2. Single Degree of Freedom Systems (1-DOFs) under Harmonic Base Excitation & Elastic Response Spectra

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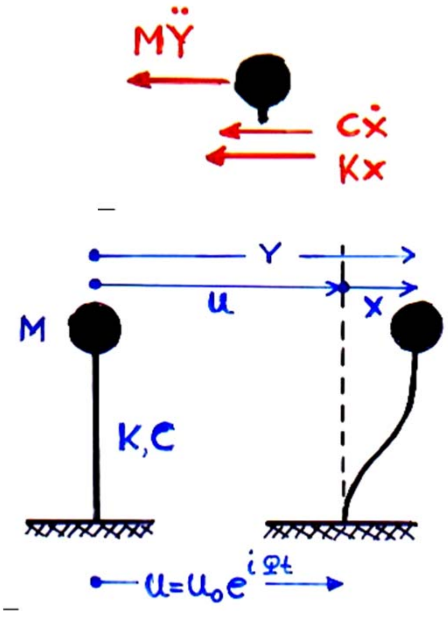
2.1 Equation of dynamic equilibrium



$$\left.\begin{aligned} M\ddot{Y} + C\dot{X} + KX &= 0 \\ Y &= X + U \end{aligned}\right|$$

$$M\ddot{X} + C\dot{X} + KX = -M\ddot{U}$$

για αρμονική διέγερση:

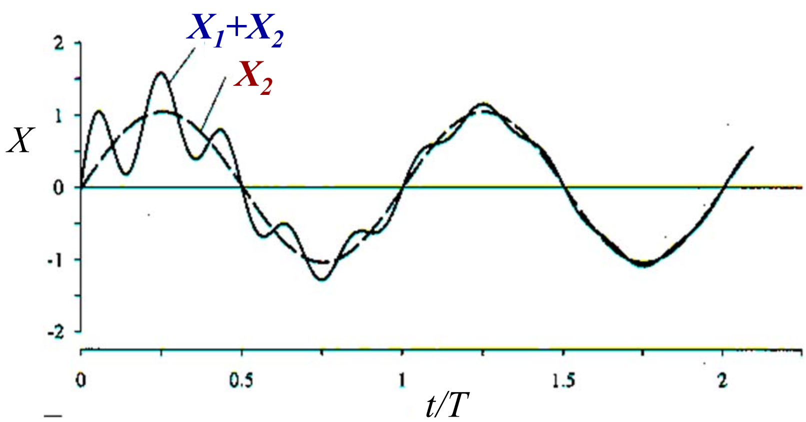
$$U = U_O e^{i\Omega t} \Rightarrow \ddot{U} = -\Omega^2 U_O e^{i\Omega t}$$

and, finally:

$$M\ddot{X} + C\dot{X} + KX = F_O e^{i\Omega t}$$

με $F_O = \Omega^2 M U_O$

Reminder: [$e^{i\Omega t}$ = cosΩt + i sin Ωt]



$X = X_1+X_2$

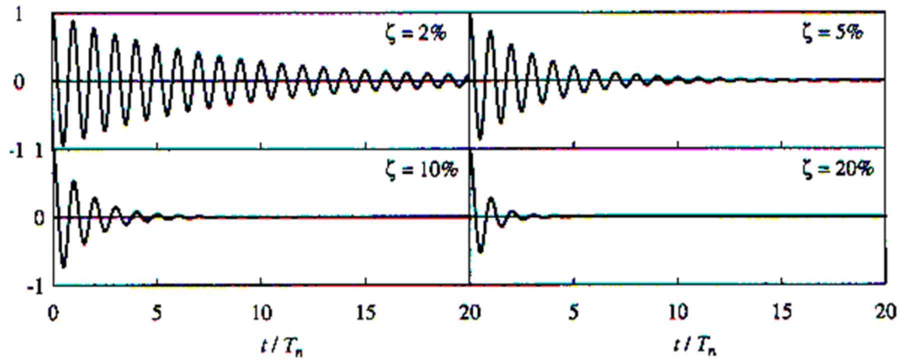
where X_1 is the «***transient***» component, and
X_2 is the «***steady state***» component of displacement which is of greater practical importance (why?)

The "***transient***" component of displacement (X_1) is a <u>damped free vibration</u> described by the equation:

$$X_1 = e^{-\xi\omega t}(A\cos\omega_D t + B\sin\omega_D t)$$

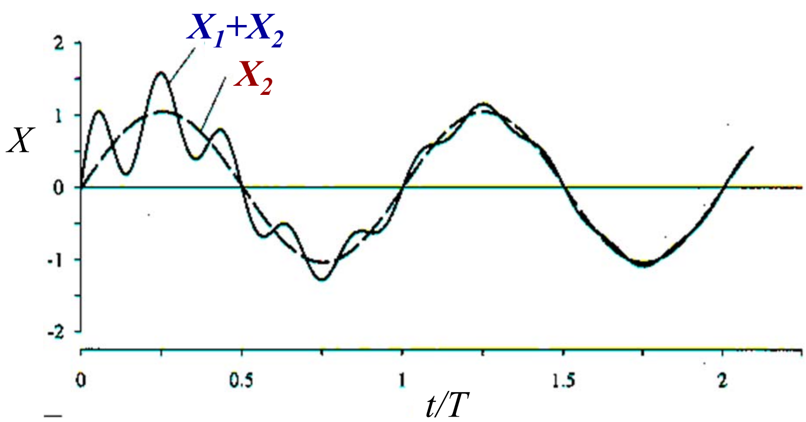
where: $\zeta = \xi = C/2\sqrt{KM}$, $\omega_n = \sqrt{\frac{K}{M}}$ and $\omega_D = \omega_n\sqrt{1-\xi^2} \approx \omega_n$

e.g.
A=1
B=0



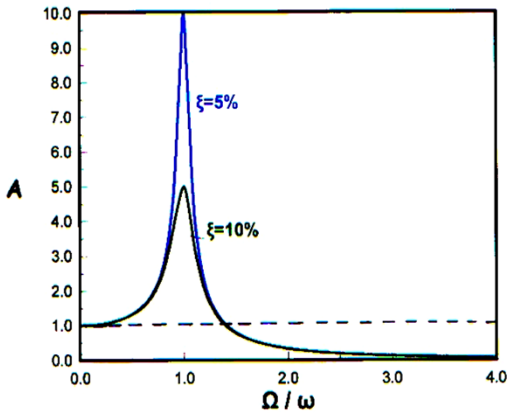
NOTE: This component has frequency equal to the fundamental frequency of vibration of the 1-DOF (NOT the excitation frequency)

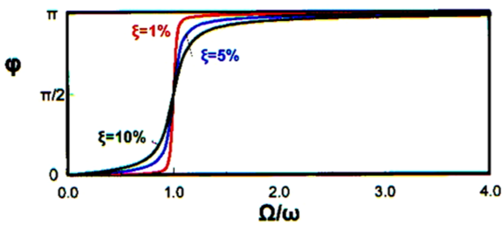
2.2 Relative displacement X



$$\mathbf{X = X_1+X_2}$$

where X_1 is the «***transient***» component, and
X_2 is the «***steady state***» component of displacement which is of greater practical importance (why?)





The "***steady state***" component of displacement (X_2) is harmonic, With frequency equal to the excitation frequency (Ω) and phase difference (φ) from the excitation:

$$X_2 = (X_{ST} \cdot A)e^{i(\Omega t - \phi)}$$

where $X_{ST} = \dfrac{F_O}{K} = \dfrac{\Omega^2 M U_o}{\omega^2 M} = \left(\dfrac{\Omega}{\omega}\right)^2 U_o$

$$A = \frac{1}{\sqrt{[1-(\Omega/\omega)^2]^2 + 4\xi^2(\Omega/\omega)^2}}$$

$$\tan\phi = \frac{2\xi(\Omega/\omega)}{1-(\Omega/\omega)^2}$$

Question for the Class:

Based on the previous definitions and the figures for the *Dynamic coefficient A* and the *angle of phase difference φ*, describe The basic mechanisms and the physical meaning of the steady state response $(X_2/U_o)_{MAX}$ in the following benchmark cases:

I. Area $\Omega/\omega \lll 1.0$
II. Area $\Omega/\omega = 1.0$
III. Area $\Omega/\omega \ggg 1.0$

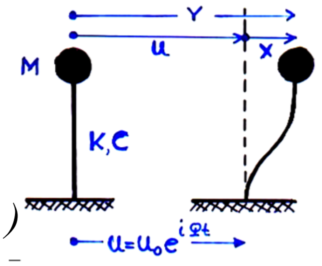
Examples from engineering practice (and/or from every day life) will contribute to better understanding and are welcome.

2.3 Total Displacement Y (velocity & acceleration)

$$Y = U + X = U_O e^{i\Omega t} + X_O e^{i(\Omega t - \phi)}$$

$$Y = (U_O + X_O e^{-i\phi})e^{i\Omega t} = Y_O e^{i\Omega t}$$

$$Y_O = (U_O + X_O \cos\phi) - i(X_O \sin\phi)$$



After normalization against the base excitation,

$$\frac{Y}{U} = \frac{Y_O}{U_O} = 1 + \frac{X_O}{U_O} e^{-i\phi} \ ,$$

$$\boxed{\frac{X_O}{U_O} = \left(\frac{\Omega}{\omega}\right)^2 A}$$

and finally,

$$\frac{Y}{U} = \frac{Y_O}{U_O} = 1 + \left(\frac{\Omega}{\omega}\right)^2 A e^{-i\phi}$$

or

[*from a .. strange coincidence that is worth proving analytically*]

$$\boxed{\left|\frac{Y}{U}\right| = \left|\frac{Y_O}{U_O}\right| \approx A}$$

Question for the Class:

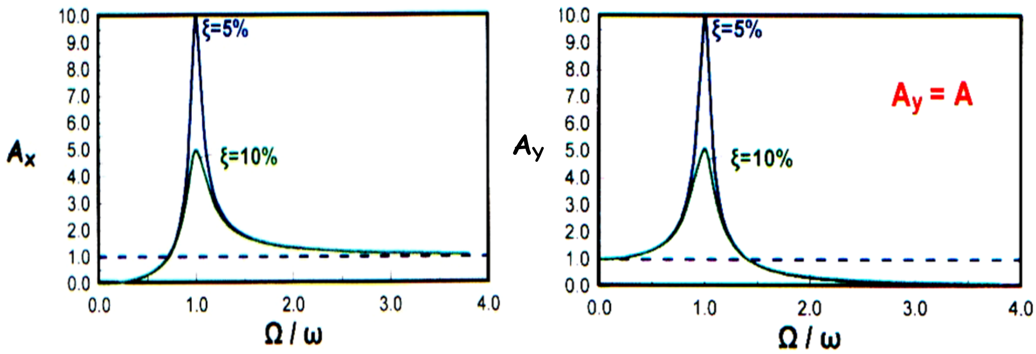
Based on the previous definitions and the figures for the *Dynamic coefficient A* and the *angle of phase difference φ*, describe The basic mechanisms and the physical meaning of the **TOTAL** 1-DOF response $(Y/U_O)_{MAX}$ in the following benchmark cases:

I. Area $\Omega/\omega \lll 1.0$
II. Area $\Omega/\omega = 1.0$
III. Area $\Omega/\omega \ggg 1.0$

Examples from engineering practice (and/or from every day life) will contribute to better understanding and are welcome.

For harmonic excitation, all previous non-dimensional expressions for relative and total displacements are also valid for the corresponding velocities and accelerations. Hence, we may define the following two "***dynamic factors***":

$$A_X = \frac{X}{U} = \frac{\dot{X}}{\dot{U}} = \frac{\ddot{X}}{\ddot{U}} \qquad\qquad A_Y = \frac{Y}{U} = \frac{\dot{Y}}{\dot{U}} = \frac{\ddot{Y}}{\ddot{U}}$$



*In this case also it is worth checking the physical meaning of the above diagrams for the characteristic cases where **Ω/ω<<<1.0**, **Ω/ω=1** and **Ω/ω>>>1**.*

2.4 The "Elastic Response Spectrum"

Harmonic Excitation

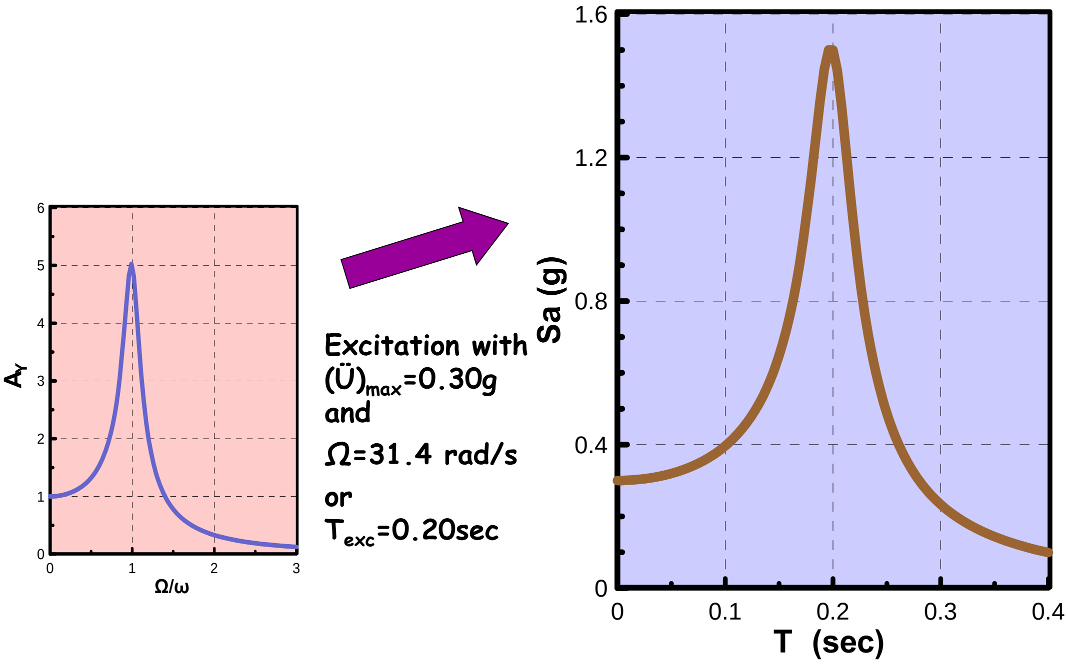
The diagram for the dynamic factor variation $A_Y - (\Omega/\omega)$ may be interpreted in two basic ways:

A. **For a given structure**, when $\omega=2\pi/T$=constant, it describes the maximum dynamic response for various excitations with frequency $\Omega=2\pi/T_{exc}$.

B. **For a given excitation**, when Ω=constant, it describes the maximum dynamic response of various structures in terms of their fundamental vibration frequency $\omega=2\pi/T_{\alpha}$, i.e. it provides the **SPECTRUM** (possible range) of the visco-elastic structural response.

The second interpretation is of particular practical interest for the seismic design of structures as it provides the maximum seismic actions (displacement, velocity, acceleration) which are applied to the mass of the structure during base excitation. Hence, it was extended to non harmonic – seismic excitations, in the form of the widely used "**Elastic Response Spectra**", initially by Biot (1932) and later by Housner.

Elastic response "SPECTRUM" for harmonic excitation



Elastic response "SPECTRUM" for harmonic excitation

The other way round…

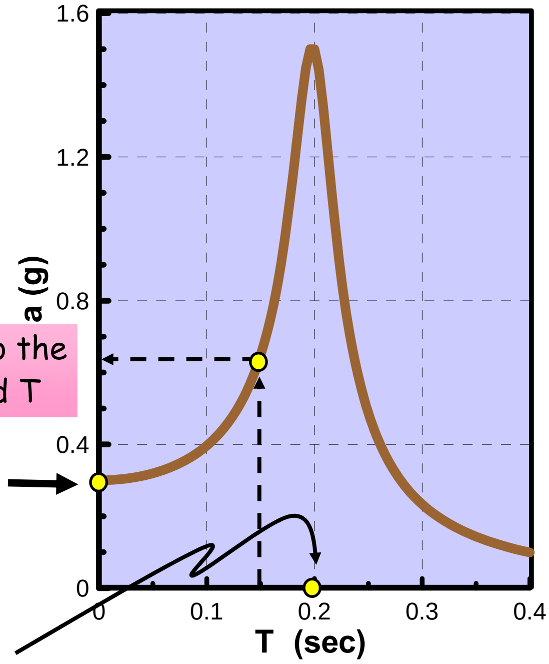
i.e.

if the spectrum of a seismic motion is given to us.....

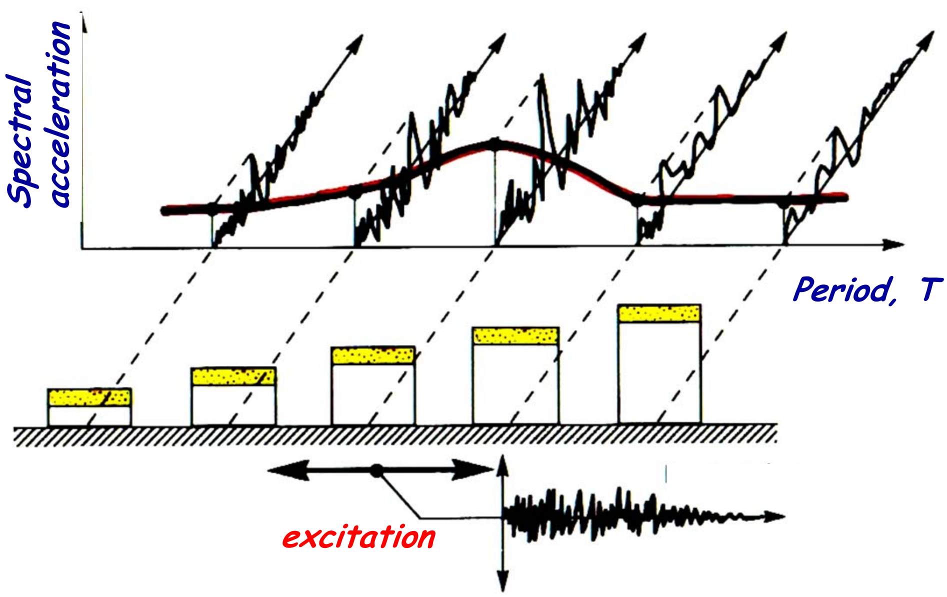
(inertia) acceleration applied to the mass of a structure with period T

PGA=$(\ddot{U})_{max}$=0.30g

T_{exc}=0.20sec

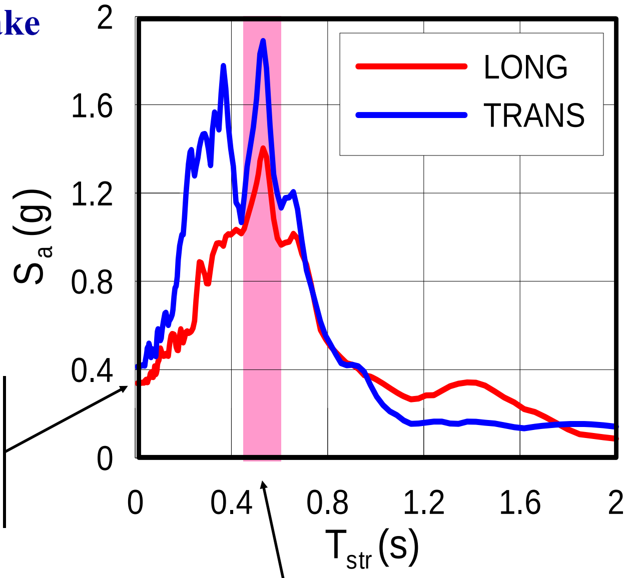


Elastic response "SPECTRUM" for SEISMIC excitation



Elastic response "SPECTRUM" for SEISMIC excitation

Lefkas earthquake (14/08/2003 M=6.4



Max. acceleration of seismic motion

Range of predominant excitation periods

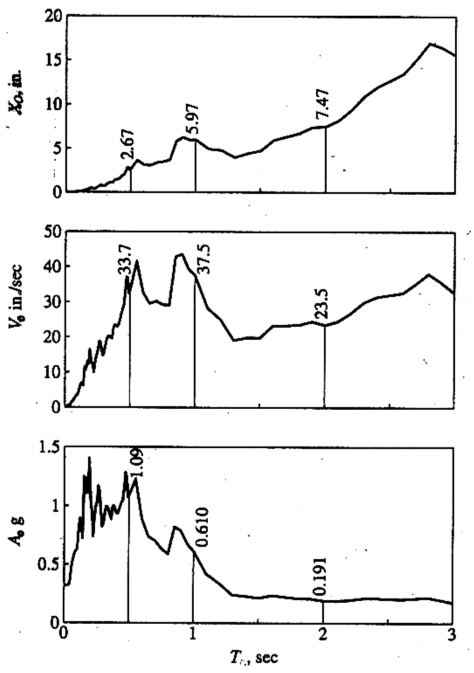
Alternative types of Elastic Response Spectra

El Centro earthquake, ξ=2%

Max. relative displacement X_O.
[Max. base shear force F_O=K X_O]

Max. PSEUDO-velocity
$V_O=\omega X_O=(2\pi/T)X_O$
[Max. stored elastic energy
$E_O= ½ (K X_O^2) = ½ (MV_O^2)$]

Max. PSEUDO-acceleration
$A_O=\omega^2 X_O=(2\pi/T)^2 X_O$
[Max. base shear force
F_O=K X_O = MA_O]



Comparison:

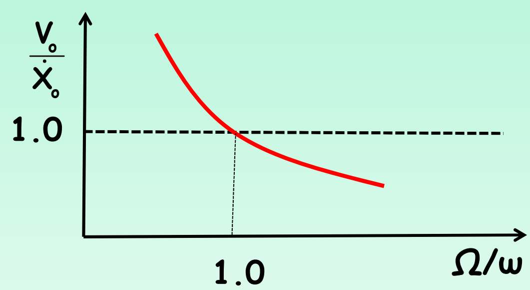
Pseudo-velocity

vs.

max. relative velocity

Harmonic base excitation

$$\left.\begin{aligned} V_o &= \omega \cdot X_o \\ \dot{X}_o &= \Omega \cdot X_o \end{aligned}\right\} \quad \frac{V_o}{\dot{X}_o} = \frac{\omega}{\Omega} = \frac{1}{(\Omega / \omega)}$$



good agreement at resonance . .

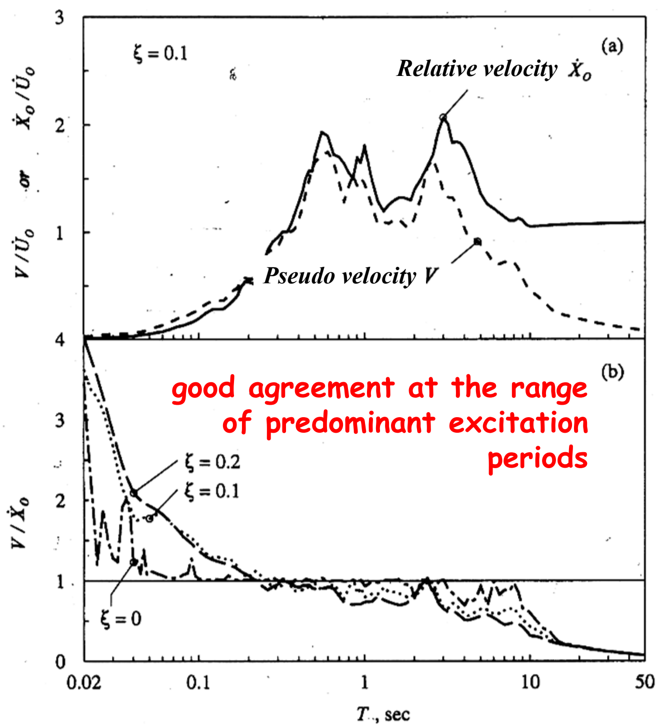
Comparison:

Pseudo-velocity

vs.

max. relative velocity

Actual seismic excitation



Comparison:

Pseudo-acceleration

vs.

max. absolute acceleration

Harmonic base excitation

$$\left.\begin{aligned} A_o &= \omega^2 \cdot X_o \\ \ddot{Y}_o &= \Omega^2 \cdot Y_o \approx \Omega^2 \cdot A \cdot U_o \end{aligned}\right\} \quad \frac{A_o}{\ddot{Y}_o} = \frac{1}{(\Omega/\omega)^2} \cdot \frac{1}{A} \cdot \frac{X_o}{U_o}$$

but $\qquad \dfrac{X_o}{U_o} = \left(\dfrac{\Omega}{\omega}\right)^2 \cdot A$

so that .... $\qquad \dfrac{A_o}{\ddot{Y}_o} \approx 1.0$!

good overall agreement ….

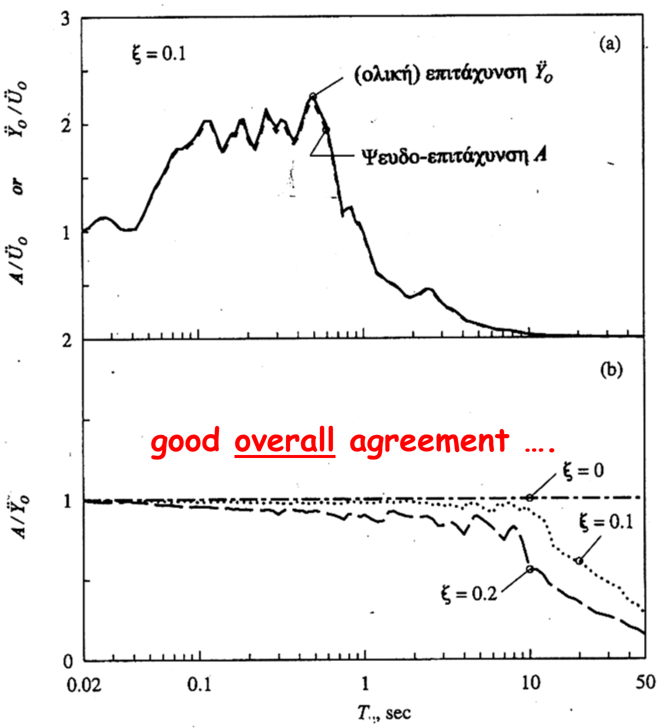
Comparison:

Pseudo-acceleration

vs.

max. absolute acceleration

Actual seismic excitation

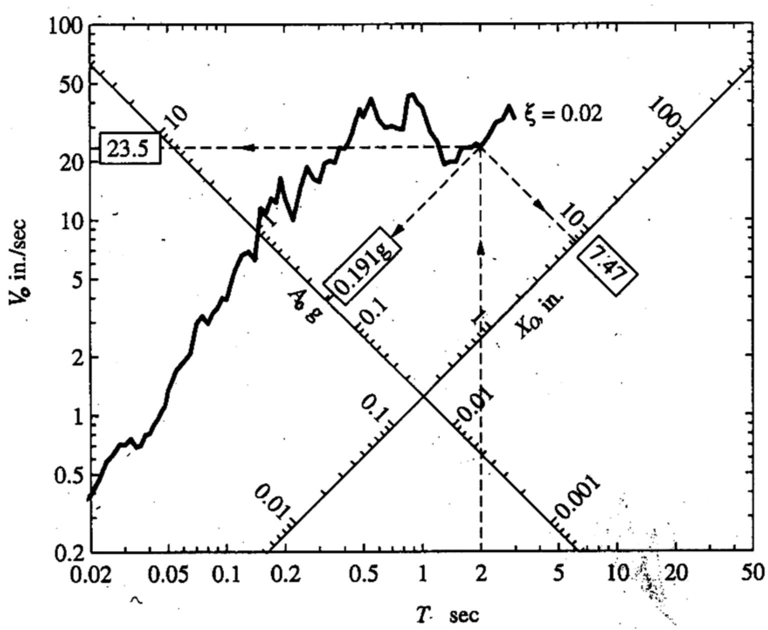


Combined THREE-LOGARITHMIC presentation

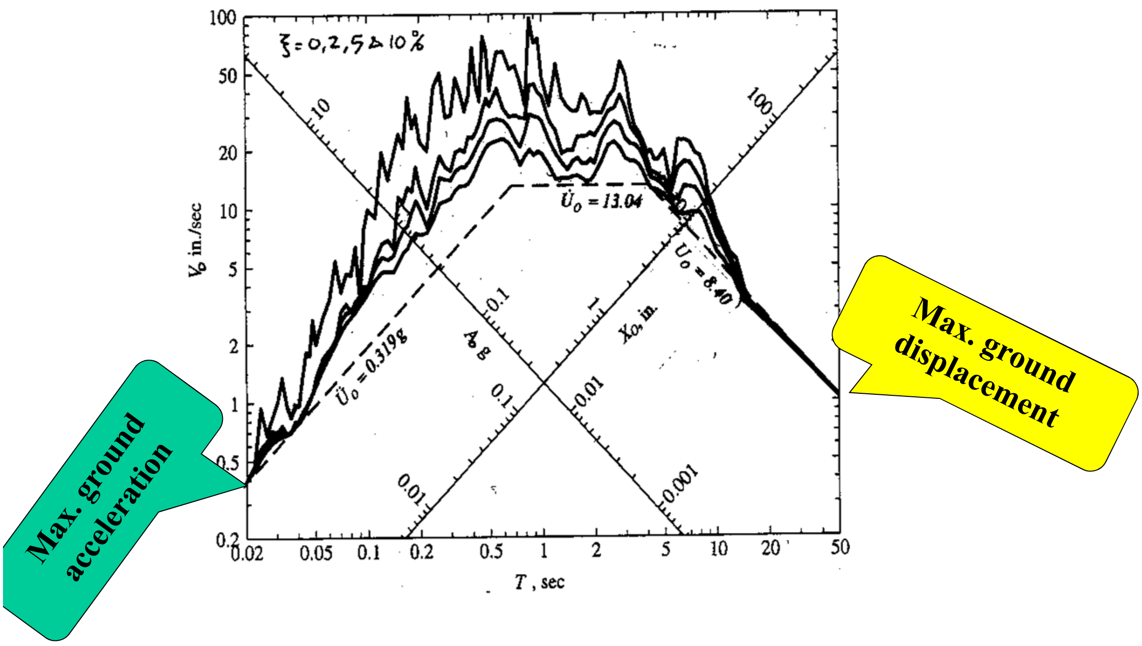
X_O

$V_O=\omega X_O$

$A_O=\omega^2 X_O$



Combined THREE-LOGARITHMIC presentation



Problem solving for Chapter 2

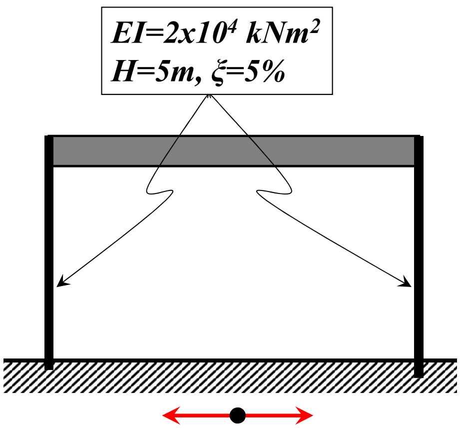
HWK 2.1:

For harmonic ground excitation, with displacement amplitude U_O=0.15m and period $T_{exc.}$=0.40sec, compute:

1) The maximum absolute acceleration, velocity and displacement at the floor level of the frame.

2) The maximum shear force T and moment M developing at the top of the two columns.

Assume that the floor is rigid with a total weight of 500 kN, while both columns are flexible without mass.



HWK 2.2:

For harmonic seismic excitation, with displacement amplitude U_O=0.15m and period $T_{διεγ.}$=0.40sec, compute the following diagrams:

1) The elastic response spectrum for the **relative** displacement, for ξ=2%.

2) The elastic response spectra for the Pseudo-velocity and the max. **relative** velocity, for ξ=2% (in the same diagram)

3) The elastic response spectra for the Pseudo-acceleration and the max. **absolute** acceleration, for ξ=2% (in the same diagram)

4) The ratio of the spectra of question 2 and the ratio of the spectra of question 3 (in the same figure)

Compare with the respective diagrams – spectra for El Centro earthquake. Discuss the similarities and the differences.

HWK 2.3:

Repeat HWK 2.1 for Lefkada Earthquake

(any one of the two components)