boldsymbol{\omega}}_{ab}\bar{\psi}\sigma^{ab} \Leftrightarrow \overline{D}_\mu\bar{\psi} = \partial_\mu\bar{\psi} + \frac{i}{4}\overline{\omega}_{\mu ab}\bar{\psi}\sigma^{ab}} \tag{2.1.2}$$

where σ^{ab} is defined in terms of the γ matrices in flat space[11]:

$$\boxed{\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]} \tag{2.1.3}$$

2.1.2 Action for Spinors in Contorted Curved Spacetime

To express a spinor field coupled to the electromagnetic field in a contorted curved spacetime background, we just need to adjust the known action of free fermion fields in Minkowski spacetime, given as,

$$S_{QED}^{Flat} = \frac{i}{2}\int (\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi)\, d^4x = \int \frac{1}{2}(i\bar{\psi}\gamma^\mu\partial_\mu\psi + h.c.)\, d^4x \tag{2.1.4}$$

[11]These are the familiar gamma matrices used in Quantum Field Theories. Gamma matrices with Greek indices, which will be encountered later in the text, are dependent on the spacetime coordinates, unlike the Latin index ones, which are constant.

to account for the coupling to the electromagnetic field and the gravitational background. This is done simply by replacing the partial derivative with a covariant derivative:

$$\boxed{\overline{\mathcal{D}}_\mu = \overline{D}_\mu - ieA_\mu} \tag{2.1.5}$$

where $\overline{\boldsymbol{D}}_\mu$ is the gravitational covariant derivative, A_μ is the photon field and $e^2 = 4\pi\alpha$, where α is the fine structure constant (the coupling constant of QED). Finally, we need to replace d^4x with $\sqrt{-g}\,d^4x$ in order to have diffeomorphism invariance. Therefore, our QED in contorted spacetime reads as

$$\boxed{S_{QED}^{Curved+Torsion} = \frac{1}{2}\int (i\bar{\psi}\gamma^\mu(\overline{\mathcal{D}}_\mu\psi) + h.c.)\sqrt{-g}\,d^4x} \tag{2.1.6}$$

When we expand this, we get

$$\begin{aligned} S_{QED}^{Curved+Torsion} &= \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu[(\overline{D}_\mu - ieA_\mu)\psi] + h.c.\right]\sqrt{-g}\,d^4x \\ &= \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu\overline{D}_\mu\psi + e\bar{\psi}\gamma^\mu\psi A_\mu + h.c.\right]\sqrt{-g}\,d^4x \\ &= \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu\overline{D}_\mu\psi - i(\overline{D}_\mu\bar{\psi})\gamma^\mu\psi\right]\sqrt{-g}\,d^4x + e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x \end{aligned}$$

We can now split the gravitational covariant derivative into the parts with and without torsion. More specifically, for $\bar{\omega}_{\mu ab} = \omega_{\mu ab} + K_{a\mu b}$ equations (2.1.1) and (2.1.2) become:

$$\boxed{\overline{D}_\mu\psi = D_\mu\psi - \frac{i}{4}K_{a\mu b}\sigma^{ab}\psi} \tag{2.1.7}$$

$$\boxed{\overline{D}_\mu\bar{\psi} = D_\mu\bar{\psi} + \frac{i}{4}K_{a\mu b}\psi\sigma^{ab}} \tag{2.1.8}$$

The resulting action after the split therefore can easily be shown to have the following three terms:

$$\begin{aligned} S_{QED}^{Curved+Torsion} =& \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu D_\mu\psi - i(D_\mu\bar{\psi})\gamma^\mu\psi\right]\sqrt{-g}\,d^4x + e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x \\ &+ \frac{1}{8}\int \bar{\psi}\{\gamma^c, \sigma^{ab}\}\psi K_{acb}\sqrt{-g}\,d^4x \end{aligned}$$

By using the identity [4]

$$\boxed{\{\gamma^c, \sigma^{ab}\} = 2\epsilon^{abc}{}_d\gamma^d\gamma^5} \tag{2.1.9}$$

we find that

$$\begin{aligned} S_{QED}^{Curved+Torsion} &= \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu D_\mu\psi - i(D_\mu\bar{\psi})\gamma^\mu\psi\right]\sqrt{-g}\,d^4x + e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x \\ &+ \frac{1}{4}\int \epsilon^{abc}{}_d\bar{\psi}\gamma^d\gamma^5\psi K_{acb}\sqrt{-g}\,d^4x \end{aligned}$$

The first two terms are the action terms for QED in curved spacetime without torsion:

$$\boxed{S_{QED}^{Curved} = \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu D_\mu\psi - i(D_\mu\bar{\psi})\gamma^\mu\psi\right]\sqrt{-g}\,d^4x + e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x} \tag{2.1.10}$$

We now focus on the third term that contains the torsion, $S_{QED}^{Torsion} = \frac{1}{4}\int \epsilon^{abc}{}_d\bar{\psi}\gamma^d\gamma^5\psi K_{acb}\sqrt{-g}\,d^4x$. The Levi-Civita symbol is contracted with the contorsion tensor and thus only the completely anti-symmetric part of the contorsion has a non-zero contribution, i.e. it is $K_{[acb]}$ that couples to the spinor. We can show that

$$\boxed{T_{[acb]} = -2K_{[acb]}} \tag{2.1.11}$$

Thus, by defining the torsion 3-form

$$\boxed{\boldsymbol{T} = \frac{1}{3!}T_{abc}\,\hat{\boldsymbol{e}}^a \wedge \hat{\boldsymbol{e}}^b \wedge \hat{\boldsymbol{e}}^c} \tag{2.1.12}$$

we observe that the quantity

$$\boxed{S_d = \frac{1}{3!}\epsilon^{abc}{}_d T_{abc}} \tag{2.1.13}$$

is a one-form that's the Hodge dual of the torsion 3-form

$$\boxed{\boldsymbol{S} = *\boldsymbol{T}} \tag{2.1.14}$$

We have that

$$\frac{1}{3!}\epsilon^{acb}{}_d T_{acb} = \frac{1}{3!}\epsilon^{acb}{}_d T_{[acb]} = -\frac{2}{3!}\epsilon^{acb}{}_d K_{[acb]} \Rightarrow \epsilon^{acb}{}_d K_{[acb]} = -3S_d$$

Having in mind all of the above, the last term in the above action can be written as

$$\frac{1}{4}\int \epsilon^{abc}{}_d \bar{\psi}\gamma^d\gamma^5\psi K_{acb}\sqrt{-g}\, d^4x = \frac{1}{4}\int \epsilon^{abc}{}_d \bar{\psi}\gamma^d\gamma^5\psi K_{[acb]}\sqrt{-g}\, d^4x = -\frac{3}{4}\int S_\mu \bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g} d^4x$$

Furthermore, we recognize

$$\boxed{(j^5)^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi} \tag{2.1.15}$$

as the fermionic axial current. Therefore, the action term that contains the coupling of the fermions to the torsion can be written as

$$-\frac{3}{4}\int S_\mu \bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\, d^4x = -\frac{3}{4}\int S_\mu (j^5)^\mu\sqrt{-g}\, d^4x = -\frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5}$$

<u>Proof</u>

We want to show that the term involving the torsion can be written in differential form notation as indicated above. To do that, we'll first use the definition for the Hodge dual (Equation (1.5.4)):

$$\begin{aligned}
-\frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5} &= -\frac{3}{4}\int S_\mu dx^\mu \wedge \frac{1}{3!}(j^5)^\rho \eta_{\rho\nu\kappa\lambda} dx^\nu \wedge dx^\kappa dx^\lambda \\
&= -\frac{3}{4}\int \frac{1}{3!} S_\mu (j^5)^\rho \underbrace{\eta_{\rho\nu\kappa\lambda} dx^\mu \wedge dx^\nu \wedge dx^\kappa dx^\lambda}_{=\sqrt{-g}\eta^{\mu\nu\kappa\lambda} d^4x} \\
&= -\frac{3}{4}\int \frac{1}{3!} S_\mu (j^5)^\rho \underbrace{\eta_{\rho\nu\kappa\lambda}\eta^{\mu\nu\kappa\lambda}}_{=6\delta^\mu_\rho}\sqrt{-g} d^4x \\
&= -\frac{3}{4}\int S_\mu (j^5)^\mu \sqrt{-g}\, d^4x
\end{aligned}$$

and we have thus proven what we wanted.

and so the action becomes

$$\boxed{S_{QED}^{Curved+Torsion} = S_{QED}^{Curved} + S_{QED}^{Torsion} = S_{QED}^{Curved} - \frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5}} \tag{2.1.16}$$

2.1.3 The Electromagnetic Field Action

We must now consider the action for the electromagnetic field. Here, we'll choose the same action as the one used for regular QED. This amounts to hypothesizing that the photon does not couple to the torsion. We are, therefore, dealing with a minimal description of the electromagnetic field. The action we consider is

$$\boxed{S_{EM} = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\, d^4x} \tag{2.1.17}$$

which can be rewritten in terms of differential forms as

$$\boxed{S_{EM} = -\frac{1}{2}\int \boldsymbol{F}\wedge *\boldsymbol{F}} \tag{2.1.18}$$

2.1.4 The Gravitational Action

In the previous chapter we saw that the torsion part of the gravitational action is expressed through the scalar Δ. Similarly, we saw that the torsion part of the fermionic action can be expressed through the tensor S_d, which is a contraction of the torsion tensor. The contorsion tensor is a four-dimensional rank-3 tensor with 48 components. Its antisymmetry in the first and third components dictates that the number of independent components of the contorsion tensor is 24. That is because, for K_{abc} there are 16 total combinations of a and c and only 6 of these are independent because of the antisymmetry in those indices. For each of those, we have 4 components (different b values) for a total of $4 \times 6 = 24$ components[12]. We can check (through Young Tableaux[13]) that the decomposition of the contorsion tensor in irreducible parts is [7]

$$\mathbf{6} \otimes \mathbf{4} = \mathbf{4_A} \oplus \mathbf{20} = \mathbf{4_A} \oplus \mathbf{4_B} \oplus \mathbf{16} \tag{2.1.19}$$

where we have labeled the two different 4 parts of the decomposition to discern them. The vector S_d turns out to be the $\mathbf{4_A}$ component and thus we can say that

$$\boxed{K_{abc} = \frac{1}{2}\epsilon_{abcd}S^d + \hat{K}_{abc}} \tag{2.1.20}$$

where $\frac{1}{2}\epsilon_{abcd}S^d$ is the 4 part and $\hat{K}_{abc}$ is the 20 part, which is not given as it will not be relevant in the future. This, in turn, results in Δ being able to be expressed as

$$\boxed{\Delta = \frac{3}{2}S_d S^d + \hat{\Delta}} \tag{2.1.21}$$

where $\hat{\Delta}$ is given by the same formula as Δ only with $\hat{K}$ replacing K, i.e. from the contractions of the 20 part of the contorsion tensor. With this in mind, the gravitational action breaks up into three components:

$$\boxed{S_G = \frac{1}{16\pi G}\int (R + \hat{\Delta})\sqrt{-g}\, d^4x + \frac{3}{32\pi G}\int S_d S^d \sqrt{-g}\, d^4x} \tag{2.1.22}$$

The last term can be rewritten as

$$\int S_d S^d \sqrt{-g}\, d^4x = \int \boldsymbol{S} \wedge *\boldsymbol{S}$$

and thus the gravitational action is

$$\boxed{S_G = \frac{1}{16\pi G}\int (R + \hat{\Delta})\sqrt{-g}\, d^4x + \frac{3}{32\pi G}\int \boldsymbol{S} \wedge *\boldsymbol{S}} \tag{2.1.23}$$

2.2 The Equations of Motion

We can now write the full action that includes the background gravity, the electromagnetic field and the fermion interacting with both:

$$\boxed{\begin{aligned} S = &\frac{1}{16\pi G}\int (R + \hat{\Delta})\sqrt{-g}\, d^4x + \frac{3}{32\pi G}\int \boldsymbol{S} \wedge *\boldsymbol{S} - \frac{1}{2}\int \boldsymbol{F} \wedge *\boldsymbol{F} \\ &+ \frac{1}{2}\int \left[i\bar{\psi}\gamma^\mu \boldsymbol{D}_\mu \psi - i(\boldsymbol{D}_\mu \bar{\psi})\gamma^\mu \psi\right]\sqrt{-g}\, d^4x + e\int (\bar{\psi}\gamma^\mu \psi A_\mu)\sqrt{-g}\, d^4x - \frac{3}{4}\int \boldsymbol{S} \wedge *\boldsymbol{j^5} \end{aligned}} \tag{2.2.1}$$

By varying with respect to $A_\mu, S_\mu, \psi, \bar{\psi}$ and the metric we get the Maxwell equations, an equation of motion for torsion, a modified Dirac equation and modified Einstein equations.

[12]Another way to consider this is to view K_{abc} as an anti-symmetric 2-tensor for each value of b. Each anti-symmetric 2-tensor has 6 independent components, and thus four of them (one for each value of b) have $4 \times 6 = 24$ independent components.

[13]Since we're decomposing in $SO(1,3)$ there are additional rules we must take into account [7].

2.2.1 The Maxwell Equations

Let us initially consider varying the action with respect to the electromagnetic field A_μ. Most terms do not contain such terms and thus their variation is zero. The only terms that contribute are

$$\begin{aligned} S_{Maxwell} &= e\int (\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x - \frac{1}{2}\int \boldsymbol{F}\wedge *\boldsymbol{F} \\ &= e\int (\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x - \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\,d^4x \end{aligned} \tag{2.2.2}$$

Variation of this action gives the usual Maxwell equations, which in differential form notation are written as [4]

$$\boxed{\boldsymbol{dF} = 0} \tag{2.2.3}$$

$$\boxed{\boldsymbol{d} * \boldsymbol{F} = *\boldsymbol{j}} \tag{2.2.4}$$

where

$$\boxed{j^\mu = e\bar{\psi}\gamma^\mu\psi} \tag{2.2.5}$$

is the four-current, i.e. the Noether current of QED.

2.2.2 The Torsion Equation of Motion

Now, let us vary the total action S with respect to the torsion S_μ. Again, only the terms containing a torsion contribute and therefore the action that has non-zero variation with respect to S_μ is

$$\begin{aligned} S_{Torsion} &= \frac{3}{32\pi G}\int \boldsymbol{S}\wedge *\boldsymbol{S} - \frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5} \\ &= \frac{3}{32\pi G}\int S_\mu S^\mu\sqrt{-g}\,d^4x - \frac{3}{4}\int S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x \\ &= \frac{3}{32\pi G}\int g^{\mu\nu}S_\mu S_\nu\sqrt{-g}\,d^4x - \frac{3}{4}\int S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x \end{aligned} \tag{2.2.6}$$

By varying this action with respect to S_μ we get the torsion equation of motion:

$$\boxed{S_\mu = 4\pi G\bar{\psi}\gamma_\mu\gamma^5\psi = 4\pi G j^5_\mu} \tag{2.2.7}$$

In differential form notation, this can be written as

$$\boxed{\boldsymbol{S} = 4\pi G\boldsymbol{j^5}} \tag{2.2.8}$$

Now, let us prove this result.

Proof

We start with the action

$$S_{Torsion} = \frac{3}{32\pi G}\int g^{\mu\nu}S_\mu S_\nu\sqrt{-g}\,d^4x - \frac{3}{4}\int S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x$$

The variation of $S_{Torsion}$ with respect to S_μ is given by

$$\delta S_{Torsion} = S_{Torsion}[S_\mu + \delta S_\mu] - S_{Torsion}[S_\mu]$$

where the square brackets indicate the functional dependence of the action on the torsion function S_μ. We, therefore, ought to calculate $S_{Torsion}[S_\mu + \delta S_\mu]$:

$$\begin{aligned} S_{Torsion}[S_\mu + \delta S_\mu] &= \frac{3}{32\pi G}\int g^{\mu\nu}(S_\mu + \delta S_\mu)(S_\nu + \delta S_\nu)\sqrt{-g}\,d^4x - \frac{3}{4}\int (S_\mu + \delta S_\mu)\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x \\ &= S_{Torsion}[S_\mu] + \frac{3}{32\pi G}\int g^{\mu\nu}(S_\mu\delta S_\nu + S_\nu\delta S_\mu)\sqrt{-g}\,d^4x \\ &\quad - \frac{3}{4}\int \delta S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x + \mathcal{O}(\delta S^2) \end{aligned}$$

We ignore terms that are of second order in the variation of the torsion, and thus we have

$$\begin{aligned}\delta S_{Torsion} &= \frac{3}{32\pi G}\int g^{\mu\nu}(S_\mu \delta S_\nu + S_\nu \delta S_\mu)\sqrt{-g}\,d^4x - \frac{3}{4}\int \delta S_\mu \bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x \\ &= \frac{3}{16\pi G}\int S^\mu \delta S_\mu \sqrt{-g}\,d^4x - \frac{3}{4}\int \delta S_\mu \bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x \\ &= \int\left[\frac{3}{16\pi G}S_\mu - \frac{3}{4}\bar{\psi}\gamma^\mu\gamma^5\psi\right]\delta S_\mu\sqrt{-g}\,d^4x\end{aligned}$$

The variational principle dictates that

$$\delta S_{Torsion} = \int \frac{\delta S_{Torsion}}{\delta S_\mu}\delta S_\mu$$

with the equations of motion being:

$$\frac{\delta S_{Torsion}}{\delta S_\mu} = 0$$

In our case,

$$\frac{\delta S_{Torsion}}{\delta S_\mu} = \frac{3}{16\pi G}S_\mu - \frac{3}{4}\bar{\psi}\gamma^\mu\gamma^5\psi$$

and thus the equation of motion for torsion is:

$$\frac{3}{16\pi G}S_\mu - \frac{3}{4}\bar{\psi}\gamma^\mu\gamma^5\psi = 0$$

This can be simplified to give

$$S_\mu = 4\pi G\bar{\psi}\gamma_\mu\gamma^5\psi = 4\pi G j^5_\mu$$

which is the desired relation.

This formula encapsulates the relation between spin and gravity that is characteristic of Einstein-Cartan theories. On the left hand of the equation we have the one-form $\boldsymbol{S}$, a geometric quantity related to torsion, and on the right hand side we have axial fermionic current, which contains spinors, i.e. the spin $\frac{1}{2}$ representations of the Lorentz group.

2.2.3 The Modified Dirac Equation

Next, varying with respect to the spinor field ought to give us the equation of motion for this kind of field. We, of course, expect this equation to be a generalized form of the Dirac equation that accounts for the curvature and torsion present. We will vary with respect to $\bar{\psi}$, as is usually done, in order to get this equation. Of course, only the terms of the action containing $\bar{\psi}$ contribute to the variation and therefore the relevant action is

$$\begin{aligned}S_{Fermion} &= \frac{1}{2}\int\left[i\bar{\psi}\gamma^\mu D_\mu\psi - iD_\mu\bar{\psi}\gamma^\mu\psi\right]\sqrt{-g}\,d^4x + e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x - \frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5} \\ &= \frac{1}{2}\int\left[i\bar{\psi}\gamma^\mu D_\mu\psi - iD_\mu\bar{\psi}\gamma^\mu\psi\right]\sqrt{-g}\,d^4x + e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x - \frac{3}{4}\int S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x\end{aligned} \tag{2.2.9}$$

By varying with respect to $\bar{\psi}$ we get the modified Dirac equation

$$\boxed{i\gamma^\mu\mathcal{D}_\mu\psi - \frac{3}{4}S_\mu\gamma^\mu\gamma^5\psi = 0} \tag{2.2.10}$$

We will now show the derivation of this equation.

Proof

Once again, variation with respect to $\bar{\psi}$ is given by

$$\delta S_{Fermion} = S_{Fermion}[\bar{\psi} + \delta\bar{\psi}] - S[\bar{\psi}]$$

and

$$\delta S_{Fermion} = \int \frac{\delta S_{Fermion}}{\delta\bar{\psi}} \delta\bar{\psi}$$

and we demand that

$$\frac{\delta S_{Fermion}}{\delta\bar{\psi}} = 0$$

Let us perform the computations. We have that

$$\begin{aligned}
S_{Fermion}[\bar{\psi} + \delta\bar{\psi}] &= \frac{1}{2} \int \left[i(\bar{\psi} + \delta\bar{\psi})\gamma^\mu D_\mu \psi - i[D_\mu(\bar{\psi} + \delta\bar{\psi})]\gamma^\mu \psi \right] \sqrt{-g}\, d^4x \\
&+ e \int (\bar{\psi} + \delta\bar{\psi})\gamma^\mu \psi A_\mu \sqrt{-g}\, d^4x - \frac{3}{4} \int S_\mu (\bar{\psi} + \delta\bar{\psi})\gamma^\mu \gamma^5 \psi \sqrt{-g}\, d^4x \\
&= S_{Fermion}[\bar{\psi}] + \frac{1}{2} \int \left[i\delta\bar{\psi}\gamma^\mu D_\mu \psi - i(D_\mu \delta\bar{\psi})\gamma^\mu \psi \right] \sqrt{-g}\, d^4x \\
&+ e \int \delta\bar{\psi}\gamma^\mu \psi A_\mu \sqrt{-g}\, d^4x - \frac{3}{4} \int S_\mu \delta\bar{\psi}\gamma^\mu \gamma^5 \psi \sqrt{-g}\, d^4x
\end{aligned}$$

For the term involving the covariant derivative of the variation $\delta\bar{\psi}$, we can do a partial integration:

$$-\frac{1}{2} \int i(D_\mu \delta\bar{\psi})\gamma^\mu \psi \sqrt{-g}\, d^4x = \frac{1}{2} \int \cancelto{0}{D_\mu(i\delta\bar{\psi}\gamma^\mu \psi)} \sqrt{-g}\, d^4x + \frac{1}{2} \int i D_\mu(\gamma^\mu \psi)\delta\bar{\psi} \sqrt{-g}\, d^4x$$

where the total derivative term cancels as it can be show to be a boundary term through Stokes' theorem, and therefore we have that

$$\begin{aligned}
\delta S_{Fermion} &= \frac{1}{2} \int \left[i\delta\bar{\psi}\gamma^\mu D_\mu \psi + i D_\mu(\gamma^\mu \psi)\delta\bar{\psi} \right] \sqrt{-g}\, d^4x \\
&+ e \int \delta\bar{\psi}\gamma^\mu \psi A_\mu \sqrt{-g}\, d^4x - \frac{3}{4} \int S_\mu \delta\bar{\psi}\gamma^\mu \gamma^5 \psi \sqrt{-g}\, d^4x
\end{aligned}$$

We have that

$$D_\mu(\gamma^\mu \psi) = (D_\mu \gamma^\mu)\psi + \gamma^\mu D_\mu \psi$$

It is possible to show, using the metric compatibility condition, that [8]

$$\boxed{D_\mu \gamma^\nu = 0} \tag{2.2.11}$$

and thus we get that

$$D_\mu(\gamma^\mu \psi) = \gamma^\mu D_\mu \psi$$

and therefore the variation of the action becomes

$$\begin{aligned}
\delta S_{Fermion} &= \int \left[i\gamma^\mu D_\mu \psi + e\gamma^\mu \psi A_\mu - \frac{3}{4} S_\mu \gamma^\mu \gamma^5 \psi \right] \delta\bar{\psi} \sqrt{-g}\, d^4x \\
&= \int \left[i\gamma^\mu D_\mu \psi - ie\gamma^\mu \psi i A_\mu - \frac{3}{4} S_\mu \gamma^\mu \gamma^5 \psi \right] \delta\bar{\psi} \sqrt{-g}\, d^4x \\
&= \int \left[i\gamma^\mu \mathcal{D}_\mu \psi - \frac{3}{4} S_\mu \gamma^\mu \gamma^5 \psi \right] \delta\bar{\psi} \sqrt{-g}\, d^4x
\end{aligned}$$

Demanding that the variation of the action vanishes we get the modified Dirac equation

$$i\gamma^\mu \mathcal{D}_\mu \psi - \frac{3}{4} S_\mu \gamma^\mu \gamma^5 \psi = 0$$

which is the desired result.

2.2.4 The Einstein Equations & the Stress-Energy Tensor

Finally, varying with respect to the metric gives us the Einstein equations

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{2.2.12}$$

with the difference with the usual General Relativity formulation being that the stress-energy tensor $T_{\mu\nu}$ now contains additional terms due to the presence of torsion. More specifically, we have [4]

$$T_{\mu\nu} = T^A_{\mu\nu} + T^\psi_{\mu\nu} + T^S_{\mu\nu}$$

where [9]

$$\boxed{T^A_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}} \tag{2.2.13}$$

is the familiar stress-energy tensor for the electromagnetic field from torsion-less gravity. Moreover,

$$\boxed{T^\psi_{\mu\nu} = -\frac{1}{2}\left[i\bar{\psi}\gamma_{(\mu}\mathcal{D}_{\nu)}\psi - i(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right] + \frac{3}{4}S_{(\mu}\bar{\psi}\gamma_{\nu)}\gamma^5\psi} \tag{2.2.14}$$

is the symmetric stress-energy tensor for an on-shell fermion, which includes a torsion contribution. We shall derive this result.

Proof

The derivation of the on-shell fermion part of the stress-energy tensor comes from the part of the action that contains the spinor ψ, i.e. from the same part that gave us the modified Dirac equation:

$$S_{Fermion} = \frac{1}{2}\int\left[i\bar{\psi}\gamma^\mu D_\mu\psi - iD_\mu\bar{\psi}\gamma^\mu\psi\right]\sqrt{-g}\,d^4x + e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x - \frac{3}{4}\int S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x$$

The stress-energy tensor for an on-shell fermion is defined as

$$\boxed{T^\psi_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{Fermion}}{\delta g^{\mu\nu}}} \tag{2.2.15}$$

and thus to derive the stress-energy tensor we must find the variation of $S_{Fermion}$ with respect to $\delta g^{\mu\nu}$. Before taking the variation of this action, we expand the gravitational covariant derivatives according to (2.1.1) and (2.1.2) and get that

$$\begin{aligned} S_{Fermion} = \frac{1}{2}\int&\left[i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\partial_\mu\bar{\psi}\gamma^\mu\psi + \frac{1}{4}\bar{\psi}\left\{\gamma^\mu, \omega_\mu\right\}\psi\right]\sqrt{-g}\,d^4x \\ &+ e\int(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g}\,d^4x - \frac{3}{4}\int S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}\,d^4x \end{aligned}$$

where $\omega_\mu = \omega_{\mu ab}\sigma^{ab}$. We must now take the variation with respect to the metric. The presence of spinors inevitably leads to a dependence on the metric through vielbeins. Therefore, to vary with the metric it is necessary to have knowledge of the variation of the vielbeins with respect to the metric. That dependence is quite complicated and given as a series in powers of the variation of the metric. In first order approximation, that dependence is [10]:

$$\boxed{\delta(\hat{e}^\mu{}_a) = \frac{1}{2}g_{\nu\rho}\hat{e}^\rho{}_a\delta g^{\mu\nu}} \tag{2.2.16}$$

$$\boxed{\delta(\hat{e}_\mu{}^a) = \frac{1}{2}g^{\nu\rho}\hat{e}_\rho{}^a\delta g_{\mu\nu}} \tag{2.2.17}$$

One might be tempted to use the metric to raise/lower an index of the varied metric. However, that would be a grave mistake as in such a process the variation of the metric used to raise/lower an index is ignored. As such, we stress that $g_{\nu\rho}\delta g^{\mu\nu} \neq \delta g^\mu{}_\rho$. This fact should become obvious if we point out that $g^\mu{}_\rho = \delta^\mu_\rho \Rightarrow \delta g^\mu{}_\rho = 0$. Thus, if the metric could lower/raise indices on the variation of the metric we would get $\delta(\hat{e}^\mu{}_a) = 0$, which of course is inconsistent. We therefore understand that only quantities that have a vanishing variation (such as Kronocker's δ and the Levi-Civita symbol ϵ) can act on the variation of the metric. Another quantity dependent on the metric is the volume element $\sqrt{-g}$, which transforms as [2]

$$\boxed{\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}} \tag{2.2.18}$$

Finally, the gamma matrices with curved indices actually have a dependence from the metric as well:

$$\gamma^\mu = \hat{e}^\mu{}_a \gamma^a$$

where γ^a are the flat spacetime gamma matrices that are constant and thus

$$\delta\gamma^\mu = \gamma^a \delta\hat{e}^\mu{}_a = \gamma^a\left(\frac{1}{2}g_{\nu\rho}\hat{e}^\rho{}_a \delta g^{\mu\nu}\right) = \frac{1}{2}\gamma_\nu \delta g^{\mu\nu}$$

and therefore the gamma matrices in curved spacetime are varied as

$$\boxed{\delta\gamma^\mu = \frac{1}{2}\gamma_\nu \delta g^{\mu\nu}} \tag{2.2.19}$$

Having the above in mind we can vary the fermionic action with respect to the metric:

$$\begin{aligned}
\delta S_{Fermion} = &\frac{1}{2}\int\left[\Big(i\bar{\psi}(\delta\gamma^\mu)\partial_\mu\psi - i\partial_\mu\bar{\psi}(\delta\gamma^\mu)\psi\Big)\sqrt{-g} + \Big(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\partial_\mu\bar{\psi}\gamma^\mu\psi\Big)(\delta\sqrt{-g})\right] d^4x \\
&+ \frac{1}{2}\int\frac{1}{4}\left[\bar{\psi}\delta(\{\gamma^\mu,\omega_\mu\})\psi\sqrt{-g} + \bar{\psi}(\{\gamma^\mu,\omega_\mu\})\psi(\delta\sqrt{-g})\right] d^4x \\
&+ e\int\left[\bar{\psi}(\delta\gamma^\mu)\psi A_\mu\sqrt{-g} + \bar{\psi}\gamma^\mu\psi A_\mu(\delta\sqrt{-g})\right] d^4x \\
&- \frac{3}{4}\int\left[S_\mu\bar{\psi}(\delta\gamma^\mu)\gamma^5\psi\sqrt{-g} + S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi(\delta\sqrt{-g})\right] d^4x
\end{aligned}$$

By expanding $(\delta\sqrt{-g})$ we notice that the terms proportional to the variation of the volume element can be expressed in terms of the fermionic lagrangian $\mathcal{L}$:

$$\begin{aligned}
\frac{1}{2}\int\mathcal{L}\,g^{\mu\nu}\delta g_{\mu\nu} = &\frac{1}{2}\int\left[\Big(i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\partial_\mu\bar{\psi}\gamma^\mu\psi\Big)(\delta\sqrt{-g})\right] d^4x + \frac{1}{2}\int\frac{1}{4}\left[\bar{\psi}(\{\gamma^\mu,\omega_\mu\})\psi(\delta\sqrt{-g})\right] d^4x \\
&+ e\int\left[\bar{\psi}\gamma^\mu\psi A_\mu(\delta\sqrt{-g})\right] d^4x - \frac{3}{4}\int\left[S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi(\delta\sqrt{-g})\right] d^4x
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[i\bar{\psi}\gamma^\mu D_\mu\psi - iD_\mu\bar{\psi}\gamma^\mu\psi\right]\sqrt{-g} + e(\bar{\psi}\gamma^\mu\psi A_\mu)\sqrt{-g} - \frac{3}{4}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g} \\
&= \frac{1}{2}\left[i\bar{\psi}\gamma^\mu(D_\mu - ieA_\mu)\psi - i(D_\mu + ieA_\mu)\bar{\psi}\gamma^\mu\psi\right]\sqrt{-g} - \frac{3}{4}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g} \\
&= \frac{1}{2}\left[i\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi - i(\mathcal{D}_\mu\bar{\psi})\gamma^\mu\psi\right]\sqrt{-g} - \frac{3}{4}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\sqrt{-g}
\end{aligned}$$

It is trivial to show that $\mathcal{L}$ vanishes "on-shell", i.e. if the equations of motion (the modified Dirac equation) hold and therefore this term is zero for on-shell fermions, which is exactly what we're interested in. Thus, the varied action (now considering on-shell fermions) is

$$\begin{aligned}
\delta S_{Fermion} = &\frac{1}{2}\int\left[i\bar{\psi}(\delta\gamma^\mu)\partial_\mu\psi - i\partial_\mu\bar{\psi}(\delta\gamma^\mu)\psi + \frac{1}{4}\bar{\psi}\delta(\{\gamma^\mu,\omega_\mu\})\psi\right]\sqrt{-g}\,d^4x \\
&+ e\int(\bar{\psi}(\delta\gamma^\mu)\psi A_\mu)\sqrt{-g}\,d^4x - \frac{3}{4}\int S_\mu\bar{\psi}(\delta\gamma^\mu)\gamma^5\psi\sqrt{-g}\,d^4x
\end{aligned}$$

The next step is calculating $\delta(\{\gamma^\mu, \omega_\mu\})$. We have that

$$\{\gamma^\mu, \omega_\mu\} = \gamma^\mu \omega_{\mu ab}\sigma^{ab} + \omega_{\mu ab}\sigma^{ab}\gamma^\mu$$

and thus its variation is

$$\begin{aligned}\delta(\{\gamma^\mu, \omega_\mu\}) &= \left[(\delta\gamma^\mu)\omega_{\mu ab}\sigma^{ab} + \gamma^\mu(\delta\omega_{\mu ab})\sigma^{ab} + (\delta\omega_{\mu ab})\sigma^{ab}\gamma^\mu + \omega_{\mu ab}\sigma^{ab}(\delta\gamma^\mu)\right] \\ &= \{\gamma^\mu, \delta\omega_\mu\} + \{\delta\gamma^\mu, \omega_\mu\}\end{aligned}$$

By replacing this expression into the varied action we get that

$$\delta S_{Fermion} = \int \left[\frac{1}{2}\left(i\bar{\psi}(\delta\gamma^\mu)\mathcal{D}_\mu\psi - i(\mathcal{D}_\mu\bar{\psi})(\delta\gamma^\mu)\psi\right) - \frac{3}{4}S_\mu\bar{\psi}(\delta\gamma^\mu)\gamma^5\psi\right]\sqrt{-g}\, d^4x + \frac{1}{8}\int \bar{\psi}\,\{\gamma^\mu, \delta\omega_\mu\}\,\psi\sqrt{-g}\, d^4x$$

We now need to calculate the variation of the spin connection, $\delta\omega_\mu = (\delta\omega_{\mu ab})\sigma^{ab}$. From Equation (1.1.20) we know that

$$\omega_{\mu ab} = \eta_{ac}\hat{e}^\lambda{}_b\,\hat{e}_\nu{}^c\,\Gamma^\nu_{\mu\lambda} - \eta_{ac}\hat{e}^\lambda{}_b\,\partial_\mu\hat{e}_\lambda{}^c$$

To take the variation of the spin connection we must know the variation of the Christoffel symbols with respect to the metric. This is given as [2]

$$\boxed{\delta\Gamma^\nu_{\mu\lambda} = \frac{1}{2}g^{\nu\rho}\left[\nabla_\mu\delta g_{\lambda\rho} + \nabla_\lambda\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\lambda}\right]} \tag{2.2.20}$$

The important information regarding to the variation of the Christoffel symbols is that that it's a tensor symmetric in its two lower indices. Having this in mind, we can go ahead and vary the spin connection:

$$\delta\omega_{\mu ab} = \eta_{ac}(\delta\hat{e}^\lambda{}_b)\hat{e}_\nu{}^c\,\Gamma^\nu_{\mu\lambda} + \eta_{ac}\hat{e}^\lambda{}_b(\delta\hat{e}_\nu{}^c)\Gamma^\nu_{\mu\lambda} + \eta_{ac}\,\hat{e}^\lambda{}_b\hat{e}_\nu{}^c\,(\delta\Gamma^\nu_{\mu\lambda}) - \eta_{ac}(\delta\hat{e}^\lambda{}_b)\,(\partial_\mu\hat{e}_\lambda{}^c) - \eta_{ac}\hat{e}^\lambda{}_b\,\partial_\mu(\delta\hat{e}_\lambda{}^c)$$

where we have used the fact that partial derivatives commute with functional derivatives. Let us focus on the last term. We have that

$$\begin{aligned}-\eta_{ac}\hat{e}^\lambda{}_b\,\partial_\mu(\delta\hat{e}_\lambda{}^c) &= -\eta_{ac}\hat{e}^\lambda{}_b\,\partial_\mu(\frac{1}{2}g^{\sigma\rho}\hat{e}_\rho{}^c\delta g_{\lambda\sigma}) \\ &= -\frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b\left[(\partial_\mu g^{\sigma\rho})\hat{e}_\rho{}^c(\delta g_{\lambda\sigma}) + g^{\sigma\rho}(\partial_\mu\hat{e}_\rho{}^c)(\delta g_{\lambda\sigma}) + g^{\sigma\rho}\hat{e}_\rho{}^c\partial_\mu(\delta g_{\lambda\sigma})\right]\end{aligned}$$

By also expanding the vielbeins variations in the rest of the terms, we get that

$$\begin{aligned}\delta\omega_{\mu ab} = &\frac{1}{2}\eta_{ac}\hat{e}^\rho{}_b\,\hat{e}_\nu{}^c\,\Gamma^\nu_{\mu\lambda}(g_{\sigma\rho}\delta g^{\lambda\sigma}) + \frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b\,\hat{e}_\rho{}^c\Gamma^\nu_{\mu\lambda}(g^{\sigma\rho}\delta g_{\nu\sigma}) + \eta_{ac}\,\hat{e}^\lambda{}_b\hat{e}_\nu{}^c\,(\delta\Gamma^\nu_{\mu\lambda}) \\ &- \frac{1}{2}\eta_{ac}\hat{e}^\rho{}_b(\partial_\mu\hat{e}_\lambda{}^c)(g_{\sigma\rho}\delta g^{\lambda\sigma}) - \frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b\,\hat{e}_\rho{}^c(\partial_\mu g^{\sigma\rho})(\delta g_{\lambda\sigma})\underbrace{-\frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b(\partial_\mu\hat{e}_\rho{}^c)(g^{\sigma\rho}\delta g_{\lambda\sigma})} \\ &- \frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b\,\hat{e}_\rho{}^c g^{\sigma\rho}\partial_\mu(\delta g_{\lambda\sigma})\end{aligned}$$

Now, we focus on the underbraced term. We will use the very useful property:

$$\boxed{\delta g_{\rho\sigma} = -g_{\sigma\mu}g_{\rho\nu}\delta g^{\mu\nu}} \tag{2.2.21}$$

This result is easy to show and follows from the fact that $\delta(g^{\mu\nu}g_{\nu\rho}) = \delta(\delta^\mu_\rho) = 0$. Therefore, we have that

$$\begin{aligned}-\frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b(\partial_\mu\hat{e}_\rho{}^c)(g^{\sigma\rho}\delta g_{\lambda\sigma}) &= +\frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b(\partial_\mu\hat{e}_\rho{}^c)(g^{\sigma\rho}g_{\sigma\xi}g_{\lambda\kappa}\delta g^{\kappa\xi}) = \frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b(\partial_\mu\hat{e}_\rho{}^c)(\delta^\rho_\xi g_{\lambda\kappa}\delta g^{\kappa\xi}) \\ &= \frac{1}{2}\eta_{ac}\hat{e}^\lambda{}_b(\partial_\mu\hat{e}_\rho{}^c)(g_{\lambda\kappa}\delta g^{\kappa\rho}) = \frac{1}{2}\eta_{ac}\hat{e}^\rho{}_b(\partial_\mu\hat{e}_\lambda{}^c)(g_{\kappa\rho}\delta g^{\kappa\lambda})\end{aligned}$$

where in the last step we renamed the indices $\lambda \leftrightarrow \rho$. Having this in mind, we rewrite the variation of the spin connection,

$$\begin{aligned}\delta\omega_{\mu ab} = &\frac{1}{2}\eta_{ac}\hat{e}^{\rho}{}_{b}\,\hat{e}_{\nu}{}^{c}\,\Gamma^{\nu}_{\mu\lambda}(g_{\sigma\rho}\delta g^{\lambda\sigma}) + \frac{1}{2}\eta_{ac}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\Gamma^{\nu}_{\mu\lambda}(g^{\sigma\rho}\delta g_{\nu\sigma}) + \eta_{ac}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{c}\,(\delta\Gamma^{\nu}_{\mu\lambda})\\ &- \cancel{\frac{1}{2}\eta_{ac}\hat{e}^{\rho}{}_{b}(\partial_{\mu}\hat{e}_{\lambda}{}^{c})(g_{\sigma\rho}\delta g^{\lambda\sigma})} - \frac{1}{2}\eta_{ac}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}(\partial_{\mu}g^{\sigma\rho})(\delta g_{\lambda\sigma}) + \cancel{\frac{1}{2}\eta_{ac}\hat{e}^{\rho}{}_{b}(\partial_{\mu}\hat{e}_{\lambda}{}^{c})(g_{\kappa\rho}\delta g^{\kappa\lambda})}\\ &- \frac{1}{2}\eta_{ac}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}g^{\sigma\rho}\partial_{\mu}(\delta g_{\lambda\sigma})\end{aligned}$$

and observe that the two terms involving the partial derivative of the vielbein cancel out. We can then regroup the remaining terms as

$$\begin{aligned}\delta\omega_{\mu ab} = &-\eta_{ac}\Bigg[\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\underbrace{\Big((\partial_{\mu}g^{\sigma\rho})(\delta g_{\lambda\sigma}) + g^{\sigma\rho}\partial_{\mu}(\delta g_{\lambda\sigma})\Big)}_{=\partial_{\mu}(g^{\sigma\rho}\delta g_{\lambda\sigma})} - \frac{1}{2}\hat{e}^{\rho}{}_{b}\,\hat{e}_{\nu}{}^{c}\,\Gamma^{\nu}_{\mu\lambda}(g_{\sigma\rho}\delta g^{\lambda\sigma}) - \frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\Gamma^{\nu}_{\mu\lambda}(g^{\sigma\rho}\delta g_{\nu\sigma})\Bigg]\\ &+ \eta_{ac}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{c}\,(\delta\Gamma^{\nu}_{\mu\lambda})\end{aligned}$$

The inverse of Equation (2.2.21) is

$$\boxed{\delta g^{\kappa\lambda} = -g^{\sigma\kappa}g^{\rho\lambda}\delta g_{\rho\sigma}} \tag{2.2.22}$$

We use this relation to lower the indices of the variation on the second term:

$$\begin{aligned}-\frac{1}{2}\hat{e}^{\rho}{}_{b}\,\hat{e}_{\nu}{}^{c}\,\Gamma^{\nu}_{\mu\lambda}(g_{\sigma\rho}\delta g^{\lambda\sigma}) &= \frac{1}{2}\hat{e}^{\rho}{}_{b}\,\hat{e}_{\nu}{}^{c}\,\Gamma^{\nu}_{\mu\lambda}(g_{\sigma\rho}g^{\lambda\kappa}g^{\sigma\xi}\delta g_{\kappa\xi}) = \frac{1}{2}\hat{e}^{\rho}{}_{b}\,\hat{e}_{\nu}{}^{c}\,\Gamma^{\nu}_{\mu\lambda}(\delta^{\xi}_{\rho}g^{\lambda\kappa}\delta g_{\kappa\xi})\\ &= \frac{1}{2}\hat{e}^{\rho}{}_{b}\,\hat{e}_{\nu}{}^{c}\,\Gamma^{\nu}_{\mu\lambda}(g^{\lambda\kappa}\delta g_{\kappa\rho})\end{aligned}$$

and therefore the variation of the spin connection is

$$\begin{aligned}\delta\omega_{\mu ab} = &-\eta_{ac}\Bigg[\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\partial_{\mu}(g^{\sigma\rho}\delta g_{\lambda\sigma}) + \underbrace{\frac{1}{2}\hat{e}^{\rho}{}_{b}\,\hat{e}_{\nu}{}^{c}\,\Gamma^{\nu}_{\mu\lambda}(g^{\lambda\kappa}\delta g_{\kappa\rho})}_{\nu\leftrightarrow\rho\,\&\,\lambda\leftrightarrow\kappa} - \underbrace{\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\Gamma^{\nu}_{\mu\lambda}(g^{\sigma\rho}\delta g_{\nu\sigma})}_{\nu\leftrightarrow\kappa}\Bigg]\\ &+ \eta_{ac}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{c}\,(\delta\Gamma^{\nu}_{\mu\lambda})\end{aligned}$$

We now make the index renaming indicated in the underbraces above. We get that

$$\begin{aligned}\delta\omega_{\mu ab} = &-\eta_{ac}\Bigg[\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\partial_{\mu}(g^{\sigma\rho}\delta g_{\lambda\sigma}) + \underbrace{\frac{1}{2}\hat{e}^{\nu}{}_{b}\,\hat{e}_{\rho}{}^{c}\,\Gamma^{\rho}_{\mu\kappa}(g^{\lambda\kappa}\delta g_{\lambda\nu})}_{\lambda\leftrightarrow\sigma} - \underbrace{\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\Gamma^{\kappa}_{\mu\lambda}(g^{\sigma\rho}\delta g_{\kappa\sigma})}_{\nu\leftrightarrow\sigma}\Bigg]\\ &+ \eta_{ac}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{c}\,(\delta\Gamma^{\nu}_{\mu\lambda})\end{aligned}$$

Another renaming of indices (indicated above) results in

$$\begin{aligned}\delta\omega_{\mu ab} = &-\eta_{ac}\Bigg[\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\partial_{\mu}(g^{\sigma\rho}\delta g_{\lambda\sigma}) + \underbrace{\frac{1}{2}\hat{e}^{\nu}{}_{b}\,\hat{e}_{\rho}{}^{c}\,\Gamma^{\rho}_{\mu\kappa}(g^{\sigma\kappa}\delta g_{\sigma\nu})}_{\nu\leftrightarrow\lambda} - \frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\Gamma^{\kappa}_{\mu\lambda}(g^{\sigma\rho}\delta g_{\kappa\sigma})\Bigg]\\ &+ \eta_{ac}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{c}\,(\delta\Gamma^{\nu}_{\mu\lambda})\end{aligned}$$

A final renaming of indices results in

$$\begin{aligned}\delta\omega_{\mu ab} = &-\eta_{ac}\underbrace{\Bigg[\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\partial_{\mu}(g^{\sigma\rho}\delta g_{\lambda\sigma}) + \frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\,\Gamma^{\rho}_{\mu\kappa}(g^{\sigma\kappa}\delta g_{\sigma\lambda}) - \frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\Gamma^{\kappa}_{\mu\lambda}(g^{\sigma\rho}\delta g_{\kappa\sigma})\Bigg]}_{=\frac{1}{2}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}\nabla_{\mu}(g^{\sigma\rho}\delta g_{\lambda\sigma})}\\ &+ \eta_{ac}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{c}\,(\delta\Gamma^{\nu}_{\mu\lambda})\end{aligned}$$

where we observe that the quantity in the brackets is the covariant derivative of $(g^{\sigma\rho}\delta g_{\lambda\sigma})$. By making use of the metric compatibility condition, the metric commutes with the covariant derivative and thus the final expression for the variation of the spin connection is

$$\boxed{\delta\omega_{\mu ab} = -\frac{1}{2}\eta_{ac}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{c}g^{\sigma\rho}\nabla_{\mu}(\delta g_{\lambda\sigma}) + \eta_{ac}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{c}\,(\delta\Gamma^{\nu}_{\mu\lambda})} \tag{2.2.23}$$

We are now ready to calculate the term $\bar{\psi}\{\gamma^{\mu},\delta\omega_{\mu}\}\psi\sqrt{-g}$. We have that

$$\bar{\psi}\{\gamma^{\mu},\delta\omega_{\mu}\}\psi\sqrt{-g} = \bar{\psi}\left\{\gamma^{c},\sigma^{ab}\right\}\psi\hat{e}^{\mu}{}_{c}\delta\omega_{\mu ab}\sqrt{-g}$$

After using identity (2.1.9) we get

$$\bar{\psi}\{\gamma^{\mu},\delta\omega_{\mu}\}\psi\sqrt{-g} = 2\epsilon^{abc}{}_{d}\bar{\psi}\gamma^{d}\gamma^{5}\psi\hat{e}^{\mu}{}_{c}\delta\omega_{\mu ab}\sqrt{-g}$$

We replace the expression for the variation of the spin connection and get

$$\begin{aligned}\bar{\psi}\{\gamma^{\mu},\delta\omega_{\mu}\}\psi\sqrt{-g} = &-\epsilon^{abc}{}_{d}\bar{\psi}\gamma^{d}\gamma^{5}\psi\hat{e}^{\mu}{}_{c}\eta_{am}\hat{e}^{\lambda}{}_{b}\,\hat{e}_{\rho}{}^{m}g^{\sigma\rho}\nabla_{\mu}(\delta g_{\lambda\sigma})\sqrt{-g}\\ &+2\epsilon^{abc}{}_{d}\bar{\psi}\gamma^{d}\gamma^{5}\psi\hat{e}^{\mu}{}_{c}\eta_{am}\,\hat{e}^{\lambda}{}_{b}\hat{e}_{\nu}{}^{m}\,(\delta\Gamma^{\nu}_{\mu\lambda})\sqrt{-g}\end{aligned}$$

It is quite easy to perform the contractions with the vielbeins and the metrics and get

$$\begin{aligned}\bar{\psi}\{\gamma^{\mu},\delta\omega_{\mu}\}\psi\sqrt{-g} = &-\underbrace{\epsilon^{\sigma\lambda\mu}{}_{d}}\bar{\psi}\gamma^{d}\gamma^{5}\psi\nabla_{\mu}\underbrace{(\delta g_{\lambda\sigma})}\sqrt{-g}\\ &+2\underbrace{\epsilon_{\nu}{}^{\lambda\mu}{}_{d}}\bar{\psi}\gamma^{d}\gamma^{5}\psi\underbrace{(\delta\Gamma^{\nu}_{\mu\lambda})}\sqrt{-g}\end{aligned}$$

We know that both the variation of the metric and the variation of the Christoffel symbols are symmetric under exchange of their indices. Therefore, contraction with the Levi-Civita symbol results in the vanishin of *both* terms,

$$\bar{\psi}\{\gamma^{\mu},\delta\omega_{\mu}\}\psi\sqrt{-g} = 0 \tag{2.2.24}$$

and thus the term that includes the variation of the spin connection does not contribute at all. We can now go back and rewrite the varied action:

$$\delta S_{Fermion} = \int\left[\frac{1}{2}\left(i\bar{\psi}(\delta\gamma^{\mu})\mathcal{D}_{\mu}\psi - i(\mathcal{D}_{\mu}\bar{\psi})(\delta\gamma^{\mu})\psi\right) - \frac{3}{4}S_{\mu}\bar{\psi}(\delta\gamma^{\mu})\gamma^{5}\psi\right]\sqrt{-g}\,d^{4}x$$

where we now replace the variation of the gamma matrices and get

$$\delta S_{Fermion} = \int\frac{1}{2}\left[\frac{1}{2}\left(i\bar{\psi}\gamma_{\nu}\mathcal{D}_{\mu}\psi - i(\mathcal{D}_{\mu}\bar{\psi})\gamma_{\nu}\psi\right) - \frac{3}{4}S_{\mu}\bar{\psi}\gamma_{\nu}\gamma^{5}\psi\right]\sqrt{-g}\delta g^{\mu\nu}\,d^{4}x$$

It is now useful to notice that if $A_{\mu\nu}$ is an arbitrary tensor and $B^{\mu\nu}$ is a *symmetric* tensor, then

$$A_{\mu\nu}B^{\mu\nu} = \frac{1}{2}\left(A_{\mu\nu}B^{\mu\nu} + A_{\mu\nu}B^{\mu\nu}\right) = \frac{1}{2}\left(A_{\mu\nu}B^{\mu\nu} + A_{\nu\mu}B^{\nu\mu}\right) = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)B^{\mu\nu} = A_{(\mu\nu)}B^{\mu\nu}$$

Therefore, since $\delta g^{\mu\nu}$ is symmetric, we have that the final form of the variation of the fermionic action is

$$\delta S_{Fermion} = \int\frac{1}{2}\left[\frac{1}{2}\left(i\bar{\psi}\gamma_{(\nu}\mathcal{D}_{\mu)}\psi - i(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right) - \frac{3}{4}S_{(\mu}\bar{\psi}\gamma_{\nu)}\gamma^{5}\psi\right]\sqrt{-g}\delta g^{\mu\nu}\,d^{4}x \tag{2.2.25}$$

We can now employ the definition of the stress-energy tensor, given in Equation (2.2.15) and find that

$$T^{\psi}_{\mu\nu} = -\frac{1}{2}\left[i\bar{\psi}\gamma_{(\mu}\mathcal{D}_{\nu)}\psi - i(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right] + \frac{3}{4}S_{(\mu}\bar{\psi}\gamma_{\nu)}\gamma^{5}\psi$$

which is the result we wanted.

Finally,

$$\boxed{T^S_{\mu\nu} = -\frac{3}{16\pi G}(S_\mu S_\nu - \frac{1}{2}g_{\mu\nu}S_\lambda S^\lambda)} \tag{2.2.26}$$

is the stress-energy torsion attributed to the torsion.

Proof

This part of the stress-energy tensor originates from the torsion-only parts of the action, i.e. from

$$S_T = \frac{3}{32\pi G}\int g^{\mu\nu}S_\mu S_\nu\sqrt{-g}\,d^4x$$

Since we consider S_μ to be an independent variable, only $g^{\mu\nu}$ and $\sqrt{-g}$ have a non-zero variation. Therefore,

$$\delta S_T = \frac{3}{32\pi G}\int\left[(\delta g^{\mu\nu})S_\mu S_\nu\sqrt{-g} + g^{\mu\nu}S_\mu S_\nu(\delta\sqrt{-g})\right]d^4x$$

We can now use the property

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}} \tag{2.2.27}$$

which is easily obtained by Equations (2.2.18) & (2.2.21). This gives us:

$$\begin{aligned}\delta S_T &= \frac{3}{32\pi G}\int\left[S_\mu S_\nu\sqrt{-g}(\delta g^{\mu\nu}) - \frac{1}{2}g^{\lambda\sigma}S_\lambda S_\sigma\sqrt{-g}g_{\mu\nu}(\delta g^{\mu\nu})\right]d^4x\\ &= \frac{3}{32\pi G}\int\left[S_\mu S_\nu - \frac{1}{2}g^{\lambda\sigma}S_\lambda S_\sigma g_{\mu\nu}\right]\sqrt{-g}(\delta g^{\mu\nu})d^4x\end{aligned}$$

Thus, the variation of the torsion action is

$$\delta S_T = \frac{3}{16\pi G}\int\frac{1}{2}\left[S_\mu S_\nu - \frac{1}{2}S_\lambda S^\lambda g_{\mu\nu}\right]\sqrt{-g}(\delta g^{\mu\nu})d^4x \tag{2.2.28}$$

We now apply the definition of the stress-energy tensor (2.2.15) and find that

$$T^S_{\mu\nu} = -\frac{3}{16\pi G}(S_\mu S_\nu - \frac{1}{2}g_{\mu\nu}S_\lambda S^\lambda)$$

which is the desired relation.

2.3 Anomaly and Axions

In the previous section, we derived the classical equations of motion for our system. From Equations (2.2.8) and (2.2.10) we get that the axial current $j^5_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi$ is conserved. This would, in turn, imply that the torsion is conserved, i.e.

$$\boxed{\boldsymbol{d} * \boldsymbol{S} = 0} \tag{2.3.1}$$

This can easily be seen as we have that

$$\boldsymbol{S} = S_\lambda dx^\lambda$$

and thus, by definition of the Hodge dual,

$$*\boldsymbol{S} = \frac{1}{(4-1)!}S^\lambda\eta_{\lambda\mu\nu\rho}dx^\mu\wedge dx^\nu\wedge dx^\rho$$

Then, by definition of the exterior derivative,

$$\boldsymbol{d} * \boldsymbol{S} = \frac{1}{3!}\partial_\sigma S^\lambda\eta_{\lambda\mu\nu\rho}dx^\sigma\wedge dx^\mu\wedge dx^\nu\wedge dx^\rho$$

and by direct substitution of Equation (2.2.7) we get that

$$\boldsymbol{d} * \boldsymbol{S} = \frac{1}{3!}4\pi G\partial_\sigma(j^5)^\lambda\eta_{\lambda\mu\nu\rho}dx^\sigma\wedge dx^\mu\wedge dx^\nu\wedge dx^\rho$$

which vanishes, due to the conservation of the axial current. Therefore, the torsion is shown to be conserved as claimed. However, it is known from Quantum Electrodynamics that the axial current is not conserved when we pass onto a quantum theory due to an anomaly that shows up in the one-loop level. More specifically, the axial current has a non-vanishing divergence given by [4]

$$\boxed{\boldsymbol{d} * \boldsymbol{j^5} = -\frac{e^2}{4\pi^2}\boldsymbol{F} \wedge \boldsymbol{F} - \frac{1}{96\pi^2}\mathrm{tr}(\overline{\boldsymbol{R}} \wedge \overline{\boldsymbol{R}})} \tag{2.3.2}$$

or, written in components,

$$\boxed{\nabla j^5 = \frac{e^2}{8\pi^2}F^{\mu\nu}(*F_{\mu\nu}) - \frac{1}{192\pi^2}\overline{R}^{\rho\sigma\mu\nu}(*\overline{R}_{\rho\sigma\mu\nu})} \tag{2.3.3}$$

where

$$\boxed{*F_{\mu\nu} = \frac{1}{2}\sqrt{-g}\,\epsilon_{\mu\nu\lambda\kappa}F^{\lambda\kappa}} \tag{2.3.4}$$

and

$$\boxed{*\overline{R}_{\rho\sigma\mu\nu} = \frac{1}{2}\sqrt{-g}\,\epsilon_{\mu\nu\lambda\kappa}\overline{R}_{\rho\sigma}{}^{\lambda\kappa}} \tag{2.3.5}$$

At this point, it must be stressed that we're dealing with a semi-classical approach to the effects of torsion in Quantum Electrodynamics. That means that, in the absence of a quantum theory of gravity itself, we use a classical theory of gravity (i.e. we consider a classical geometric background) and consider its effects to the matter fields. Hence, while the axial current non-conservation implies that the torsion field is not conserved either, we do not actually know the quantum behavior of $\boldsymbol{S}$. This means that there might be more contributions due to quantum effects we aren't aware of. A possible route to take is to hypothesize that there are additional action terms such that torsion is conserved (i.e. Equation (2.3.1) holds) even though the axial current is not conserved. This hypothesis can then be shown [4] to lead to the replacement of torsion by a (scalar) axion field ϕ, to which the fermion field couples. To show this, we will follow the path integral formalism and apply the conservation of torsion (Equation (2.3.1)) as a constraint [4]. The full path integral for the quantization of the system is

$$Z = \int \mathcal{D}\boldsymbol{g}\,\mathcal{D}\psi\,\mathcal{D}\overline{\psi}\,\mathcal{D}\boldsymbol{S}\,e^{iS[\boldsymbol{g},\psi,\overline{\psi},\boldsymbol{S}]} \tag{2.3.6}$$

where $S[\boldsymbol{g},\psi,\overline{\psi},\boldsymbol{S}]$ is the action functional dependent on the metric, the spinor and the torsion. We are only interested on the torsion part of the path integral:

$$Z_S = \int \mathcal{D}\boldsymbol{S}\exp\Big[i\int\big(\frac{3}{32\pi G}\int \boldsymbol{S}\wedge *\boldsymbol{S} - \frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5}\big)\Big] \tag{2.3.7}$$

By making use of the constraint (Equation (2.3.1)) the torsion is replaced by an axion and we get the following result:

$$\boxed{Z_S^C = \int \mathcal{D}\phi\exp\Big[i\int\Big(-\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2f_\phi^2}j^5_\mu(j^5)^\mu - \frac{1}{f_\phi}j^5_\mu(\partial^\mu\phi)\Big)\sqrt{-g}\,d^4x\Big]} \tag{2.3.8}$$

Proof

To enforce Equation (2.3.1) as a constraint, we introduce a functional delta function to the path integral:

$$Z_S^C = \int \mathcal{D}\boldsymbol{S}\,\delta(\boldsymbol{d}*\boldsymbol{S})\exp\Big[i\int\big(\frac{3}{32\pi G}\int \boldsymbol{S}\wedge *\boldsymbol{S} - \frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5}\big)\Big] \tag{2.3.9}$$

This delta functional can be written in integral form as[14]

$$\delta(\boldsymbol{d}*\boldsymbol{S}) = \int \mathcal{D}\Phi e^{i\int \Phi \boldsymbol{d}*\boldsymbol{S}} \tag{2.3.10}$$

Therefore, the torsion path integral becomes

$$Z_S^C = \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi\exp\Big[i\int\Big(\frac{3}{32\pi G}\int \boldsymbol{S}\wedge *\boldsymbol{S} - \frac{3}{4}\int \boldsymbol{S}\wedge *\boldsymbol{j^5} + \Phi\boldsymbol{d}*\boldsymbol{S}\Big)\Big] \tag{2.3.11}$$

We can write this path integral in index notation as

$$Z_S^C = \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi \exp\left[i\int\left(\frac{3}{32\pi G}S_\mu S^\mu - \frac{3}{4}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi + \Phi\partial_\mu S^\mu\right)\sqrt{-g}\,d^4x\right] \tag{2.3.12}$$

Now, we integrate the last term by parts and get

$$\begin{aligned} Z_S^C &= \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi \exp\left[i\int\left(\frac{3}{32\pi G}S_\mu S^\mu - \frac{3}{4}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi - (\partial_\mu\Phi)S^\mu\right)\sqrt{-g}\,d^4x\right] \\ &= \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi \exp\left[i\int\left(\frac{3}{32\pi G}S_\mu S^\mu - \left[\frac{3}{4}j^5_\mu + (\partial_\mu\Phi)\right]S^\mu\right)\sqrt{-g}\,d^4x\right] \\ &= \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi \exp\left[i\int\frac{3}{32\pi G}\left(S_\mu S^\mu - \frac{32\pi G}{3}\left[\frac{3}{4}j^5_\mu + (\partial_\mu\Phi)\right]S^\mu\right)\sqrt{-g}\,d^4x\right] \\ &= \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi \exp\left[i\int\frac{3}{32\pi G}\left(S_\mu S^\mu - \underbrace{\left[8\pi G j^5_\mu + \frac{32\pi G}{3}(\partial_\mu\Phi)\right]}_{=2b_\mu(x)}S^\mu\right)\sqrt{-g}\,d^4x\right] \end{aligned}$$

Now, it is possible to complete the square:

$$\begin{aligned} Z_S^C &= \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi \exp\left[i\int\frac{3}{32\pi G}\left(S_\mu S^\mu - 2b_\mu S^\mu + b_\mu b^\mu - b_\mu b^\mu\right)\sqrt{-g}\,d^4x\right] \\ &= \int \mathcal{D}\boldsymbol{S}\,\mathcal{D}\Phi \exp\left[i\int\frac{3}{32\pi G}\left((S_\mu - b_\mu)^2 - b_\mu b^\mu\right)\sqrt{-g}\,d^4x\right] \end{aligned}$$

We therefore have a gaussian path integral over the torsion field, which gives a constant which can be absorbed by the measure $\mathcal{D}\Phi$. Therefore,

$$Z_S^C = \int \mathcal{D}\Phi \exp\left[-i\int\frac{3}{32\pi G}b_\mu b^\mu\sqrt{-g}\,d^4x\right]$$

Now, all that's left is to calculate $b_\mu b^\mu$. We have that

$$\begin{aligned} b_\mu b^\mu &= \frac{1}{4}\left[8\pi G j^5_\mu + \frac{32\pi G}{3}(\partial_\mu\Phi)\right]\left[8\pi G (j^5)^\mu + \frac{32\pi G}{3}(\partial^\mu\Phi)\right] \\ &= \frac{1}{4}\left[(8\pi G)^2 j^5_\mu (j^5)^\mu + \left(\frac{32\pi G}{3}\right)^2(\partial_\mu\Phi)(\partial^\mu\Phi) + j^5_\mu\frac{16\cdot 32\pi^2G^2}{3}(\partial^\mu\Phi)\right] \end{aligned}$$

We also rescale Φ by setting $\Phi = \sqrt{\frac{3}{16\pi G}}\phi$ and get

$$Z_S^C = \int \mathcal{D}\phi \exp\left[i\int -\left(\frac{3\pi G}{2}j^5_\mu(j^5)^\mu + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \sqrt{3\pi G}j^5_\mu(\partial^\mu\phi)\right)\sqrt{-g}\,d^4x\right]$$

Finally, we set the constant $f_\phi = \frac{1}{\sqrt{3\pi G}}$ and get that the path integral for the torsion has become

$$Z_S^C = \int \mathcal{D}\phi \exp\left[i\int\left(-\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2f_\phi^2}j^5_\mu(j^5)^\mu - \frac{1}{f_\phi}j^5_\mu(\partial^\mu\phi)\right)\sqrt{-g}\,d^4x\right] \tag{2.3.13}$$

which is the desired result.

We recognize the presence of a pseudoscalar field ϕ with a kinetic term and a coupling to fermions. These are the exact characteristics that define an axionic field. Effectively, we see that QED on contorted spacetime is shown to be equivalent to QED in a spacetime without torsion coupled to an axion.

[14]This is in complete analogy with the more familiar delta function

$$\delta(x) = \int\frac{dk}{2\pi}e^{ikx}$$

2.4 Summary

Up to this point, we have built up the formalism of Einstein-Cartan theory for curved spacetime with torsion and have considered a minimal model of Quantum Electrodynamics in such a background. Classically, torsion is conserved and it has no dynamical character. However, we can argue that at the quantum level, torsion can acquire a dynamical character by being turned into an axion due to the anomaly in the axial current of QED. This is a very interesting result for a minimal model. However, models beyond the minimal one can be considered. For example, we can consider a non-minimal model where the torsion couples classically to the electromagnetic field. This would, in turn, require a modification to the gauge transformation properties of the vector potential and a mass to be given to torsion (see [4] for further references).

In general, Einstein-Cartan theory has many more points of interest besides those discussed in this text. For example, it is worth exploring exactly how spin affects the Einstein equations. As it turns out, since torsion is not classically dynamical, it does not propagate and the Einstein equations are the same as in General Relativity in vacuum. In fermionic matter, spin does play a role in altering the geometry of spacetime, but the contributions are important only when the spin densities are extreme, such as in neutron stars and, of course, black holes. This, in turn, raises a big interest in Cosmological models that include torsion and are based in Einstein-Cartan theories (as discussed, for example, here [11]). An important characteristic of such models is the avoidance of singularities, both in the Big Bang and in black holes. More specifically, the Big Bang is replaced by a so-called Big Bounce [12], which happens after a period of contraction of the universe. Similarly, black holes do not collapse into a singularity, but reach a bounce and a new, growing universe is formed on the other side of the event horizon. Thus, the physics at the center of a black hole, which in the context of General Relativity is an unknown parameter, is restored to a classical level.

Chapter 3

String-inspired Inflation due to Torsion

In the previous chapters, we developed a theory for classical gravity with torsion and examined how contorted gravity gives rise to a dynamical axion when interactions with QED are considered. In this chapter, we'll consider a string-inspired model and show how the torsion arising in this model affects the inflation of the universe.

3.1 The Effective String Action

In this section, we'll outline our string-inspired model, which consists of an effective string action in four spacetime dimensions. Our main point of interest is the Kalb-Ramond field. We argue that this field plays the role of torsion and gives rise to an axion, which couples to both gravitational and Yang-Mills anomalies, in complete analogy with the case examined in the previous chapters. This, in turn, leads to a modification in our definition of the stress-energy tensor.

3.1.1 Kalb-Ramond Field and Torsion

For completeness' sake, we give a small summary of the origin of torsion in string theory and present the model we'll work with. A deeper dive in string theory is out of the scope of this text and the derivation of the results presented in this section can be found in standard textbooks such as [13] and [14]. It is known that any string theory model predicts the existence of three massless gravitational fields which form the so-called *gravitational multiplet* [4]: the spin-0 (scalar) Dilaton Φ, the spin-1 antisymmetric Kalb-Ramond field $B_{\mu\nu}$ and the spin-2 symmetric graviton field $g_{\mu\nu}$. We devote our attention to the Kalb-Ramond field, which has a $U(1)$ gauge symmetry [15]

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\theta_{\nu]} \tag{3.1.1}$$

As a consequence, the action (at least in the low energy regime) depends on the field strength of the Kalb-Ramond field, rather than the field itself. The field strength is defined as

$$\boxed{H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}} \tag{3.1.2}$$

which is completely antisymmetric and thus forms a 3-form, i.e.

$$\boxed{\boldsymbol{H} = \boldsymbol{dB}} \tag{3.1.3}$$

It is, therefore, an immediate consequence that the following Bianchi identity holds:

$$\boxed{\boldsymbol{dH} = 0} \tag{3.1.4}$$

or, in index notation,

$$\boxed{\partial_{[\mu}H_{\nu\rho\sigma]} = 0} \tag{3.1.5}$$

Anomaly cancellation considerations dictate that the field strength of the Kalb-Ramond field has to be modified by the addition of so-called Lorentz and gauge Chern-Simons 3-form terms:

$$\boldsymbol{H} = \boldsymbol{dB} + \frac{a'}{8\kappa}(\Omega_{3L} - \Omega_{3Y}) \tag{3.1.6}$$

where a is the *Regge slope*, $\kappa = \sqrt{8\pi G}$ and Ω_{3L}, Ω_{3Y} are the Lorentz and gauge Chern-Simons terms respectively. The Bianchi identity is thus modified and becomes

$$\boldsymbol{dH} = \frac{a'}{8\kappa} Tr\left(\boldsymbol{R} \wedge \boldsymbol{R} - \boldsymbol{F} \wedge \boldsymbol{F}\right) \tag{3.1.7}$$

where $\boldsymbol{R}$ is the curvature and $\boldsymbol{F}$ is the Yang-Mills field strength. The effective action for the bosonic string in four spacetime dimensions can be written as an expansion in powers of the Regge slope a'. In zeroth-order, and after assuming that the Dilaton is irrelevant, i.e. $\Phi \approx 0$, we get that the relevant action has the form [15][15]

$$S_B = \int \left(\frac{1}{2\kappa^2} R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right)\sqrt{-g}\, d^4x \tag{3.1.8}$$

where $\mathcal{H}_{\lambda\mu\nu} = \kappa^{-1} H_{\lambda\mu\nu}$. Direct comparison with the action (1.5.15) indicates that the Kalb-Ramond field strength plays the role of torsion in this effective field theory. We can thus define the contorted connection to be

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \frac{\kappa}{\sqrt{3}}\mathcal{H}^{\lambda}{}_{\mu\nu} \tag{3.1.9}$$

and therefore the effective torsion tensor is proportional to the (modified) Kalb-Ramond field strength. The Bianchi identity (3.1.7) considered above can be written in index notation as [4]

$$\eta_{abc}{}^{\mu}\nabla_{\mu}\mathcal{H}^{abc} = \frac{a'}{16\kappa}\sqrt{-g}\left(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \equiv \sqrt{-g}\,\mathcal{G}(\omega, \boldsymbol{A}) \tag{3.1.10}$$

where the covariant derivative is taken with the torsion-free Christoffel symbols (hence the absence of an overbar), $\mathcal{G}(\omega, \boldsymbol{A})$ is the anomaly and the quantities with the "tilde" over them are the dual quantities, defined as

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\eta_{\mu\nu\lambda\kappa}R^{\lambda\kappa}{}_{\rho\sigma} \tag{3.1.11}$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{3.1.12}$$

3.1.2 Torsion Induced Axion

In the previous section, we showcased a string inspired model that gives rise to contorted gravity in four spacetime dimensions. These results tell us that this theory is completely analogous with the one studies in the first two chapters. Here, the Kalb-Ramond field strength $\mathcal{H}_{\lambda\mu\nu}$ plays the role of torsion and the Bianchi identity (3.1.10) replaces Equation (2.3.1) as the constraint we use. Therefore, in a completely analogous procedure, we can enforce the constraint given by the Bianchi equation in the path integral formulation. We have the partition function:

$$Z = \int \mathcal{D}\boldsymbol{g}\,\mathcal{D}\boldsymbol{\mathcal{H}}\, e^{iS[\boldsymbol{g},\boldsymbol{\mathcal{H}}]} = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\mathcal{H}_{\lambda\mu\nu} e^{i\int\left(\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right)\sqrt{-g}d^4x} \tag{3.1.13}$$

We focus our attention to the Kalb-Ramond field strength part of the path integral and impose the constraint as a delta function:

$$Z_H = \int \mathcal{D}\mathcal{H}_{\lambda\mu\nu}\delta(\eta^{\mu\nu\rho\sigma}\nabla_{\mu}\mathcal{H}_{\nu\rho\sigma} - \mathcal{G}(\omega, \boldsymbol{A}))e^{-i\int\frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\sqrt{-g}\,d^4x} \tag{3.1.14}$$

[15]This article uses the opposite sign convention for the metric. In this text, the results have been adjusted to our convention, which is the $- + ++$ one.

The constraint can be written as

$$\delta(\eta^{\mu\nu\rho\sigma}\nabla_\mu\mathcal{H}_{\nu\rho\sigma} - \sqrt{-g}\mathcal{G}(\omega,\boldsymbol{A})) = \int \mathcal{D}b e^{i\int b(x)[\epsilon^{\mu\nu\rho\sigma}\nabla_\mu\mathcal{H}_{\nu\rho\sigma} - \mathcal{G}(\omega,\boldsymbol{A})]\sqrt{-g}\,d^4x} \tag{3.1.15}$$

We now use the property $\nabla_\mu\mathcal{H}_{\nu\rho\sigma} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\mathcal{H}_{\nu\rho\sigma})$ and get that

$$\int \mathcal{D}b e^{i\int\left[b(x)\sqrt{-g}\epsilon^{\mu\nu\rho\sigma}\nabla_\mu\mathcal{H}_{\nu\rho\sigma} - b(x)\sqrt{-g}\mathcal{G}(\omega,\boldsymbol{A})\right]d^4x} = \int \mathcal{D}b e^{i\int\left[b(x)\cancel{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}\frac{1}{\cancel{\sqrt{-g}}}\partial_\mu(\sqrt{-g}\,\mathcal{H}_{\nu\rho\sigma}) - b(x)\sqrt{-g}\mathcal{G}(\omega,\boldsymbol{A})\right]d^4x}$$

We can now integrate by parts and get that

$$\delta(\eta^{\mu\nu\rho\sigma}\nabla_\mu\mathcal{H}_{\nu\rho\sigma} - \sqrt{-g}\mathcal{G}(\omega,\boldsymbol{A})) = \int \mathcal{D}b e^{-i\int\left[\partial_\mu b(x)\epsilon^{\mu\nu\rho\sigma}\,\mathcal{H}_{\nu\rho\sigma} + b(x)\mathcal{G}(\omega,\boldsymbol{A})\right]\sqrt{-g}\,d^4x} \tag{3.1.16}$$

and thus the partition function is

$$Z_H = \int \mathcal{D}b\mathcal{D}\mathcal{H}_{\lambda\mu\nu} e^{-i\int\left[\frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} + \partial_\mu b(x)\epsilon^{\mu\nu\rho\sigma}\,\mathcal{H}_{\nu\rho\sigma} + b(x)\mathcal{G}(\omega,\boldsymbol{A})\right]\sqrt{-g}\,d^4x} \tag{3.1.17}$$

We can now complete the squares:

$$Z_H = \int \mathcal{D}b\mathcal{D}\mathcal{H}_{\lambda\mu\nu} e^{-i\int\left[\frac{1}{6}\left(\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} + 6\partial_\mu b(x)\eta^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma} \pm 9\partial_\mu b\,\partial^\mu b\,\eta^{\mu\nu\rho\sigma}\eta_{\mu\nu\rho\sigma}\right) + b\,\mathcal{G}(\omega,\boldsymbol{A})\right]\sqrt{-g}\,d^4x} \tag{3.1.18}$$

and carry out the integration on $\mathcal{H}$, absorbing the resulting constant in the measure $\mathcal{D}b$:

$$Z_H = \int \mathcal{D}b e^{-i\int\left[-\frac{3}{2}\partial_\mu b\,\partial^\mu b\,\eta^{\mu\nu\rho\sigma}\eta_{\mu\nu\rho\sigma} + b\,\mathcal{G}(\omega,\boldsymbol{A})\right]\sqrt{-g}\,d^4x} \tag{3.1.19}$$

We have that $\eta_{\mu\nu\kappa\lambda}\eta^{\mu\nu\kappa\lambda} = -24$ and thus

$$Z_H = \int \mathcal{D}b e^{-i\int\left[36\partial_\mu b\,\partial^\mu b + b\,\mathcal{G}(\omega,\boldsymbol{A})\right]\sqrt{-g}\,d^4x} \tag{3.1.20}$$

In order to normalize the kinetic term of the b field, we redefine it to be $b \to \frac{1}{6\sqrt{2}}b$ and thus

$$Z_H = \int \mathcal{D}b e^{-i\int\left[\frac{1}{2}\partial_\mu b\,\partial^\mu b + \frac{1}{6\sqrt{2}}b\,\mathcal{G}(\omega,\boldsymbol{A})\right]\sqrt{-g}\,d^4x} \tag{3.1.21}$$

and thus the full action becomes:

$$\boxed{S_B = \int\left(\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu b\,\partial^\mu b - \frac{a'\sqrt{2}}{192\kappa}b\left(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\right)\sqrt{-g}\,d^4x} \tag{3.1.22}$$

3.1.3 The Hirzebruch-Pontryagin Topological Density

It is quite obvious that the term of interest in the resulting action is the term that couples the axion with the gravitational and gauge anomalies. Therefore, we focus our attention to the anomalies term, which is called the *Hirzebruch-Pontryagin topological density* [15]:

$$\sqrt{-g}\left(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu}\tilde{F}^{\mu\nu}\right) = \sqrt{-g}\nabla_\mu\mathcal{K}^\mu_{mixed}(\omega,A) = \partial_\mu\left(\sqrt{-g}\,\mathcal{K}^\mu_{mixed}(\omega,A)\right) \tag{3.1.23}$$

As we can see, this term is a total derivative. The term $\mathcal{K}^\mu_{mixed}(\omega,A)$ is called the mixed (gravitational and gauge) anomaly current density and can be expressed as a function of the spin connection ω and the gauge fields A. The purpose of this chapter is to study the inflation period of the universe, and therefore we may assume that the gauge fields are vanishing. Thus, we are left with the pure gravitational anomaly and the corresponding current density $\mathcal{K}^\mu(\omega)$ which we can express in terms of the spin connection:

$$\sqrt{-g}\left(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right) = \sqrt{-g}\nabla_\mu\mathcal{K}^\mu(\omega) = \partial_\mu\left(\sqrt{-g}\mathcal{K}^\mu(\omega)\right) = 2\partial_\mu\left[\epsilon^{\mu\nu\lambda\rho}\omega_\nu{}^{ab}\left(\partial_\lambda\omega_{\rho ab} + \frac{2}{3}\omega_{\lambda a}{}^c\omega_{\rho cb}\right)\right] \tag{3.1.24}$$

This is also called the gravitational Chern-Simons term. The effective string action at before and during the inflation era can thus be written as

$$S_B = \int \left(\frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu b \, \partial^\mu b + \frac{a' \sqrt{2}}{192\kappa} (\partial_\mu b) \, \mathcal{K}^\mu \right) \sqrt{-g} d^4 x \tag{3.1.25}$$

where we did a partial integration on the last term. Therefore, the action can be split into three parts:

$$S_B = S_{grav} + S_b + S_{b-grav} \tag{3.1.26}$$

where $S_{grav} = -\int \frac{1}{2\kappa^2} R\sqrt{-g} \, d^4x$ is the standard Einstein-Hilbert action, $S_b = -\int \frac{1}{2} \partial_\mu b \, \partial^\mu b \sqrt{-g} \, d^4x$ is the axion kinetic energy term and the last term, S_{b-grav}, which is the axion & gravitational anomaly term given by

$$S_{b-grav} = \int \frac{a' \sqrt{2}}{192\kappa} (\partial_\mu b) \, \mathcal{K}^\mu \sqrt{-g} \, d^4 x \tag{3.1.27}$$

3.1.4 The Cotton & Stress-Energy Tensors

Having this action, we can examine what the stress-energy tensor is. For the axions, the "matter" stress-energy tensor is found using the standard definition:

$$\boxed{T^b_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_b}{\delta g^{\mu\nu}} = \partial_\mu b \partial_\nu b - \frac{1}{2} g_{\mu\nu} \partial_\rho b \partial^\rho b} \tag{3.1.28}$$

In the absence of the gravitational Chern-Simons term, we'd get the usual Einstein equations. However here, this new term can also be varied with respect to the metric and give a non-trivial result. More specifically, variation of the Chern-Simons term gives us the *Cotton tensor*, which is defined as

$$\boxed{\mathcal{C}_{\mu\nu} = -\frac{1}{4\sqrt{-g}} \frac{\delta S_C}{\delta g^{\mu\nu}}} \tag{3.1.29}$$

where

$$S_C = \int b R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \sqrt{-g} \, d^4 x \tag{3.1.30}$$

such that $S_{b-grav} = \frac{a'\sqrt{2}}{192\kappa} S_C$. After calculating the variation, we get that the Cotton tensor is

$$\begin{aligned} \mathcal{C}^{\mu\nu} &= -\frac{1}{2} \left[\partial_\sigma b \left(\eta^{\sigma\mu\rho\lambda} \nabla_\rho R^\nu{}_\lambda + \eta^{\sigma\nu\rho\lambda} \nabla_\rho R^\mu{}_\lambda \right) + \partial_\sigma \partial_\tau b \left(\tilde{R}^{\tau\mu\sigma\nu} + \tilde{R}^{\tau\nu\sigma\mu} \right) \right] \\ &= -\frac{1}{2} \left[\nabla_\lambda \left(\partial_\sigma b \tilde{R}^{\lambda\mu\sigma\mu} \right) + (\mu \leftrightarrow \nu) \right] \end{aligned} \tag{3.1.31}$$

An important property of the Cotton tensor is that it is traceless:

$$g_{\mu\nu} \mathcal{C}^{\mu\nu} = 0 \tag{3.1.32}$$

The Einstein equations then take the form:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{a' \kappa \sqrt{2}}{24} \mathcal{C}^{\mu\nu} + \kappa^2 T^{\mu\nu}_b \tag{3.1.33}$$

In standard gravity theories, it is the matter stress-energy tensor that gets conserved, i.e. $\nabla_\mu T^{\mu\nu}_b = 0$. However, it is easy to check that the Cotton tensor is not conserved. In fact, we get that

$$\nabla_\mu \mathcal{C}^{\mu\nu} = -\frac{1}{8} \partial^\nu b R^{\rho\lambda\sigma\kappa} \tilde{R}_{\rho\lambda\sigma\kappa} \tag{3.1.34}$$

It becomes obvious, then, that the conservation of the matter stress-energy tensor breaks down in this scenario. This, indeed, makes sense, as the matter field (the axions) now exchange energy with the gravitational field, something that doesn't appear in standard General Relativity. Instead, there is a modified, more general stress-energy tensor that gets conserved and is defined as

$$\kappa^2 \tilde{T}^{\mu\nu}_{total} = \frac{a' \kappa \sqrt{2}}{24} \mathcal{C}^{\mu\nu} + \kappa^2 T^{\mu\nu}_b \tag{3.1.35}$$

with the conservation law then being

$$\nabla_\mu \tilde{T}^{\mu\nu}_{total} = 0 \tag{3.1.36}$$

3.2 Gravitational Wave Condensate Induced Inflation

In this section, we'll see how gravitational waves in the early universe can be a suitable explanation for a period of inflation for the universe. To that end, we'll use the so-called Running Vacuum Model as our Cosmological model of the universe. Then, we'll consider gravitational wave perturbations and quantize them on a classical background in order to calculate the vacuum expectation value of the gravitational Chern-Simons term. Our endgoal will be the calculation of the vacuum energy density of this model, which we'll show that in the context of the Running Vacuum Model can explain the inflation of our universe without the need for additional fields commonly considered, such as the inflaton field.

3.2.1 Running Vacuum Model

It is widely known that matter only makes up about 5% of the energy in the universe. Another 26% is attributed to what is known as "dark matter", i.e. a form of matter that does not appear to interact with ordinary matter but has a gravitational effect. This dark matter is widely assumed to be "cold" or, in other words, moving slowly compared to the speed of light. The final 68% of the universe's energy is the so-called "dark energy", which is thought to be responsible for the observed accelerating expansion of the universe. This dark energy arises from the cosmological constant Λ of the Einstein equations. The mathematical model that is considered to be the "standard model" of cosmology, currently used in experiments, takes all of the above into account and is called the ΛCDM model (Λ-Cold Dark Matter). However, in recent years, the ΛCDM model has proven to not be perfect. The biggest threat to the ΛCDM is the discrepancy present in the measurements of the Hubble constant, known as the "Hubble tension". While the possibility of the Hubble tension being of statistical nature has not been excluded, it is instructive, if not necessary, to consider other cosmological models that can explain the current experimental data. One of these models is the "Running Vacuum Model" (from here on RVM). In the ΛCDM model, the vacuum energy density is given as

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \tag{3.2.1}$$

and is a constant. In the RVM, it is assumed that the vacuum energy density "runs" smoothly with cosmic time. Hence, we have a "running" vacuum energy density $\rho_{RVM}(t)$. The RVM model originates from arguments and calculations made in the context of the Renormalization Group in Quantum Field Theory in curved spacetime. However, for our purposes it suffices to consider it as a purely phenomenological model. In this model, we can assume that the running vacuum energy density can be expressed as a perturbative expansion of even powers of the Hubble parameter [16]:

$$\boxed{\rho_{RVM}(H) = \frac{\Lambda(H^2)}{8\pi G} = \frac{3}{8\pi G}\left(c_0 + \nu H^2 + \beta\frac{H^4}{H_I^2} + \cdots\right)} \tag{3.2.2}$$

where c_0, ν, β[16] are real constants and $H_I \sim 10^{-5} M_{Pl}$ is the inflationary scale, with M_{Pl} being the Planck mass.

3.2.2 Gravitational Wave Condensate

Lets consider tensor perturbations of the FLRW metric:

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{3.2.3}$$

It is well known that these tensor perturbations are gravitational waves, which come in two different polarizations. In the so-called linear polarization basis, the tensor perturbation can be expressed as [17]

$$h_{ij} = h_+ \epsilon_{ij}^{(+)} + h_\times \epsilon_{ij}^{(\times)} \tag{3.2.4}$$

The polarization tensors are defined as

$$\epsilon_{ij}^{(+)} = [e_1(\vec{k})]_i [e_1(\vec{k})]_j - [e_2(\vec{k})]_i [e_2(\vec{k})]_j \tag{3.2.5}$$

$$\epsilon_{ij}^{(\times)} = [e_1(\vec{k})]_i [e_2(\vec{k})]_j - [e_1(\vec{k})]_j [e_2(\vec{k})]_i \tag{3.2.6}$$

[16] In the original source, α is used instead of β for the coefficient of the H^4 term. The change in this text has been made to avoid confusion, as this parameter appears alongside the scale factor a.

where

$$e_3(\vec{k}) = \frac{\vec{k}}{|\vec{k}|} \tag{3.2.7}$$

and the three unit vectors e_1, e_2, e_3 are orthogonal to each other. We have the freedom to choose the z-axis as the direction of the propagation of the gravitational wave and thus the unit vectors take the values

$$e_1 = (1, 0, 0) \tag{3.2.8}$$
$$e_2 = (0, 1, 0) \tag{3.2.9}$$
$$e_3 = (0, 0, 1) \tag{3.2.10}$$

As such, the perturbation tensor now can be written as a traceless, symmetric tensor:

$$h = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.2.11}$$

Assuming an action of the form

$$S = \int \left(\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - AbR_{CS} \right) \sqrt{-g}\, d^4x \tag{3.2.12}$$

where $R_{CS} = R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ is the gravitational Chern-Simons term, the linearized Einstein equations take the form:

$$\Box h_+ = +\frac{4A\kappa^2}{a^2}\left(2\dot{a}\dot{b} + a\ddot{b}\right)\partial_t\partial_z h_\times + \frac{4A\kappa^2\dot{b}}{a}\partial_t^2\partial_z h_\times - \frac{4A\kappa^2\dot{b}}{a^3}\partial_z^3 h_\times \tag{3.2.13}$$
$$\Box h_\times = -\frac{4A\kappa^2}{a^2}\left(2\dot{a}\dot{b} + a\ddot{b}\right)\partial_t\partial_z h_+ - \frac{4A\kappa^2\dot{b}}{a}\partial_t^2\partial_z h_+ + \frac{4A\kappa^2\dot{b}}{a^3}\partial_z^3 h_+ \tag{3.2.14}$$

where the box operator is defined as

$$\Box = -\partial_t^2 - 3\frac{\dot{a}}{a}\partial_t + \frac{1}{a^2}\partial_z^2 \tag{3.2.15}$$

We can easily see that these two polarizations are coupled to each other due to the presence of the KR-axion field that couples to the CS-term. We can get a pair of decoupled polarizations by switching to the chiral basis:

$$h_{L,R} = \frac{1}{\sqrt{2}}\left(h_+ \pm i h_\times\right) \tag{3.2.16}$$

In this basis, the wave equations (3.2.13) & (3.2.14) become

$$\Box h_L = -\frac{4iA\kappa^2}{a^2}\left(2\dot{a}\dot{b} + a\ddot{b}\right)\partial_t\partial_z h_L - \frac{4iA\kappa^2\dot{b}}{a}\partial_t^2\partial_z h_L + \frac{4iA\kappa^2\dot{b}}{a^3}\partial_z^3 h_L \tag{3.2.17}$$
$$\Box h_R = +\frac{4iA\kappa^2}{a^2}\left(2\dot{a}\dot{b} - a\ddot{b}\right)\partial_t\partial_z h_R + \frac{4iA\kappa^2\dot{b}}{a}\partial_t^2\partial_z h_R - \frac{4iA\kappa^2\dot{b}}{a^3}\partial_z^3 h_R \tag{3.2.18}$$

Furthermore, we can calculate the gravitational Chern-Simons term in terms of the perturbation h up to second order [17]:

$$R_{CS} = \frac{4i}{a^3}\left[(\partial_z^2 h_L\partial_z\partial_t h_R + a^2\partial_t^2 h_L\partial_z\partial_t h_R + a\dot{a}\partial_t h_L\partial_z\partial_t h_R) - (L \leftrightarrow R)\right] + \mathcal{O}(h^4) \tag{3.2.19}$$

As we can see, this term would vanish if the two polarizations satisfied the same equation. However, that is not the case for these gravitational waves as the wave equations of the two polarizations differ in sign, as seen above, due to contributions from the Cotton tensor. Hence, the CS-term survives. This phenomenon is called "cosmological birefringence" [17]. We can now treat the gravitational wave perturbations as quantum operators, i.e. we can proceed into a second quantization scheme for the tensor perturbation. We saw that this perturbation can be analyzed into two scalar polarization fields and thus we only need to quantize those. This process is done in detail in [17][17].

[17]The authors here find a factor of two difference in their result compared to earlier efforts, as they include more terms. This correction will be applied in this text when referencing the value of $\langle R_{CS}\rangle_I$ from older sources.

After quantization, we're in position to calculate the vacuum expectation value of the gravitational CS-term during inflation. In the inflationary era, we have that the scale factor is approximately

$$a(t) \sim exp(H_I t) \tag{3.2.20}$$

where H_I is the Hubble parameter, which is approximately constant. By doing the calculations, we find that [17]

$$\langle R_{CS} \rangle_I = -\mathcal{N}_I \frac{A\kappa^4 \mu^4}{\pi^2} \dot{\bar{b}}_I H_I^3 \tag{3.2.21}$$

where $\mathcal{N}_I$ is the density of gravitational wave sources during inflation and μ is the UV energy cutoff of the effective field theory we're working with, while $\dot{b}_I$ symbolizes the axion field during the inflation era, which can be approximately calculated to be [18]

$$\dot{\bar{b}}_I \sim \sqrt{2\epsilon} M_{Pl} H_I \tag{3.2.22}$$

where ϵ is a phenomenological parameter that we fix as

$$\epsilon \sim 10^{-2} \tag{3.2.23}$$

This means that, in total, the condensate $\langle R_{CS} \rangle_I$ is proportional to H_I^4. Integrating this we get an expression for the axion field:

$$\bar{b}_I(t) = \bar{b}_I(0) + \sqrt{2\epsilon} H t M_{Pl} \tag{3.2.24}$$

The initial condition $\bar{b}(0)$ cannot be predicted in the context of our effective field theory, but requires the full string theory model. Nonetheless, it is possible [15] to find a range of phenomenologically acceptable values. A suitable choice is [18]

$$\bar{b}(0) \sim 10 M_{Pl} \tag{3.2.25}$$

3.2.3 Vacuum Energy Density

We now want to calculate the various contributions to the vacuum energy density. First of all, there are two contributions coming from the axion field and the gravitational anomaly. This can be seen from Equation (3.1.33), where the modified stress energy tensor is comprised of two parts: one for the axionic matter, and the Cotton tensor for the gravitational anomaly. Each of these parts contributes to the vacuum energy density. In our gravitational quantization scheme, a rough approach to calculate the vacuum energy density is to calculate the modified stress-energy tensor over the quantized gravitational wave perturbations. This can be done for the Cotton tensor, which is given in terms of the Riemann tensor and its derivatives. This is done in [18] and the resulting vacuum energy density related to the Cotton tensor is found to be

$$\rho_{gCS} = -1.484 \epsilon M_{Pl}^2 H_I^2 \tag{3.2.26}$$

Then, the conservation law (3.1.36) relates the Cotton tensor to the axionic stress-energy tensor, leading us to the relation

$$\rho_b \simeq -\frac{2}{3} \rho_{gCS} \tag{3.2.27}$$

where ρ_b is the vacuum energy density related to the axionic matter. Thus, we find that [18]

$$\rho_b \simeq \epsilon M_{Pl}^2 H_I^2 \tag{3.2.28}$$

and, in total,

$$\rho_b + \rho_{gCS} = \frac{1}{3} \rho_{gCS} \simeq -0.496 \epsilon M_{Pl}^2 H_I^2 \tag{3.2.29}$$

i.e. a negative quantity. However, this is not the full picture, the reason being that we had not proceeded into our quantization scheme when we produced Equation (3.1.33). The proper approach is to expand the gravitational Chern-Simons term around its vacuum expectation value and write the action as [15]:

$$S = \int \left(\frac{R}{2\kappa^2} - \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - \frac{a'\sqrt{2}}{192\kappa} \bar{b}(x) \langle R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \rangle_I - \frac{a'\sqrt{2}}{192\kappa} : b(x) R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} : \right) \sqrt{-g}\, d^4x \tag{3.2.30}$$

We can see that this adds an extra term to the action, a linear potential term for the axion that contains the gravitational wave condensate $\langle R_{CS}\rangle_I$ [19]:

$$S_\Lambda = -\int \left(\frac{a'\sqrt{2}}{192\kappa}\bar{b}(x)\langle R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\rangle_I\right)\sqrt{-g}\,d^4x = -\int \left(8.6\times 10^10 0\sqrt{\epsilon}\frac{|\bar{b}(0)|}{M_{Pl}}H_I^4\right)\sqrt{-g}\,d^4x \tag{3.2.31}$$

Therefore, another vacuum energy density term is produced, one that was initially overlooked:

$$\rho_\Lambda = 8.6\times 10^{10}\sqrt{\epsilon}\frac{|\bar{b}(0)|}{M_{Pl}}H^4 \tag{3.2.32}$$

The full vacuum energy density expression is the sum of all the above:

$$\boxed{\rho_{vac}(H) = \rho_b + \rho_{gCS} + \rho_\Lambda = -\frac{1}{2}\epsilon M_{Pl}^2 H^2 + 8.6\times 10^{10}\sqrt{\epsilon}\frac{|\bar{b}(0)|}{M_{Pl}}H^4} \tag{3.2.33}$$

3.2.4 Inflation

We can easily see that the final expression of the vacuum energy density is consistent with the Running Vacuum Model with constants $c_0 = 0$, $v < 0$ and $\beta > 0$. In this subsection we'll show how such an expression for the vacuum energy density leads to inflation. Let us assume a vacuum energy density of the form (3.2.2) with $c_0 = 0$, $v < 0$ and $\beta > 0$. The conservation of the total stress-energy tensor of vacuum matter and radiation [19] leads to the differential equation

$$\dot{H} + \frac{3}{2}(1+\omega_m)H^2\left(1 - v - \beta\frac{H^2}{H_I^2}\right) = 0 \tag{3.2.34}$$

where $\omega_m = \frac{p_m}{\rho_m}$ and the subscript "m" refers to both matter and radiation. Solving this equation, we get a solution for the Hubble parameter as a function of the FLRW scale factor $a(t)$:

$$H(a) = \left(\frac{1-v}{\beta}\right)^{\frac{1}{2}}\frac{H_I}{\sqrt{Da^{3(1-v)(1+\omega_m)}+1}} \tag{3.2.35}$$

where $D > 0$ is an integration constant.

Proof

We will prove this final expression by solving the differential equation (3.2.34). We have that:

$$\dot{H} + \frac{3}{2}(1+\omega_m)H^2\left(1 - v - \beta\frac{H^2}{H_I^2}\right) = 0 \Rightarrow \frac{dH}{dt} + \frac{3}{2}(1+\omega_m)(1-v)H^2\left(1 - \frac{\beta}{1-v}\frac{H^2}{H_I^2}\right) = 0$$

We can then use the fact that $\frac{d}{dt} = \frac{da}{dt}\frac{d}{da} = \dot{a}\frac{d}{da}$ and get that

$$\frac{dH}{dt} + \frac{3}{2}(1+\omega_m)(1-v)H^2\left(1 - \frac{\beta}{1-v}\frac{H^2}{H_I^2}\right) = 0 \Rightarrow \dot{a}\frac{dH}{da} + \frac{3}{2}(1+\omega_m)(1-v)H^2\left(1 - \frac{\beta}{1-v}\frac{H^2}{H_I^2}\right) = 0$$

We know that the Hubble parameter H is defined as $H = \frac{\dot{a}}{a}$, so we can replace $\dot{a}$ in the above equation, and also proceed to divide by H^2:

$$\dot{a}\frac{dH}{da} + \frac{3}{2}(1+\omega_m)(1-v)H^2\left(1 - \frac{\beta}{1-v}\frac{H^2}{H_I^2}\right) = 0 \Rightarrow \frac{a}{H}\frac{dH}{da} + \frac{3}{2}(1+\omega_m)(1-v)\left(1 - \frac{\beta}{1-v}\frac{H^2}{H_I^2}\right) = 0$$

Proceeding, we can define a normalized Hubble parameter, $\tilde{H} = \frac{H}{H_I}$, thus getting the equation:

$$\frac{a}{\tilde{H}}\frac{d\tilde{H}}{da} + \frac{3}{2}(1+\omega_m)(1-v)\left(1 - \frac{\beta}{1-v}\tilde{H}^2\right) = 0$$

This is the equation we need to solve. This is a first order differential equation, so we just need to separate the two variables:

$$\frac{d\tilde{H}}{\tilde{H}\left(1-\frac{\beta}{1-v}\tilde{H}^2\right)} = -\frac{3}{2}(1+\omega_m)(1-v)\frac{da}{a}$$

We can then write the left part as:

$$\frac{1}{\tilde{H}\left(1-\frac{\beta}{1-v}\tilde{H}^2\right)} = \frac{1-\frac{\beta}{1-v}\tilde{H}^2+\frac{\beta}{1-v}\tilde{H}^2}{\tilde{H}\left(1-\frac{\beta}{1-v}\tilde{H}^2\right)} = \frac{1}{\tilde{H}} + \frac{\beta}{1-v}\frac{\tilde{H}}{\left(1-\frac{\beta}{1-v}\tilde{H}^2\right)}$$

and thus the differential equation becomes:

$$\frac{d\tilde{H}}{\tilde{H}} + \frac{\beta}{1-v}\frac{\tilde{H}}{\left(1-\frac{\beta}{1-v}\tilde{H}^2\right)}d\tilde{H} = -\frac{3}{2}(1+\omega_m)(1-v)\frac{da}{a}$$

In this form, we can easily perform the integration of both sides:

$$\int\frac{d\tilde{H}}{\tilde{H}} + \frac{\beta}{1-v}\int\frac{\tilde{H}}{\left(1-\frac{\beta}{1-v}\tilde{H}^2\right)}d\tilde{H} = -\frac{3}{2}(1+\omega_m)(1-v)\int\frac{da}{a}$$

To solve this integral, it is conveniant to make the following change of variable:

$$\frac{\beta}{1-v}\tilde{H}^2 = \hat{H}^2 \Rightarrow \tilde{H} = \sqrt{\frac{1-v}{\beta}}\hat{H} \Rightarrow d\tilde{H} = \sqrt{\frac{1-v}{\beta}}d\hat{H}$$

Then, the differential equation becomes:

$$\int\frac{d\hat{H}}{\hat{H}} + \int\frac{\hat{H}}{\left(1-\hat{H}^2\right)}d\hat{H} = -\frac{3}{2}(1+\omega_m)(1-v)\int\frac{da}{a}$$

We can also write $\hat{H}d\hat{H} = \frac{1}{2}d\hat{H}^2$ and thus get the differential equation:

$$\int\frac{d\hat{H}}{\hat{H}} + \frac{1}{2}\int\frac{1}{\left(1-\hat{H}^2\right)}d\hat{H}^2 = -\frac{3}{2}(1+\omega_m)(1-v)\int\frac{da}{a}$$

In this form, the integration can be performed immediately, giving us the solution:

$$\ln\hat{H} + \ln\left(\frac{1}{\sqrt{1-\hat{H}^2}}\right) = -\frac{3}{2}(1+\omega_m)(1-v)\ln a + C$$

where C is a constant of integration. We can write $C = \ln D$ and thus the equation above can be rewritten as:

$$\ln\left(\frac{\hat{H}}{\sqrt{1-\hat{H}^2}}\right) = \ln\left(Da^{-\frac{3}{2}(1+\omega_m)(1-v)}\right)$$

Thus, we get that:

$$\frac{\hat{H}}{\sqrt{1-\hat{H}^2}} = Da^{-\frac{3}{2}(1+\omega_m)(1-v)} \Rightarrow \frac{1}{\sqrt{\frac{1}{\hat{H}^2}-1}} = Da^{-\frac{3}{2}(1+\omega_m)(1-v)} \Rightarrow \frac{1}{\hat{H}^2} - 1 = Da^{3(1+\omega_m)(1-v)}$$

Finally, we revert back to the original Hubble parameter, which is given as $\hat{H}^2 = \frac{\beta}{1-v}\left(\frac{H}{H_I}\right)^2$ and thus get:

$$\frac{1-v}{\beta}\frac{1}{\left(\frac{H}{H_I}\right)^2} = Da^{3(1+\omega_m)(1-v)} + 1 \Rightarrow \left(\frac{H}{H_I}\right)^2 = \frac{1-v}{\beta}\frac{1}{Da^{3(1+\omega_m)(1-v)}+1}$$

Finally, by simply taking the square root in both sides of the equation we get the final result:

$$H = \sqrt{\frac{1-v}{\beta}} \frac{H_I}{\sqrt{Da^{3(1+\omega_m)(1-v)} + 1}}$$

We have, therefore, proved the desired equation.

Since v is negative and $\omega_m = 0$ in the vacuum, the power of a in the superscript is positive. If we consider $a \ll 1$, as is the case in the early universe, we get that $Da^{3(1-v)(1+\omega_m)} \ll 1$ also, i.e. it is neglectable, resulting in a mostly constant Hubble parameter $H \simeq H_I$.

3.3 Summary & Conclusion

In this chapter we considered a string-inspired model in which the field strength of the Kalb-Ramond field plays the role of torsion. We showed that, in a completely analogous way to the Einstein-Cartan theory, this torsion field induces an axionic field. The big difference, however, is the presence of a gravitational anomaly (with a string-theoretical origin), which couples to this new axionic field, giving us a new term. This term gives rises to the Cotton tensor and leads to the modification of the Einstein equations. The Cotton tensor is not conserved, as it expresses the exchange of energy between the axions and the gravitational field itself. Thus, it is necessary to define a new, generalized stress-energy tensor which includes the Cotton tensor.

Having all of the above in mind, we then explored how such a string-inspired theory can lead to a cosmological model that explains inflation without the need to introduce a special field, such as the inflaton. To do this, we worked in the so-called Running Vacuum Model of cosmology, which assumes that the vacuum energy density is a function of even powers of the Hubble parameter. Then, working on our string-inspired model, we examined the presence of gravitational waves in the early universe. The reason for this specific choice is the difference in behavior exhibited by the two different polarizations of gravitational waves. This left/right asymmetry leads to a non-vanishing anomaly term (Chern-Simons term), and thus, in a second quantization scheme these gravitational waves form a condensate, which contributes a higher order term (H^4) to the vacuum energy density, besides another H^2 one. This RVM-type vacuum energy density is then showed to lead to inflation in a natural way.

Of course, this string-inspired theory and the resulting cosmological model can be extended beyond the inflationary era of the universe, up until the modern era. In [15], the later eras of the universe are explored in this kind of string-inspired theory and RVM cosmology. At the end of inflation, fermionic matter is generated. The presence of the Kalb-Ramond axion field (which is induced by the presence of torsion) can be shown to explain the matter/antimatter asymmetry observed in the universe, and also to break both Lorentz and CPT symmetries. Furthermore, it is also possible for the axion to acquire a mass in later stages of the universe, making it a candidate for Dark Matter. Therefore, in this model of the universe, torsion plays a central role in its evolution. This is in complete disagreement with our conventional, standard theories, which are based on General Relativity and assume a vanishing torsion. Unfortunately, the presence of torsion is not something that has been experimentally detected yet. However, if that day ever comes, the implications of its presence have already been shown to be of utmost importance to the universe as we know it today and our presence itself.

Bibliography

[1] J. Schwichtenberg, *Physics from symmetry* (Springer International Publishing, 2018).

[2] S. M. Carroll, *Spacetime and geometry: an introduction to general relativity* (Cambridge University Press, 2019).

[3] J. Yepez, "Einstein's vierbein field theory of curved space", arXiv: General Relativity and Quantum Cosmology (2011).

[4] M. Duncan, N. Kaloper, and K. Olive, "Axion hair and dynamical torsion from anomalies", Nuclear Physics B **387**, 215–235 (1992).

[5] M. Nakahara, *Geometry, topology and physics, second edition* (Taylor & Francis Incorporated, 2003).

[6] M. Gasperini, *Theory of gravitational interactions* (Springer International Publishing, 2017).

[7] P. Cvitanović, *Group theory: birdtracks, lie's, and exceptional groups* (Princeton University Press, 2008).

[8] D. R. Brill and J. M. Cohen, "Cartan frames and the general relativistic dirac equation", Journal of Mathematical Physics **7**, 238–243 (1966).

[9] V. Ferrari, L. Gualtieri, and P. Pani, *General relativity and its applications: black holes, compact stars and gravitational waves* (CRC Press, Dec. 2020).

[10] G. de Berredo-Peixoto, L. Freidel, I. Shapiro, and C. de Souza, "Dirac fields, torsion and barbero-immirzi parameter in cosmology", Journal of Cosmology and Astroparticle Physics **2012**, 017 (2012).

[11] N. E. Mavromatos, P. Pais, and A. Iorio, "Torsion at different scales: from materials to the universe", Universe **9**, 516 (2023).

[12] N. Popławski, "Nonsingular, big-bounce cosmology from spinor-torsion coupling", Physical Review D **85**, 10.1103/physrevd.85.107502 (2012).

[13] J. Polchinski, *String theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1998).

[14] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory: 25th anniversary edition*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2012).

[15] S. Basilakos, N. E. Mavromatos, and J. Solà Peracaula, "Gravitational and chiral anomalies in the running vacuum universe and matter-antimatter asymmetry", Phys. Rev. D **101**, 045001 (2020).

[16] N. E. Mavromatos, J. S. Peracaula, and A. Gómez-Valent, "String-inspired running-vacuum cosmology, quantum corrections and the current cosmological tensions", arXiv: General Relativity and Quantum Cosmology (2023).

[17] P. Dorlis, N. E. Mavromatos, and S.-N. Vlachos, "Condensate-induced inflation from primordial gravitational waves in string-inspired chern-simons gravity", arXiv: General Relativity and Quantum Cosmology (2024).

[18] S. Basilakos, N. E. Mavromatos, and J. Solà Peracaula, "Quantum anomalies in string-inspired running vacuum universe: inflation and axion dark matter", Physics Letters B **803**, 135342 (2020).

[19] N. E. Mavromatos, *Lorentz symmetry violation in string-inspired effective modified gravity theories*, 2022.