3. HARMONIC

(P, S, Rayleigh & LOVE)

WAVES

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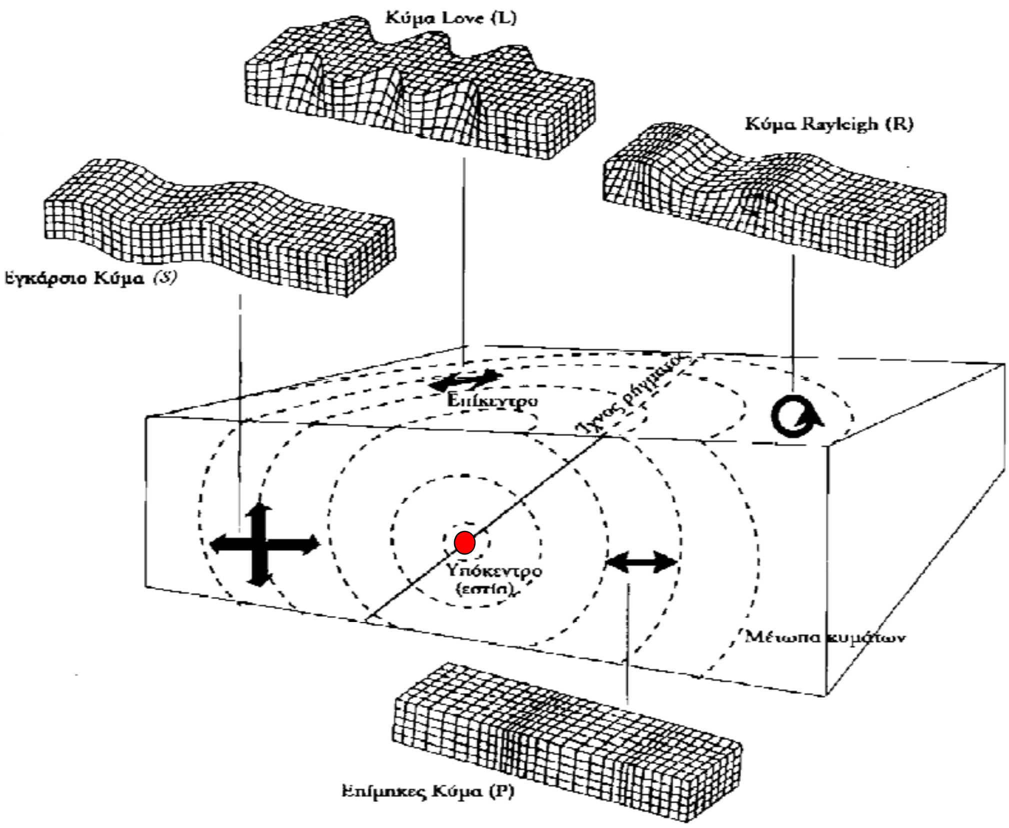
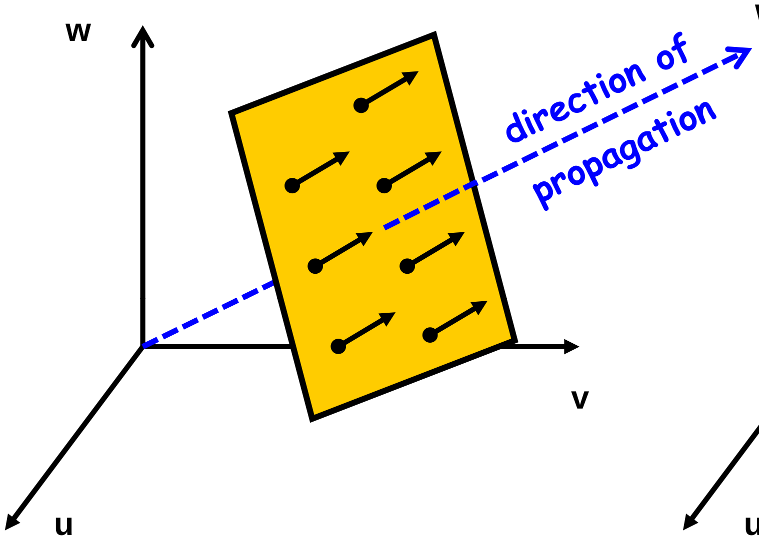
The Seismic motion is due to …….



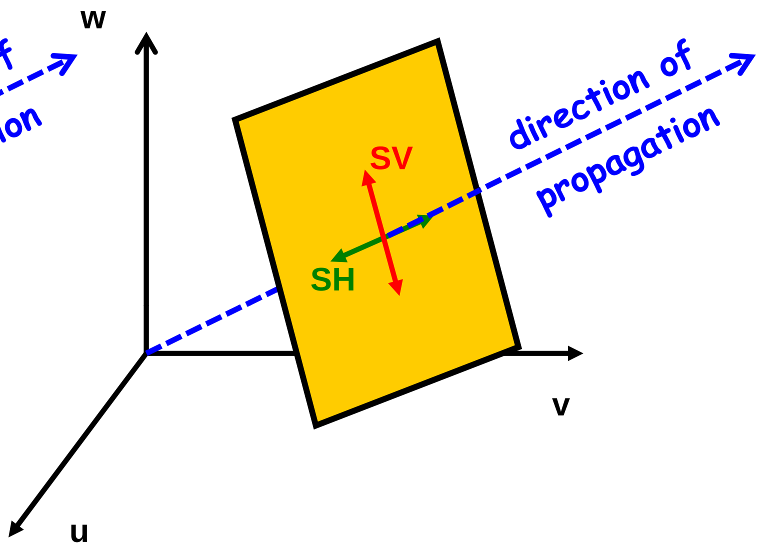
Pressure (P), Shear (S), Rayleigh (R) and other wave types propagating through the ground (soil & bedrock) ...

3.1 ELASTIC WAVES (P & S) in UNIFORM MEDIA

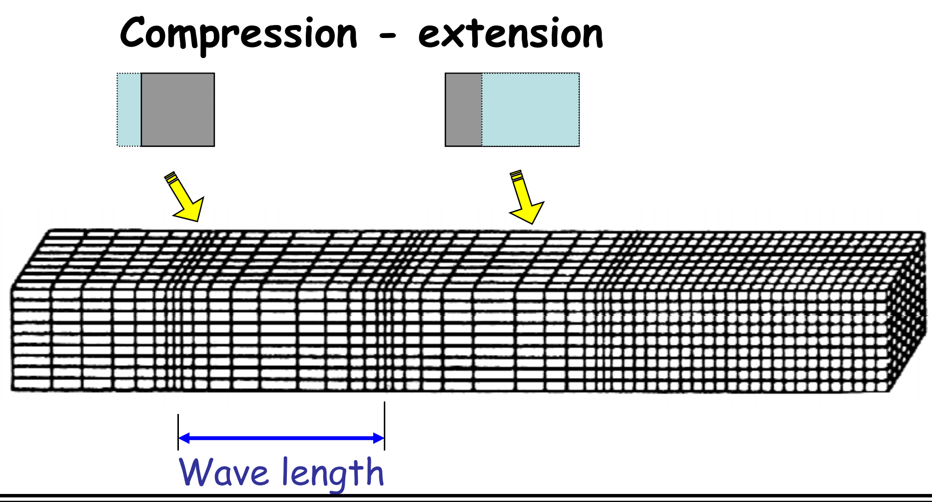
P- WAVE with planar front



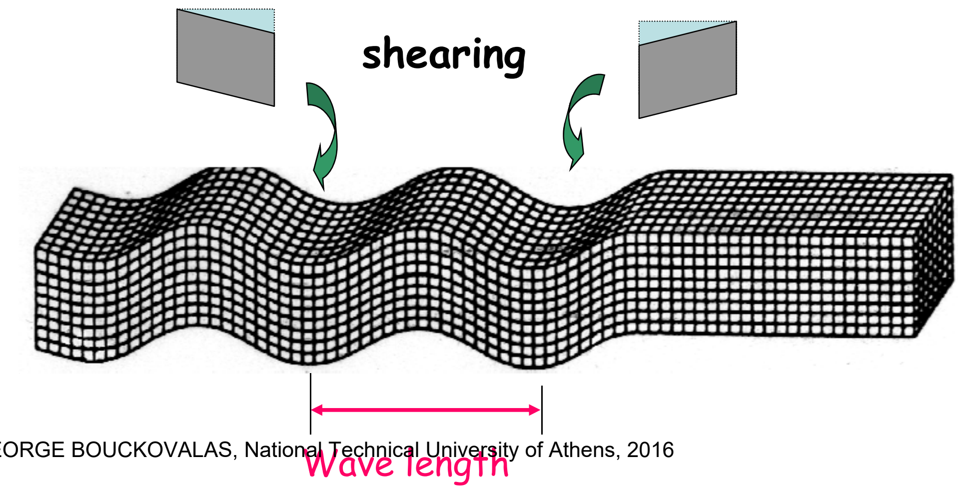
S (SH, SV) - WAVE with planar front



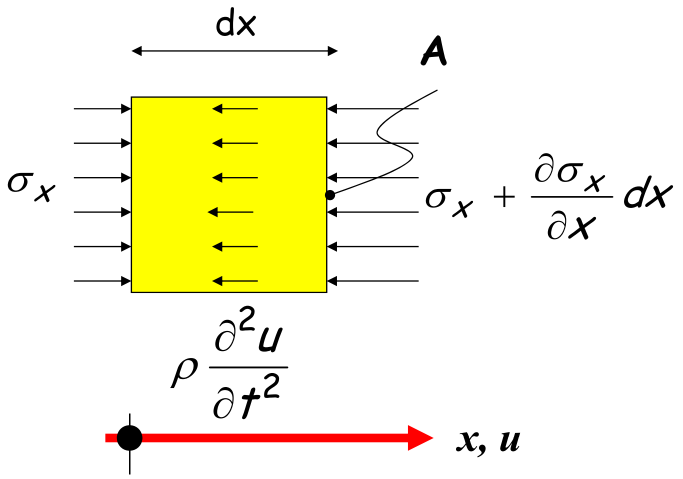
P-waves



S-waves



Wave equation . . .



$$\sigma_x A = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) A + \rho \frac{\partial^2 u}{\partial t^2} dV$$

$$\ldots\ldots \quad \frac{\partial \sigma_x}{\partial x} = -\rho \frac{\partial^2 u}{\partial t^2}$$

$$Όμως \quad \sigma_x = D\varepsilon_x = -D \frac{\partial u}{\partial x}$$

$$και \quad \frac{\partial \sigma_x}{\partial x} = -D \frac{\partial^2 u}{\partial x^2} = -\rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \left(C^2 \frac{\partial^2 u}{\partial x^2}\right)$$

οπου $C = \sqrt{D/\rho}$

General solution...

$$u = f\left(Ct \pm x\right)$$

why;

$$Y = Ct \pm x$$

$$\frac{\partial^2 u}{\partial t^2} = (C)^2 \frac{\partial^2 f(Y)}{\partial Y^2} = (C)^2 f''$$

ενώ $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f(Y)}{\partial Y^2} = f''$$

άρα πράγματι

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$$

Physical meaning...

The problem variables are NOT two independent ones ('x' and 't') but a combined one: $Y = x \pm Ct$

If $Y=Ct-x$

then, $t=0 \Rightarrow Y=-x_o$

$t \neq 0 \Rightarrow Y=Ct-x_t=-x_o$

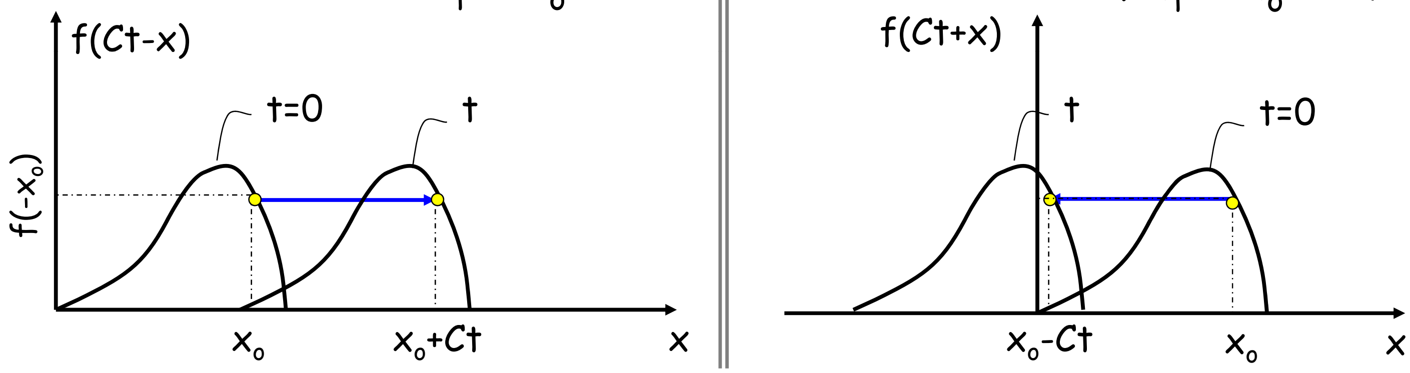
$\& \; x_t = x_o + Ct$

If $Y=Ct+x$

then, $t=0 \Rightarrow Y=x_o$

$t \neq 0 \Rightarrow Y=Ct+x_t=x_o$

$\& \; x_t = x_o - Ct$



P & S - wave propagation velocities in soil & rock formations

S- waves: $V_s = \sqrt{G/\rho}, \quad G = \dfrac{E}{2(1+\nu)}$

P- waves: $V_p = \sqrt{D/\rho}, \quad D = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \dfrac{2G(1-\nu)}{1-2\nu}$

	V_S (m/s)	V_P (m/s)	
		dry	saturated
loose-recent SOIL deposits	<400	<1000	≈1500
stiff SOILS - soft ROCKS	400-800	800-1600	1500-2000
Rocks	>800	>1600	>2000

Vibration velocity (at a point) V

$$u = f(x \pm Ct)$$

$$V = \dot{u} = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial X^*}\frac{\partial X^*}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial X^*}\frac{\partial X^*}{\partial t} \Rightarrow$$

$$V = \left(-\varepsilon_x\right)\left(\pm C\right)$$

$$\acute{\alpha}\rho\alpha \quad V = \mp \frac{\sigma_x}{D} C \neq C$$

Vibration velocity (at a point) V

ATTENTION:

The velocity of VIBRATION is different

(2 to 3 orders of magnitude lower!)

than the velocity of PROPAGATION

$$V = \left(-\varepsilon_x\right)\left(\pm C\right)$$

$$\acute{\alpha}\rho\alpha \quad V = \mp \frac{\sigma_x}{D} C \neq C$$

Special Case: Harmonic waves

$$f_a\left(Ct+x\right)=Ae^{i\frac{\omega}{C}(Ct+x)}$$

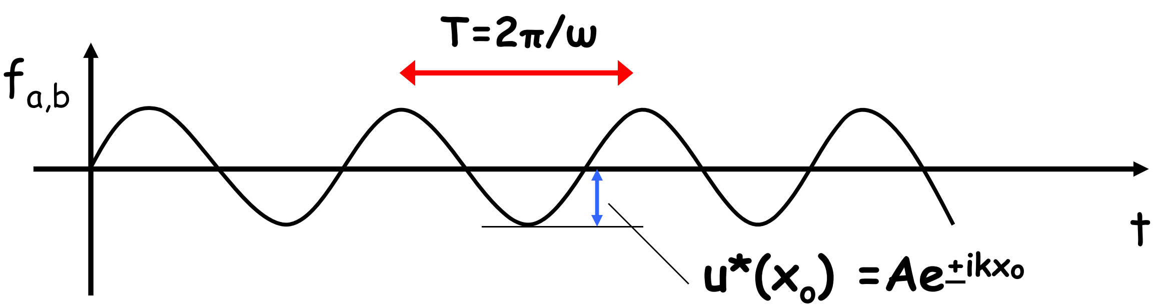
$$f_b\left(Ct-x\right)=Ae^{i\frac{\omega}{C}(Ct-x)}$$

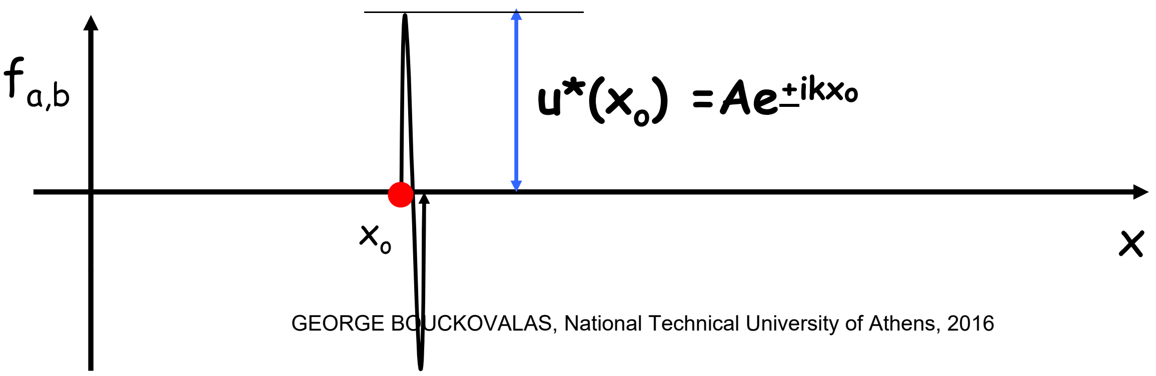
where

k =ω/C = 2π/(T·C) = 2π/λ = wave number

Ground VIBRATION at a given POINT (i.e. $x=x_o$)

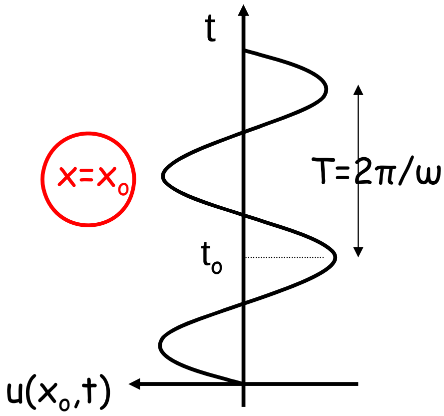
$$f_{a,b}=Ae^{\pm ikx_o}e^{i\omega t}=u^*(x_o)e^{i\omega t}$$ ⇒ Harmonic vibration with frequency ω=2π/T





Ground DISPLACEMENT at a given INSTANT OF TIME (i.e. $t=t_o$)

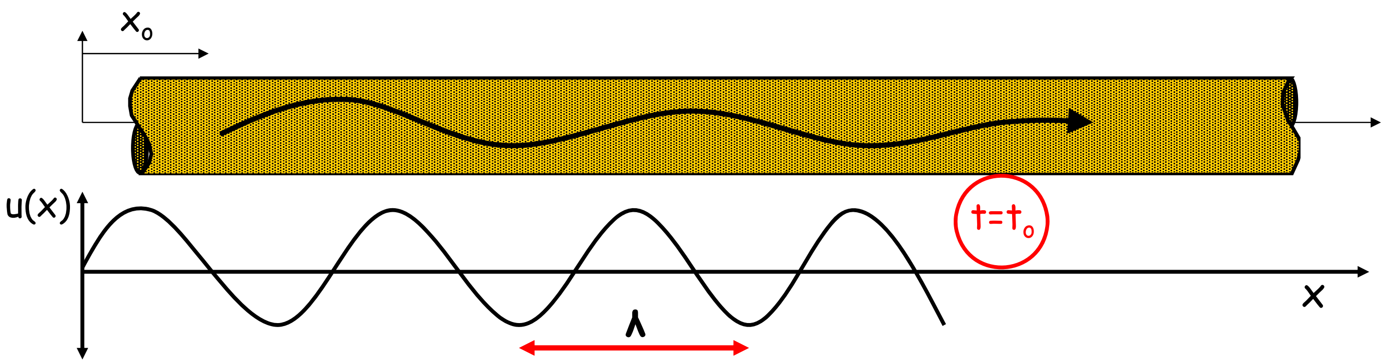
$$f_{a,b} = Ae^{i\omega t_o}e^{\pm ikx} = u^*(t_o)e^{\pm ikx}$$



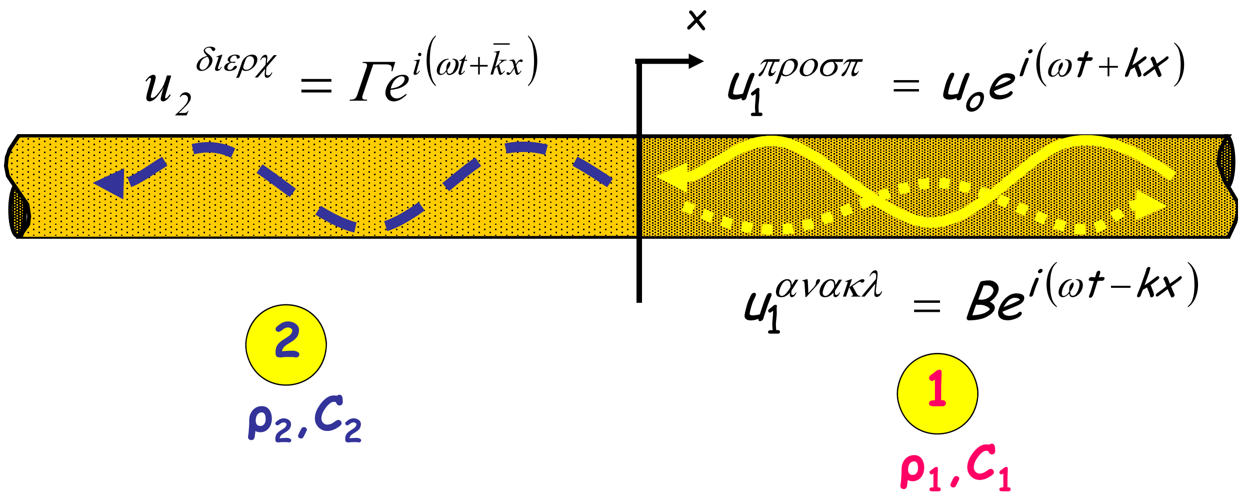
As before, but now the role of frequency is played by the wave number k

$k\Lambda = 2\pi \Rightarrow \Lambda = 2\pi/k$

$[k = \omega/C \Rightarrow \Lambda = 2\pi/\omega C = T \cdot C]$



EXAMPLE: Two rods in contact



$$u_2 = \Gamma e^{i(\omega t + \bar{k}x)}$$

$$\sigma_2 = \overline{M}\varepsilon_x = -\overline{M}\frac{\partial u_2}{\partial x} =$$

$$= -\overline{M}\Gamma\bar{k}e^{i(\omega t + \bar{k}x)}$$

$$u_1 = u_o e^{i(\omega t + kx)} + Be^{i(\omega t - kx)}$$

$$\sigma_1 = M\varepsilon_x = -M\frac{\partial u_1}{\partial x} =$$

$$= -Mu_o ke^{i(\omega t + kx)} + MBke^{i(\omega t - kx)}$$

EXAMPLE: Two rods in contact

Boundary conditions:

x=0 $u_1=u_2$, $\sigma_1=\sigma_2$ } displacement & stress compatibility

$$\sigma_1^{(o)} = \sigma_2^{(o)} \Rightarrow \overline{M}\Gamma\overline{k}e^{i\omega t} = Mu_o k e^{i\omega t} + MBke^{i\omega t} \Rightarrow$$

$$\Rightarrow u_o - B = \frac{\overline{k}}{k}\frac{\overline{M}}{M}\Gamma = \frac{C_2\rho_2}{C_1\rho_1}\Gamma$$

where $k=\omega/C$, $M=C^2\rho$ and $kM=\omega C\rho$

EXAMPLE: Two rods in contact

$$u_1(o) = u_2(o) \Rightarrow u_o + B = \Gamma$$

$$u_o - B = \frac{\rho_2 C_2}{\rho_1 C_1}\Gamma$$

$$\Longrightarrow \quad B = \frac{1 - \dfrac{\rho_2 C_2}{\rho_1 C_1}}{1 + \dfrac{\rho_2 C_2}{\rho_1 C_1}} u_o$$

$$\Gamma = \frac{2}{1 + \dfrac{\rho_2 C_2}{\rho_1 C_1}} u_o$$

Special cases:

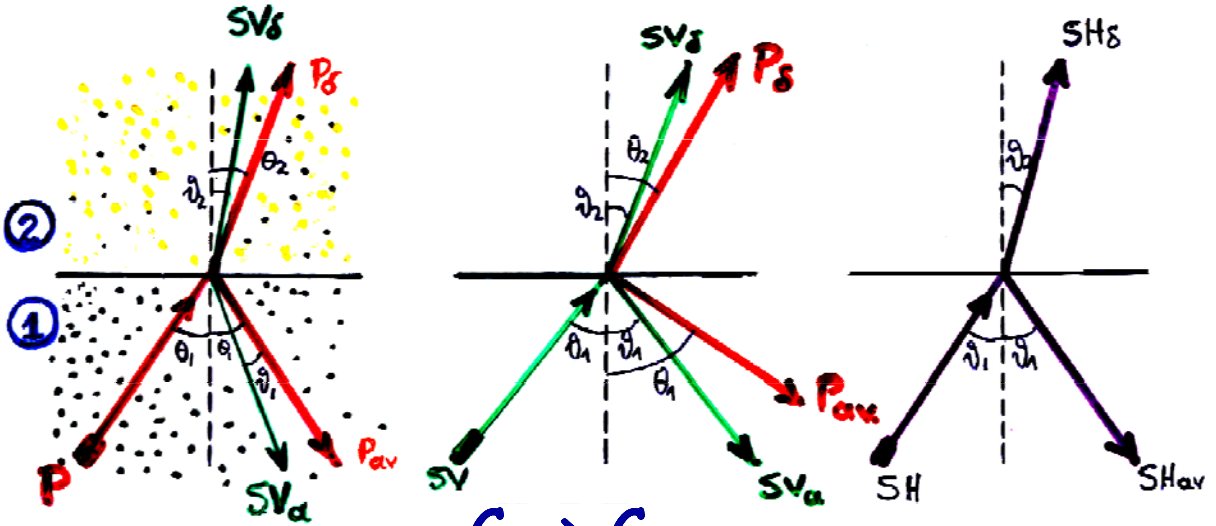
Free end: $\rho_2 C_2 = 0$ $B=u_o$ & $\Gamma=2u_o$

Fixed end: $\rho_2 C_2 = \infty$ $B=-u_o$ & $\Gamma=0$

3.2 ELASTIC WAVES IN non-UNIFORM MEDIA (with interfaces)

SNELL's LAW:

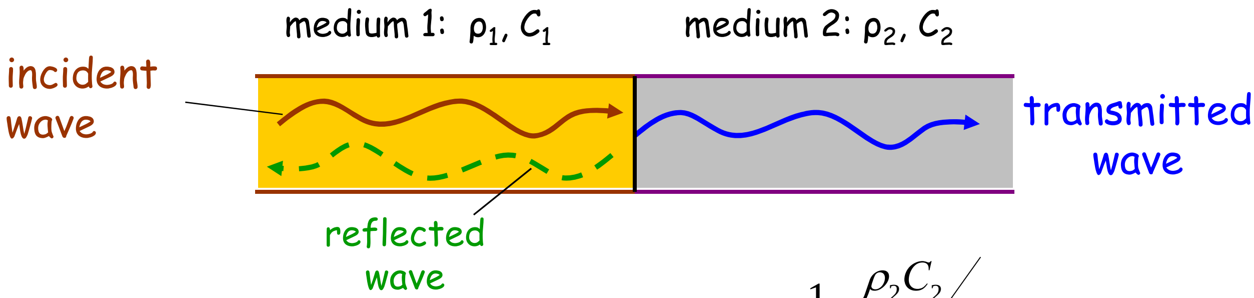
$$\frac{\sin\theta_1}{C_{p_1}} = \frac{\sin\theta_2}{C_{p_2}} = \frac{\sin\vartheta_1}{C_{s_1}} = \frac{\sin\vartheta_2}{C_{s_2}}$$



$C_1 > C_2$

VIBRATION AMPLITUDE of reflected & transmitted- refracted waves

In the already known, simple case of 1-D wave propagation:



Vibration amplitude u:

$$u_{refl.} = \frac{1 - \rho_2 C_2 / \rho_1 C_1}{1 + \rho_2 C_2 / \rho_1 C_1} u_{inc.}$$

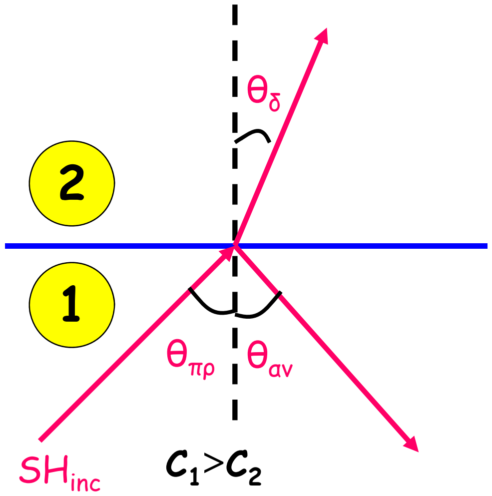
$$u_{transm.} = \left(1 + \frac{u_{refl.}}{u_{inc.}}\right) u_{inc.} =$$

$$= \frac{2}{1 + \rho_2 C_2 / \rho_1 C_1} u_{inc.}$$

VIBRATION AMPLITUDE
of reflected & transmitted- refracted waves

The amplitude of vibration of the reflected and refracted waves in 2-D problems, is computed based on stress equilibrium and strain compatibility (continuity) at the interface. In this case, the amplitude is also a function of the angle of wave incidence.

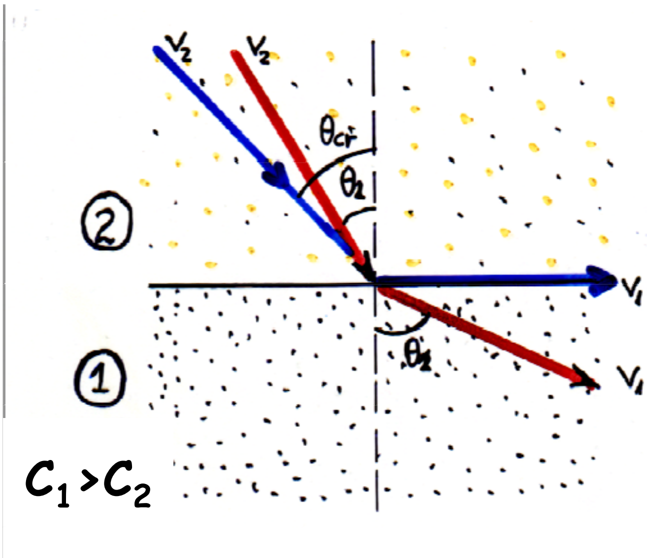
EXAMPLE: SH waves (*Richter 1958*)



$$\frac{u_{refl.}}{u_{inc.}} = \frac{1 - \dfrac{\rho_2 C_2 \cos\theta_\delta}{\rho_1 C_1 \cos\theta_\pi}}{1 + \dfrac{\rho_2 C_2 \cos\theta_\delta}{\rho_1 C_1 \cos\theta_\pi}}$$

$$\frac{u_{refr.}}{u_{inc.}} = 1 + \frac{u_{refl.}}{u_{inc.}} = \frac{2}{1 + \dfrac{\rho_2 C_2 \cos\theta_\delta}{\rho_1 C_1 \cos\theta_\pi}}$$

Critical refraction angle:



$$\frac{\sin\theta_1}{C_1} = \frac{\sin\theta_2}{C_2} \Rightarrow \sin\theta_1 = \sin\theta_2 \frac{C_1}{C_2}$$

$$\sin\theta_1 \le 1 \Rightarrow \sin\theta_2 \frac{C_1}{C_2} \le 1$$

$$\Rightarrow \sin\theta_2 \le \sin\theta_{cr} = \frac{C_2}{C_1}$$

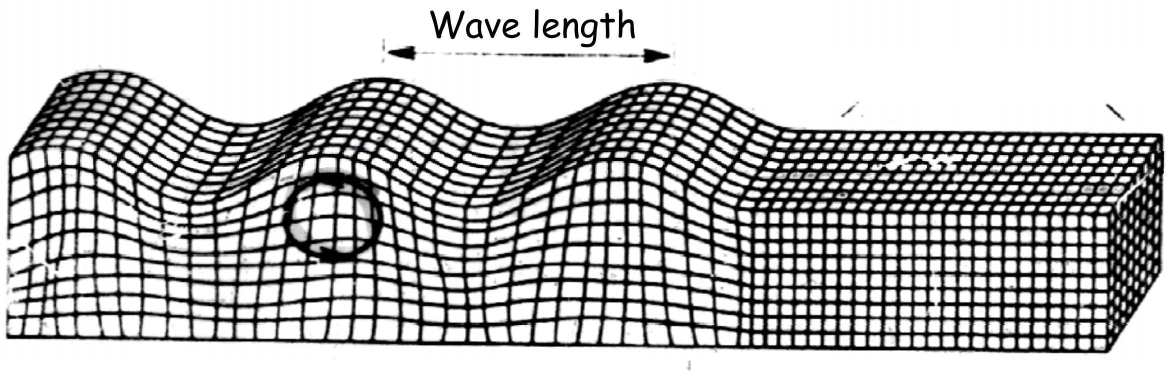
$$\eta \quad \theta_2 \le \theta_{cr} = \sin^{-1}\frac{C_2}{C_1}$$

What happens for $\theta_2 > \theta_{cr}$?

Application:

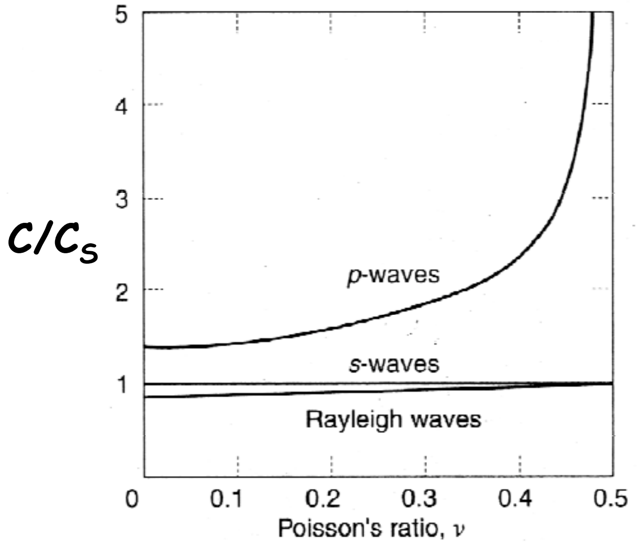
Seismic refraction method for geophysical exploration ……….

3.3 SURFACE WAVES



P-wave + SV-wave + FREE SURFACE → **Rayleigh wave**

$C_R = 0.94 C_S$

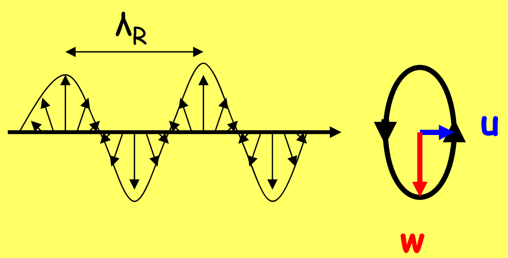


Ground displacements due to R-waves

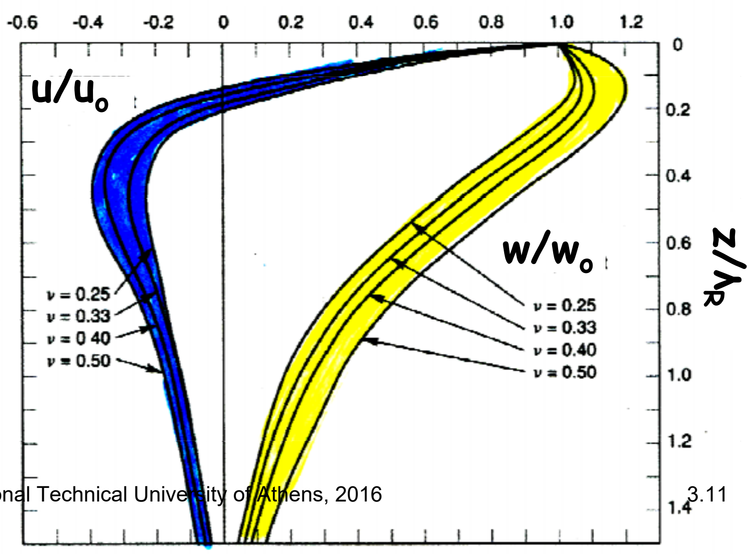
$$u = u(z,x,t) = iB\left\{0.55e^{-2\,z/\lambda_R} - e^{-5.8\,z/\lambda_R}\right\}e^{i(\omega t - k_R x)}$$

$$w = w(z,x,t) = B\left\{1.66e^{-2\,z/\lambda_R} - 0.92e^{-5.8\,z/\lambda_R}\right\}e^{i(\omega t - k_R x)}$$

$k_R = \omega / C_R$ wave number
$\lambda_R = C_R T$ wave length
B = constant

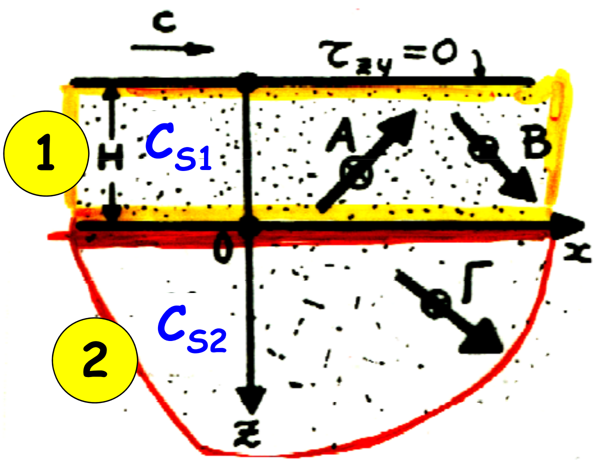


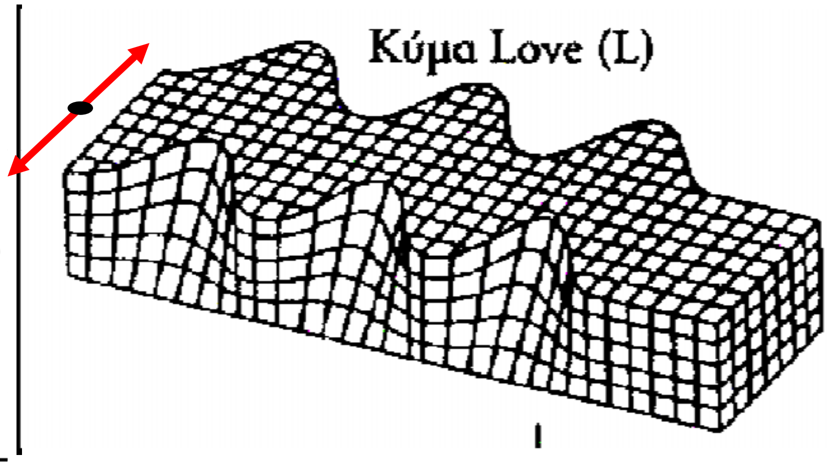
$$\frac{|w(0)|}{|u(0)|} = 1.6$$



LOVE WAVES (in layered soil profile)

They are SH waves "trapped" within the upper soft layer of a non-uniform soil profile.





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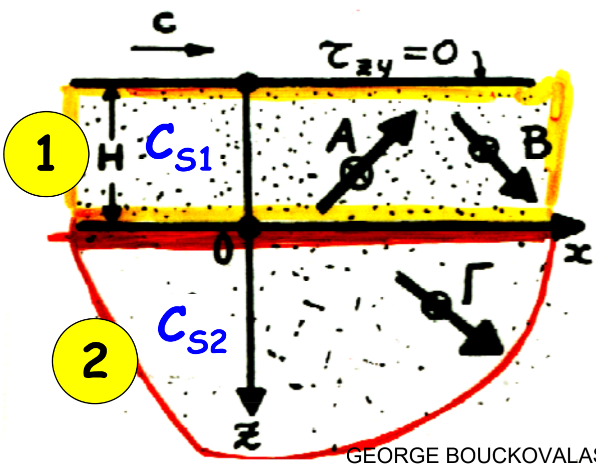
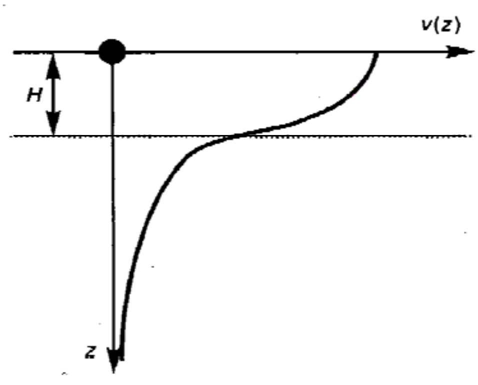
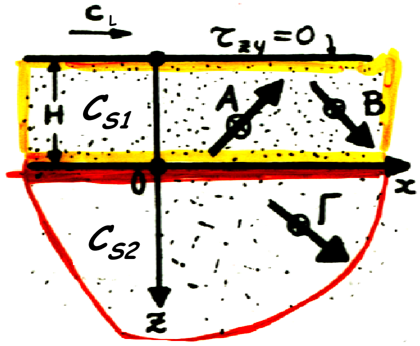


Ground displacement variation with depth



$$u_1 = \left(Ae^{ir_1 k_L z} + Be^{-ir_1 k_L z} \right) e^{i(\omega t - k_L x)}$$

$$u_2 = \Gamma e^{-ir_2 k_L z} e^{i(\omega t - k_L x)}$$

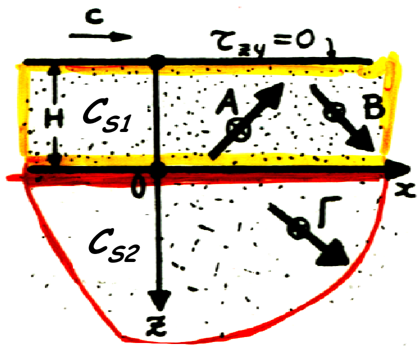


$$k_L = \omega / C_L$$

$$r_1 = \sqrt{\frac{C_L^2}{C_{s1}^2} - 1}, \qquad r_2 = \sqrt{\frac{C_L^2}{C_{s2}^2} - 1}$$

$$u_1 = \left(Ae^{ir_1 k_L z} + Be^{-ir_1 k_L z} \right) e^{i(\omega t - k_L x)}$$

$$u_2 = \Gamma e^{-ir_2 k_L z} e^{i(\omega t - k_L x)}$$



$$\begin{cases} z = -H \quad (\varepsilon\lambda\varepsilon\acute{\upsilon}\theta\varepsilon\rho\eta\ \varepsilon\pi\iota\varphi\acute{\alpha}\nu\varepsilon\iota\alpha) : \tau_1 = 0 \\ z = 0 \quad (\delta\iota\varepsilon\pi\iota\varphi\acute{\alpha}\nu\varepsilon\iota\alpha) : \tau_1 = \tau_2, u_1 = u_2 \end{cases}$$

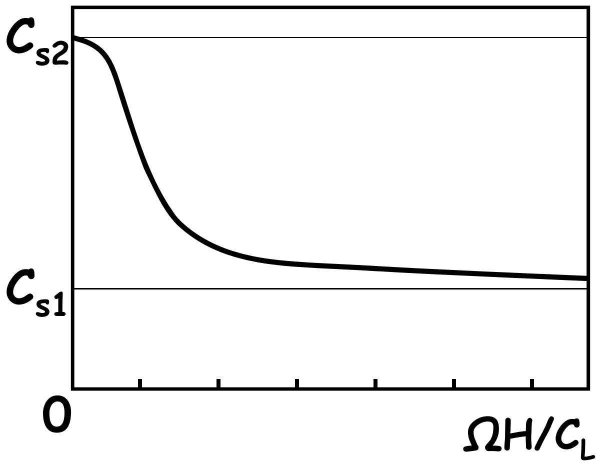
$$\left\{ \begin{array}{llll} Ae^{-ir_1 kH} & -Be^{ir_1 kH} & & = 0 \\ A & +B & -\Gamma & = 0 \\ A(G_1 r_1) & +B(G_1 r_1) & -\Gamma(G_2 r_2) & = 0 \end{array} \right\} \begin{array}{c} substituting \quad for \\ \mathrm{A, B, \Gamma} \\ gives: \end{array}$$

$$tan\left[\frac{\omega H}{C_L} \sqrt{\frac{C_L^2}{C_{S1}^2} - 1} \right] = -\frac{G_2}{G_1} \frac{\sqrt{1 - \frac{C_L^2}{C_{S2}^2}}}{\sqrt{\frac{C_L^2}{C_{S1}^2} - 1}}$$

«Dispersion» relationship

Observe that, since tan(*) must be a real number, the following condition must be satisfied for the previous equation to have a real solution:

$$C_{S1} < C_{LOVE} < C_{S2}$$



$$\tan\left[\frac{\omega H}{C_L}\sqrt{\frac{C_L^2}{C_{S1}^2}-1}\right] = -\frac{G_2}{G_1}\frac{\sqrt{1-\frac{C_L^2}{C_{S2}^2}}}{\sqrt{\frac{C_L^2}{C_{S1}^2}-1}}$$

In addition, LOVE waves are not possible unless the surface layer is softer than the underlying medium (i.e. $C_{S1} < C_{S2}$).

Dispersion: Code name for LOVE (and other) waves which exhibit a frequency dependent propagation velocity (even for a uniform medium).

Implications of dispersion:

The propagation velocity C_{LOVE} decreases with increasing frequency ω (*decreasing period T*).

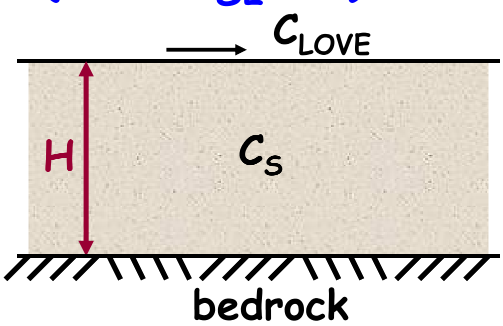
As a result….. the low-frequency components of the excitation get separated from the high-frequency components, as they propagate faster. Thus:

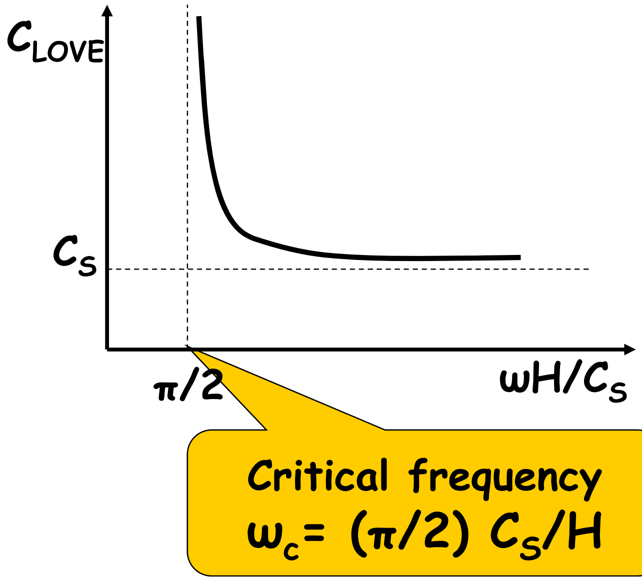
- The duration of shaking increases with distance from the source
- The frequency content of the motion (e.g. the elastic response spectrum) changes with distance from the source

Special Case:

Soft soil layer upon BEDROCK (i.e. $C_{S2}=\infty$)

$$C_{LOVE} = \frac{\omega H}{\left\{\left(\frac{\omega H}{C_s}\right)^2 - \left[\frac{\pi}{2}\right]^2\right\}^{1/2}}$$





- Finally, LOVE waves cannot exist for frequencies lower than the fundamental vibration frequency of the soft soil layer, i.e. when

$\omega < \omega_C= (\pi/2)\ C_s/H$

PROBLEM SOLVING FOR CHAPTER 3

HWK 3.1:

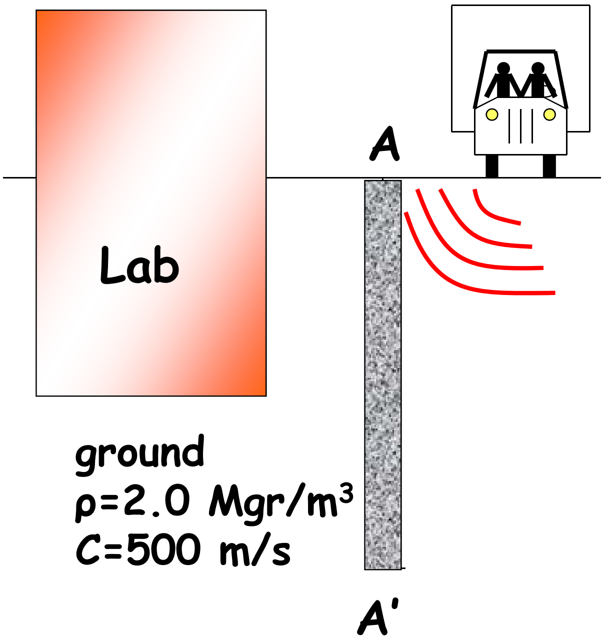
The free end of an infinite rod is displaced according to the following relationship:

U=0	for	$t \le 0$
U=t (cm)	for	$0 \le t \le 0.1s$
U=0.2-t	for	$0.1s \le t \le 0.2s$
U=0	for	$0.2s \le t$

Find the displacement variation along the rod at t=0.3s. Assume that the wave propagation velocity is 300m/s.

(*από «Σημειώσεις Εδαφοδυναμικής» Γ. Γκαζέτα)*

HWK 3.2:



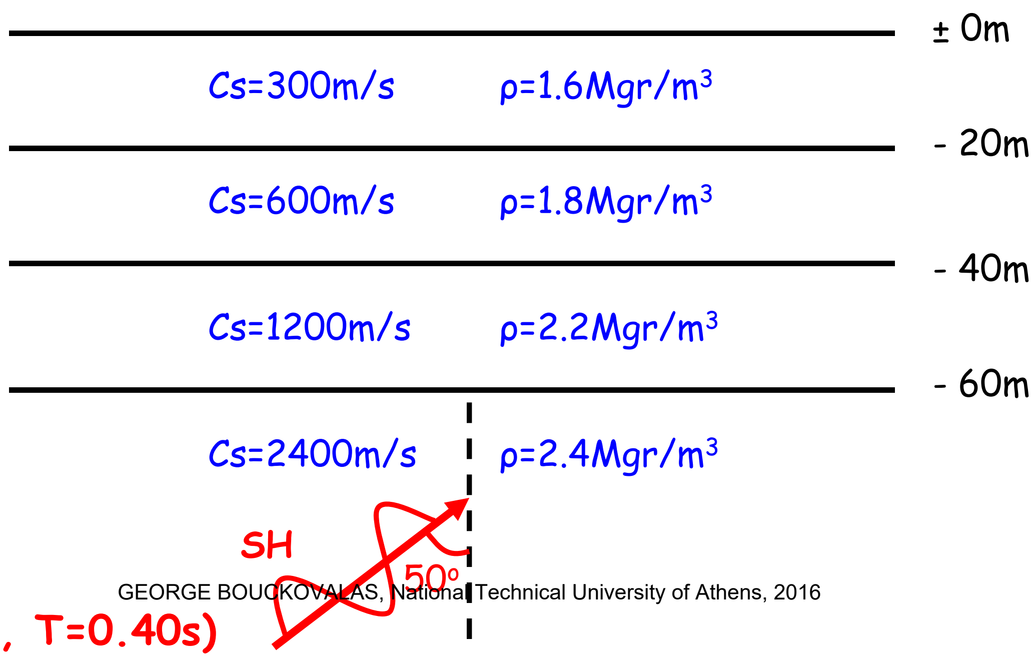
To isolate a precision measurement laboratory (e.g. a geotech lab.) from traffic vibrations, it is proposed to construct the trench AA'. Which of the following two fill materials is preferable :

(a) concrete, with ρ=2.5 Mgr/m^3 &
C=2000 m/s, or

(b) pumice, with ρ=0.8 Mgr/m^3
C=100 m/s ?

HWK 3.3:

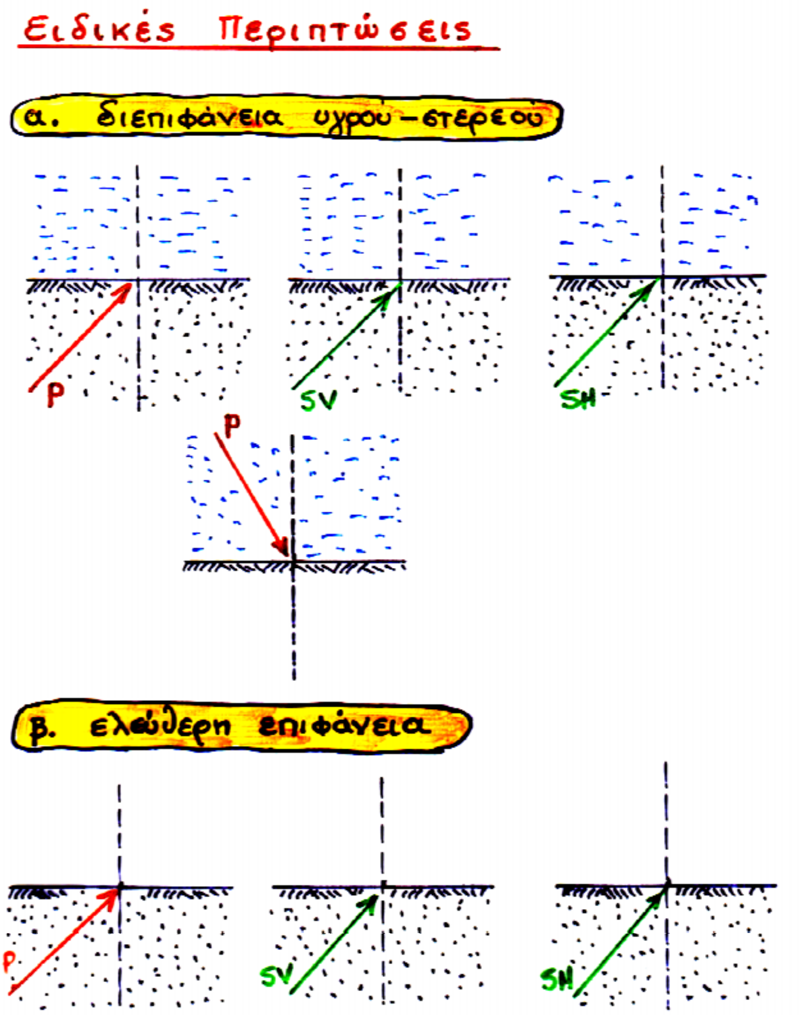
Draw the SH wave propagation path through the soil profile shown below, and compute the horizontal acceleration applied to each soil layer.



(u_o=1.6cm, T=0.40s)

HWK 3.4:

Draw the reflected and refracted waves (type and propagation direction) in the special cases shown in the figure .



HWK 3.5:

To get a feeling of the depth (from the free ground surface) which is affected by R-waves,

1) Compute the normalized depth z/λ_R where ground displacements are reduced to 10% of the value at the free ground surface.

2) What is the value of the above critical depth (range of variation) in the case of soft soil, stiff soil-soft rock and rock;

(*Από «Σημειώσεις Εδαφοδυναμικής» Γ. Γκαζέτα)*