# Efficiency in an Overlapping Generations model with Endogenous Population.*

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28 February 2002

**Abstract**

In this paper we have undertaken an optimality study in an overlapping generation model with endogenous fertility and exogenous endowments. The contribution of this paper were twofold. First, we present two definitions of Pareto dominance and, consequently, of Pareto efficiency. It is shown that conceptual problems (which could also be said "ethical") arises when fertility decisions are analyzed. Second, the necessary (static) and sufficient (dynamic) efficient conditions have been shown.

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# Introduction

The study of Pareto efficient allocations with endogenous fertility decisions seems crucial to understand the consequences on loosing parental links after implementing government policies and opening new financial markets. The existence of a suitable way to transfer wealth to the future may account for fertility, both through the developing of security products and some governmental welfare programs. An example of the latter is Becker and Murphy (1988) which presents a reformulation of the welfare state, later modeled by Boldrin and Montes (2002) in an exogenous fertility set-up. These latter authors have justified a welfare state as a mechanism to complete financial markets that, otherwise, do not exist in real world. On the other hand, Cigno (1992) and Conde-Ruiz et al (2002) and Miles (1997) among others, have pointed out that the expansion of the Welfare State may explain for the decrease of fertility.

However, an additional source of inefficiency due to incompleteness of markets arises with endogenous fertility, since there is no market where offspring may bargain with their own parents the right to be born. In this sense it is interesting to provide a suitable framework to explore the consequences of the welfare state in a world of endogenous population.

The model we are dealing with consists of an overlapping generations model with endogenous fertility and incomplete financial markets. Agents live for three periods: child, young and old. When child, the agent only accumulates human capital. When old, the agent only consumes. The young generation plays the main role in the model and takes three economic decisions: the number of children; how much human capital invest on them and, at the same time, they are the financial support of their parents. When they reach old age, parents become dependent on their immediate offspring from material support, consisting basically in financial transfers.

We emphasis that old-age security is likely to be an important motive for fertility. Because of the incompleteness of the markets, parents can only rely on their own children for supporting at old times.[1] Children are dependent on their parents for

[1] In this sense our model departures from previous literature where parents are assumed to be purely altruistic with their children. Hence our demand of children is explained without any assumption on parents' utility function.

The characterization of the motives for childbearing account to understand the fertility decline. First, the consumption motive is diminished because there is a substitution effect either with higher costs of growing up children or with better labor force opportunities for women, i.e. higher opportunity cost of women (higher real wage rates) substituting time for child-bearing with working time (see Barro and Lee, 1994). But, unless we consider that offspring is an inferior good, income effect should increase fertility, since the population of development countries have been experienced an important increase in income (see Cigno, 1992, and Boldrin and Jones, 2001). This reinforces the importance of other motives for childbearing. For instance, the importance of the "income motive" since children are not used to participate in labor activities in development countries. However, we claim the relevance of the old-age security motive on the fertility fall in a way shown to be essential in the present paper. The existence and availability of a set of statements, jointly with commitments, to transfer wealth into the future allow agents to risk hedge on old times. The both their material needs and the acquisition of productive knowledge. Accumulation of human capital thus occurs in our model through implicit contracts involving intergenerational trade and companionship. If parents have some certainty that their children are going to support them at elderhood, parents may be strongly interested in invest higher on their own children's human capital, since more income is possible for children.

The literature have provided some criteria for an allocation to be efficient, after a notion of efficiency must be shown. First, Cass (1972) presents a sufficient efficient conditions for a production growing economy, where higher aggregate consumption is needed for improving optimality. Galor and Ryder (1991) rely their notion of Pareto efficient as an improvement on utility. They establish sufficient technological conditions for dynamic efficiency at the economy'svsteady-steate equilibria in a OLG with production. Balasko and Shell (1980) also study efficiency in an overlapping generation model with endowments. They characterize some notions of Pareto efficiency and show that, although an allocation can be *short-run* efficient (or statically efficient), i.e., individually all agents are maximizing their individual welfare, it could not be *long-run* efficient (or dynamically efficient); that is, that a lower rate of savings in all generation can achieve higher levels of welfare. This dichotomy arises very often in overlapping generations model.

Both papers, however, deals with exogenous fertility. Some attempts has been made to study the case where an optimal number of populations yields the higher level of welfare, e.g. Samuelson (1975 and 1976) and Deardorff (1976) or, at least, the case of endogenous fertility. Examples of the latter, like Eckstein and Wolpin (1985), Bental (1989) orAbio and Patxot (2001), are restricted, however, to the golden rule, so a wide range of optimal allocations are not identified.

The present paper undertakes a study on optimality in an overlapping generations model with endogenous fertility. First, we present two definitions of Pareto dominance and, consequently, of Pareto efficiency. Some conceptual problems (which could also be said "ethical") arises when fertility decisions are analyzed. For example an allocation with a given number of individuals in each of the infinite periods, could not be Pareto comparable with another with lower number of individuals at least in one period. The reason is that the "disappeared" (or killed) individuals are, obviously, worst. In consequence, we provide two definitions: a *weak* Pareto dominance, where the case mentioned are considered, and the *strong* Pareto dominance. In the latter, the notion of efficiency is period dependent, in the sense that we can compare allocations where at any given period some agent previously born may be improved without considering all the contingent non-born descendants still not born at that given period.

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more developed private financial markets is, the lesser the fertility. This is the case of societies with financial institutions, property rights on land property or the existence of large families with strong siblings relationships. Analogous result is achieved if there is a well developed public welfare state, like the social security. These types of institutions are mainly associated with developed countries, so the fertility drop is reinforced.

Second, we present both necessary (static) efficient conditions and a sufficient (dynamic) efficient condition for an allocation to be efficient. That is, in order to show up that efficiency is fulfilled, we do not only require to equalize the rate of return of all investments, both in quantity and in quality of children. We show that the criteria with endogenous fertility is analogous to that provided by Cass (1972).

The paper develops along the following lines. First the model is presented. Next we characterize Pareto dominance and Pareto efficiency. In section 3 we present the necessary conditions for an allocation to be efficient, while in section 4 the sufficient conditions are provided. The paper concludes with some straight forward extensions.

# 1 The model: the primitives

At each date $t = 0, 1, 2...$and, for each dinasty or family, there exist $n_{t-2}$ *old adults* (that is, $n_{t-2}$ agents born at date $t-2$), $n_{t-1}$ *young adults* (that is, $n_{t-1}$agents born at date $t-1$) and $n_t$ *children* (that is, $n_t$ agents born at $t$). The set of agents is endogenous.

Given $\overline{c}_0^o \in \Re_+$, $\overline{n}_{-1} \in \Re_+$, and $\overline{d}_{-1} \in \Re_+$, an allocation is feasible if, for each $t = 0, 1, 2, ...$,one has

$$c_t^o + n_{t-1}\left(c_t^m + b(n_t) + d_t n_t\right) \leq n_{t-1} y_t(d_{t-1}); \tag{1.1}$$

and

$$c_0^o = \overline{c}_0^o;\ n_{-1} = \overline{n}_{-1};\ d_{-1} \leq \overline{d}_{-1}. \tag{1.2}$$

Condition (1.1), written in *per capita* terms, establishes that consumption and investment made by all agents cannot exceed the total amount of the perishable good, which is assumed to be produced out of human capital accumulated in the previous period according to a non-increasing returns to scale technology $y_t : \Re_+ \to \Re$. Condition (1.2) establishes that, by the time the economy starts (that is, at time $t = 0$), consumption made by people who are old at that time is fixed at $c_0^o = \overline{c}_0^o$; the number of agents who are at their mature age at that time is fixed at $n_{-1} = \overline{n}_{-1}$; and, finally, *per capita* investment in education of each of these agents is fixed at $d_{-1} = \overline{d}_{-1}$.

Preferences of each agent born at time $\tau = -2$ are represented by a utility function $U_{-2} : A \to \Re$ defined, for each $a = \{(n_{t-1}, c_t^o, c_t^m, d_{t-1})\}_{t=0}^{\infty}$ by

$$U_{-2}(a) = u(c_0^o)$$

and, for each $t \geq 0$, preferences of each agent born at date $t-1$ are represented by a utility function $U_{t-1} : A \to \Re$ defined, for each $a = \{(n_{t-1}, c_t^o, c_t^m, d_{t-1})\}_{t=0}^{\infty}$ , by

$$U_{t-1}(a) = U(c_t^m, c_{t+1}^o) = u_m(c_t^m) + u_o(c_{t+1}^o),$$

where $u_m : \Re_+ \to \Re_+$ and $u_o : \Re_+ \to \Re_+$ are strictly increasing, concave functions[2] satisfying

$$u_m(0) = u_o(0) = 0.$$

# 2 Efficiency with endogenous population

In this section, we discuss several extensions of the Pareto criteria to economies with endogenous population. Throughout the section, we focus on symmetric allocations, that is, on allocations in which living agents belonging to the same generation are treated equally. Thus, we focus on allocations of the form

$$a = \{x_t\}_{t=1}^{\infty},$$

where , for each $t = 1, 2, ..., \tau, \tau + 1, ...$, the vector $x_t = (n_t, c_t^o, c_t^m, d_t)$ specifies

- the number of children, $n_t$ born at date $t$;
- the amount of a perishable good, $c_t^m$, consumed at date $t$ by each agent born at date $t - 1$,
- the amount of the perishable good, $c_t^o$, consumed at date $t$ by each agent born at date $t - 2$,
- the amount of the perishable good, $d_t$, invested on the education of each agent born at date $t - 1$.

Given a vector of initial conditions $\left(\overline{n}_0, \overline{d}_0\right) \in \Re_+^2$, an allocation $a = \{(n_t, c_t^o, c_t^m, d_t)\}_{t=1}^{\infty}$ is said to be feasible if, for each $t = 1, 2, ...,$ one has

$$c_t^o \leq n_{t-1} \left[y_t(d_{t-1}) - c_t^m - b(n_t) - d_t n_t\right] \tag{2.1}$$

and

$$(n_0, d_0) = \left(\overline{n}_0, \overline{d}_0\right). \tag{2.2}$$

Denote by $A$ the set of al feasible allocations.

## 2.1 Pareto-dominance criteria

In order to extend the notion of Pareto-dominance to a framework in which population is endogenous, one needs to decide first how agents that never get to be born should be taken into account when making social welfare judgements. Here, we propose two different extensions of the notion of Pareto-dominance.

[2] This formulation is slightly more general than your current formulation. The gains in generality can be made at no cost and might bring out some advantages in modeling agents' behavior as a cooperative game. For example, quasilinear preferences allows one the problem of deciding the level of education as a cooperative game with transferable utility.

### 2.1.1 Strong dominance

The first dominance criterium, which will be referred to as strong dominance, takes into account all potential agents in the economy, and it is based on the assumption that all these potential agents prefer to be alive. Thus, according to this criterium, an allocation $a \in A$ *dominates* an allocation $a' \in A$ if both the number of individuals and per capita utility obtained by each individual under the allocation $a$ are never lower than those corresponding to the allocation $a'$; and there exists at least one period in which either the number of individuals or per capita utility obtained by each individual is strictly higher under $a$ than it is under $a'$. Formally, an allocation $\widehat{a} = \left\{ \left( \widehat{n}_{t-1}, \widehat{c}_t^o, \widehat{c}_t^m, \widehat{d}_{t-1} \right) \right\}_{t=1}^{\infty} \in A$ **strongly dominates** an allocation $a' \in A$ if the following conditions are satisfied:

$i)$ for all $t = 1, 2, ....$one has

$$\widehat{n}_t \geq n_t$$

and

$$u_m(\widehat{c}_t^m) + u_o(\widehat{c}_{t+1}^o) \geq u_m(c_t^m) + u_o(c_{t+1}^o);$$

and

$ii)$ there exists at least one period $\tau$ such that either

$$\widehat{n}_\tau > n_\tau$$

or

$$u_m(\widehat{c}_t^m) + u_o(\widehat{c}_{t+1}^o) > u_m(c_t^m) + u_o(c_{t+1}^o)$$

is satisfied.

Observe that taking into account all potential agents might be too demanding. Note also that allocation $\hat{a}$ only can be compared with allocations with the same or less number of children, i.e. $\hat{n}_t \geq n_t$. The allocation $\hat{a}$ is not Pareto-comparable with allocations with more children at the same $t$.

### 2.1.2 Weak dominance

The second criterium, referred to as weak dominance, is based exclusively on preferences of those individuals who get to be born. Formally, an allocation $\widehat{a} \in A$ **weakly dominates** an allocation $a$ en $A$ if the following conditions are satisfied.

$i)$ for all $t = 1, 2, ....$one has

$$u_m(\widehat{c}_t^m) + u_o(\widehat{c}_{t+1}^o) \geq u_m(c_t^m) + u_o(c_{t+1}^o);$$

and

*ii*) there exists at least one period $\tau$ such that

$$u_m(\widehat{c}_t^m) + u_o(\widehat{c}_{t+1}^o) > u_m(c_t^m) + u_o(c_{t+1}^o)$$

is satisfied.

This two dominance criteria defined above have associated two efficiency criteria. An allocation $a \in A$ is said to be weakly efficient if it is not strongly dominated by any other allocation. Analogously, an allocation $a \in A$ is said to be strongly efficient if it is not weakly dominated by any other allocation. Every strongly efficient allocation is, therefore, weakly efficient, although the converse might not be true.

# 3 Weak efficiency. Necessary conditions

In order to obtain necessary conditions to achieve weak efficiency in this framework, some additional notation is now introduced. Given an allocation $a = \{(n_t, c_t^o, c_t^m, d_t)\}_{t=1}^{\infty}$ and, for each $t$, let

$$Z_t = \begin{cases} 0, & \text{if } n_{t-1} = 0 \\ c_t^o / n_{t-1}, & \text{otherwise.} \end{cases}$$

That is, $Z_t$ represents the amount of the consumption good transferred by each agent born at date $t-1$ to those agents born at date $t-2$. Note that for each $t \geq 1$, the feasibility condition in (2.1) can be equivalently written as

$$c_t^m \leq y_t(d_{t-1}) - Z_t - b(n_t) - d_t n_t. \tag{3.1}$$

Let $t$ be arbitrary. Given an efficient allocation $\widehat{a} = \left\{ \left( \widehat{n}_{t-1}, \widehat{c}_t^o, \widehat{c}_t^m, \widehat{d}_{t-1} \right) \right\}_{t=1}^{\infty} \in A$, let

$$\widehat{U}_{t-1} = U_{t-1}(\widehat{a}) = u_m(\widehat{c}_t^m) + u_o(\widehat{c}_{t+1}^o),$$

that is,

$$\widehat{U}_{t-1} = u_m \left[ y_t(\widehat{d}_{t-1}) - \widehat{Z}_t - b(\widehat{n}_t) - \widehat{d}_t \widehat{n}_t \right] + u_o(\widehat{n}_t \widehat{Z}_{t+1}).$$

Also $\widehat{U}_{-1} = u_o(\widehat{n}_0 \widehat{Z}_1)$.

Now let $P_t \subseteq \Re_+^2$ be defined as the set containing all pairs $(U_{t-1}, c_{t+1}^m) \in \Re_+^2$ such that

$$U_{t-1} > \widehat{U}_{t-1}$$

and

$$c_{t+1}^m > \widehat{c}_{t+1}^m.$$

Also, let $F_t \subseteq \Re^2_+$ be defined as the set containing all pairs $(U_{t-1}, c^m_{t+1}) \in \Re^2_+$ for which there exists a triple $(n_t, d_t, Z_{t+1}) \in \Re^3_+$, with $b(n_t) + d_t n_t \leq y_t(\widehat{d}_{t-1}) - \widehat{Z}_t$ and $b(\widehat{n}_{t+1}) + \widehat{d}_{t+1}\widehat{n}_{t+1} \leq y_{t+1}(d_t) - Z_{t+1}$, such that

$$U_{t-1} \leq u_m\left(y_t(\widehat{d}_{t-1}) - \widehat{Z}_t - b(n_t) - d_t n_t\right) + u_0\left(n_t Z_{t+1}\right)$$

and

$$c^m_{t+1} \leq y_{t+1}(d_t) - Z_{t+1} - b(\widehat{n}_{t+1}) - \widehat{d}_{t+1}\widehat{n}_t.$$

Observe that since $\widehat{a} = \left\{\left(\widehat{n}_{t-1}, \widehat{c}^o_t, \widehat{c}^m_t, \widehat{d}_{t-1}\right)\right\}^\infty_{t=0}$ is efficient, the sets $P_t$ and $F_t$ are disjoint sets. Also, since $U$ and $y_t$ are concave functions and the function $b$ is convex, the sets $P_t$ and $F_t$ are convex sets, both having non-empty interior. It follows from the Separating Hyplerplane Theorem that there exists a pair of welfare weights $\lambda_t = (\lambda^m_t, \lambda^y_t) \neq 0$ and a number $r$ such that, for every $(U^P_{t-1}, c^{mP}_{t+1}) \in P_t$ and every $(U^F_{t-1}, c^{mF}_{t+1}) \in F_t$ one has

$$\lambda^m_t U^P_{t-1} + \lambda^y_t c^{mP}_{t+1} \geq r$$

and

$$\lambda^m_t U^F_{t-1} + \lambda^y_t c^{mF}_{t+1} \leq r.$$

Morover, it can be shown that $\lambda^m_t$ and $\lambda^y_t$ are both non-negative. To see this, note first that the vector $(U^F_{t-1}, c^{mF}_{t+1}) = (U(0), 0) \in F_t$ and, hence, one has $r > 0$. Therefore, if either $\lambda^m_t < 0$ or $\lambda^y_t < 0$ were satisfied, then it would be possible to find a pair $(U^p_{t-1}, c^{mP}_{t+1}) \in P_t$ such that $\lambda_{t-1t}U^P_{t-1} + \lambda_{t-1t}c^{mP}_{t+1} < r$, a contradiction that establishes that both $\lambda^m_t \geq 0$ and $\lambda^y_t \geq 0$ are satisfied. Taking this into account, together with the fact that $\widehat{a}$ is efficient one obtains

$$\lambda^m_t U^P_{t-1} + \lambda^y_t c^{mP}_{t+1} \geq \lambda^m_t \widehat{U}_{t-1} + \lambda^y_t \widehat{c}^m_{t+1} \geq \lambda^m_t U^F_{t-1} + \lambda^m_t c^{mF}_{t+1}, \tag{3.2}$$

that is,

$$\frac{\lambda^m_t}{\lambda^m_t + \lambda^y_t} U^P_{t-1} + \frac{\lambda^y_t}{\lambda^m_t + \lambda^y_t} c^{mP}_{t+1} \geq \frac{\lambda^m_t}{\lambda^m_t + \lambda^y_t} \widehat{U}_{t-1} + \frac{\lambda^y_t}{\lambda^m_t + \lambda^y_t} \widehat{c}^m_{t+1} \geq \frac{\lambda^m_t}{\lambda^m_t + \lambda^y_t} U^F_{t-1} + \frac{\lambda^y_t}{\lambda^m_t + \lambda^y_t} c^{mF}_{t+1}.$$

Let $S^1$ denote the simplex in $\Re^2_+$. Also, given a vector $\left(\widehat{d}_{t-1}, \widehat{Z}_t\right) \in \Re^2_+$, a vector $\lambda_t = (\lambda^m_t, \lambda^y_t) \in S^1$, and for each triple $(n_t, d_t, Z_{t+1}) \in \Re^3_+$, let

$$\begin{aligned} W_t\left(n_t, d_t, Z_{t+1}; \lambda_t, \widehat{d}_{t-1}, \widehat{Z}_t\right) &= \lambda^m_t\left[u_m\left(y_t(\widehat{d}_{t-1}) - \widehat{Z}_t - b(n_t) - d_t n_t\right) + u_o\left(n_t Z_{t+1}\right)\right] \\ &\quad + \lambda^y_t\left([y_{t+1}(d_t) - Z_{t+1}]]\right]. \end{aligned}$$

With this notation, and taking into account that the inequalities in (3.2) hold for every $(U^P_{t-1}, c^{mP}_{t+1}) \in P_t$ and every $(U^F_{t-1}, c^{mF}_{t+1}) \in F_t$, it follows that each triple $(\widehat{n}_t, \widehat{d}_t, \widehat{Z}_{t+1})$ corresponding to an efficient allocation can be rationalized as the solution to a welfare maximization problem, as summarized in the following Proposition.

**Proposition 2.1.** *For every strongly efficient allocation* $\widehat{a}$, *there exists a sequence* $\lambda = \left\{(\lambda_t^m, \lambda_t^y) \in \Re_+^2\right\}_{t=0}^{\infty}$ *such that, for every* $t \geq 1$, *the triple* $(\widehat{n}_t, \widehat{d}_t, \widehat{Z}_{t+1})$ *maximizes* $W_t\left(n_t, d_t, Z_{t+1}; \lambda_t, \widehat{d}_{t-1}, \widehat{Z}_t\right)$ *among those triples* $(n_t, d_t, Z_{t+1})$ *satisfying*

$$b(n_t) + d_t n_t \leq y_t(\widehat{d}_{t-1}) - \widehat{Z}_t \qquad (P.1.1)$$

$$b(\widehat{n}_{t+1}) + \widehat{d}_{t+1}\widehat{n}_{t+1} \leq y_{t+1}(d_t) - Z_{t+1}. \qquad (P.1.2)$$

*and*

$$(n_t, d_t, Z_{t+1}) \geq 0. \qquad (P.1.3)$$

Therefore, following from Kuhn Tucker Theorem that for every efficient allocation $\widehat{a}$ and for each $t \geq 0$, there exists a pair$(\lambda_t^m, \lambda_t^y) \in S_+^2$, a vector $\mu_t =(\mu_t^1, \mu_t^2) \geq 0$, and a vector $\alpha_t = (\alpha_t^n, \alpha_t^d, \alpha_t^R) \geq 0$ such that the maximum satisfies the following conditions:

$$\lambda_t^m u_o'(\widehat{n}_t \widehat{Z}_{t+1})\widehat{Z}_{t+1} - \left[\lambda_t^m u_m'\left(y_t(\widehat{d}_{t-1}) - \widehat{Z}_t - b(\widehat{n}_t) - \widehat{d}_t\widehat{n}_t\right) + \mu_t^1\right]\left[b'(\widehat{n}_t) + \widehat{d}_t\right] + \alpha_t^n = 0; \qquad (3.3)$$

$$-\lambda_t^m u_m'\left(y_t(\widehat{d}_{t-1}) - \widehat{Z}_t - b(\widehat{n}_t) - \widehat{d}_t\widehat{n}_t\right)\widehat{n}_t + \left[\lambda_t^y + \mu_t^2\right] y_{t+1}'(\widehat{d}_t) + \alpha_t^d = 0; \qquad (3.4)$$

$$\lambda_t^m u_o'(\widehat{n}_t \widehat{Z}_{t+1})\widehat{n}_t - \left[\lambda_t^y + \mu_t^2\right] + \alpha_t^Z = 0; \qquad (3.5)$$

$$\mu_t^1\left[b(\widehat{n}_t) - \widehat{d}_t\widehat{n}_t - y_t(\widehat{d}_{t-1}) - \widehat{Z}_t\right] = 0; \qquad (3.6)$$

$$\mu_t^2\left[y_{t+1}(\widehat{d}_t) - \widehat{Z}_{t+1} - b(\widehat{n}_{t+1}) - \widehat{d}_{t+1}\widehat{n}_{t+1}\right] = 0; \qquad (3.7)$$

and

$$\alpha_t^n \widehat{n}_t = 0; \alpha_t^d \widehat{d}_t = 0; \alpha_t^R \widehat{Z}_{t+1} = 0. \qquad (3.8)$$

**Remark 2.1.** It is straightforward to check out that Proposition 2.1 can be deduced by arguing that for every efficient allocation $\widehat{a}$ and every $t \geq 1$, the triple $(\widehat{n}_t, \widehat{d}_t, \widehat{Z}_{t+1}) \in \Re_+$ should maximize

$$U_{t-1} = u_m\left(y_t(\widehat{d}_{t-1}) - \widehat{Z}_t - b(\widehat{n}_t) - d_t n_t\right) + u_o\left(n_t Z_{t+1}\right)$$

subject to the constraints in $(P.1.1)$, $(P.1.2,)$ $(P.1.3)$, and the additional constraint

$$u_m\left(y_{t+1}(d_t) - Z_{t+1} - b(\widehat{n}_{t+1}) - \widehat{d}_{t+1}\widehat{n}_{t+1}\right) \geq u_m\left(y_{t+1}(\widehat{d}_t) - \widehat{Z}_{t+1} - b(\widehat{n}_{t+1}) - \widehat{d}_{t+1}\widehat{n}_{t+1}\right)$$

which, given monotonicity of preferences, is equivalent to

$$y_{t+1}(d_t) - Z_{t+1} \geq y_{t+1}(\widehat{d}_t) - \widehat{Z}_{t+1}.$$

**Remark 2.2.** Consider an strongly efficient allocation $\widehat{a} = \{\widehat{x}_t\}_{t=1}^{\infty}$ and select an arbitrary period $\tau$. Then, there exists a sequence $\{\gamma_t\}_{t=0}^{\tau}$ such that the sequence $\{\widehat{x}_t\}_{t=1}^{\tau-1}$ solves the welfare maximization problem

$$\max_{\{x_t\}_{t=1}^{\tau}} \left\{ \gamma_{-1} c_t^o + \sum_{t=1}^{\tau-1} \gamma_{t-1} \left[ u_m(c_t^m) + u_o(c_{t+1}^o) \right] + \gamma_\tau u_m(c_\tau^m) \right\}$$

subject to the constraints

$$c_t^o + n_t \left[ c_t^m + b(n_t) + d_t n_t \right] \leq n_t y_t(d_{t-1}) \text{ for } t = 1, 2, ...\tau. \tag{3.9}$$

and

$$y_{\tau+1}(d_\tau) - Z_t \leq -b(n_t) u_m(c_t^m)$$

must be a solution of the optimization problem.

Consider a weakly efficient allocation $\widehat{a} = \left\{ \left( \widehat{n}_{t-1}, \widehat{c}_t^o, \widehat{c}_t^m, \widehat{d}_{t-1} \right) \right\}_{t=0}^{\infty}$ satisfying, for each $t \geq 0$,

$$(\widehat{c}_t^m, \widehat{c}_t^o) > (0, 0).$$

Observe that for each $t \geq 0$, the triple $(\widehat{n}_t, \widehat{d}_t, \widehat{Z}_{t+1})$ correponding to such allocation such allocation $\widehat{a}$ satisfies (with strict inequality) the constraints in $(P.1.1)$, $(P.1.2)$ and $(P.1.3)$. Then it follows from the above conditions that the triple $(\widehat{n}_t, \widehat{d}_t, \widehat{Z}_{t+1})$ satisfies

$$\frac{\widehat{Z}_{t+1}}{b'(\widehat{n}_t) + \widehat{d}_t} = \frac{u_m'(\widehat{c}_t^m)}{u_o'(\widehat{c}_{t+1}^o)} = y_t'(\widehat{d}_t). \tag{3.10}$$

Proposition 2.1 above summarizes this result.

**Proposition 2.2.** *Let* $\widehat{a} = \left\{ \left( \widehat{n}_{t-1}, \widehat{c}_t^o, \widehat{c}_t^m, \widehat{d}_{t-1} \right) \right\}_{t=1}^{\infty} \in A$ *be an strongly efficient allocation satisfying, for every* $t \geq 1$,[3]

$$(\widehat{c}_t^m, \widehat{c}_t^o) > (0, 0). \tag{P.2.1}$$

*Then, there exists a sequence* $\widehat{p} = \{p_t\}_{t=0}^{\infty}$ *of strictly positive real numbers satisfying, for every* $t \geq 0$,

$$\frac{\widehat{Z}_{t+1}}{b'(\widehat{n}_t) + \widehat{d}_t} = \frac{u_m'(\widehat{c}_t^m)}{u_o'(\widehat{c}_{t+1}^o)} = y_{t+1}'(\widehat{d}_t) = \frac{p_t}{p_{t+1}}. \tag{P.2.2}$$

That is, every efficient allocation can be associated to a sequence of relative prices defined implicitly by the marginal rate of return to any investment (either on quantity or in quality of human capital), which in turn must be equal to the marginal

[3]Why strictly positive is required in (P.2.1)? In order for the solution to be interior (Look for primitives to support this result.)

rate of substitution between current and future consumption. Observe that the two equalities at the left hand side of (3.10), together with the feasibility condition

$$\widehat{c}_t^m = y_t(\widehat{d}_{t-1}) - \widehat{Z}_t - b(\widehat{n}_t) - \widehat{d}_t\widehat{n}_t$$

and the identity

$$\widehat{n}_t\widehat{Z}_{t+1} = \widehat{c}_{t+1}^o$$

form a system with 4 equations and 5 unknowns. Thus, there might be infinitely many triples that solve the system in (1.10). Each of this solutions can be rationalized as the solution to the welfare maximization problem in the statement of proposition 2.1, where the system of welfare weights is defined by

$$\frac{\lambda_t^y}{\lambda_t^m} = u_m'(\widehat{c}_{t+1}^o)\widehat{n}_t.$$

Finally, the following results allow us to identify certain properties of those allocations that, although verifying the Proposition 2.2, they are not inefficient. An example is the following result:

**Corollary 2.3.** *Let $\widehat{a}$ be an allocation satisfying, for each $t = 1, 2, ...,$ conditions $(P.2.1)$ and $(P.2.2)$, and suppose there exists an allocation $a$ that strongly dominates the allocation $\widehat{a}$ satisfying the following inequalities $\triangle n_t \leq 0$, $\triangle c_t^m \leq 0$ and $\triangle d_t \leq 0$, for each $t = 1, 2, ...$ Then for each $t = 1, 2, ..$ one has*

$$\triangle c_{t+1}^o \geq \frac{p_t}{p_{t+1}} \nabla c_t^m.$$

The proof is immediate from (5.1), (5.2) and $(P.2.2)$. This results indicate that there would be always an improvement whenever the reduction on consumption for middle age agents is far from being balanced, in welfare terms, by the increase with consumption in old times.[4]

# 4 Efficiency. Sufficient conditions

In this section we provide conditions that guarantee that an allocation $\widehat{a} = \left\{ \left( \widehat{n}_t, \widehat{c}_t^o, \widehat{c}_t^m, \widehat{d}_t \right) \right\}_{t=1}^{\infty}$ $\in A$ satisfying conditions $(P.2.1)$ and $(P.2.2)$ is efficient.

The main result is presented as follows:

[4]For example, take the case for a feasible allocation $a$ such that $\triangle n_t = 0$ and $\triangle d_t = 0$ for all $t$. Then, the allocation $a$ will Pareto-dominate the allocation $\widehat{a}$ if the reduction on consumption when young $\triangle c_t^m \leq 0$ is balanced in utility terms with the increase in consumption when old, i.e. $\triangle c_t^o = \widehat{n}_{t-1}\nabla c_t^m$ at (5.1).

**Proposition 3.1.** *Let $\widehat{a}$ be an allocation satisfying, for each $t = 1, 2, ...$, conditions (P.1.1) and (P.1.2,). Let be the sequence of strictly positive real numbers $\widehat{p} = \{p_t\}_{t=0}^{\infty}$ defined in Proposition 2.2. Then, the condition*

$$\lim_{T \to \infty} \left[ p_T \prod_{\tau=1}^{T} n_\tau \right] = 0.$$

*is sufficient for an allocation satisfying (P.2.1) to be efficient.*

This proposition is very close to Proposition 5.3 and Lemma 5.5 in Balasko and Shell (1980) for an exogenous endowment economy. We could also think on an extension to Proposition 1 and Theorems 2 and 3 in Cass (1972) for an capital accumulation growth economy with exogenous population growth, as a consequence of this proposition.

**Corollary 3.2.** *If feasible path $a$ is innefficient then*

$$\lim_{T \to \infty} \left[ p_T \prod_{\tau=1}^{T} n_\tau \right] = \infty.$$

Two comments on this result, proved in the Appendix. First, observe that Balasko and Shell (1980) assume that $\triangle d_t = 0$ and $n_t = 1$ for all $t$, so their proposition 5.3 is a particular case of ours.

Second, to show that for all $t$, both the required condition

$$\triangle n_{t-1} \triangle Z_t - \widehat{n}_{t-1} \triangle d_t \triangle n_t \leq 0$$

holds and the term $h_t = \triangle c_t^o - \widehat{n}_{t-1} \frac{p_{t-1}}{p_t} \triangle d_{t-1} - \triangle n_{t-1} \widehat{Z}_t$ is always non-negative, it is sufficient to show that there is no loss of generality in assuming that for any allocation $a$ strongly dominating $\widehat{a}$, one has

$$\triangle n_t \leq 0; \quad \triangle c_t^m \leq 0; \quad \triangle d_t \leq 0.$$

In order to prove our previous assumption we are going to proceed by steps by proving the following auxiliary results.

The following Lemma 3.4 is in line with the argument presented in Corollary 2.3 which indicates that there would be always an improvement whenever the reduction on consumption for middle age agents is far from being balanced, in welfare terms, by the increase with consumption in old times. Observe, finally, that concavity conditions for preferences on individuals' intertemporal consumption are crucial for this result.

**Lemma 3.4.** *Let $\widehat{a}$ be an allocation satisfying, for each $t = 1, 2, ...,$ conditions (P.1.1) and (P.1.2), and suppose there exists an allocation $\widetilde{a}$ that weakly dominates the allocation $\widehat{a}$. Then for each $t = 1, 2, ..$ one has*

$$\widetilde{c}_t^m + b(\widetilde{n}_t) + \widetilde{n}_t\widetilde{d}_t \leq \widehat{c}_t^m + b(\widehat{n}_t) + \widehat{n}_t\widehat{d}_t.$$

That is, allocations that weakly dominates those who verifies the necessary conditions for optimality are such that the total resources not devoted to feed the old generation must be lower. In other words, the old generation must improve in an efficient allocation. The Lemma is proved in the appendix.

**Lemma 3.5.** *Let $\widehat{a}$ be an allocation satisfying, for each $t = 1, 2, ...,$ conditions (P.2.1) and (P.2.2,) and suppose $\widehat{a}$ is not strongly efficient. Then there exists an allocation $a$ that weakly dominates the allocation $\widehat{a}$ and satisfies the following inequalities, for each $t = 1, 2, ..$ ,*

$$\triangle n_t \leq 0; \triangle c_t^m \leq 0; \triangle d_t \leq 0.$$

Lemma 3.5, proved in the Appendix, means that an allocation that verifies necessary conditions for efficiency given in Proposition 2.1 could not be efficient, if and only if there exists another allocation that also verifies necessary conditions given by Proposition 2.1 but with at least with one of the following: less number of children, lower investment in children's human capital, or with lower middle age agents' consumption.

# 5 Conclusions and Extensions

In this paper we have undertaken an optimality study in an overlapping generation model with endogenous fertility and exogenous endowments. The contribution of this paper were twofold. First, two definitions of Pareto dominance and, therefore, Pareto efficiency has been presented. Second, the necessary (static) and sufficient (dynamic) efficient conditions have been shown.

This work is the starting point for several lines of research. Theoretically, there are two straight forward extensions. First, an study on efficiency can be carried out in a model where it is considered that offspring are not only childbeared by old-age security motive but also by consumption motive. Notice that in our framework the old-age security motive arises as a necessary condition for having children. However, that would not be the case if parents enhance welfare with the mere fact of being born them. In this case, would be worth exploring of the results are robusted or changed. The incompleteness of markets derived in the paper may still cause inefficiency.

Second, we are ready to explore efficiency, even in the presence of free-access to capital markets there might be markets could be incomplete due to parents and their own children may not be able to trade on the right to existence. So further research will study the role of the family and the Welfare State in the completeness of this market.

Third, it is interesting to find contracts between parents and their children in decentralized economies that reproduce optimal allocations. Some contracts have been proposed in the literature, Ehrlich and Lui (1991), Cigno (1993) and Conde-Ruiz et al (2002) These would lead us to extend Becker and Murphy's ideas on how the state reproduces family roles at aggregate level and, then, to proceed Boldrin and Montes's study in an endogenous fertility set-up.

# Appendix

**Proof of Proposition 3.1**. Some additional notation is now introduced. Consider an allocation $\widehat{a} = \{\widehat{x}_t\}_{t=1}^{\infty}$ satisfying conditions $(P.2.1)$ and $(P.2.2)$. For every other feasible allocation $a = \{x_t\}_{t=1}^{\infty}$ and let $\triangle x_t = (\triangle n_t, \triangle c_t^o, \triangle c_t^m, \triangle d_t)$ each $t = 1, 2, 3, ...$ be defined as difference

$$\triangle x_t = x_t - \widehat{x}_t.$$

By feasibility both allocations $\widehat{a}$ and $a$ verifies (2.1); that is, one has

$$\widehat{c}_t^o = \widehat{n}_{t-1}\widehat{Z}_t = \widehat{n}_{t-1}\left[y_t(\widehat{d}_{t-1}) - \widehat{c}_t^m - b(\widehat{n}_t) - \widehat{d}_t\widehat{n}_t\right]$$

and

$$\begin{aligned}\widehat{c}_t^o + \triangle c_t^o &= [\widehat{n}_{t-1} + \triangle n_{t-1}]\Big[y_t(\widehat{d}_{t-1} + \triangle d_{t-1}) - \widehat{c}_t^m + \nabla c_t^m - b(\widehat{n}_t + \triangle n_t) - \\ &\quad - \left(\widehat{d}_t + \triangle d_t\right)(\widehat{n}_t + \triangle n_t)\Big]\end{aligned}$$

where $\nabla c_t^m = -\Delta c_t^m$. Subtracting the latter from the former one obtains

$$\begin{aligned}\triangle c_t^o &= \widehat{n}_{t-1}\Big[\Big(y_t(\widehat{d}_{t-1} + \triangle d_{t-1}) - y_{t-1}(\widehat{d}_{t-1})\Big) + \nabla c_t^m - (b(\widehat{n}_t + \triangle n_t) - b(\widehat{n}_t)) - \\ &\quad -\widehat{d}_t\triangle n_t - \triangle d_t\widehat{n}_t - \triangle d_t\triangle n_t\Big] + \triangle n_{t-1}Z_t \qquad (5.1)\end{aligned}$$

By concavity of preferences and production function, and convexity of children cost function one has[5]

$$\begin{aligned}\triangle c_t^o &\leq \widehat{n}_{t-1}\left[y_t'(\widehat{d}_{t-1})\triangle d_{t-1} + \frac{u_o'(\widehat{c}_{t+1}^o)}{u_m'(\widehat{c}_t^m)}\triangle c_{t+1}^o - \left(b'(\widehat{n}_t) + \widehat{d}_t\right)\triangle n_t - \triangle d_t n_t - \triangle d_t\triangle n_t\right] \\ &\quad +\triangle n_{t-1}\left(\widehat{Z}_t + \triangle Z_t\right)\end{aligned}$$

[5]First, the concavity of preferences implies that $\Delta c_{t+1}^o \Delta c_t^m \geq 0$. Then if $\Delta c_t^m < 0$ then $\frac{u_m'(\widehat{c}_t^m)}{u_o'(\widehat{c}_{t+1}^o)} \leq -\frac{\triangle c_{t+1}^o}{\Delta c_t^m}$, and if $\Delta c_t^m > 0$ then $\frac{u_m'(\widehat{c}_t^m)}{u_o'(\widehat{c}_{t+1}^o)} \geq -\frac{\triangle c_{t+1}^o}{\triangle c_t^m}$; that is, $\nabla c_t^m \frac{u_m'(\widehat{c}_t^m)}{u_o'(\widehat{c}_{t+1}^o)} \leq \triangle c_{t+1}^o$ . Second, the concavity of production function means that $y'(\widehat{d}) \geq \frac{y(\widehat{d}+\triangle d)-y(\widehat{d})}{\triangle d}$. Finally, the convexity of children cost function means that $b'(\widehat{n}) \geq \frac{b(\widehat{n}+\triangle n)-b(\widehat{n})}{\triangle n}$.

Since $\widehat{a}$ satisfies the necessary conditions in $(P.2.1)$ and $(P.2.2)$ one has

$$\triangle c_t^o \leq \widehat{n}_{t-1}\left[\frac{p_{t-1}}{p_t}\triangle d_{t-1} + \frac{p_{t+1}}{p_t}\triangle c_{t+1}^o - \frac{p_{t+1}}{p_t}\widehat{Z}_{t+1}\triangle n_t - \triangle d_t \widehat{n}_t - \triangle d_t \triangle n_t\right] + +\triangle n_{t-1}\left(\widehat{Z}_t + \triangle Z_t\right) \tag{5.2}$$

That is,

$$\triangle c_t^o - \widehat{n}_{t-1}\frac{p_{t-1}}{p_t}\triangle d_{t-1} - \triangle n_{t-1}\widehat{Z}_t \leq \widehat{n}_{t-1}\frac{p_{t+1}}{p_t}\left[\triangle c_{t+1}^o - \widehat{n}_t \frac{p_t}{p_{t+1}}\triangle d_t - \triangle n_t \widehat{Z}_{t+1}\right] - -\widehat{n}_{t-1}\triangle d_t \triangle n_t + \triangle n_{t-1}\triangle Z_t$$

In Lemma 3.5 we show that there is no loss of generality in assuming that for every allocation $a$ that weakly dominates a given allocation $\hat{a}$, one has, for all $t$,

$$\triangle n_{t-1}\triangle Z_t - \widehat{n}_{t-1}\triangle d_t \triangle n_t \leq 0$$

Let us denote $h_t = \triangle c_t^o - \widehat{n}_{t-1}\frac{p_{t-1}}{p_t}\triangle d_{t-1} - \triangle n_{t-1}\widehat{Z}_t$. Lemma 3.5 also implies that this term is non negative. Then applying the argument recursively one has

$$h_t \leq \widehat{n}_{t-1}\frac{p_{t+1}}{p_t}h_{t+1} \leq \widehat{n}_{t-1}\widehat{n}_t \frac{p_{t+1}}{p_t}\frac{p_{t+2}}{p_{t+1}}h_{t+2} < \ldots < h_{t+T}\frac{p_{t+T}}{p_t}\prod_{\tau=1}^{T} n_{t+\tau-2}$$

Given that there exist other feasible allocation this inequality must be strict at least for some $t$. Thus, given that $h_{t+T}$ is bounded by feasibility for all $T$, a sufficient condition for an allocation satisfying $(P.2.1)$

$$\lim_{T\to\infty}\left[p_T \prod_{\tau=1}^{T} n_\tau\right] = 0.\square$$

**Proof of Lemma 3.4.**. Let $t > 0$, $A > 0$ and $B > 0$, and define

$$W_t(A,B) = \max_{(n_t,d_t,Z_{t+1})\geq 0}\{u_m(A - b(n_t) - d_t n_t) + u_o(n_t Z_{t+1}) \ : \ b(n_t) + d_t n_t \leq A, \quad y_{t+1}(d_t) - Z_{t+1} \geq B\}$$

Consider now and allocation $\widehat{a} = \left\{\left(\widehat{n}_t, \widehat{c}_t^o, \widehat{c}_t^m, \widehat{d}_t\right)\right\}_{t=1}^{\infty} \in A$ satisfying conditions $(P.1.1)$ and $(P.1.2)$. It is straightforward to check out that, given $\overline{d}_0$,

$$W_1(y_1(\overline{d}_0) - \widehat{Z}_1, y_2(\widehat{d}_1) - \widehat{Z}_2) = u_m(y_1(\overline{d}_0) - \widehat{Z}_1 - b(\widehat{n}_1) - \widehat{d}_1\widehat{n}_1) + u_o(\widehat{n}_1\widehat{Z}_2) = U_1(\widehat{a}).$$

Consider now an allocation $\widetilde{a} = \left\{\left(\widetilde{n}_t, \widetilde{c}_t^o, \widetilde{c}_t^m, \widetilde{d}_t\right)\right\}_{t=1}^{\infty} \in A$ that weakly dominates the allocation $\widehat{a}$. From the definition of weak dominance one has that at least some agent at generation $t-2$ or at $t-1$ (old and middle age agents at $t=1$, respectively) is better and none is worst; that is,

$$\widetilde{Z}_1 \geq \widehat{Z}_1 \text{ and } U_1(\widetilde{a}) \geq U_1(\widehat{a}),$$

which taking into account that each function is increasing in $A$ yields

$$U_1(\widetilde{a}) \geq U_1(\widehat{a}) = W_1\left(y_1(\overline{d}_0) - \widehat{Z}_1, y_2(\widehat{d}_1) - \widehat{Z}_2\right) \geq W_1\left(y_1(\overline{d}_0) - \widetilde{Z}_1, y_2(\widehat{d}_1) - \widehat{Z}_2\right).$$

¿From the definitions, this inequality can only be preserved if the inequality

$$y_2(\widetilde{d}_1) - \widetilde{Z}_2 \leq y_2(\widehat{d}_1) - \widehat{Z}_2.$$

is satisfied. Suppose that it is not true. Since, $\widetilde{Z}_1 \geq \widehat{Z}_1$ then the triple $\left(\widetilde{n}_t, \widetilde{c}_t^o, \widetilde{c}_t^m, \widetilde{d}_t\right)$ verifies both $b(n_t) + d_t n_t \leq A$, and $y_{t+1}(d_t) - Z_{t+1} \geq B$, for the given $A$ and $B$. This contradicts, therefore, that the triple $\left(\widehat{n}_t, \widehat{c}_t^o, \widehat{c}_t^m, \widehat{d}_t\right)$ solves the optimization problem above stated.

By applying an entirely analogous argument one obtains

$$U_2(\widetilde{a}) \geq U_2(\widehat{a}) = W_2\left(y_2(\widehat{d}_1) - \widehat{Z}_2, y_3(\widehat{d}_2) - \widehat{Z}_3\right) \geq W_2\left(y_2(d_1) - \widetilde{Z}_2, y_3(\widehat{d}_2) - \widehat{Z}_3\right),$$

which in turn implies that the inequality

$$y_3(d_2) - \widetilde{Z}_3 \leq y_3(\widehat{d}_2) - \widehat{Z}_3.$$

must be satisfied. By proceeding recursively one obtains for all $t$,

$$\widetilde{c}_t^m + b(\widetilde{n}_t) + \widetilde{n}_t\widetilde{d}_t = y_t(\widetilde{d}_{t-1}) - \widetilde{Z}_t \leq y_t(\widehat{d}_{t-1}) - \widehat{Z}_{t-1} = \widehat{c}_t^m + b(\widehat{n}_t) + \widehat{n}_t\widehat{d}_t,$$

which establishes Lemma 3.4.□

**Proof of Lemma 3.5.**. Let $\widehat{a}$ be an allocation satisfying, for each $t = 1, 2, ...,$ conditions $(P.2.1)$ and $(P.2.2)$, and suppose $\widehat{a}$ is not strongly efficient. The proof proceeds by selecting arbitrarily a feasible allocation $\widetilde{a}$ that weakly dominates $\widehat{a}$ and use it to identify a third allocation $a$ satisfying the required property in the statement of Lemma 3.5.

We now proceed to identify such allocation $a = \{(n_t, c_t^m, c_t^o, d_t)\}_{t=1}^{\infty}$, with $n_t = \widehat{n}_t + \triangle n_t$, $d_t = \widehat{d}_t + \triangle d_t$ and $c_t^m = \widehat{c}_t^m + \triangle c_t^m$. Given an allocation $\widetilde{a}$ that weakly dominates $\widehat{a}$, select a vector $(\eta_t, \delta_t, \gamma_t) \in [0,1]^3$ such that

$$\triangle n_t = \triangle\widetilde{n}_t(1-\eta_t) \leq 0; \quad \triangle c_t^m = \triangle\widetilde{c}_t^m(1-\gamma_t) \leq 0; \quad \triangle d_t = \triangle\widetilde{d}_t(1-\delta_t) \leq 0, \quad (A.3.1)$$

and

$$[\widetilde{c}_t^m + \gamma_t\nabla\widetilde{c}_t^m] + b(\widetilde{n}_t + \eta_t\nabla\widetilde{n}_t) + \left[\widetilde{d}_t + \delta_t\nabla\widetilde{d}_t\right][\widetilde{n}_t + \eta_t\nabla\widetilde{n}_t] = \widetilde{c}_t^m + b(\widetilde{n}_t) + \widetilde{n}_t\widetilde{d}_t \quad (A.3.2)$$

are satisfied.

That is, the vector $(\eta_t, \delta_t, \gamma_t)$ is used to modify the vector $\left(\widetilde{n}_t, \widetilde{c}_t^m, \widetilde{d}_t\right)$ in such a way that *i*) the required property in the statement of Lemma 3.5 is satisfied, condition $(A.3.1)$;[6] and *ii*) total expenditures on current consumption of young adults and current investments in children and education is preserved, condition $(A.3.2)$, i.e., the resources in the economy after making the transfers to the old generation at each period $t$ is the same to those existed in the allocation $\widetilde{a}$ at each period $t$, so that the condition in Lemma 3.4 is verified.

To show such vector $(\eta_t, \delta_t, \gamma_t)$ exists, it is useful to consider a particular case. First, let $e_t : [0,1]^3 \rightarrow \Re$ be a function defined, for each $(\eta_t, \delta_t, \gamma_t) \in [0,1]^3$, by

$$\begin{aligned} e_t(\eta_t, \delta_t, \gamma_t) &= [\widetilde{c}_t^m + \gamma_t \nabla \widetilde{c}_t^m] + b(\widetilde{n}_t + \eta_t \nabla \widetilde{n}_t) + \left[\widetilde{d}_t + \delta_t \nabla \widetilde{d}_t\right] [\widetilde{n}_t + \eta_t \nabla \widetilde{n}_t] - \\ &\quad - \widetilde{c}_t^m - b(\widetilde{n}_t) - \widetilde{n}_t \widetilde{d}_t = \\ &= \gamma_t \nabla \widetilde{c}_t^m + [b(\widetilde{n}_t + \eta_t \nabla \widetilde{n}_t) - b(\widetilde{n}_t)] + \left[\widetilde{d}_t + \delta_t \nabla \widetilde{d}_t\right] \eta_t \nabla \widetilde{n}_t + \delta_t \nabla \widetilde{d}_t \widetilde{n}_t. \end{aligned}$$

Note that condition $(A.3.2)$ can be written as

$$e_t(\eta_t, \delta_t, \gamma_t) = 0.$$

Then, consider, for example, the case in which $\triangle \widetilde{n}_t > 0$, $\triangle \widetilde{c}_t^m > 0$ and $\triangle \widetilde{d}_t < 0$ is satisfied. Observe that, in this particular case, condition $(A.3.1)$ imposes that $\eta_t = 1$ and $\gamma_t = 1$ must be satisfied. Also, observe that in this case the function $e_t((1, \delta_t, 1)$ is strictly increasing in $\delta_t$. Therefore $e_t(1,0,1) < 0$ must be satisfied. Note finally that given by assumption of this Lemma $\widetilde{a}$ Pareto dominates $\hat{a}$, then Lemma 3.4 implies[7] $e_t(1,1,1) > 0$. Since the function $e_t(\cdot)$ is continuous, there must exist a number $\delta_t^* \in [0,1]$ such that $e_t(1, \delta_t^*, 1) = 0$. Therefore the vector $(\eta_t, \delta_t, \gamma_t) = (1, \delta^*, 1)$ satisfies $(A.3.1)$ and $(A.3.2)$. By applying analogous arguments to all possible cases, one can show that a vector $(\eta_t, \delta_t, \gamma_t)$ satisfying the required conditions always exists.

Given a sequence $\{(\eta_t, \delta_t, \gamma_t)\}_{t=0}^{\infty}$ of vectors satisfying conditions $(A.3.1)$ and $(A.3.2)$, let $a = \{(n_t, c_t^m, c_t^o, d_t)\}_{t=1}^{\infty}$ be a feasible allocation defined, for each $t > 0$,

[6] This could be understood geometrically. Let be an hypercube where allocations $\widehat{a}$ and $\underline{\widetilde{a}} = \left(- \mid \widetilde{n}_t \mid, - \mid \widetilde{c}_t^m \mid, - \mid \widetilde{d}_t \mid\right)$ are set at the far opposite vortex. Then, *i*) means that by construction, the new allocation $a$ is chosen such that it belongs to the adherence of this hypercube.

[7] Observe that, given the definition of the function $e_t(\cdot)$, it can only be non-negative $e_t(1,1,1)$ if at least one of the following $\triangle \widetilde{n}_t$, $\triangle \widetilde{c}_t^m$ or $\triangle \widetilde{d}_t$ is non-positive. Geometrically, this means that no allocation $\widetilde{a}$ that strongly dominates $\widehat{a}$ is placed at the positive orthant of a hypercube with center $\widehat{a} = \left(\widehat{n}_t, \widehat{c}_t^m, \widehat{d}_t\right)$, and where allocations $\overline{\widetilde{a}} = \left(\mid \widetilde{n}_t \mid, \mid \widetilde{c}_t^m \mid, \mid \widetilde{d}_t \mid\right)$ and $\underline{\widetilde{a}} = \left(- \mid \widetilde{n}_t \mid, - \mid \widetilde{c}_t^m \mid, - \mid \widetilde{d}_t \mid\right)$ are set at the far opposite vortex. This is a direct application of Lemma 3.4, where the hiperplane defined in that Lemma $\widetilde{c}_t^m + b(\widetilde{n}_t) + \widetilde{n}_t \widetilde{d}_t = \widehat{c}_t^m + b(\widehat{n}_t) + \widehat{n}_t \widehat{d}_t$, which intersects with the vortex $\widehat{a}$ of the hypercube preclude that the positive vortex verifies this relation with inequality.

by:

$$\begin{aligned} n_t &= \widetilde{n}_t - \eta_t \triangle \widetilde{n}_t = \widehat{n}_t + (1-\eta_t) \triangle \widetilde{n}_t, \\ d_t &= \widetilde{d}_t - \delta_t \triangle \widetilde{d}_t = \widehat{d}_t + (1-\delta_t) \triangle \widetilde{d}_t, \\ c_t^m &= \widetilde{c}_t^m - \gamma_t \triangle \widetilde{c}_t^m = \widehat{c}_t^m + (1-\gamma_t) \triangle \widetilde{c}_t^m; \end{aligned}$$

and[8]

$$\begin{aligned} c_{t+1}^o &= \widetilde{c}_{t+1}^o + (\widetilde{n}_t - \eta_t \triangle \widetilde{n}_t) \left[ y_{t+1}(\widetilde{d}_t - \delta_t \triangle \widetilde{d}_t) - y_{t+1}(\widetilde{d}_t) \right] - \eta_t \triangle \widetilde{n}_t \widetilde{Z}_{t+1} \\ &= \widetilde{c}_{t+1}^o + n_t \left[ y_{t+1}(\widetilde{d}_t - \delta_t \triangle \widetilde{d}_t) - y_{t+1}(\widetilde{d}_t) \right] - \eta_t \triangle \widetilde{n}_t \widetilde{Z}_{t+1} \\ &= \widetilde{c}_{t+1}^o + \widetilde{n}_t \left[ y_{t+1}(\widetilde{d}_t - \delta_t \triangle \widetilde{d}_t) - y_{t+1}(\widetilde{d}_t) \right] - \eta_t \triangle \widetilde{n}_t Z_{t+1} \end{aligned} \tag{5.3}$$

It is straightforward to show that $a$ is feasible. Also, since $a$ satisfies $(A.3.2)$, one has that $e_t(\eta_t, \delta_t, \gamma_t) = 0$ for every $t > 0$. That is,

$$0 = \gamma_t \nabla \widetilde{c}_t^m + [b(\widetilde{n}_t + \eta_t \nabla \widetilde{n}_t) - b(\widetilde{n}_t)] + \left[ \widetilde{d}_t + \delta_t \nabla \widetilde{d}_t \right] \eta_t \nabla \widetilde{n}_t + \delta_t \nabla \widetilde{d}_t \widetilde{n}_t,$$

which taking into account the strict convexity of the cost function $b(\cdot)$ yields

$$\gamma_t \nabla \widetilde{c}_t^m + [b'(n_t) + d_t] \eta_t \nabla \widetilde{n}_t + \widetilde{n}_t \delta_t \nabla \widetilde{d}_t > 0 > \gamma_t \nabla \widetilde{c}_t^m + \left[ b'(\widetilde{n}_t) + \widetilde{d}_t \right] \eta_t \nabla \widetilde{n}_t + n_t \delta_t \nabla \widetilde{d}_t \tag{A.3.3}$$

In order to complete the proof, we will prove that the allocation $\widehat{a}$, although verifying conditions $(P.2.1)$ and $(P.2.2)$, is not efficient and the allocation $a$ strongly dominates it. Hence, it is sufficient to show Corollary 2.3; that is, for every $t > 0$,

$$\triangle c_{t+1}^o \geq \frac{p_t}{p_{t+1}} \nabla c_t^m,$$

or equivalently,[9]

$$\triangle \widetilde{c}_{t+1}^o + \left( c_{t+1}^o - \widetilde{c}_{t+1}^o \right) \geq \frac{p_t}{p_{t+1}} \left[ \nabla \widetilde{c}_t^m + \gamma_t \Delta \widetilde{c}_t^m \right] \tag{A.3.4}$$

is satisfied. Taking into account that, by assumption, allocation $\widetilde{a}$ strongly dominates the allocation $\widehat{a}$, so it verifies Corollary 2.3, i.e. $\triangle \widetilde{c}_{t+1}^o \geq \frac{p_t}{p_{t+1}} \nabla \widetilde{c}_t^m$, it is sufficient to show that

$$\left( c_{t+1}^o - \widetilde{c}_{t+1}^o \right) \geq \frac{p_t}{p_{t+1}} \gamma_t \Delta \widetilde{c}_t^m \tag{A.3.5}$$

is satisfied.

[8] The following result is found by subtracting $c_{t+1}^o - \widetilde{c}_{t+1}^o$, substitution by (2.1) and considering the condition $(A.3.2)$.

[9] Remember that $\triangle x_t = x_t - \widehat{x}_t$. Subtracting $c_{t+1}^o - \widetilde{c}_{t+1}^o$, and taking into account $(A.3.1)$.

To show $(A.3.5)$ is satisfied, use (5.3) and the strict concavity of the production function to obtain

$$\begin{aligned} c_{t+1}^o - \widetilde{c}_{t+1}^o &= \widetilde{n}_t \left[ y_{t+1} \left( \widetilde{d}_t + \delta_t \nabla \widetilde{d}_t \right) - y_{t+1}(\widetilde{d}_t) \right] + Z_{t+1} \eta_t \nabla \widetilde{n}_t \qquad (A.3.6) \\ &> y'_{t+1}(d_t) \widetilde{n}_t \delta_t \nabla \widetilde{d}_t + Z_{t+1} \eta_t \nabla \widetilde{n}_t. \end{aligned}$$

By combining the inequalities in $(A.3.3)$ and $(A.3.6)$ one can find different lower bounds for the term $c_{t+1}^o - \widetilde{c}_{t+1}^o$. More precisely, considering that from the first inequality in $(A.3.3)$, $\widetilde{n}_t \delta_t \nabla \widetilde{d}_t > \gamma_t \triangle \widetilde{c}_t^m - [b'(n_t) + d_t] \eta_t \nabla \widetilde{n}_t$, yields

$$c_{t+1}^o - \widetilde{c}_{t+1}^o > y'_{t+1}(d_t) \gamma_t \triangle \widetilde{c}_t^m + \eta_t \nabla \widetilde{n}_t \left[ Z_{t+1} - y'_{t+1}(d_t) [b'(n_t) + d_t] \right]. \qquad (A.3.7)$$

Analogously, substituting the term $\eta_t \nabla \widetilde{n}_t$ in $(A.3.6)$ by its lower bound obtained from the first inequality in $(A.3.3)$, yields

$$c_{t+1}^o - \widetilde{c}_{t+1}^o > \left[ \frac{Z_{t+1}}{b'(n_t) + d_t} \right] \gamma_t \triangle \widetilde{c}_t^m + \widetilde{n}_t \delta_t \nabla \widetilde{d}_t \left[ y'_{t+1}(d_t) - \frac{Z_{t+1}}{b'(n_t) + d_t} \right] \qquad (A.3.8)$$

In what follows, it is useful to consider all possible cases separately.[10]

**Case** *i*) $\triangle n_t < 0$, $\triangle c_t^m < 0$ and $\triangle d_t < 0$, i.e., $\triangle \widetilde{n}_t < 0$, $\triangle \widetilde{c}_t^m < 0$ and $\triangle \widetilde{d}_t < 0$. In this case, $(A.3.5)$ is trivially satisfied for $\eta_t = \delta_t = \gamma_t = 0$.

**Case** *ii*) $\triangle d_t = 0$. In this case, it follows from $(A.3.1)$ that either $\delta_t = 1$ or $\triangle \widetilde{d}_t = 0$ must be satisfied; that is, $d_t = \widetilde{d}_t$. If $\widetilde{a}$ weakly dominates $\widehat{a}$ then Lemma 3.4, jointly with the strictly concavity of the production function, will yield

$$\left( Z_{t+1} - \widehat{Z}_{t+1} \right) > y'_{t+1}(d_t) \triangle d_t = 0$$

Taking this into account, together with the fact that $d_t = \widehat{d}_t$ and $\triangle n_t \leq 0$ are satisfied (so $b'(\widehat{n}_t) > b'(n_t)$), and that the allocation $\widehat{a}$ verifies $(P.2.1)$ and $(P.2.2)$ by assumption of this Lemma 3.5, write the inequality $(A.3.7)$ as

$$\begin{aligned} c_{t+1}^o - \widetilde{c}_{t+1}^o &> y'_{t+1}\left(\widehat{d}_t\right) \gamma_t \triangle \widetilde{c}_t^m + \eta_t \nabla \widetilde{n}_t \left[ \widehat{Z}_{t+1} - y'_{t+1}\left(\widehat{d}_t\right) \left[ b'(\widehat{n}_t) + \widehat{d}_t \right] \right] = \\ &= \frac{p_t}{p_{t+1}} \gamma_t \triangle \widetilde{c}_t^m + \eta_t \nabla \widetilde{n}_t \left[ b'(\widehat{n}_t) + \widehat{d}_t \right] \left[ \frac{p_t}{p_{t+1}} - \frac{p_t}{p_{t+1}} \right] = \frac{p_t}{p_{t+1}} \gamma_t \triangle \widetilde{c}_t^m, \end{aligned}$$

[10] The following proof can be understood geometrically. First, think on an usual three dimensional cube with eight vortex, the huge cube defined above by $\left\{ \underline{\widetilde{a}}, \overline{\widetilde{a}} \right\}$. The center of the cube is placed with the allocation $\hat{a}_t$. Second, each vortex and each center of each side and edge represents the twenty-seven possible situations of the allocation $\widetilde{a}$. (We saw above that, given that by definition $\widetilde{a}$ strongly dominates $\widehat{a}$, the four possible allocations at the positive orthant are ruled out.) The new allocation $a$ was chosen such that it belongs to the adherence of the hypercube $\{\underline{\widetilde{a}}, \widehat{a}\}$. That is, the allocation $a$ is situated in the sub-hypercube where all coordinates are lower or equal than those in allocation $\widehat{a}$. This sub-hypercube can be represented by a cube of 1 unit of length, where the coordinates are given by $e_t(\eta_t, \delta_t, \gamma_t)$. The origin, $e_t(0,0,0)$, is $\underline{\widetilde{a}}$, while the opposite far vortex $e_t(1,1,1)$ is the allocation $\widehat{a}$. Therefore, only four cases are possible: where all coordiantes are negative (inside the cube), and where at least one of the three coordinates is zero, i.e., the sides and edges of the cube where at least one coordinate is set to zero.

which establishes that $(A.3.5)$ is satisfied.

**Case** *iii*) $\triangle n_t = 0$, so $n_t = \widehat{n}_t$. In this case, either $\triangle \widetilde{n}_t = 0$ or $\eta_t = 1$ must be satisfied, as well as it follows from $(A.3.1)$ $\nabla c_t^m \geq 0$ and $\nabla d_t > 0$ (if $\triangle d_t = 0$ we return to Case *ii*)). The latter implies that $\nabla \widetilde{d}_t > 0$, $d_t < \widetilde{d}_t$ and, therefore, $y'_{t+1}(d_t) \geq y'_{t+1}(\widehat{d}_t)$.
First, $\triangle \widetilde{n}_t = 0$ implies, from $(A.3.7)$ that

$$c_{t+1}^o - \widetilde{c}_{t+1}^o > y'_{t+1}\left(d_t\right)\gamma_t \triangle \widetilde{c}_t^m \geq y'_{t+1}(\widetilde{d}_t)\gamma_t \triangle \widetilde{c}_t^m = \frac{p_t}{p_{t+1}}\gamma_t \triangle \widetilde{c}_t^m,$$

which establishes that $(A.3.5)$ is satisfied.
Second, if $\eta_t = 1$, i.e. $n_t = \widetilde{n}_t$, then

$$e_t(1, \delta_t, \gamma_t) \quad = \quad \gamma_t \nabla \widetilde{c}_t^m + [b(\widehat{n}_t) - b(\widetilde{n}_t)] + \widetilde{d}_t \nabla \widetilde{n}_t + \delta_t \nabla \widetilde{d}_t \widehat{n}_t$$

which, by Lemma 3.4, implies that $e_t(1,1,1) = \left\{[b(\widehat{n}_t) - b(\widetilde{n}_t)] + \widetilde{d}_t \nabla \widetilde{n}_t\right\} + \nabla \widetilde{c}_t^m + \nabla \widetilde{d}_t \widehat{n}_t \geq 0$.

Now, in the case that $\triangle \widetilde{n}_t > 0$, i.e. $\widetilde{n}_t > \widehat{n}_t$, then $e_t(1,0,0) = [b(\widehat{n}_t) - b(\widetilde{n}_t)] + \widetilde{d}_t \nabla \widetilde{n}_t < 0$. Then, for this case, $e_t(1,0,0) < 0$ and $e_t(1,1,1) \geq 0$ implies that the function $e_t(1, \delta_t, \gamma_t) = \gamma_t \nabla \widetilde{c}_t^m + \delta_t \nabla \widetilde{d}_t \widehat{n}_t + \left\{[b(\widehat{n}_t) - b(\widetilde{n}_t)] + \widetilde{d}_t \nabla \widetilde{n}_t\right\}$ is increasing in $\delta_t$. If $\triangle \widetilde{c}_t < 0$ it is also increasing in $\gamma_t$, and there exists some $\delta_t^*$ and $\gamma_t^*$ such that represents an allocation verifying $(A.3.1)$ and $(A.3.2)$, i.e., $e_t(1, \delta_t^*, \gamma_t^*) = 0$, for which $\gamma_t^* \nabla \widetilde{c}_t^m + \delta_t^* \nabla \widetilde{d}_t \widehat{n}_t \geq 0$, that is, $\delta_t^* \nabla \widetilde{d}_t \widehat{n}_t \geq \gamma_t^* \triangle \widetilde{c}_t^m$. Due to $\widetilde{n}_t > \widehat{n}_t$ then $\delta_t^* \nabla \widetilde{d}_t \widetilde{n}_t > \gamma_t^* \triangle \widetilde{c}_t^m$. This implies, from $(A.3.8)$ that

$$c_{t+1}^o - \widetilde{c}_{t+1}^o > y'_{t+1}\left(d_t\right)\gamma_t^* \triangle \widetilde{c}_t^m > y'_{t+1}(\widetilde{d}_t)\gamma_t^* \triangle \widetilde{c}_t^m = \frac{p_t}{p_{t+1}}\gamma_t^* \triangle \widetilde{c}_t^m,$$

which establishes that $(A.3.5)$ is satisfied.

Finally, in the case that $\triangle \widetilde{n}_t < 0$ then $e_t(1,0,0) = [b(\widehat{n}_t) - b(\widetilde{n}_t)] + d_t \nabla \widetilde{n}_t > 0$. Then, for this case, given that $e_t(1,0,0) > 0$ and $e_t(0,0,0) = 0$ then the allocation where $\eta_t = 0$ also verifies the conditions $(A.3.1)$ and $(A.3.2)$ returning to Case *i*).

**Case** *iv*) $\triangle c_t^m = 0$, so $c_t^m = \widehat{c}_t^m$. In this case, it follows from $(A.3.1)$ that $\nabla d_t > 0$ and $\nabla n_t > 0$ (if $\triangle d_t = 0$ or $\triangle n_t = 0$ we return to Cases *ii*) and *iii*), respectively). These imply that $\nabla \widetilde{n}_t > 0$, $\nabla \widetilde{d}_t > 0$, so $d_t < \widetilde{d}_t$ and, therefore, $y'_{t+1}(d_t) \geq y'_{t+1}(\widehat{d}_t)$.

Now, notice that there are two cases. First, $\triangle \widetilde{c}_t^m \geq \widetilde{n}_t \nabla \widetilde{d}_t$. In this case, since $\nabla \widetilde{d}_t > 0$, this means that $\triangle \widetilde{c}_t^m > 0$, so $\gamma_t = 1$ and, therefore, $e_t(0,1,1) = \nabla \widetilde{c}_t^m + \widetilde{n}_t \nabla \widetilde{d}_t < 0$. Hence, given that $e_t(1,1,1) > 0$ there must exist some $\eta_t^*$

such that represents an allocation verifying $(A.3.1)$ and $(A.3.2)$, i.e., $e_t(\eta_t^*, 1, 1) = 0$. But this allocation implies that $\triangle d_t = 0$, so the Case $ii)$ applies again.

Second, $\triangle\widetilde{c}_t^m < \widetilde{n}_t \nabla \widetilde{d}_t$. In this case, if $\triangle\widetilde{c}_t^m > 0$ then $e_t(0,0,1) = \nabla\widetilde{c}_t^m < 0$. Hence for continuity of the function $e_t(\eta_t, \delta_t, \gamma_t)$ and given that $e_t(1,1,1) > 0$, there must exist a set $(\eta_t^*, \delta_t^*, \gamma_t^*)$ such that represents an allocation verifying $(A.3.1)$ and $(A.3.2)$. Then, there must exist a particular values such that i.e., $e_t(0, \delta_t^{**}, \gamma_t^{**}) = \gamma_t^{**}\nabla\widetilde{c}_t^m + \delta_t^{**}\widetilde{n}_t\nabla\widetilde{d}_t = 0$. This implies, from $(A.3.8)$ that

$$c_{t+1}^o - \widetilde{c}_{t+1}^o > y_{t+1}'(d_t)\,\gamma_t^{**}\triangle\widetilde{c}_t^m > y_{t+1}'(\widetilde{d}_t)\gamma_t^{**}\triangle\widetilde{c}_t^m = \frac{p_t}{p_{t+1}}\gamma_t^{**}\triangle\widetilde{c}_t^m,$$

which establishes that $(A.3.5)$ is satisfied.
Finally, if $\triangle\widetilde{c}_t^m < 0$ then $e_t(0, \delta_t, 1) = \nabla\widetilde{c}_t^m + \delta_t\widetilde{n}_t\nabla\widetilde{d}_t > 0$, and in particular $e_t(0,0,1) > 0$. We will prove that, in this case, there is no allocation that verifies $\gamma_t = 1$, for any $\eta_t$ and $\gamma_t$ and, then, this case refers to the previous case $i)$. It would be sufficient to prove that $e_t(1,0,1) > 0$. To prove this, observe first that $e_t(1,0,1) = e_t(0,0,1) + e_t(1,0,0)$. The first term is positive. Then, given that this case $iv)$ specifies $\nabla\widetilde{n}_t > 0$, so $\widehat{n}_t > \widetilde{n}_t$, and $\nabla\widetilde{d}_t > 0$, we can find $e_t(1,0,0) = [b(\widehat{n}_t) - b(\widetilde{n}_t)] + \widetilde{d}_t\nabla\widetilde{n}_t > 0$. So this completes the proof.

This concludes the proof of Lemma 3.5.□

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