Equilibrium with Indivisible Goods and Paper Money *

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November 7, 2001

Résumé

Nous étudions une économie dont les biens sont indivisibles au niveau individuel, mais parfaitement divisibles au niveau agrégé. Un nouveau concept d'équilibre concurrentiel est introduit. Afin de faciliter les échanges, il est possible d'utiliser une monnaie-papier qui n'influence pas les préférences. Nous démontrons l'existence d'un équilibre avec un prix de monnaie strictement positif. Un théorème d'équivalence avec le noyau et un premier et deuxième théorème de bien être sont établis. Finalement, nous étudions le comportement asymptotique quand le niveau d'indivisiblité des biens tend vers zéro.

Mots clés : équilibre concurrentiel, biens indivisibles, monnaie-papier, optimum de Pareto, noyau.

Abstract

We study economies where all commodities are indivisible at the individual level, but perfectly divisible at the aggregate level. We introduce a new competitive equilibrium concept. Paper money (fiat money) which does not influence agents preferences may be used to facilitate exchange. We prove existence with a strictly positive price of fiat money. We establish a core equivalence result, and first and second welfare theorems for weak Pareto optima. Later, we study the asymptotic behavior when indivisibilities become small.

Keywords: competitive equilibrium, indivisible goods, fiat money, Pareto optimum, core.

JEL Classification: C62, D50, E40

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*This work was supported by FONDECYT - Chile, Project nr. 1000766-2000, ECOS, Regional French Cooperation and CMM, UMR - CNRS, Universidad de Chile.
We want to thank Jacques Drèze and Olivier Gossner for helpful comments.

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1 Introduction

1.1 Indivisible Commodities

Perfect divisibility of commodities is one of the crucial assumptions in general equilibrium theory. This corresponds to an idealized representation of a commodity space. The rational is that the commodities one considers are "almost perfectly" divisible in the sense that the indivisibilities are small and insignificant enough so that they can be neglected. It is well known that a Walras equilibrium may fail to exist in the presence of indivisible goods (Henry (1970)). Even the core may be empty (Shapley and Scarf (1974)).

Following Henry (1970), numerous authors (e.g. Broome (1972), Mas Colell (1977), Kahn and Yamazaki (1981), Quinzii (1984), see Bobzin (1998) for a survey) consider economies with indivisible commodities and one perfectly divisible commodity. All these contributions suppose that the divisible commodity satisfies overriding desirability, i.e. it is so desirable by the agents that it can replace the consumption of indivisible goods. Moreover, every agent initially owns an important quantity of this good. The non-emptiness of the core and existence of a Walras equilibrium is then established.

Dierker (1971) proposed an equilibrium concept existing without a perfectly divisible good. However, according to his notion, at an equilibrium agents do not necessarily receive an individually rational commodity bundle.

There thus remains the question of what would happen in a competitive economy where a Walras equilibrium fails to exist because all commodities are indivisible. If an equilibrium concept can be established, one could study the asymptotic behavior when indivisibilities become small, i.e. if consumption and production sets converge to convex sets. If the limit corresponds to a competitive equilibrium, this could formally justify the approximation of "small" indivisibilities by perfectly divisible goods.

1.2 Money in General Equilibrium

Since Hahn (1965) it is well known that there may be problems introducing money into a general equilibrium model with a finite horizon. If the price of money is positive in the last period, all consumers sell their money holding at the end. So in the last period the price of money must be zero and by induction it will be zero in all periods.

In order to ensure a positive price of money there exist several approaches in the literature. The infinite horizon approach with overlapping generations (Samuelson (1958), Balasko, Cass and Shell (1981)) or with infinitely lived agents (e.g. Bewley (1980, 1983), Gale and Hellwig (1984)).

In a static or finite horizon model, one may consider money lump-sum taxation with a zero total money supply (Lerner (1947), Balasko and Shell (1986)). Clower (1967) proposed a cash in advance constraint (e.g. Dubey and Geanakoplos (1992)). Nevertheless, in all these approaches an equilibrium with worthless money exists as well.

However, an introduction of money into the Arrow-Debreu model may be necessary in a much simpler setting. If the non-satiation assumption does not hold, for any given price, some consumer may wish to consume a commodity bundle in the interior of his budget set. Therefore a Walras equilibrium may fail to exist.

Without the non-satiation assumption, one may establish existence of an equilibrium by allowing for the possibility that some agents spend more than the value of their initial endowment. This generalization of the Walras equilibrium is called dividend equilibrium or equilibrium with slack (Makarov (1981), Aumann and Drèze (1986), Mas-Colell (1992)). It was first introduced in a fixed price setting by Drèze and Müller (1980).

Kajii (1996) showed that this dividend approach is equivalent to considering Walras equilibria with an additional commodity called paper money. Paper money can be consumed in positive quantities, but preferences are independent of the consumption of it. If local non-satiation holds, paper money has price zero and we are back in the Arrow-Debreu setting. However, if satiation problems occur, an equilibrium with price zero of paper money may fail to exist. Then, paper money must have a positive price in equilibrium. In fact, if a consumer does not want to spend his entire income on consumption goods, he can satisfy his budget constraint as an equality by buying paper money, if this paper money has a positive price.

1.3 Indivisible Commodities and Paper Money

We introduce a new competitive equilibrium concept for economies without a perfectly divisible good. We work with a finite set of types of agents, but a continuum of agents per type. This implies that commodities are indivisible at the individual level, but perfectly divisible at the aggregate level. So whether a consumer has a house or not is not negligible for him. Whether a house is constructed or not has however a negligible impact on the economy as a whole. This is natural. If some consumer owns a commodity which may not be considered negligible at the level of the entire economy, it would be hard to justify that this consumer acts as a price taker. His impact on the economy would be quite important and modelling such a situation by a competitive approach might be inappropriate.

By the discreteness of the consumption sets, local non-satiation cannot hold. As in Kajii (1996), we introduce paper money (i.e. fiat money) which does not enter the consumers preferences, but it may be used to facilitate exchange. Unlike goods, paper money is assumed to be perfectly divisible. This is natural. If fiat money can be produced at zero cost by an external agent, then if the minimal unit would become non-negligible, one could easily start to issue smaller coins. Equivalently one could easily start to account in smaller units, if one thinks of a bank account.

Existence of equilibrium with a strictly positive price of paper money is ensured, provided all consumers initially have a strictly positive amount of paper money. This differs from the case of convex consumption sets and possible satiation of the preferences. There paper money may have a positive price, but a positive price is not ensured.

The equilibrium is weakly Pareto optimal and in Konovalov's (1998) rejective core (which is a refinement of the weak core). However, strong Pareto optimality fails. This is due to the fact that some consumers may own commodities which are worthless to them as a consumption good, or they own more than they need. The value of these commodities may be so small that selling them does not enable to buy more of the goods they are interested in. Thus, they may waste these commodities. These commodities may however be very useful and expensive for poorer agents. So the market is not as efficient as in the standard Arrow-Debreu setting (Arrow and Debreu (1954)). We offer an equivalent of the second welfare theorem. A core equivalence result for the rejective core holds.

Later, we study the asymptotic behavior when the level of indivisibility converges to zero. Without a survival assumption and a local non-satiation hypothesis on the limit economy, a Walras equilibrium will not exist. However a hierarchic equilibrium (Marakulin (1990), Florig (2001)) exists. At a hierarchic equilibrium, consumers are partitioned according to their level of wealth. Poorer consumers have not access to all the expensive commodities to which the richer have access. Such access restrictions occur easily if the commodities are not perfectly divisible, as we described above. So the same phenomena occur as in the case of indivisible economies, and for the same reason only weak Pareto optimality holds. When the level of indivisibility vanishes, an equilibrium converges to a hierarchic equilibrium. This formally confirms the interpretation of hierarchic equilibria in terms of small indivisibilities given in Florig (2001). In the absence of the survival assumption, indivisibilities, even if they are small, may thus remain significant. In particular, we do not approach a (strong) Pareto optimum as indivisibilities become small. The failure of strong Pareto optimality is thus not related to the level of indivisibility of the commodities.

If the survival assumption holds, then a hierarchic equilibrium is a dividend equilibrium. So if local non-satiation does not hold at the limit economy, the price of money does not converge to zero along with the indivisibilities. If moreover, a local non-satiation assumption on the limit economy holds then the limit corresponds to a Walras equilibrium and the price of money at the limit is zero. So when a survival assumption holds indivisibilities become indeed insignificant when they are small enough.

2 Model

We set $L \equiv \{1, \ldots, L\}$ to denote the finite set of commodities. Let $I \equiv \{1, \ldots, I\}$ and $J \equiv \{1, \ldots, J\}$ be finite sets of types of identical consumers and producers respectively.

We assume that each type $k \in I, J$ of agents consists of a continuum of identical individuals represented by a set $T_k \subset I\!R$ of finite Lebesgue measure.[1] We set $\mathcal{I} = \cup_{i \in I} T_i$ and $\mathcal{J} = \cup_{j \in J} T_j$. Of course, $T_t \cap T_{t'} = \emptyset$ if type t and t' are different.

Each firm of type $j \in J$ is characterized by a **finite** production set[2] $Y_j \subset I\!R^L$. Every consumer of type $i \in I$ is characterized by a **finite** consumption set $X_i \subset R^L$, an initial endowment $e_i \in I\!R^L$ and a preference correspondence $P_i : X_i \to 2^{X_i}$. Let $e = \sum_{i \in I} \lambda(T_i) e_i$ be the aggregate initial endowment of the economy. For $(i, j) \in I \times J$, $\theta_{ij} \geq 0$ is the share of type i consumers in type j firms. For all $j \in J$, $\sum_{i \in I} \lambda(T_i)\theta_{ij} = 1$.

We introduce a parameter for each type of consumer $m_i \geq 0$ which may be interpreted as *fiat money*.

An *economy* $\mathcal{E}$ is a collection

$$\mathcal{E} = ((X_i, P_i, e_i, m_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(ij) \in I \times J}).$$

An allocation (or consumption plan) is an element of[3]

$$X = \{x \in L^1(\mathcal{I}, \cup_{i \in I} X_i) \,|\, x_t \in X_t \, for \, a.e. \, t \in \mathcal{I}\},$$

and a production plan is an element of

$$Y = \{y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j) \,|\, y_t \in Y_t \, for \, a.e. \, t \in \mathcal{J}\}.$$

Feasible consumption-production plans are

$$A(\mathcal{E}) = \left\{(x, y) \in X \times Y \mid \int_{\mathcal{I}} x_t = \int_{\mathcal{J}} y_t + e\right\}.$$

Given $p \in I\!R^L$, the *weak supply* of a firm of type $j \in J$ and their *aggregate profit* are, respectively,

$$S_j(p) = \mathrm{argmax}_{y \in Y_j} \, p \cdot y \qquad \pi_j(p) = \lambda(T_j)\mathrm{sup}_{y \in Y_j} \, p \cdot y.$$

[1] We note by λ the Lebesgue measure.

[2] The aggregate production set of the firms of type j is the convex hull of $\lambda(T_j)Y_j$.

[3] We note $L^1(T, Z)$ the Lebesgue integrable functions from $T \subset I\!R$ to $Z \subset I\!R^L$.

Given $(p,q) \in I\!\!R^L \times I\!\!R_+$, we denote the *budget set* of a type $i \in I$ consumer by

$$B_i(p,q) = \{x \in X_i \,|\, p \cdot x \leq p \cdot e_i + qm_i + \sum_{j \in J} \theta_{ij}\pi_j(p)\}$$

and we note the set of *maximal elements* in type i's budget set by

$$d_i(p,q) = \{x_i \in B_i(p,q) \,|\, B_i(p,q) \cap P_i(x) = \emptyset\}.$$

Definition 2.1 *A collection $(x,y,p,q) \in A(\mathcal{E}) \times I\!\!R^L \times I\!\!R_+$ is a* **Walras equilibrium** *(with money) of $\mathcal{E}$ if:*
(i) for a.e. $t \in \mathcal{I}$, $x_t \in d_t(p,q)$;
(ii) for a.e. $t \in \mathcal{J}$, $y_t \in S_t(p)$.

In our framework, a Walras equilibrium (with money) may fail to exist (cf. Section 6). In general, the correspondence d_i is not upper semi continuous. This leads us to a regularized notion of demand. The *weak demand* of type $i \in I$ consumers is defined by

$$D_i(p,q) = \limsup_{(p',q') \to (p,q)} d_i(p',q').$$

Definition 2.2 *A collection $(x,y,p,q) \in A(\mathcal{E}) \times I\!\!R^L \times I\!\!R_+$ is a* **weak equilibrium** *of $\mathcal{E}$ if:*
(i) for a.e. $t \in \mathcal{I}$, $x_t \in D_t(p,q)$;
(ii) for a.e. $t \in \mathcal{J}$, $y_t \in S_t(p)$.

The following aspects of our model are illustrated by Examples 1 and 2 below.

- Without a positive price of money, the market may not be viable, i.e. the only weak equilibrium price with $q = 0$ may be $p = 0$. Of course, if $p = 0, q = 0$, then there is no real market anymore. In fact, if $J = \emptyset$, $e_i \in X_i$ and if preferences are discrete-convex, then $((e_t)_{t \in \mathcal{I}}, p = 0, q = 0)$ is a "trivial" weak equilibrium.

- The equilibrium depends on the distribution of money. However, a multiplication of the total money supply by $\gamma > 0$, without changing its distribution, just changes the price of goods to γp.

- Weak equilibria are in a certain sense unstable when the consumers know more than just their own characteristics and the market price. If they have information on other's preferences and equilibrium allocations, trade could continue once the weak equilibrium is realized. For this reason, we will introduce a stronger notion of equilibrium which has not this inconvenient (cf. Section 5).

Example 1. *Without money markets may be non viable.*
Consider an exchange economy with three types of consumers (with $\lambda(T_1) = \lambda(T_2) = \lambda(T_3)$) and one commodity: for all $i \in I$, $X_i = \{0, 1, 2\}$, $u_1(x) = -x$, $u_2(x) = u_3(x) = x$, $e_1 = 2, e_2 = e_3 = 0$.

Without money, if $p < 0$ demand will be above supply. If $p > 0$, supply is above demand.

Suppose $m_2 = m_3 > 0$, then $p = m_2, q = 1$, $x_1(t)^4 = 0, x_2(t) = x_3(t) = 1$ is a weak equilibrium and its the only one with $p \neq 0$.

Suppose $m_2 = 3m_3 > 0$, then $p = m_2/2, q = 1$, $x_1(t) = 0, x_2(t) = 2, x_3(t) = 0$ is the unique weak equilibrium with $p \neq 0$.

Example 2. *Weak equilibria may be unstable.*
Consider an exchange economy with three types of consumers (with $\lambda(T_1) = \lambda(T_2) = \lambda(T_3)$) and two commodities: for all $i \in I$, $X_i = \{0, 1, 2\}^2$, $u_1(x) = -x^1 - x^2$, $u_2(x) = 2x^1 + x^2$, $u_3(x) = x^1 + 2x^2$, $e_1 = (1, 1), e_2 = e_3 = (0, 0)$ (cf. Konovalov 1998).

Suppose $m_1 = m_2 = m_3 = 1$. Then (x, p, q) with $x_1(t) = (0, 0)$, $x_2(t) = (0, 1)$, $x_3(t) = (1, 0)$ for all t and $p = (1, 1)$, $q = 1$ is a weak equilibrium and it is even in the weak core. However, once the allocation is realized, consumers two and three wish to swap their allocations leading to $\xi_1(t) = (0, 0)$, $\xi_2(t) = (1, 0)$, $\xi_3(t) = (0, 1)$.

In the remaining part of this section, we introduce a stronger equilibrium notion than the weak equilibrium. An interpretation of both concepts will be given in the next section. So given a vector $p \in \mathbb{R}^L$, we note $\mathcal{C}$ the set of closed convex cones $K \subset \mathbb{R}^L$ such that $-K \cap K = \{0\}$. Let $(p, q, K) \in \mathbb{R}^L \times \mathbb{R}_+ \times \mathcal{C}$, then we define the *demand* of type $i \in I$ consumers by

$$\delta_i(p, q, K) = \{x \in D_i(p, q) \,|\, P_i(x) - x \subset K\}.$$

The *supply* of a firm of type $j \in J$ is

$$\sigma_j(p, K) = \{y \in S_j(p) \,|\, Y_t - y \subset -K\}.$$

Definition 2.3 *A collection $(x, y, p, q, K) \in A(\mathcal{E}) \times \mathbb{R}^L \times \mathbb{R}_+ \times \mathcal{C}$ is a* **rationing equilibrium** *of $\mathcal{E}$ if:*

(i) for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p, q, K)$;

(ii) for a.e. $t \in \mathcal{J}$, $y_t \in \sigma_t(p, K)$.

Remark. For $q > 0$ the *demand for money* of consumer $t \in \mathcal{I}$ is

$$\mu_t = \frac{1}{q}(p \cdot e_t + q m_t + \sum_{j \in J} \theta_{tj} \pi_j(p) - p \cdot x_t).$$

[4] We note $x_i(t)$ for x_t with $t \in T_i$.

Walras law implies that the money market is in equilibrium at an equilibrium. A Walras equilibrium with money is of course a rationing equilibrium and a rationing equilibrium is a weak equilibrium. We refer to Kajii (1996) for the links among Walras equilibrium, Walras equilibrium with money and dividend equilibrium (cf. Section 8).

3 Demand: Characterization and Interpretation

We first characterize the (weak) demand in the most important case when $qm_i > 0$ for all $i \in I$. The proof will be given at the end of the section where we also give a complete characterization of the (weak) demand for the sake of completeness. For convenience, for all $(p,q) \in I\!R^L \times I\!R_+$, we note $w_i(p,q) = p \cdot e_i + qm_i + \sum_{j \in J} \theta_{ij} \pi_j(p)$.

Proposition 3.1 *Suppose $qm_i > 0$. Then* [5]

$$D_i(p,q) = \left\{ x \in B_i(p,q) \,\middle|\, \begin{array}{c} p \cdot P_i(x) \geq w_i(p,q), \\ x \notin \mathrm{co}P_i(x) \end{array} \right\}.$$

Remark. In the previous proposition and the forthcoming, the condition $x \notin \mathrm{co}P_i(x)$ is redundant, if one considers the demand as defined for the rationing equilibrium.

Interpretation. First of all note that in our model a consumer might be unable to obtain a maximal element within his budget set. Should he be unable to buy $\xi \in B_i(p,q)$ with $p \cdot \xi < w_i(p,q)$, then he could try to pay this bundle at a higher price than the market price in order to be "served first". There is some pressure on the price of the bundle ξ and its price would rise, if a non-negligible set of consumers is rationed in this sense. So at equilibrium, no consumer obtains a bundle of goods $x \in B_i(p,q)$ such that a strictly preferred bundle ξ with $p \cdot \xi < w_i(p,q)$ exists.

As explained in Example 2, this notion of demand could lead to an unstable situation, if the agents have more information than their own characteristics and the market price. To eliminate this instability it is however not necessary that the agents have a precise information on their trading partners. It is enough that they know which kind of net-trades are difficult

[5] For a set $Z \subset R^L$, we denote the convex hull of Z by

$$coZ = \{\sum_{n=1}^{m} \mu_n z_n \mid z_n \in Z,\, \mu_n \geq 0,\, \sum_{n=1}^{m} \mu_n = 1, m \in I\!N\}.$$

to realize on the market (which is the "short" side of the market) when formulating their demand. This is summarized by the cone K in Definition 2.3. It is natural to consider only cones which do not contain straight lines, i.e. if a direction of net-trade is difficult to realize, the opposite direction is easy to realize. One could think of the demand for the rationing equilibrium as follows. First agents perceive the market price and the cone K and then they compute their budget set. They try to find out for which type of allocations they could find a counterpart. So an allocation is not acceptable, if there exists a preferred one in the budget set which costs less than their total wealth. Moreover, they do not accept an allocation x, if a preferred allocation x' exists which is contained in the budget set and such that $x' - x \notin K$. In fact, it should not be difficult to find a counterpart for the net-exchange $x' - x$. Alternatively think that they first accept the allocation x, but then they make another net-exchange $x' - x$ leading to x' and so on, until they are at an allocation ξ such that $P_i(\xi) - \xi \subset K$. At this stage, obtaining a preferred allocation would require a net-exchange of a direction for which it is difficult to find a counterpart.

As for the firms, in their supply, as defined here, they do not only maximize profit as in the weak (or standard) supply, but amongst the profit maximizing production plans, they choose the one which should be the most "easy" to sell according to the cone K.

Proposition 3.2 *(i) Suppose $m_i > 0$. Then*

$$D_i(p,q) = \left\{ x \in B_i(p,q) \,\middle|\, \begin{array}{c} p \cdot P_i(x) \geq w_i(p,q), \\ \mathrm{co}P_i(x) \cap \mathrm{co}\{x, e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j\} = \emptyset \end{array} \right\}.$$

(ii) Suppose $m_i = 0$. Then

$$D_i(p,q) = \left\{ x \in B_i(p,q) \,\middle|\, \begin{array}{c} p \cdot P_i(x) \geq w_i(p,q), \\ \mathrm{co}P_i(x) \cap C(p,x) = \emptyset \end{array} \right\}$$

where $C(p,x) = \mathrm{co}\{tx + (1-t)\left[e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) \mathrm{argmax}\, \pi_j(p)\right] \mid t \geq 0\}$.

Proof of Proposition 3.2 *Proof of (i).* Let

$$A(p,q) = \left\{ x \in B_i(p,q) \,\middle|\, \begin{array}{c} p \cdot P_i(x) \geq w_i(p,q), \\ \mathrm{co}P_i(x) \cap \mathrm{co}\{x, e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j\} = \emptyset \end{array} \right\}.$$

Step 1. $A(p,q) \subset D_i(p,q)$.

Let $x \in A(p,q)$. Thus, there exists p' such that

$$p' \cdot P_i(x) > p' \cdot \{x, e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j\}.$$

For all $\varepsilon > 0$, let $p^\varepsilon = p + \varepsilon p'$. Thus, for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_i(x) > p^\varepsilon \cdot \left\{ x, e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j \right\},$$

$$p^\varepsilon \cdot P_i(x) > w_i(p^\varepsilon, q).$$

Let[6]

$$q^\varepsilon = q + \left[\frac{p^\varepsilon \cdot x - w_i(p^\varepsilon, q)}{m_i} \right]_+ .$$

Note that $\lim_{\varepsilon \to 0}(p^\varepsilon, q^\varepsilon) = (p, q)$ and moreover for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_i(x) > w_i(p^\varepsilon, q^\varepsilon) \geq p^\varepsilon \cdot x.$$

Thus, $x \in D_i(p, q)$.

Step 2. $D_i(p, q) \subset A(p, q)$:

For all $x \in D_i(p, q)$, there exists sequences (p^n, q^n) converging to (p, q), such that for all n

$$p^n \cdot P_i(x) > w_i(p^n, q^n) \geq p^n \cdot x.$$

Thus $p \cdot P_i(x) \geq w_i(p, q)$ and

$$\mathrm{co}P_i(x) \cap \mathrm{co}\{x, e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j\} = \emptyset$$

which ends the proof of *(i)*.

Proof of (ii). Let

$$c(p) = \left\{ x \in B_i(p, q) \,\middle|\, \begin{array}{c} p \cdot P_i(x) \geq w_i(p, q), \\ \mathrm{co}P_i(x) \cap C(p, x) = \emptyset \end{array} \right\}.$$

Step 1. $c(p) \subset D_i(p, q)$:

Given $x \in c(p)$ there exists p' such that

$$p' \cdot \mathrm{co}P_i(x) > p' \cdot \left(e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) \mathrm{argmax}\, \pi_j(p) \right) \geq p' \cdot x.$$

Thus, for all $\varepsilon > 0$, given $p^\varepsilon = p + \varepsilon p'$ it follows that

$$\min p^\varepsilon \cdot P_i(x) > \max p^\varepsilon \cdot \left(e_i + \sum_{j \in J} \theta_{ij} \lambda(T_j) \mathrm{argmax}\, \pi_j(p) \right),$$

[6]For $x \in I\!R$, we note $[x]_+ = \max\{x, 0\}$.

$$\min p^\varepsilon \cdot \left(e_i + \sum_{j\in J} \theta_{ij}\lambda(T_j)\text{argmax}\,\pi_j(p)\right) \geq p^\varepsilon \cdot x.$$

Moreover, since for all $j \in J$, Y_j is finite we may check that for all $\varepsilon > 0$ small enough and all $j \in J$, $\text{argmax}\,\pi_j(p^\varepsilon) \subset \text{argmax}\,\pi_j(p)$. Thus for all small $\varepsilon > 0$, $\min p^\varepsilon \cdot P_i(x) > w_i(p^\varepsilon, q) \geq p^\varepsilon \cdot x$. Thus, $x \in D_i(p, q)$.

Step 2. $D_i(p, q) \subset c(p)$:

Let $x \in D_i(p, q)$. Then there exists a sequence p^n converging to p such that for all n,

$$p^n \cdot P_i(x) > w_i(p^n, q) \geq p^n \cdot x.$$

Thus $p \cdot P_i(x) \geq w_i(p, q)$ and p^n separates strictly $\text{co}P_i(x)$ and $\text{co}\{tx + (1 - t)[e_i + \sum_{j\in J} \theta_{ij}\lambda(T_j)Y_j] \,|\, t \geq 0\}$. Since

$$C(p, x) \subset \text{co}\left\{ tx + (1-t)[e_i + \sum_{j\in J} \theta_{ij}\lambda(T_j)Y_j] \,|\, t \geq 0 \right\}$$

we can conclude that $x \in c(p)$. □

Proof of Proposition 3.1 Let

$$a(p, q) = \left\{ x \in B_i(p, q) \,\middle|\, \begin{array}{c} p \cdot P_i(x) \geq w_i(p, q), \\ x \notin \text{co}P_i(x) \end{array} \right\}.$$

Note first, that by definition $A(p, q) \subset a(p, q)$.

Let $x \in a(p, q)$. If $p \cdot x < w_i(p, q)$, then for all small enough $\varepsilon > 0$, $x \in d_i(p, q - \varepsilon)$ and hence $x \in D_i(p, q)$. Otherwise, note that there exists p' such that $p' \cdot P_i(x) > p' \cdot x$. For all $\varepsilon > 0$, let $p^\varepsilon = p + \varepsilon p'$ and let

$$q^\varepsilon = \left[\frac{p^\varepsilon \cdot (x - e_i) - \sum_{j\in J} \theta_{ij}\pi_j(p^\varepsilon)}{m_i} \right].$$

Note that $\lim_{\varepsilon\to 0}(p^\varepsilon, q^\varepsilon) = (p, q)$. Moreover for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_i(x) > p^\varepsilon \cdot x = w_i(p^\varepsilon, q^\varepsilon).$$

Since for $\varepsilon > 0$ small enough, $q^\varepsilon > 0$, we have $x \in D_i(p, q)$. Thus $a(p, q) \subset D_i(p, q) = A(p, q)$. □

4 Existence

The strongest condition we use to ensure existence of equilibrium is the finiteness of the consumption and production sets. The rest of our assumptions are quite weak. In particular, we do not need a strong survival assumption, that is, our consumers may not own initially a strictly positive quantity of every good and the interior of the convex hull of the consumption sets may be empty (cf. Arrow and Debreu (1954)).

Assumption C. For all $i \in I$, P_i is irreflexive and transitive.

Assumption S. (Weak survival assumption). For all $i \in I$,

$$0 \in \mathrm{co}X_i - \{e_i\} - \sum_{j \in J} \theta_{ij} \lambda(T_j) \mathrm{co}Y_j.$$

Theorem 4.1 *For every economy $\mathcal{E}$ satisfying Assumptions C, S, there exists a weak equilibrium with $q > 0$.*

Theorem 4.2 *For every economy $\mathcal{E}$ satisfying Assumptions C, S and $m_i > 0$ for all $i \in I$, there exists a rationing equilibrium with $q > 0$.*

We prepare the proof of Theorem 4.1 by the following lemmata. Lemma 4.1 is an extension of the well known Debreu-Gale-Nikaido Lemma. The proof of Theorem 4.2 is given in the Appendix.

Lemma 4.1 *Let $\varepsilon \in]0,1]$ and φ be an upper semi continuous correspondence from $\mathbb{B}(0,\varepsilon)$[7] to $\mathbb{R}^L$ with nonempty, convex, compact values. If for some $k > 0$,*

$$\forall p' \in \mathbb{B}(0,\varepsilon), \quad \|p'\| = \varepsilon \quad \Longrightarrow \quad \sup p' \cdot \varphi(p') \leq k(1-\varepsilon),$$

then there exists $p \in \mathbb{B}(0,\varepsilon)$ such that, either:

- $0 \in \varphi(p)$;

or

- $\|p\| = \varepsilon$ *and* $\exists\ \xi \in \varphi(p)$ *such that* ξ *and* p *are collinear and* $\|\xi\| \leq k\frac{1-\varepsilon}{\varepsilon}$.[8]

Proof. From the properties of φ, one can select a convex compact subset $K \subset \mathbb{R}^L$ such that for all $p \in \mathbb{B}(0,\varepsilon)$, $\varphi(p) \subset K$. Consider the correspondence $F : \mathbb{B}(0,\varepsilon) \times K \to \mathbb{B}(0,\varepsilon) \times K$ defined by

$$F(p,z) = \{q \in \mathbb{B}(0,\varepsilon) \mid \forall q' \in \mathbb{B}(0,\varepsilon),\ q \cdot z \geq q' \cdot z\} \times \varphi(p).$$

From Kakutani Theorem, F has a fixed point (p,ξ). If $\|p\| < \varepsilon$, then $\xi = 0$. If $\|p\| = \varepsilon$, then from the definition of F, p and ξ are collinear. Therefore, $\|\xi\| \leq k\frac{1-\varepsilon}{\varepsilon}$. □

For simplicity we note $D_i(p)$ for $D_i(p, 1 - \|p\|)$. The following Lemma is easy to proof.

[7] $\mathbb{B}(0,\varepsilon) = \{x \in \mathbb{R}^L \mid \|x\| \leq \varepsilon\}$.

[8] For the Euclidean norm.

Lemma 4.2 *For all $\varepsilon \in [0,1]$, all $i \in I$, and all $j \in J$ the set-valued mappings $\mathrm{co}D_i : I\!\!B(0,\varepsilon) \to \mathrm{co}X_i$, $\mathrm{co}S_j : I\!\!B(0,\varepsilon) \to \mathrm{co}Y_j$ are upper semi-continuous, nonempty, convex and compact valued.*

Proof of Theorem 4.1. Note first that if

$$(\bar{x}, \bar{y}, p, q) \in \prod_{i\in I} \lambda(T_i)\mathrm{co}D_i(p,q) \times \prod_{j\in J} \lambda(T_j)\mathrm{co}S_j(p) \times I\!\!R^L \times I\!\!R_+$$

such that

$$\sum_{i\in I} \bar{x}_i = \sum_{j\in J} \bar{y}_j + e,$$

then there exist $(x,y) \in X \times Y$ such that

- for all $i \in I$, $\bar{x}_i = \int_{T_i} x_t$ and for all $t \in T_i$, $x_t \in D_t(p,q)$;
- for all $j \in J$, $\bar{y}_j = \int_{T_j} y_t$ and for all $t \in T_j$, $y_t \in S_t(p)$.

Moreover, we then have a weak equilibrium (x,y,p,q).

Now, define the excess demand mapping

$$\varphi : I\!\!B\left(0, 1 - \frac{1}{n}\right) \to \sum_{i\in I} \lambda(T_i)(\mathrm{co}X_i - e_i) - \sum_{j\in J} \lambda(T_j)\mathrm{co}Y_j$$

by

$$\varphi(p) = \sum_{i\in I} \lambda(T_i)(\mathrm{co}D_i(p) - e_i) - \sum_{j\in J} \lambda(T_j)\mathrm{co}S_j(p).$$

Obviously φ is nonempty, convex and compact valued and also upper semi continuous. Moreover, for each $n \in I\!\!N$ and each $p \in I\!\!B(0, 1-1/n)$ we have that

$$p \cdot \varphi(p) \leq (1 - \|p\|) \sum_{i\in I} \lambda(T_i) m_i.$$

So we may apply Lemma 4.1 to conclude that for all $n > 1$ there exists

$$(x^n, y^n, p^n) \in \prod_{i\in I} \lambda(T_i)\mathrm{co}D_i(p^n) \times \prod_{j\in J} \lambda(T_j)\mathrm{co}S_j(p^n) \times I\!\!B\left(0, 1 - \frac{1}{n}\right)$$

such that either $0 \in \varphi(p^n)$ or $\| \varphi(p^n) \| \leq \frac{1}{n-1} \sum_{i\in I} \lambda(T_i) m_i$. Therefore, taking a subsequence, we may suppose that (x^n, y^n) converges to $(\bar{x}, \bar{y})$ such that $\sum_{i\in I} \bar{x}_i = \sum_{j\in J} \bar{y}_j + e$. Moreover, since the consumption and production sets are finite, we may suppose that there exist sets $(S_j)_{j\in J}$, $(D_i)_{i\in I}$ such that for all n and for all $i \in I$, $j \in J$, $S_j(p^n) = S_j$ and $D_i(p^n) = D_i$. Let $\bar{p} = p^{\bar{n}}$ for some fixed $\bar{n}$. Then, for all n,

$$(x^n, y^n) \in \prod_{i\in I} \lambda(T_i)\mathrm{co}D_i(\bar{p}) \times \prod_{j\in J} \lambda(T_j)\mathrm{co}S_j(\bar{p}).$$

Since these sets are compact,

$$(\bar{x}, \bar{y}) \in \prod_{i \in I} \lambda(T_i) \mathrm{co} D_i(\bar{p}, \bar{q}) \times \prod_{j \in J} \lambda(T_j) \mathrm{co} S_j(\bar{p})$$

with $\bar{q} = 1 - \|\bar{p}\| > 0$. □

5 Core

In this section, we will study the core properties of our equilibrium notions. In particular, we will establish a core equivalence result for rationing equilibria.

Definition 5.1 *A collection $(x, y) \in A(\mathcal{E})$ is in the weak core if there does not exist $(x', y') \in X \times Y$ and a measurable set $T \subset \mathcal{I}$ with $\lambda(T) > 0$ such that:*

(i) for a.e. $t \in T$, $x'_t \in P_t(x_t)$;

(ii) $\int_T x'_t - e_t = \sum_{i \in I} \sum_{j \in J} \lambda(T \cap T_i) \theta_{ij} \lambda(T_j) y'_j$.

Proposition 5.1 *Let (x, y, p, q) be a weak equilibrium such that for all $i \in I$, $qm_i > 0$, then (x, y) is in the weak core.*

Proof. We proceed by contraposition.

Let T, x', y' as described in the definition. So for a.e. $t \in T$,

$$p \cdot x'_t > p \cdot e_t + \sum_{j \in J} \theta_{tj} \pi_j(p) \geq p \cdot e_t + \sum_{j \in J} \theta_{tj} \lambda(T_j) p \cdot y'_j.$$

Thus $p \cdot \int_T x'_t - e_t > p \cdot \sum_{i \in I} \sum_{j \in J} \lambda(T \cap T_i) \theta_{ij} \lambda(T_j) y'_j$ contradicting (ii) of Definition 5.1. □

Example 3. *The weak core cannot be decentralized.*

Adapting an example from Konovalov (1998), consider an exchange economy with two types of consumers (with $\lambda(T_1) = \lambda(T_2)$) and two commodities. Let $X_1 = X_2 = \{0, 1, 2\}^2$, $u_1(x) = -x_1 + x_2$, $u_2(x) = \min\{x_1, x_2\}$, $e_1 = (2, 0)$, $e_2 = (0, 2)$. The type-symmetric allocation $x_1 = (0, 2)$, $x_2 = (2, 0)$ is in the weak core (in fact, it is even in the strong core, i.e. the one using weak blocking). By Proposition 3.2, for all $p, q \in \mathbb{R}^L \times \mathbb{R}_+$, $x_2 \notin \mathrm{co} D_2(p, q)$. So this allocation cannot be decentralized. One may check that we have a unique weak equilibrium allocation with $qm_i > 0$ for all $i \in I$ which is in fact type symmetric: $x_1 = (0, 0), x_2 = (2, 2)$.

We already saw in Example 2 (Section 2) that weak equilibrium allocations and weak core allocations may be unstable. For this reason, we use the following refinement of the weak core due to Konovalov (1998).

Definition 5.2 *The coalition $T \subset \mathcal{I}$* **rejects** *$(x, y) \in A(\mathcal{E})$, if there exist a measurable partition U, V of T, and an allocation $x' \in X$ such that the following holds:*

(i) $\int_T x'_t \in \int_U [x_t + \sum_{j \in J} \theta_{tj} \int_{T_j} (Y_j - y_\tau) d\tau] dt + \int_V [e_t + \sum_{j \in J} \theta_{tj} \lambda(T_j) Y_j] dt$;

(ii) for a.e. $t \in T$, $x'_t \in P_t(x_t)$.

The **rejective core** *$\mathcal{RC}(\mathcal{E})$ of $\mathcal{E}$ is the set of $(x, y) \in A(\mathcal{E})$ which cannot be rejected by a non-negligible coalition.*

Interpretation. The interpretation of this core concept could be as follows. An allocation x is proposed; group V refuses this allocation and stays with the initial endowment; group U realizes the proposed exchange and once they obtained the allocation x, they meet with group V leading them to the allocation x'.

Allocation x' could be infeasible, if groups U and V were too big. However, one can always construct from U and V smaller groups U' and V' such that x' is feasible for them. It is sufficient to choose them such that for all $i \in I$, $\lambda(U' \cap T_i) = \frac{1}{2}\lambda(U \cap T_i)$ and $\lambda(V' \cap T_i) = \frac{1}{2}\lambda(V \cap T_i)$. Now if V' refuses to exchange, then a proportion larger than $1/2$ of the set of agents can establish x. The complement fails to establish x since V' refused. They stay with their initial endowment. Then, U' and V' can indeed establish x' together.

Example 4. *Rationing equilibria without money may be rejected.*
Consider an exchange economy with three types of consumers (with $\lambda(T_1) = \lambda(T_2) = \lambda(T_3)$) and two commodities: for all $i \in I$, $X_i = \{0, 1, 2\}^2$, $u_1(x) = -x^1 - x^2$, $u_2(x) = -\|x - (1, 1)\|_1$, $u_3(x) = -\|x - (0, 1)\|_1$, $e_1 = (0, 4)$, $e_2 = (0, 0)$, $e_3 = (1, 0)$. The type symmetric allocation $x_1 = (0, 0)$, $x_2 = (1, 2)$, $x_3 = (0, 2)$ is a rationing equilibrium with $p = q = 0$, $K = \{t(0, -1) \mid t \geq 0\}$. However it is not in the rejective core since the players of type 2 and 3 may reject this leading them to $\xi_2 = (1, 1)$ and $\xi_3 = (0, 1)$ (type 2 agents accepts x_2 and type 3 agents stay with their initial endowment).

Proposition 5.2 *Let (x, y, p, q, K) be a rationing equilibrium such that for all $i \in I$, $qm_i > 0$, then (x, y) is in the rejective core.*

Proof. Let $T \subset \mathcal{I}$ with $\lambda(T) > 0$ and a measurable partition U, V of T and $x' \in X$ such that for a.e. $t \in T$, $x'_t \in P_t(x_t)$. Thus $\int_U x'_t - x_t \in K \setminus \{0\}$. First note that

$$p \cdot \int_T x'_t = p \cdot \int_T e_t + q \int_T m_t + \sum_{i \in I} \lambda(T \cap T_i) \sum_{j \in J} \theta_{ij} \pi_j(p).$$

Thus, if condition *(i)* of Definition 5.2 is satisfied, we necessarily have $\lambda(V) = 0$. Note that for every $j \in J$, $\int_{T_j}(Y_\tau - y_\tau)d\tau \subset -K$. Thus

$$\int_U x'_t \in \int_T \left[x_t + \sum_{j \in J} \theta_{tj} \int_{T_j} (Y_j - y_\tau)d\tau \right] dt \subset \int_T x_t - K.$$

Thus, $\int_T x'_t - x_t \in -K$ and this contradicts $\int_T x'_t - x_t \in K \setminus \{0\}$. □

The absence of some local non-satiation property would entail the existence of rejective core allocations which cannot be decentralized. This is due to the fact that a consumer at a satiation point does not care whether a firm he entirely owns chooses an efficient production plan or not (cf. Florig (2001)).

Proposition 5.3 *Suppose $J = \emptyset$. Then, for every $x \in \mathcal{RC}(\mathcal{E})$ there exists $(p, m') \in I\!R^L \setminus \{0\} \times L^1(\mathcal{I}, I\!R_{++})$ such that $(x, p, q = 1)$ is a Walras equilibrium with money of the economy $\mathcal{E}$ when replacing m by m'.*

Proof. Let $x \in \mathcal{RC}(\mathcal{E})$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \ldots, A\}$ satisfying the following:

(i) $(T_a)_{a \in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_t = x_a$.

Set

$$H_a = \lambda(T_a)(\mathrm{co}P_a(x_a) - x_a),\ G_a = \lambda(T_a)(\mathrm{co}P_a(x_a) - e_a),$$

$$\mathcal{K} = \mathrm{co} \cup_{a \in A} (G_a \cup H_a).$$

Claim 5.1 $0 \notin \mathcal{K}$.

Proof of Claim. Otherwise there exist $(\lambda_a), (\mu_a) \in [0,1]^A$ with $\sum_{a \in A}(\lambda_a + \mu_a) = 1$ and $\xi_a \in \mathrm{co}P_a(x_a)$ for all $a \in A$ such that

$$\sum_{a \in A} [\lambda_a \lambda(T_a)(\xi_a - x_a) + \mu_a \lambda(T_a)(\xi_a - e_a)] = 0.$$

Thus there exists $T \subset \mathcal{I}$, a measurable partition U, V of T, $\xi \in X$ such that for a.e. $t \in T$, $\xi_t \in P_t(x_t)$ and for all $a \in A$, $\lambda(U \cap T_a) = \lambda_a \lambda(T_a)$ and $\lambda(V \cap T_a) = \mu_a \lambda(T_a)$. Thus $\int_T \xi_t = \int_U x_t + \int_V e_t$ contradicting $x \in \mathcal{RC}$. □

Since $\mathcal{K}$ is compact, there exists $p \in I\!R^L \setminus \{0\}$ and $\varepsilon > 0$ such that $\varepsilon < \min p \cdot \mathcal{K}$. For every $a \in A$, let $m'_a = p \cdot (x_a - e_a) + \varepsilon/2$ and set $q = 1$. Then, of course for every $t \in \mathcal{I}$, $p \cdot x_t < p \cdot e_t + qm'_i < \min p \cdot P_t(x_t)$. □

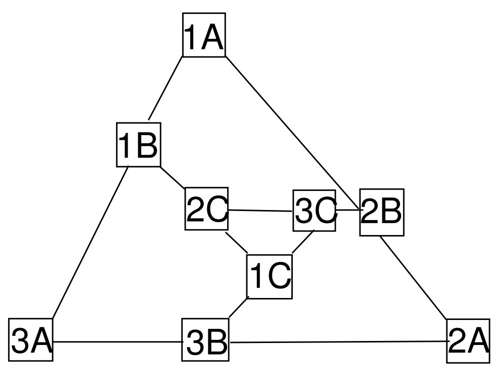
6 Shapley - Scarf's Example

Shapley and Scarf (1974) gave the following example in order to show that the core may be empty when commodities are indivisible. We consider an economy with three types of agents $I = \{1, 2, 3\}$ nine commodities $L = \{1_A, 1_B, 1_C, \ldots, 3_C\}$, commodity sets $X_i = \{0, 1\}^9$ and concave utility functions for $i \in I$

$$u_i(x) = \max\{2 \min\{x_{i_A}, x_{i+1_A}, x_{i+1_B}\}; \min\{x_{i_C}, x_{i+2_B}, x_{i+2_C}\}\}.$$

The indices are module 3. Initial endowments are $e_i = (e_{ih}) \in X_i$ with $e_{ih} = 1$ if and only if $h \in \{i_A, i_B, i_C\}$.

The following picture illustrates endowments and preferences. Each consumer would like to have three commodities on a straight line containing only one of his commodities. The best bundle is to own a long line containing his commodity i_A and $i + 1_B, i + 1_A$ and the second best would be to own a short line containing his commodity i_C and $i + 2_B, i + 2_C$.



If there is only one agent per type this reduces indeed to Shapley and Scarf's (1974) setting. In this case, at any feasible allocation for some $i \in I$, agent i obtains utility zero and agent $i + 2$ at most utility one. However, if they form a coalition it is possible to give utility one to i and two to $i + 2$. Thus, the core is empty.

With an even number of agents per type or a continuum of measure one per type the weak and the rejective core correspond to the allocations such that half of the consumers of type i consume $x_{ih} = 1$ for all $h \in \{i_A, i + 1_A, i + 1_B\}$ and the other half consumes $x_{ih} = 1$ for all $h \in \{i_C, i + 2_B, i + 2_C\}$. So every consumer obtains at least his second best allocation. It is not possible to block an allocation in the sense that all consumers who block are better off. Indeed, they would all need to obtain their best allocation and this is not feasible for any group. To see that this is the only allocation in the core, note that at any other allocation at least one consumer say a consumer of type 1 (or a non-negligible group of a given type) would necessarily get an allocation which yields zero utility. Then by feasibility, a consumer of type 3 (or a non-negligible group of type 3) obtain only their second best choice. The consumer of type 1 can propose the commodities $1_A, 1_B$ in exchange for $3_B, 3_C$ making everybody strictly better off.

Allocations in the core are supported by a uniform distribution of paper money $m_i = m > 0$ for all $i \in I$ and the price $p = (2,1,1,2,1,1,2,1,1)$, $q = 1/m$. Thus, a Walras equilibrium with money does not exist for a uniform distribution of paper money. A rationing equilibrium, however, exists. If half of each type obtains one unit of paper money and the other half strictly less than one unit, then the core allocation is a Walras equilibrium allocation with the same price p and $q = 1$.

7 Welfare Analysis

We will first study Pareto optimality of weak equilibria and then of rationing equilibria. Then, we will show that every Pareto optimum can be decentralized.

Definition 7.1 *A collection $(x, y) \in A(\mathcal{E})$ is a:*

(i) **feeble Pareto** *optimum if there does not exist $(x', y') \in A(\mathcal{E})$ such that $x'_t \in P_t(x_t)$ for a.e. $t \in \mathcal{I}$;*

(ii) **weak Pareto** *optimum if there does not exist $(x', y') \in A(\mathcal{E})$ and a non-negligible set $T \subset \mathcal{I}$ such that for a.e. $t \in T$, $x'_t \in P_t(x_t)$ and for a.e. $t \in \mathcal{I}$, $x'_t \neq x_t$ if and only if $t \in T$.*

Remark. A feeble Pareto optimum is usually called weak Pareto optimum in the literature. If for some consumer type $i \in I$, $P_i(x) = \emptyset$ for all $x \in X_i$, then any feasible allocation is a feeble Pareto optimum. The present definition of weak Pareto optimum has not this inconvenient. Hence, our change of terminology. According to our definitions, the set of weak Pareto optima is included in the set of feeble Pareto optima.

Example 5. *Weak equilibria with $q = 0$ need not be feeble Pareto optima.* Consider an economy with two types of agents $I = \{1, 2\}$, $L = \{A, B\}$, $X = \{0,1\}^2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$ $u_1(x) = x_B$, $u_2(x) = x_A$. Then (x, p, q) with $x_t = e_t$ for all t, $p = (1, 1), q = 0$ is a weak equilibrium. However, type one agents consuming $(0, 1)$ and type two agents $(1, 0)$ increases the utility of all agents. Moreover, as we saw in Example 2 (Section 2), a weak equilibrium with $qm_i > 0$ for all i need not be a weak Pareto optimum.

In spite of all foregoing, from Proposition 5.1 we can readily deduce the following assertion.

Proposition 7.1 *Let (x, y, p, q) be a weak equilibrium with $qm_i > 0$ for all $i \in I$. Then (x, y) is a feeble Pareto optimum.*

Proposition 7.2 *Let (x, y, p, q, K) be a rationing equilibrium for all $i \in I$. Then (x, y) is a weak Pareto optimum.*

Proof. Let $(x', y') \in A(\mathcal{E})$ Pareto dominating (x, y) in the weak sense. Thus,

$$e + \int_{\mathcal{J}} y'_t = \int_{\mathcal{I}} x'_t \in \int_{\mathcal{I}} x_t + K \setminus \{0\} = e + \int_{\mathcal{J}} y_t + K \setminus \{0\}.$$

Hence, $\int_{\mathcal{J}} y'_t - y_t \in K \setminus \{0\}$. This yields a contradicting, since at a rationing equilibrium we have $\int_{\mathcal{J}} y'_t - y_t \in -K$. □

For similar reasons as in Section 5, we restrict ourselves to exchange economies when studying the Second Welfare Theorem.

Proposition 7.3 *Let $\mathcal{E}$ be an economy with $J = \emptyset$. Let x be a weak Pareto optimum. Then there exists $p \in I\!R^L \setminus \{0\}$ and $e' \in X$ such that (x, p) is a Walras equilibrium of $\mathcal{E}'$ which is obtained from $\mathcal{E}$, replacing the initial endowment e by e'.*

Proof. For all $t \in \mathcal{I}$ set $e'_t = x_t$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \ldots, A\}$ satisfying the following:

(i) $(T_a)_{a \in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_t = x_a$.

Set

$$H_a = \lambda(T_a)(\mathrm{co}P_a(x_a) - x_a) \text{ and } \mathcal{H} = \mathrm{co} \cup_{a \in A} H_a.$$

Note that $0 \notin \mathcal{H}$. Otherwise there exist $(\lambda_a) \in [0, 1]^A$ with $\sum_{a \in A} \lambda_a = 1$ and $\xi_a \in \mathrm{co}P_a(x_a)$ for all $a \in A$ such that $\sum_{a \in A} \lambda_a \lambda(T_a)(\xi_a - x_a) = 0$. Thus there exists $\xi \in X$ such that for all $a \in A$, $\lambda(\{t \in T_a \mid \xi_t \in P_t(x_t)\}) = \lambda_a \lambda(T_a)$ and $\lambda(\{t \in T_a \mid \xi_t = x_t\}) = (1 - \lambda_a)\lambda(T_a)$ contradicting the weak Pareto optimality of x.

As $\mathcal{H}$ is compact, there exists $p \in I\!R^L \setminus \{0\}$ and $\varepsilon > 0$ such that for all $z \in \mathcal{H}$, $p \cdot z > \varepsilon$. Hence for a.e. $t \in \mathcal{I}$, $P_t(x_t) \cap \{\xi \in X_t \mid p \cdot \xi \leq p \cdot x_t + \varepsilon\} = \emptyset$. So (x, p) is indeed a Walras equilibrium of $\mathcal{E}'$. Setting $q > 0$ such that for all i, $qm_i < \varepsilon/2$, (x, p, q) would also be a Walras equilibrium with a positive value of paper money. □

Example 6. *Feeble Pareto optima cannot always be decentralized by $p \neq 0$*[9]. Consider an exchange economy with three consumers and two commodities

[9] Or $(p, q) \in I\!R^L \times I\!R_+$ with $qm_i > 0$.

$L = \{A, B\}$: for all $i \in I$, $X_i = \{0, 1, 2\}^2$, $u_1(x) = 0$, $u_2(x) = x_A$, $u_3(x) = x_B$, $x_1 = (0, 0)$, $x_2 = (0, 2)$, $x_3 = (2, 0)$. Decentralizing this allocation by $p \in I\!\!R^L \setminus \{0\}$ (or $(p, q) \in I\!\!R^L \times I\!\!R_+$ with $qm_i > 0$) implies that $p \in I\!\!R^L_{++}$. For $p_A \geq p_B$, $x \in D_2(p, q)$ implies $x_A \geq 1$ and for $p_A \leq p_B$, $x \in D_3(p, q)$ implies $x_B \geq 1$.

Remark. Under the assumptions of the previous proposition, we could also decentralize any Pareto optimum x by a bonafide fiscal policy. Collecting taxes $\tau_t = p \cdot (x_t - e_t) + m_t$ from agent $t \in \mathcal{I}$ payable in monetary units, x becomes an equilibrium together with $q = 1$ and p as in the previous proof.

8 Arbitrarily Small Indivisibilities

Perfect divisibility of goods, usually assumed in general equilibrium models, should obviously be seen as an approximation of commodities with a "small" enough level of indivisibility. Considering a sequence of economies with only indivisible goods, but consumption and production sets converging to convex sets, the limit of our weak equilibrium should thus be a Walras, dividend or an hierarchic equilibrium, depending whether local non-satiation and the survival assumption are satisfied or not.

We will now consider economies with finite consumption and production sets, as well as economies with convex consumption and production sets, which should be seen as the limit case, if the level of indivisibility is arbitrarily small.

Following Florig (2001), we will now introduce the notion of hierarchic equilibrium[10]. Let $\bar{I\!\!R} = (I\!\!R \cup \{+\infty\})$. For any $n \in N$, let $\preceq$ be the lexicographic order[11] on $I\!\!R^n$. Extrema will be taken with respect to the lexicographic order. We adopt the convention $0(+\infty) = 0$.

Definition 8.1 *A finite ordered family $\mathcal{P} = \{p^1, \ldots, p^k\}$ of vectors of $I\!\!R^L$ is called a* **hierarchic price***.*

Remark. If $k = 1$, this reduces to the standard case. We denote by $\mathcal{HP}$ the set of hierarchic prices. The number k is determined at the equilibrium. We will see that k never needs to be greater than L.[12]

[10] Marakulin (1990) introduced a similar notion for exchange economies, using non-standard analysis.

[11] For $(s, t) \in I\!\!R^n \times I\!\!R^n$, $s \preceq t$, if $s_r > t_r$, $r \in \{1, \ldots, n\}$ implies that $\exists \rho \in \{1, \ldots, r-1\}$ such that $s_\rho < t_\rho$. We write $s \prec t$ if $s \preceq t$, but not $[t \preceq s]$.

[12] The forthcoming definitions will depend for any $r \in \{2, \ldots, k\}$ only on the non-zero part of p^r which is orthogonal to $p^1, \ldots, p^{r-1}$. Therefore by an inductive argument we can always transform a hierarchic price into an equivalent one consisting of two by two orthogonal vectors (thus of at most L).

For $\mathcal{P} \in \mathcal{HP}$ and $x \in I\!R^L$, we define the *value* of x to be

$$\mathcal{P}x = (p^1 \cdot x, \dots, p^k \cdot x) \in I\!R^k.$$

The *supply* of firm $j \in J$ at the price $\mathcal{P}$ is

$$S_j(\mathcal{P}) = \{y \in Y_j \mid \forall z \in Y_j,\ \mathcal{P}z \preceq \mathcal{P}y\}.$$

Given a hierarchic price, firms are thus assumed to maximize the profit lexicographically. The aggregate *profit* of firms of type $j \in J$ is

$$\pi_j(\mathcal{P}) = \lambda(T_j)\mathrm{sup}_{y \in Y_j} \mathcal{P}y.$$

A *hierarchic revenue* is a vector $w \in \bar{I\!R}^k$. For all $i \in I$, all $\mathcal{P} \in \mathcal{HP}$, all $w \in \bar{I\!R}^k$ let

$$r_i(\mathcal{P}, w) = \min \{r \in \{1, \dots, k\} \mid \exists x \in X_i, (p^1 \cdot x, \dots, p^r \cdot x) \prec (w^1, \dots, w^r)\},$$

$$v_i(\mathcal{P}, w) = (w^1, \dots, w^{r_i(\mathcal{P},w)}, +\infty, \dots, +\infty) \in \bar{I\!R}^k.$$

The *budget set* of consumer i, with respect to $\mathcal{P} \in \mathcal{HP}$ and $w \in \bar{I\!R}^k$ will be

$$B_i(\mathcal{P}, w) = \{x \in X_i \mid \mathcal{P}x \preceq v_i(\mathcal{P}, w)\}.$$

Definition 8.2 [13] *A collection $(x, y, \mathcal{P}, w) \in A(\mathcal{E}) \times \mathcal{HP} \times (I\!R^k)^I$ is a* **hierarchic equilibrium** *of the economy $\mathcal{E}$ if:*

(i) for a.e. $t \in \mathcal{I}$, $x_t \in B_t(\mathcal{P}, w_t)$ and $P_t(x_t) \cap B_t(\mathcal{P}, w_t) = \emptyset$;

(ii) for all $i \in I$, $\mathcal{P}e_i + \sum_{j \in J} \theta_{ij}\pi_j(\mathcal{P}) \preceq w_i$;

(iii) for a.e. $t \in \mathcal{J}$, $y_t \in S_t(\mathcal{P})$.

Remark. The difference between revenue and the value of the initial endowment plus the value of shares in the firms may also be interpreted as the positive value of paper money held by the consumers (cf. Florig (2001)).

Definition 8.3 *A dividend equilibrium (resp. Walras equilibrium) is a hierarchic equilibrium $(x, y, \mathcal{P}, w)$ with $k = 1$ (resp. $k = 1$ and $\mathcal{P}e_i + \sum_{j \in J} \theta_{ij}\pi_j(\mathcal{P}) = w_i$).*

[13] If we note L_i the linear space of the positive hull generated by consumer i's net trade set and c_i the codimension of L_i, then we may reduce any hierarchic price into an equivalent one with $k \leq 1 + \min_{i \in I} c_i$. Indeed, either 0 is an equilibrium price or we may assume the prices two by two orthogonal and all non-zero (cf. Footnote 12). The rank of consumer i is smaller than or equal to the index of the first vector which is not orthogonal to L_i. The prices of a higher index are irrelevant to this consumer.

Remark. The definition of a dividend equilibrium was introduced in a fixed price setting by Drèze and Müller (1980). It is in fact equivalent to the definition of Walras equilibrium with money (Kajii (1996)). A Walras equilibrium is of course a Walras equilibrium with money, but where paper money is worthless.

Assumption C′. For all $i \in I$, X_i is a compact, convex polyhedron, $P_i : X_i \to 2^{X_i}$ is irreflexive, transitive and has an open graph in $X_i \times X_i$.

Assumption P. For every $j \in J$, Y_j is a compact, convex polyhedron.

Assumption SS. For all $i \in I$, $0 \in \mathrm{int}(X_i - \{e_i\} - \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j)$.

Remark. Assumption SS is the standard survival assumption. It states that every consumer is initially endowed with a strictly positive quantity of every existing commodity. Typically, consumers have however a single commodity to sell - their labor. This assumption is thus highly unrealistic. It implies that all agents have the same level of income at equilibrium in the sense that they have all access to the same commodities. A hierarchic equilibrium exists without such an assumption (Marakulin (1990), Florig (2001)).

For every $n = (n_1, \dots n_L) \in I\!N^L$, let

$$M^n = \{z \in I\!R^L \,|\, (n_1 z_1, \dots, n_L z_L) \in Z^L\}.$$

We say that a sequence $n \subset I\!N^L$ converges to ∞, if for all $h \in L$, n_h converges to ∞. Note that in the sense of Kuratowski-Painlevé, $\lim_{n \to \infty} M^n = I\!R^L$. Given an economy $\mathcal{E}$ (with convex consumption and production sets) and $n \in I\!N^L$, we note $\mathcal{E}^n$ the economy obtained by intersecting the consumption and production sets with M^n. We note the weak supply, budget and weak demand in the economy $\mathcal{E}^n$ by S_j^n, B_i^n, D_i^n.

Theorem 8.1 *Suppose $\mathcal{E}$ satisfies Assumptions C', P, S and $m_i > 0$ for all $i \in I$. Consider a sequence $n \subset I\!N^L$ converging to ∞ such that for all n, for all $i \in I$, $X_i = \mathrm{co}(X_i \cap M^n)$ and for all $j \in J$, $Y_j = \mathrm{co}(Y_j \cap M^n)$.*

Let (x^n, y^n, p^n, q^n) be a weak equilibrium of $\mathcal{E}^n$ with $q^n = (1- \parallel p^n \parallel) > 0$. Then, there exists a hierarchic equilibrium $(x, y, \mathcal{P}, w)$ with $\mathcal{P} = \{p^1, \dots, p^k\}$ and a subsequence such that:

- *For a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$,*[14]

$$x_t \in \mathrm{cl}\{x_t^n\} \text{ and } y_{t'} \in \mathrm{cl}\{y_{t'}^n\};$$

[14]We note cl Z for the closure of Z.

- $p^n = \sum_{r=1}^{k} \varepsilon_r^n p^r$, *with* $\varepsilon_{r+1}^n = \varepsilon_r^n o(\varepsilon_r^n) > 0$[15] *and* $\lim_{n\to+\infty} \varepsilon_1^n = 1$*;*

- $w_i = \mathcal{P}e_i + \sum_{j\in J} \theta_{ij}\pi_j(\mathcal{P}) + \{q_1, \ldots, q_k\}m_i$ *with* $q^r = \lim_{n\to+\infty} \frac{1-\|p^n\|}{\varepsilon_r^n}$.

Remark. Thus under the assumptions of the theorem, core equivalence for the rejective core and weak Pareto optimality is asymptotically established even for the weak equilibrium. Strong Pareto optimality however generally fails, even at the limit (cf. Florig (2001)).

Corollary 8.1 *(i) Suppose moreover Assumption SS, then* (x, y, p^1, w^1) *is a dividend equilibrium.*

(ii) If furthermore for a.e. $t \in \mathcal{I}$*,* $x_t \in \mathrm{cl}P_t(x_t)$ *(local non-satiation holds at* x*), then* (x, y, p^1) *is a Walras equilibrium.*

Proof of Theorem 8.1. Let (x^n, y^n, p^n, q^n) be a weak equilibrium of $\mathcal{E}^n$ with $q^n = (1- \| p^n \|) > 0$. For all $i \in I$, $j \in J$ note $\bar{x}^n = \int_{T_i} x_i^n/\lambda(T_i)$ and $\bar{y}^n = \int_{T_j} y_j^n/\lambda(T_j)$ the average consumption and production plans per type at x^n, y^n respectively. We note

$$\beta_i(p) = \{x \in X_i \,|\, p \cdot x \leq p \cdot e_i + (1 - \|p\|)m_i + \sum_{j\in j} \theta_{ij}\pi_j(p)\}.$$

As in Florig (2001), we can extract a subsequence such that:

- $p^n = \sum_{r=1}^{k} \varepsilon_r^n p^r$, with $\varepsilon_{r+1}^n = \varepsilon_r^n o(\varepsilon_r^n) > 0$ for $r \in \{1, \ldots, k-1\}$, and $\lim_{n\to\infty} \varepsilon_1^n = 1$. Let $\mathcal{P} = \{p^1, \ldots, p^k\}$;

- for all large enough n, for all $j \in J$, $\mathrm{co}S_j^n(p^n) = S_j(\mathcal{P})$ and thus $\bar{y}_j^n \in S_j(\mathcal{P})$ and $\bar{y}_j \in S_j(\mathcal{P})$;

- for all $i \in I$, $\beta_i(p^n)$ converges to $B_i(\mathcal{P}, w_i)$.

For the last two points, we use the fact that for all n, for all $j \in J$, $\mathrm{co}Y_j^n = Y_j$ and for all $i \in I$, $\mathrm{co}X_i^n = X_i$, in order to apply the arguments from Florig (2001).

Since the consumption sets are compact and for all n, $(x^n, y^n) \in A(\mathcal{E}^n)$, there exists by Fatou's lemma (Arstein (1979)) $(x, y) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$,

$$x_t \in \mathrm{cl}\{x_t^n\} \text{ and } y_{t'} \in \mathrm{cl}\{y_{t'}^n\}.$$

Thus, by the second point above for a.e. $t \in \mathcal{J}$, $y_t \in S_t(\mathcal{P})$. Obviously, for a.e. $t \in \mathcal{I}$, $x_t \in \lim_{n\to\infty} B_t^n(p^n, q^n)$. Moreover, $\lim_{n\to\infty} B_t^n(p^n, q^n) \subset B_t(\mathcal{P}, w_t)$.

[15] Let $o : \mathbb{R}_+ \to \mathbb{R}_+$ such that $o(0) = 0$ and o is continuous in 0.

It remains to be proven that for a.e. $t \in \mathcal{I}$, $P_t(x_t) \cap B_t(\mathcal{P}, w_t) = \emptyset$. We will proceed by contraposition. Let $\mathcal{N}$ be the negligible subset of $\mathcal{I}$ containing all $t \in \mathcal{I}$ such that either for some n, $x_t^n \notin D_t^n(p^n, q^n)$ or such that $x_t \notin \mathrm{cl}\{x_t^n\}$. This set is negligible since it is a countable union of negligible sets. Let $t \in \mathcal{I} \setminus \mathcal{N}$ such that there exists $\xi_t \in P_t(x_t) \cap B_t(\mathcal{P}, w_t)$. If the budget set is reduced to a single point then $x_t = \xi_t$. Thus the budget set has a non-empty interior in some facet F of X_t. By the continuity of P_t, we may assume that $\xi_t \in \mathrm{int}_F(F \cap B_t(\mathcal{P}, w_t))$. If $F \subset B_t(\mathcal{P}, w_t)$, then since for all n, $\mathrm{int}_{X_t}\beta_t(p^n) \neq \emptyset$, we have for all large n,

$$p^n \cdot \xi_t < w_t^n = p^n \cdot e_i + (1 - \|p^n\|)m_i + \sum_{j \in j} \theta_{tj}\pi_j(p^n).$$

Otherwise there exists $\xi_t' \in \mathrm{int}_F(F \cap B_t(\mathcal{P}, w_t))$ such that $\mathcal{P}\xi_t \prec \mathcal{P}\xi_t'$. Thus for all large n, $p^n \cdot \xi_t < p^n \cdot \xi_t'$. Thus for all large enough n, $p^n \cdot \xi_t < w_t^n$. Moreover, since $F \cap M^n$ converges to F, we may assume that $\xi_t \in M^n$ for all n larger than some n_t. Again by the continuity of P_t, for a subsequence, we have $\xi_t \in P_t(x_t^n)$. Thus $x_t^n \notin D_t^n(p^n, w_t^n)$, a contradiction. □

9 Appendix

In this section, we will use notations introduced in Section 8 (lexicographic order, hierarchic price and value, supply with respect to a hierarchic price).

Proof of Theorem 4.2. For every $i \in I$ choose $(m_i^1, \ldots, m_i^L) \subset I\!R_{++}^L$ with $m_i^1 = m_i$ and let (x^1, y^1, p^1, q^1) be a weak equilibrium of $\mathcal{E}^1 = \mathcal{E}$. By induction, we construct a sequence $(x^1, y^1, p^1, q^1), \ldots, (x^r, y^r, p^r, q^r)$ of weak equilibria of the economies $\mathcal{E}^1, \ldots, \mathcal{E}^r$ as follows.

Since the number of types is finite and the consumption sets are finite, for every $r \geq 1$, we can define a finite set of consumer types $A^{r+1} \equiv \{1, \ldots, A^{r+1}\}$ with $A^1 = I$ satisfying the following:

(i) $(T_a)_{a \in A^{r+1}}$ is a finer partition of $\mathcal{I}$ than $(T_a)_{a \in A^r}$,

(ii) for every $a \in A^{r+1}$, there exists x_a^r such that for every $t \in T_a$, $x_t^r = x_a^r$.

Set $X_a^{r+1} = (P_a(x_a^r) \cup x_a^r) \cap (x_a^r + (p^r)^\perp)$, $e_a^{r+1} = x_a^r$ and P_a^{r+1} is the restriction of P_a to X_a^{r+1}.

Since there is also a finite number of types of producers and production sets are finite, for every $r \geq 1$ we can define a finite set of producer types $B^{r+1} \equiv \{1, \ldots, B^{r+1}\}$ with $B^1 = J$ satisfying the following:

(i) $(T_b)_{b \in B^{r+1}}$ is a finer partition of $\mathcal{J}$ than $(T_b)_{b \in B^r}$,

(ii) for every $b \in B^{r+1}$, there exists y_b^r such that for every $t \in T_b$, $y_t^r = y_a^r$.

Set $Y_b^{r+1} = ((Y_b^r - y_b^r) \cap (p^r)^\perp)$. Denote the new economy by $\mathcal{E}^{r+1}$ where m_i^{r+1} are the initial endowments in money.

By induction, for all $r \in \{1, \ldots, L\}$ the consumption and production sets of the economy $\mathcal{E}^r$ are non-empty and Assumptions C, S are satisfied. So by the previous theorem, for all $r \in \{1, \ldots, L\}$ there exists a weak equilibrium (x^r, y^r, p^r, q^r) with $q^r > 0$ for the economy $\mathcal{E}^r$. Let

$$k = \min \{L, \{r \in \{1, \ldots, L\} \mid p^r \in \operatorname{span}\{p^1, \ldots, p^{r-1}\}\}.$$

Set $\mathcal{P} = \{p^1, \ldots, p^k\}$.

Claim 9.1 *For all $t \in \mathcal{I}$, $\mathcal{P}x_t^k \preceq w_t$ with $w_t \in I\!R^k$ and for all $r \in \{1, \ldots, k\}$, $w_t^r = p^r \cdot x_t^{r-1} + q^r m_i^r + \sum_{b \in B^r} \theta_{tb} \lambda(T_b) p^r \cdot y_b^r$ with $x_t^0 = e_t$.*

Note that by the construction of $X_t^2, \ldots, X_t^k$, we have for all $t \in \mathcal{I}$, for every $r \in \{1, \ldots, k\}$, $p^r \cdot x_t^r = \ldots = p^r \cdot x_t^k$. Since for every $r \in \{1, \ldots, k\}$, $p^r \cdot x_t^r \leq w_t^r$ we have for all $t \in \mathcal{I}$, $\mathcal{P}x_t^k \preceq w_t$.

Claim 9.2 *For all $t \in \mathcal{J}$, $y_t = \sum_{r=1}^k y_t^r \in S_t(\mathcal{P})$.*

It is sufficient to show by induction that for all $r, r' \in \{1, \ldots, k\}$ with $r \leq r'$, $\sum_{\rho=1}^{r'} y_t^\rho \in S_t(\{p^1, \ldots, p^r\})$.

Claim 9.3 *For all $t \in \mathcal{I}$, $\xi_t \in P_t(x_t^k)$ implies $\mathcal{P}x_t^k \prec \mathcal{P}\xi_t$.*

If $w_t \prec \mathcal{P}\xi_t$, the claim is trivially satisfied by Claim 9.1. Now, by transitivity of the preferences, $\xi_t \in P_t(x_t^k)$ implies that $\xi_t \in P_t(x_t^r)$ for every $r \in \{1, \ldots, k\}$.

If $\mathcal{P}\xi_t \preceq w_t$, then by Proposition 3.1 we must then have $\mathcal{P}\xi_t = w_t$. Thus by Claim 9.1, $\mathcal{P}x_t^k \preceq \mathcal{P}\xi_t$. Now we may distinguish two cases. Either $k = L$ and $\{p^1, \ldots, p^k\}$ forms a basis of R^L. Then, we must have $\mathcal{P}x_t^k \prec \mathcal{P}\xi_t$. Or p^k is in the linear span of $\{p^1, \ldots, p^{k-1}\}$. If $\mathcal{P}x_t^k = \mathcal{P}\xi_t$, then $\xi_t \in X_t^r$ for all $r \in \{1, \ldots, k\}$. Since $q^k m_t^k > 0$ and since Assumption S is satisfied, we have $p^k \cdot \xi_t < w_t^k$. This contradicts Proposition 3.1. Hence, $\mathcal{P}x_t^k \prec \mathcal{P}\xi_t$.

Set $(x, y, p, q) = (x^k, y, p^1, q^1)$. Let $K' = \{x \in I\!R^L \mid (0, \ldots, 0) \prec \mathcal{P}x\} \cup \{0\}$. This is a convex cone. Note that $-K' \cap K' = \{0\}$. Since for all $t \in \mathcal{J}$, $y_t \in S_t(\mathcal{P})$, we have for all $t \in \mathcal{J}$, $Y_t - y_t \subset -K'$. For all $t \in \mathcal{J}$, let K_t be the positive hull of $K' \cap (y_t - Y_t)$. Note that for all $t \in \mathcal{I}$, if $x_t' \in P_t(x_t)$, then $(0, \ldots, 0) \prec \mathcal{P}(x_t' - x_t)$. For all $t \in \mathcal{I}$, let K_t be the positive hull of $K' \cap (P_t(x_t) - x_t)$. Let $K = \operatorname{cl}\{\operatorname{co} \cup_{t \in \mathcal{I} \cup \mathcal{J}} K_t\}$. Of course K is a convex cone and by the finiteness of the consumption and production sets $K \subset K'$. Thus, $-K \cap K = \{0\}$. For all $t \in \mathcal{I}$, $P_t(x_t) - x_t \subset K$, for all $t \in \mathcal{J}$, $Y_t - y_t \subset -K$. □

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