Regular Economies with Non-Ordered Preferences

Jean-Marc Bonnisseau*

Novembre 2001

Résumé

Nous considérons une économie d'échange avec des préférences non-ordonnées et des effets externes. Premièrement, nous montrons une formule de l'index sous une hypothèse de comportement au bord des préférences qui implique l'existence d'équilibres pour toutes les dotations initiales. De plus, nous montrons la semi continuité supérieure des la correspondance de Walras. Nous énonçons ensuite une hypothèse de différentiabilité sur les préférences qui nous permet de montrer que la variété des équilibres is une sous-variété différentiable non vide d'un espace euclidien. Nous définissons les économies régulières de façon usuelle comme les valeurs régulières de la projection naturelle. Nous en déduisons que l'ensemble des économies régulières est ouvert, dense et de mesure pleine pour la mesure de Lebesgue. Une économie régulière a un nombre fini, impair d'équilibres et pour chacun d'eux, il existe une sélection différentiable locale. Donc, de telles économies ont des propriétés similaires à celles où les préférences sont représentées par des fonctions d'utilités différentiables.

Mots-Clés : équilibre de Walras, économie régulière, préférences non-ordonnées, effets externes, unicité locale des équilibres.

Abstract

We consider an exchange economies with non-ordered preferences and external effects. We first prove an index formula under an assumtion on the boundary behavior of the preferences, which implies the existence of equilibria for all initial endawments. Furthermore, one shows the upper semi-continuity of the Walras correspondence. We then posit a differentiability assumption on the preferences which allows us to prove that the equilibrium manifold is a nonempty differentiable submanifold of an Euclidean space. As usual, we define regular economies as the regular value of the natural projection. We deduce that the set of regular economies is open, dense and of full Lebesgue measure. A regular economy has a finite odd number of equilibria and for each of them, there exists a local differentiable selection. So, such economies have the same properties as the one with differentiable utility functions.

Key Words: Walras equilibrium, regular economy, non-ordered preferences, external effects, local uniqueness of equilibria.

JEL Classification Code: C61, C62, D50

*CERMSEM, Université de Paris 1 Panthéon-Sorbonne, 106 boulevard de l'Hôpital, 75647 Paris cedex 13, France, e-mail : jmbonni@univ-paris1.fr

1 Introduction

This work is in the line of works that dates back to the seminal paper of Debreu (1970) on economies with a finite set of equilibria. The literature on this subject and related topics is huge. We can find surveys on it in Balasko (1988), Debreu (1976), Dierker (1974, 1982), Mas-Colell (1985), Smale (1981).

From the Debreu-Mantel-Sonnenschein Theorem (Debreu (1974), Mantel (1974), Sonnenschein (1973, 1974)), we know that we cannot hope to get a finite set of equilibria for each economy only under strong assumptions. Thus, the concept of regular economies appears as the right one. In this case, the equilibria are locally unique and thus, finite. Furthermore, there exists a differentiable selection of the equilibrium price around each equilibrium. These properties are very important to analyze the behavior of the markets.

In Bonnisseau-Florig-Jofré (2001a,b), a notion of regular economies is defined with linear preferences and it is proven that they have the same nice properties as those where the utility functions are strictly quasiconcave. The purpose of this work is to deal with another weakening of the standard assumption on the preferences. We consider an exchange economy with non-ordered preferences and external effects. This means that preferences are not assumed to be complete or transitive and the preferred set of an agent depends on the consumption bundles of the others.

This is less demanding on the rationality of the agents. An interpretation is that they only take into account consumption bundles in a neighborhood of the actual consumption plan and they have a local preference relation. The presence of external effects is also recognized in economics since a long time and it is interesting to incorporate this feature as it has been done in the existence problem.

A major difficulty comes from the fact that the demand is no more single valued. Thus, we characterize the maximal element in the budget set by considering the normal cone to the preferred set. When the preferences are represented by a differentiable utility function, this merely means that relative prices equal marginal rates of substitution.

Consequently, only the normal cone of the preferred set at a consumption bundle matters for equilibrium purposes. This means that if the preference relations are different but the normal cones are the same, then the equilibria are the same. Since the normal cone is actually a local concept and since we do not assume the existence of a utility function, this means that the behavior of the consumers is reduced to a local rationality in the following sense. At each consumption bundle x, the consumer has a concave homogeneous valuation of the bundles around and she/he is at a maximal element in the budget set if she/he cannot find a consumption bundle in the budget set close to x and with a higher personal valuation than x. The valuation can be described by a set of vectors of weights on the commodities which is the set of normalized vectors in the normal cone to the preferred set at x.

We first posit an assumption on the preferences, which is not stated in the usual way. Nevertheless, we show that usual conditions imply it. In particular, we need to know the boundary behavior of the preferences. Usually, this is a consequence of the fact that it is assumed that the better than sets are closed in the space of commodities, together with a strict monotony of utility functions. Here, it is not possible to translate this assumption. Consequently, we assume that, if the quantity of one commodity tends to 0, then the consumer puts a strong weights on this commodity. More precisely, the individual valuation of a bundle tends to 0 if the quantity of one commodity tends to 0. Later, we show that it is weaker than the usual assumption when preferences are representable by utility functions.

Under this assumption, we prove an index formula, which is an extension of what is done in the usual framework (see, Mas-Colell (1985)). More precisely, we consider a correspondence Φ, which characterizes the equilibria in the following sense. Up to translations, an equilibrium is a zero of Φ. We prove that the degree of Φ with respect to the value 0 on an open convex set containing 0 is 1. Later, one deduces from this formula that a regular economy has an odd number of equilibria. This also implies that for each initial endowments there exists an equilibrium. We also show that the boundary condition implies that the equilibrium correspondence is upper semicontinuous.

Then, we consider a differentiability assumption on the preferences. It says that the normal cone to the preferred set is reduced to a half line and the selection obtained by normalization is differentiable. Furthermore, one posits an assumption on the range of the differential. It is weaker than assuming that it is onto. This assumption extends the one given in Smale (1974 a, b), since we do not assume that this mapping is a normalized gradient vector of a utility function and we allow for external effects. It is closely related to the one in Jouini (1992, 1993). Note that the assumption is adapted to take into account the external effects.

In the appendix, we give an example of a preference relation, which satisfies the differentiability assumption and we show that the demand is a finite set of isolated points for some prices and incomes and the number of elements in the demand varies. This shows that our assumption allows considering much more general demand than in the standard model. Indeed, in this later case, the demand is always a singleton and it is a diffeomorphism on the consumption set.

We also prove that our assumption holds true for a preference relation, which is representable by a utility function satisfying the standard assumption. Note that our boundary condition is weaker than the usual one since the indifference curves may be tangent to the boundary of the consumption set in our setting and this is excluded in the usual models. Furthermore, we encompass the case where the preferences of one consumer are not strictly quasi concave. This is possible since we never use the demand function, which may be multi-valued in this case.

We are now able to prove that the equilibrium manifold, which is the graph of the equilibrium correspondence, also called Walras correspondence, is actually a differentiable submanifold of an Euclidean space. This is deduced from the fact that it is the inverse image of 0 by a submersion. To define the submersion, we follow the method of Smale. Note that the same approach is used in Jouini (1992, 1993) to deal with economies with non-convex production sets. The two problems are related since, with non-convex production sets, the supply function is no more single valued when the behavior of the producers is represented by pricing rules.

We define, as usual, the natural projection, which associates the initial endowments to an equilibrium. Then, an economy is regular if the initial endowments are a regular value of the natural projection. Since the equilibrium correspondence is upper semi-continuous, the natural projection is proper in the sense that the inverse image of a compact set is compact. Consequently, with the usual arguments, one obtains the following results : the set of regular economies is open, dense and of full Lebesgue measure; each regular economy has a finite odd number of equilibria; for each of them, there exists a local differentiable selection.

Note that the results in the standard case have been extended to non-smooth demand functions by using the method of Debreu (1970) (See, Rader (1973), Shannon (1994)). Until now, we are not able to deal with such generalization with non-ordered preferences.

To conclude, one remarks that the nice properties of the equilibrium manifold and of the regular economies remain true even if we consider a wide class of non-ordered preferences with external effects. This shows that the coordination of agents' actions through the market is relatively strong since almost everywhere, one gets only a finite number of states which clear the market and where all consumers are at an optimal allocation.

2 The model

We[1] consider a class of exchange economies. The parameters of these economies are the same but the initial endowments. L and I are the nonempty finite sets of commodities and of consumers. The consumption set X_i of consumer i, $i \in I$ is $\mathbb{R}^L_{++}$. The preferences of the agent i are represented by a correspondence P_i from $\prod_{i\in I} X_i$ to X_i. For each $x \in \prod_{i\in I} X_i$, $P_i(x) \subset X_i$ is the set of consumption plan which are strictly preferred to x_i if the consumption bundles of the other consumers are $(x_j)_{j\neq i}$. This presentation allows us to encompass the case of external effects among the agents. In the following, we

[1] Notations. If x and y are vectors of $\mathbb{R}^J$ where J is a finite set, then $x \cdot y = \sum_{j\in J} x_j y_j$ is the canonical inner product of x and y. The norm of x is $\|x\| = \sqrt{x \cdot x}$. For all $r > 0$, $B(x,r) = \{x' \in \mathbb{R}^J \mid \|x' - x\| < r\}$. $x^\perp$ is the linear subspace of $\mathbb{R}^J$ defined by $\{x' \in \mathbb{R}^J \mid x \cdot x' = 0\}$ and $\mathrm{proj}_{x^\perp}$ is the orthogonal projection on $x^\perp$. 1_J is the vector of $\mathbb{R}^J$ whose coordinates are all equal to 1, $\mathbb{R}^J_+ = \{x \in \mathbb{R}^J \mid x_j \geq 0, \forall j \in J\}$ and $\mathbb{R}^J_{++} = \{x \in \mathbb{R}^J \mid x_j > 0, \forall j \in J\}$. If X is a subset of $\mathbb{R}^J$, then $\overline{X}$ is the closure of X and ∂X the boundary of X. If φ is a linear mapping, then $\mathrm{Im}\varphi$ is the range of φ. $\sharp L$ denotes the cardinality of L.

never assume that the preference relations are complete or transitive. Thus, we cannot represent them by utility functions.

Taking these elements as given, an economy is completely determined by the initial endowments, which are a collection $e = (e_i)_{i\in I} \in \prod_{i\in I} X_i$. The class of economies is then $\{\mathcal{E}_e = \big((X_i, P_i, e_i)_{i\in I}\big) \mid e_i \in X_i,\ \forall i \in I\}$.

The set of normalized prices is $S = \{p \in \mathbb{R}^L_+ \mid 1_L \cdot p = 1\}$ where 1_L is the vector of $\mathbb{R}^L$ whose coordinates are all equal to 1. In the following, $A = \{p \in \mathbb{R}^L \mid 1_L \cdot p = 1\}$ denotes the affine subspace spanned by S. H is the linear subspace of $(1_L^\perp)^I$ defined by $H = \{s \in (1_L^\perp)^I \mid s_i = s_j,\ \forall (i,j) \in I^2\}$.

Definition 2.1 For each $e \in \prod_{i\in I} X_i$, an equilibrium of the economy $\mathcal{E}_e$ is an element $(p^*, (x_i^*)) \in \mathbb{R}^L_+ \setminus \{0\} \times \prod_{i\in I} X_i$ such that :

a) For all i, $x_i^* \in B_i^* = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\}$ and $B_i^* \cap P_i(x^*) = \emptyset$;

b) $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$.

For each $e \in \prod_{i\in I} X_i$, $W(e) \subset S \times \prod_{i\in I} X_i$ denotes the set of equilibria of the economy $\mathcal{E}_e$ with a normalized equilibrium price. We now posit some remarks on this definition, which will be useful in the following to understand our characterization of equilibria. First note, that (b) and $x_i^* \in B_i^*$ for all i, implies that $p^* \cdot x_i^* = p^* \cdot e_i$.

We recall that the normal cone of the convex analysis to a subset X (not necessarily convex) of $\mathbb{R}^L$ at a point $x \in X$, $N_X(x)$, is defined by $N_X(x) = \{y \in \mathbb{R}^L \mid y \cdot (x' - x) \leq 0, \forall x' \in X\}$. If the local nonsatiation holds true at x_i^* ($x_i^* \in \overline{P}_i(x^*)$) and if the preferred set $P_i(x^*)$ is open, then $B_i^* \cap P_i(x^*) = \emptyset$ is equivalent to $p^* \in -N_{\overline{P}_i(x^*)}(x_i^*)$. This means that only the normal cone to the preferred set matters for equilibrium purposes. Its precise shape is not relevant.

Consequently, one has the following proposition.

Proposition 2.1 *If for each agent and for each consumption bundle, the preferences are locally nonsatiated with open values, then for each $e \in \prod_{i\in I} X_i$, $(p^*, (x_i^*)) \in W(e)$ if and only if*

a) For all i, $x_i^* \in X_i$, $p^* \cdot x_i^* = p^* \cdot e_i$ and $p^* \in -N_{\overline{P}_i(x^*)}(x_i^*) \cap S$;

b) $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$.

We now posit an assumption on the preferences, which allows us to prove a generalization of the index formula and, then, to deduce the existence of equilibrium. We first define the correspondence G_i as follows : for all $x \in \prod_{i\in I} X_i$,

$$G_i(x) = -N_{\overline{P}_i(x)}(x_i) \cap S$$

Since the normal cone is always convex valued, G_i has always convex values.

Assumption P. For each $i \in I$,

a) for each $x \in \prod_{j\in I} X_j$, $P_i(x)$ is open, $x_i \in \overline{P}_i(x)$;

b) the correspondence G_i is upper semi-continuous with nonempty, compact values;

c) for each sequence (x^ν, g_i^ν) of $\prod_{j\in I} X_j \times S$, such that $g_i^\nu \in G_i(x^\nu)$, if (x^ν) converges to $x \in \prod_{j\in I} \bar{X}_j$ and $x_i \in \partial X_i$, then $(g_i^\nu \cdot x_i^\nu)$ converges to 0.

Assumption P implies that the standard irreflexivity and convexity assumption (x_i does not belong to the convex hull of $P_i(x)$) is satisfied by the preferences relation P_i. Indeed, it is a consequence of the fact that $G_i(x_i)$ has nonempty values and that $P_i(x)$ is open. Condition (c) deals with the boundary behavior of the preferences. It means that if the quantity of one commodity becomes very small, then the consumer essentially wants to increase the quantity of this commodity. An important consequence of this assumption is that the set of equilibria remains in a compact set if the initial endowments remain in a compact set. It is also a key argument in the proof of the index formula.

To be more precise, let us consider the following correspondence and preference relation : for all $i \in I$, for all $x \in \prod_{j\in I} X_j$,

$$G_i^+(x) = \{u \in \mathbb{R}^L \mid g_i \cdot u > 0, \ \forall g_i \in G_i(x)\}$$

for all $x_i \in X_i$,

$$\tilde{P}_i(x) = (\{x_i\} + G_i^+(x)) \cap \mathbb{R}^L_{++}$$

Proposition 2.2 *If P_i satisfies Assumption P, then for all $x \in \prod_{j\in I} X_j$, $P_i(x) \subset \tilde{P}_i(x)$, the preferences relation $\tilde{P}_i$ satisfies Assumption P(a) and*

b') the correspondence $\tilde{P}_i$ is convex valued, it has open lowersection and for all $x \in \prod_{i\in I} X_i$, $x_i \notin \tilde{P}_i(x)$ and $\tilde{P}_i(x) + \mathbb{R}^L_+ = \tilde{P}_i(x)$;

c') for each sequence (x^ν) of $\prod_{j\in I} X_j$, such that (x^ν) converges to $x \in \prod_{j\in I} \bar{X}_j$ and $x_i \in \partial X_i$, for all $\xi_i \in X_i$, then $\xi_i \in \tilde{P}_i(x^\nu)$ for all ν large enough.

Furthermore, $(p,(x_i))$ is an equilibrium with the preferences P_i if and only if it is an equilibrium with the preferences $\tilde{P}_i$.

Conversely, if the preference correspondence P_i satisfies Assumption P(a) and conditions (b') and (c') above, then it satisfies Assumption P(b) and (c).

The proof of this proposition is given in Appendix. Note that the end of the proposition gives sufficient condition directly on the preference relation, which are sufficient to satisfy Assumption P. In the following, we use an extension Γ_i of G_i which is defined on the closure of the consumption sets as follows : for all $x \in (\mathbb{R}^L_+)^I$,

$$\Gamma_i(x) = \text{co}\{p \in S \mid \exists (x^\nu, g_i^\nu) \subset \prod_{j\in J} X_j \times S, \lim_\nu x^\nu = x, \lim_\nu g_i^\nu = p, \ g_i^\nu \in G_i(x^\nu) \forall \nu\}.$$

One easily checks that $\Gamma_i(x) = G_i(x)$ if $x \in \prod_{j\in J} X_j$ since G_i is upper semi-continuous with convex values. Furthermore, the graph of Γ_i is closed and, since it takes their values in the compact set S, it is upper semi-continuous with nonempty convex compact values. The boundary behavior of G_i implies that $p \cdot x_i = 0$ for all $p \in \Gamma_i(x)$ if $x_i \in \partial\mathbb{R}^L_+$. We may define directly the preference relations on the closed positive orthant with this last property but it does not change anything for the equilibrium analysis since the allocations are never on the boundary.

3 An index formula

In this section, we provide an index formula, which generalizes the well known result for exchange economies with ordered preferences (see, Mas-Colell (1985)). Note also that this result is directly related to the computation of the degree of the natural projection as in Balasko (1988). The two main consequences are that each economy has at least one equilibrium and the number of equilibria is odd if the economy is regular.

Even if the result is not stated in these words, the basic fact behind the index formula, involves the degree of the excess demand function or, more precisely, of a projection of the excess supply function. Actually, it is well known that the Walras law implies that there is a redundant equation in the excess demand function, which can be eliminated by considering a projection. The proof is based on the fact that the excess supply function is homotopic to the identity and this is due to the boundary behavior of the excess demand. Then, the computation of the degree for differentiable mappings, leads to the index theorem for regular economies.

In our framework, the excess demand function is not defined and, thus, we need to work with additional variables. We consider a correspondence Φ, which depends on the price and the allocation. To work on a bounded set, we consider the initial endowment as an upper bound of the individual allocation. The two following propositions shows that the set of zeros of Φ is directly linked with the set of equilibria and the degree of Φ is equal to one. One then deduces that each economy has at least one equilibrium. It appears that this result essentially rests on the boundary behavior of the correspondences G_i (Assumption P(c)).

Let $(e_i)_{i\in I}$ an initial endowment and $e = \sum_{i\in I} e_i$ the total initial endowment. Let for $\rho \geq 0$, $\Sigma_\rho = \{q \in 1_L^\perp \mid q_h \geq -\frac{1}{\sharp L} - \rho, \forall h \in L\}$ and $K_i = \{\xi_i \in \mathbb{R}^L \mid -e_i \leq \xi_i \leq e - e_i\}$. We consider the correspondence Φ from $\Sigma_\rho \times \prod_{i\in I} K_i$ to $1_L^\perp \times (\mathbb{R}^L)^I$ defined by

$$\Phi(q,(\xi_i)) = (-\mathrm{proj}_{1_L^\perp}(\sum_{i\in I}\xi_i), (-\Gamma_i(\xi_i+e_i) + \{p + p\cdot\xi_i 1_L\})_{i\in I})$$

where Γ_i is defined at the end of the previous section and $p = q + \frac{1}{\sharp L}1_L$.

Proposition 3.1 *If $0 \in \Phi(q,(\xi_i))$, then $p = q + \frac{1}{\sharp L}1_L \in S$, $x_i = \xi_i + e_i \in \mathbb{R}^L_{++}$ for all i, and $(p,(x_i))$ is an equilibrium of the economy $\mathcal{E}_e$. Conversely, if $(p,(x_i))$ is an equilibrium of the economy $\mathcal{E}_e$, then $0 \in \Phi(p - \frac{1}{\sharp L}1_L, (x_i - e_i))$.*

Proof. If $0 \in \Phi(q,(\xi_i))$, then there exists $(g_i) \in \prod_{i\in I}\Gamma_i(\xi_i+e_i)$ such that $\sum_{i\in I}\xi_i = \alpha 1_L$, and $p - g_i + p\cdot\xi_i 1_l = 0$. Since p and g_i belong to A, $p - g_i$ is orthogonal to 1_L. Thus, $p = g_i$ and $p\cdot\xi_i = 0$ for all i. Consequently, $\alpha = p\cdot\sum_{i\in I}\xi_i = 0$ and $\sum_{i\in I}\xi_i = 0$. We now show that $\xi_i \gg -e_i$ for all i. If it is not true, then, from the definition of Γ_i and Assumption P, one has $g_i\cdot\xi_i = g_i\cdot -e_i$. But this contradicts the fact that $g_i\cdot\xi_i = p\cdot\xi_i = 0$. Finally, since $\sum_{i\in I}\xi_i = 0$, one deduces that $\xi_i \ll \sum_{j\neq i} e_j = e - e_i$. Hence, $\xi_i \in \mathrm{int}K_i$. Since, $G_i(\xi + e_i) = \Gamma_i(\xi + e_i)$ when $\xi + e_i \gg 0$, one deduces that $p \in G_i(\xi_i + e_i) \subset S$. Consequently, $(p,(x_i))$ is an equilibrium of the economy $\mathcal{E}_e$. The converse assertion is obvious.□

In the following, we choose $\rho > 0$ small enough such that for all $q \in \Sigma_\rho$, for all i, $(q + \frac{1}{\sharp L}1_L)\cdot e_i > 0$.

Proposition 3.2 *Under Assumption P,* $\deg(\Phi, ir\Sigma_\rho \times \prod_{i\in I} intK_i, 0) = 1$.

Proof. Since $\mathrm{ir}\Sigma_\rho \times \prod_{i\in I}\mathrm{int}K_i$ is an open subset of $1_L^\perp \times (\mathbb{R}^L)^I$, which contains 0, it is sufficient to check that Φ is homotopic to the identity. This means that there exists a mapping H from $[0,1] \times \Sigma_\rho \times \prod_{i\in I} K_i$ to $1_L^\perp \times (\mathbb{R}^L)^I$ such that $H(0,.) = \Phi$, $H(1,.)$ is the identity and $H(t,(q,(\xi_i))) \neq 0$ for all $(t,(q,(\xi_i))) \in [0,1] \times \partial(\Sigma_\rho \times \prod_{i\in I} K_i)$. Since $\Sigma_\rho \times \prod_{i\in I} K_i$ is convex, it is enough to prove[2] that $\Phi(q,(\xi_i)) \cap -N_{\Sigma_\rho\times\prod_{i\in I}K_i}(q,(\xi_i)) = \emptyset$ for all $(q,(\xi_i)) \in \partial(\Sigma_\rho \times \prod_{i\in I} K_i)$.

We now prove this assertion. Let $(q,(\xi_i)) \in \Sigma_\rho \times \prod_{i\in I} K_i$. Let us assume that

$$\Phi(q,(\xi_i)) \cap -N_{\Sigma_\rho\times\prod_{i\in I}K_i}(q,(\xi_i)) \neq \emptyset$$

Then, there exists $(g_i) \in \prod_{i\in I}\Gamma_i(\xi_i + e_i)$ such that :

a) $\mathrm{proj}_{1_L^\perp}(\sum_{i\in I}\xi_i) \in N_{\Sigma_\rho}(q)$,

b) for all $i \in I$, $g_i - p - p\cdot\xi_i 1_L \in N_{K_i}(\xi_i)$,

with $p = q + \frac{1}{\sharp L}1_L$.

From a), one deduces that there exists $\alpha \in \mathbb{R}$ such that

$$z = \sum_{i\in I}\xi_i \leq \alpha 1_L \text{ and } (z - \alpha 1_L)\cdot(p + \rho 1_L) = 0$$

From b), one deduces that for all $i \in I$,

[2]For the sake of completeness, we give a detailed proof in Proposition 5.1 in Appendix.

$$
\text{(I)}\begin{cases} \text{if } \xi_{ih} = -e_{ih}, \text{ then} & p_h \geq g_{ih} - p \cdot \xi_i \\ \text{if } -e_{ih} < \xi_{ih} < e_h - e_{ih}, \text{ then} & p_h = g_{ih} - p \cdot \xi_i \\ \text{if } \xi_{ih} = e_h - e_{ih}, \text{ then} & p_h \leq g_{ih} - p \cdot \xi_i \end{cases}
$$

We recall that Assumption P implies that, if $\xi_i + e_i \in \partial \mathbb{R}^L_{++}$, then $g_i \cdot (\xi_i + e_i) = 0$ since $g_i \in \Gamma_i(\xi_i + e_i)$.

Claim 1. For all $i \in I$, $p \cdot \xi_i \leq 0$ and $p \cdot \xi_i < 0$ if $\xi_i + e_i \in \partial \mathbb{R}^L_{++}$.

If $\xi_i + e_i \in \mathbb{R}^L_{++}$, condition (I) implies that $p_h \leq g_{ih} - p \cdot \xi_i$ for all h. Since $\sum_{h \in L} p_h = \sum_{h \in L} g_{ih} = 1$, one gets $0 \leq -p \cdot \xi_i$.

If $\xi_i + e_i \in \partial \mathbb{R}^L_{++}$, then $g_i \cdot (\xi_i + e_i) = 0$. Since $g_i \in S$ and $\xi_i + e_i \geq 0$, one has $g_{ih} = 0$ if $\xi_{ih} + e_{ih} > 0$. If $p \cdot \xi_i \geq 0$, from condition (I), one deduces that $p_h \leq 0$ if $\xi_{ih} + e_{ih} > 0$. Hence, $p_h \xi_{ih} \leq p_h(-e_{ih})$. Thus,

$$
0 \leq p \cdot \xi_i = \sum_{h | \xi_{ih} = -e_{ih}} p_h(-e_{ih}) + \sum_{h | \xi_{ih} > -e_{ih}} p_h \xi_{ih} \leq \sum_{h | \xi_{ih} = -e_{ih}} p_h(-e_{ih}) + \sum_{h | \xi_{ih} > -e_{ih}} p_h(-e_{ih}) = p \cdot (-e_i) < 0
$$

Consequently, one gets a contradiction. Note that the last inequality comes from the fact that we have chosen ρ small enough.□

Claim 2. Let $z = \sum_{i \in I} \xi_i$. Then, $z \leq 0$ and $\alpha \leq p \cdot z \leq 0$.

From the previous claim, $p \cdot z \leq 0$. Furthermore, since $z \leq \alpha 1_L$ and $(z - \alpha 1_L) \cdot (p + \rho 1_L) = 0$, one has $p \cdot z - \alpha = -\rho 1_L \cdot (z - \alpha 1_L) \geq 0$. The last inequality comes from the fact that $z \leq \alpha 1_L$ and $\rho 1_L \geq 0$. Consequently, $\alpha \leq p \cdot z \leq 0$ and $z \leq 0$.□

Claim 3. For all $i \in I$, $\xi_i \ll e - e_i$.

If it is not true, there exists $j \in I$ and $h \in L$ such that $\xi_{jh} = e_h - e_{jh}$. Since $z \leq 0$, one has $\sum_{i \neq j} \xi_{ih} + e_h - e_{jh} \leq 0$. Since $\xi_{ih} \geq -e_{ih}$ for all $i \neq j$, $\sum_{i \neq j} \xi_{ih} \geq -\sum_{i \neq j} e_{ih} = -e_h + e_{jh}$. Hence $z_h = \sum_{i \in I} \xi_{ih} = 0$ and $\xi_{ih} = -e_{ih}$ for all $i \neq j$. This implies that $\alpha = 0$ and $p \cdot z = 0$. Consequently, $p \cdot \xi_i = 0$ for all $i \in I$. This contradicts the first claim, since for all $i \neq j$, $\xi_i + e_i \in \partial \mathbb{R}^L_{++}$.□

Claim 4. For all $i \in I$, $\xi_i \in \mathrm{int} K_i$ and $p \in S \subset \mathrm{ir}\Sigma_\rho$.

From condition (I) and the previous claim, for all $i \in I$, for all $h \in L$, $p_h \geq g_{ih} - p \cdot \xi_i$, which implies $p \cdot \xi_i \geq 0$ since $\sum_{h \in L} p_h = \sum_{h \in L} g_{ih} = 1$. From the Claim 1, one deduces that $p \cdot \xi_i = 0$ for all $i \in I$ and $\xi_i + e_i \in \mathbb{R}^L_{++}$. Consequently, For all $i \in I$, $\xi_i \in \mathrm{int} K_i$. From condition (I), $p = g_i$ for all $i \in I$ and $p \in S$ since Γ_i takes its values in S.□

The previous claims prove that $\Phi(q, (\xi_i)) \cap -N_{\Sigma_\rho \times \prod_{i \in I} K_i}(q, (\xi_i)) \neq \emptyset$ implies $(q, (\xi_i)) \in \mathrm{ir}\Sigma_\rho \times \prod_{i \in I} \mathrm{int} K_i$, thus $\Phi(q, (\xi_i)) \cap -N_{\Sigma_\rho \times \prod_{i \in I} K_i}(q, (\xi_i)) = \emptyset$ if $(q, (\xi_i)) \in \partial(\Sigma_\rho \times \prod_{i \in I} K_i)$. □

Theorem 3.1 *For each $(e_i) \in \prod_{i \in I} X_i$, the economy $\mathcal{E}_e$ has an equilibrium. For each compact K of $(\mathbb{R}^L_{++})^I$, $\cup_{e \in K} W(e)$ is a compact subset of $S \times \prod_{i \in I} X_i$. In particular, $W(e)$ is compact for each e. Furthermore, the correspondence W has a closed graph and hence, it is upper semi-continuous.*

Proof. The existence of an equilibrium is a direct consequence of the two above propositions and of the following property of the degree : $\deg(\Phi, \mathrm{ir}\Sigma_\rho \times \prod_{i \in I} \mathrm{int} K_i, 0) \neq 0$ implies that there exists $(q, (\xi_i)) \in \mathrm{ir}\Sigma_\rho \times \prod_{i \in I} \mathrm{int} K_i$ such that $0 \in \Phi(q, (\xi_i))$.

The remaining of the proposition is a consequence of the following result. Let (p^ν, x^ν, e^ν) be a sequence of $S \times \prod_{i \in I} X_i \times (\mathbb{R}^L_{++})^I$ such that (e^ν) converges to $\bar{e} \in (\mathbb{R}^L_{++})^I$ and $(p^\nu, x^\nu) \in W(e^\nu)$ for all ν. Then, the sequence (p^ν, x^ν) has a subsequence which converges to $(\bar{p}, \bar{x}) \in W(e)$. Indeed, since the sequence (e^ν) is converging, it is bounded. Consequently, since $0 \leq x_i^\nu \leq \sum_{i \in I} e_i^\nu$ for all i, the sequence $(x_i^\nu)_{i \in I}$ is also bounded. Hence, the sequence (p^ν, x^ν) has a subsequence, again denoted (p^ν, x^ν), which converges to $(\bar{p}, \bar{x}) \in S \times \prod_{i \in I} \bar{X}_i$. For all i, $\bar{p} \cdot \bar{x}_i = \lim_{\nu \to \infty} p^\nu \cdot x_i^\nu = \lim_{\nu \to \infty} p^\nu \cdot e_i^\nu = \bar{p} \cdot \bar{e}_i > 0$. Since $p^\nu \in G_i(x^\nu)$ for all ν, the last inequality and Assumption P(c) imply that $\bar{x}_i \in X_i$ for all i. Thus, since G_i is upper semi-continuous on $\prod_{j \in I} X_j$, one has $\bar{p} = \lim_{\nu \to \infty} p^\nu \in G_i(\bar{x})$ for all i. Finally, $\sum_{i \in I} \bar{x}_i = \lim_{\nu \to \infty} \sum_{i \in I} x_i^\nu = \lim_{\nu \to \infty} \sum_{i \in I} e_i^\nu = \sum_{i \in I} \bar{e}_i$. This shows that $(\bar{p}, \bar{x}) \in W(e)$.□

4 Equilibrium manifold and regular economies

4.1 Differentiable Preferences

To apply tools of differential geometry, we need to assume further properties on the preference relations like in the case where they are represented by utility functions. Nevertheless, contrary to the usual presentation, the following assumption involves directly the preference relation without any reference to a representation of it.

Assumption DP. For each $i \in I$, G_i is single valued and the mapping $G = (G_j)$ from $\prod_{j\in I} X_j$ to S^I is continuously differentiable and for each $x \in \prod_{j\in I} X_j$, $\mathrm{Im}DG(x) + H = (1_L^\perp)^I$;

Remark. The existence and the uniqueness of $G_i(x)$ may also be formulated in this way. There exists a unique open half space, which contains $P_i(x) + \mathbb{R}^L_+$ and not x_i. The vector $G_i(x)$ can be interpreted as the direction along which the welfare of the ith consumer increases the most rapidly.

Since G_i takes its value in S, $DG(x)$ takes its value in $(1_L^\perp)^I$. Consequently, the range of $DG(x)$ is included in $(1_L^\perp)^I$. Note that our assumption is weaker than assuming that $DG(x)$ is onto.

If there is no external effects, the preference correspondence P_i depends only on x_i. Thus, G_i depends only on x_i. In this case, for all $v \in \prod_{j\in I} G_j(x)^\perp$, $DG(x)(v) = (DG_i(x_i)(v_i))_{i\in I}$. Consequently, the assumption is satisfied if $DG_i(x_i)$ is onto on $1_L^\perp$ for each i. Usually, when the preferences are represented by strictly quasi-concave utility functions, a stronger assumption is made which says that $D^2u_i(x_i)$ is negative definite on $G_i(x_i)^\perp$. Since G_i is the normalized gradient of u_i, one shows that $DG_i(x_i)$ is then onto. Our assumption holds true even if all but one mappings $DG_i(x_i)$ are onto (see the proof of Proposition 4.1). This means that one consumer may have non strictly quasi-concave utility function.

In the appendix, we give an example of a mapping G_i in a two goods economy which satisfies Assumptions P and DP. G_i is not the gradient mapping of a utility function and the demand associated with G_i is a finite set of isolated points. The number of elements in the demand varies from 1 to 3 with respect to the price and the income. Furthermore, there is no differentiable selection of the demand correspondence. This example shows that Assumptions P and DP allow to consider a larger class of preference relations than the one representable by differentiable utility functions.

The next proposition shows that the usual assumptions on the utility function imply that the preference correspondence satisfies Assumption P.

Proposition 4.1 *We assume that the preferences are represented by a utility function u_i from X_i to $\mathbb{R}$ which means that for each $x_i \in X_i$, $P_i(x_i) = \{x_i' \in X_i \mid u_i(x_i) < u_i(x_i')\}$. If u_i is a twice continuously differentiable mapping which satisfies for each $x_i \in X_i$,*

a) $\nabla u_i(x_i) \in \mathbb{R}^L_{++}$;

b) $\{x_i' \in X_i \mid u_i(x_i) \leq u_i(x_i')\}$ *is a closed subset of* $\mathbb{R}^L$;

c) the restriction of $D^2u_i(x_i)$ *to* $\nabla u_i(x_i)^\perp$ *is negative definite;*

then, the preferences satisfy Assumptions P and DP.

The proof of this proposition is given in Appendix. Note that we actually do not need the strong version of the boundary condition stated in (b) to prove that P_i satisfies the boundary condition of Assumption P(c). Actually, it is sufficient to assume that for each sequence (x_i^ν) of X_i which converges to $x_i \in \partial X_i$, then, the sequence $(\frac{1}{1_L \cdot \nabla u_i(x_i^\nu)} \nabla u_i(x_i^\nu) \cdot x_i^\nu)$ converges to 0. For example, a utility function like $u(a, b) = \sqrt{a} + \sqrt{b}$ satisfies the later condition but not (b) of Proposition 4.1.

4.2 The Equilibrium Manifold

As usual, the equilibrium manifold is defined as the graph of the correspondence W. With our notations, $EM = \{(p,x,e) \in S \times \prod_{i\in I} X_i \times (\mathbb{R}^L_{++})^I \mid (p,x) \in W(e)\}$. Theorem 3.1 implies that EM is nonempty. Actually, it means that for each $e \in (\mathbb{R}^L_{++})^I$, there exists at least on element (p,x,e) in EM. We now characterize EM to prove that it is a differentiable submanifold of an Euclidean space. We first define a mapping F from $1_L^\perp \times \prod_{i\in I} X_i \times (\mathbb{R}^L_{++})^I$ to $1_L^\perp \times (\mathbb{R}^L)^I$:

$$F(q,x,e) = (\mathrm{proj}_{1_L^\perp} \sum_{i\in I}(x_i - e_i), (-G_i(x_i) + p + p \cdot (x_i - e_i)1_L)_{i\in I})$$

where $p = q + \frac{1}{\sharp L}1_L$. One remarks that, for all $(q,(x_i)) \in \Sigma_0 \times \{\xi \in \mathbb{R}^L_{++} \mid \xi \ll e\}^I$, $F(q,x,e) = \Phi(q,(x_i - e_i))$ where Φ is defined in Section 3. Consequently, $(p,x,e) \in EM$ if and only if $F(p - \frac{1}{\sharp L}1_L, x, e) = 0$. In other words, up to a translation, the equilibrium manifold is the set of zeros of F. We use this characterization to prove that EM is a differentiable submanifold.

Proposition 4.2 *Under Assumptions P and DP, F is a submersion, which means that the differential of F is onto everywhere. Consequently, EM is a differentiable submanifold of dimension $|L||I|$ of $A \times \prod_{i\in I} X_i \times (\mathbb{R}^L_{++})^I$.*

In the case where the preferences are represented by utility functions, we can use the demand functions to define a global parameterization of the equilibrium manifold. This is not possible in our framework. The proof of this proposition is given in Appendix. Note that it is essential to know that EM is nonempty to conclude, that is, the result of Theorem 3.1.

4.3 Regular Economies

We can now define the notion of regular economies. For this, we consider the natural projection from EM to $(\mathbb{R}^L)^I$ which is the restriction to EM of the canonical projection from $\mathbb{R}^L \times (\mathbb{R}^L)^I \times (\mathbb{R}^L)^I$ to its third component $(\mathbb{R}^L)^I$. The natural projection associates the initial endowments to equilibria.

Definition 4.1 Under Assumptions P and DP, an economy $\mathcal{E}_e$ is regular if e is a regular value of the natural projection. We denote by $\mathcal{E}^r$ the set of regular economies.

Applying Sard's lemma to the natural projection and the implicit function theorem, we deduce the following result, which generalizes the main result of Debreu (1970). Note that the finiteness is due to the compactness of $W(e)$ which is proved in Theorem 3.1.

Theorem 4.1 *Under Assumptions P and DP, the set $\mathcal{E}^r$ of regular economies is open, dense and of full Lebesgue measure in the set of economies. Each regular economy $\mathcal{E}_e \in \mathcal{E}^r$ has an odd number of equilibria. Furthermore, for each $(p^*,(x_i^*)) \in W(e)$, there exists a neighborhood $\mathcal{N}$ of e, a neighborhood $\mathcal{N}'$ of $(p^*,(x_i^*))$ and a differentiable mapping Φ from $\mathcal{N}$ to $\mathcal{N}'$ such that :*

a) $\Phi(e) = (p^*,(x_i^*))$;

b) for each $e' \in \mathcal{N}$, $\Phi(e') = W(e') \cap \mathcal{N}'$.

Thus, the economy with non-ordered preferences has the same properties under Assumptions P and DP than the one where the preferences are represented by differentiable utility functions. Nevertheless, it remains to study more thoroughly the structure of the equilibrium manifold.

Proof. All the arguments are standard but the one, which shows that the number of equilibria is odd. For this, we give an explicit formula of the degree of Φ for regular economies. This is possible since the economy $\mathcal{E}_e$ is regular if and only if 0 is a regular value of Φ.

From the definition of EM and Proposition 4.2, the tangent space to EM at (p,x,e) is $\mathrm{Ker}DF(q,x,e)$, the kernel of $DF(q,x,e)$, with $q = p - \frac{1}{\sharp L}1_L$. Thus, an economy $\mathcal{E}_e$ is regular if and only if for all equilibrium $(p,(x_i))$, $(\xi, y, 0) \in \mathrm{Ker}DF(q,x,e)$ implies $\xi = 0$ and $y = 0$. Since $F(q,x,e) = \Phi(q,(x_i - e_i))$, $(\xi, y, 0) \in \mathrm{Ker}DF(q,x,e)$ if and only if $D\Phi(q,(x_i - e_i))(\xi, y) = 0$. Consequently, an economy $\mathcal{E}_e$ is regular if and only if for all equilibrium $(p,(x_i))$, $D\Phi(q,(x_i - e_i))$ is regular, which means that 0 is a regular value of Φ. In this case, since the degree of Φ is 1, one deduces that 0 has an odd number of preimage. Thus, the economy $\mathcal{E}_e$ has an odd number of equilibria.□

5 Appendix

Proof of Proposition 2.2 $P_i(x) \subset \tilde{P}_i(x)$ is a direct consequence of the definition of the normal cone and of the fact that $P_i(x)$ is open. $\tilde{P}_i(x)$ is open since $G_i(x)$ compact, which implies $G_i^+(x)$ open. x_i belongs to the closure of $\tilde{P}_i(x)$ since 0 belongs to $\overline{G}_i^+(x)$. $\tilde{P}_i(x)$ is convex since $G_i^+(x)$ is so. x_i does not belong to $\tilde{P}_i(x)$ since $0 \notin G_i^+(x)$. $\tilde{P}_i(x) + \mathbb{R}^L_+ = \tilde{P}_i(x)$, since $G_i(x) \subset \mathbb{R}^L_+$ and consequently, $G_i^+(x) + \mathbb{R}^L_+ = G_i^+(x)$.

We now prove that $\tilde{P}_i$ has open lowersection. If it is not true at x, there exists $\xi_i \in \tilde{P}_i(x)$ and a sequence (x^ν), which converges to x and such that $\xi_i \notin \tilde{P}_i(x^\nu)$ for all ν. This means that there exists $g_i^\nu \in G_i(x^\nu)$ such that $g_i^\nu \cdot (\xi_i - x_i^\nu) \leq 0$. Since G_i takes its values in the compact set S, the sequence (g_i^ν) has a converging subsequence and its limit g_i belongs to $G_i(x)$ since G_i is upper semi-continuous. At the limit, one gets $g_i \cdot (\xi_i - x_i) \leq 0$, which contradicts $\xi_i \in \tilde{P}_i(x^\nu)$.

We show that $\tilde{P}_i$ satisfies assertion (c'). If the result is not true, there exists $\xi_i \in \mathbb{R}^L_{++}$ and a sequence (x^ν), which converges to x such that $\xi_i \notin \tilde{P}_i(x^\nu)$ for all ν and $x_i \in \partial\mathbb{R}^L_{++}$. This implies that there exists $g_i^\nu \in G_i(x^\nu)$ such that $g_i^\nu \cdot (\xi_i - x_i^\nu) \leq 0$. From Assumption P(c), the sequence $(g_i^\nu \cdot x_i^\nu)$ converges to 0. This implies that the sequence $(g_i^\nu \cdot \xi_i)$ converges to 0 since it is a nonnegative sequence. But this is impossible since $\xi_i \in \mathbb{R}^L_{++}$ and g_i^ν is in the simplex, hence $g_i^\nu \cdot \xi_i \geq \inf_{h \in L}\{\xi_{ih}\} > 0$.

To prove that the equilibria are the same with the preferences P_i or $\tilde{P}_i$, from Proposition 2.1 , it suffices to show that $-N_{P_i(x)}(x_i) \cap S = G_i(x) = -N_{\tilde{P}_i(x)}(x_i) \cap S$. But $N_{\tilde{P}_i(x)}(x_i) = N_{G_i^+(x)}(0)$. Since $G_i^+(x)$ is a convex cone, $N_{G_i^+(x)}(0)$ is the negative polar cone of $G_i^+(x)$. From the bipolar theorem, this is the closed convex cone generated by $-G_i(x)$. Since $-G_i(x)$ is compact and does not contain 0, one deduces that $N_{G_i^+(x)}(0) = -\mathbb{R}_+ G_i(x)$ and consequently, $G_i(x) = -N_{\tilde{P}_i(x)}(x_i) \cap S$.

We now suppose that the preference correspondence P_i satisfies conditions (b') and (c'). We first prove that it satisfies P(b). Since $x_i \notin P_i(x)$ and $P_i(x) + \mathbb{R}^L_+ = P_i(x)$, one deduces that $G_i(x) = -N_{P_i(x)}(x_i) \cap S$ is nonempty and compact. We now show that G_i is upper semi-continuous. Since it takes their values in S, which is compact, it suffices to show that it has a closed graph. Let (x^ν, g_i^ν) be a sequence of $\prod_{j \in I} X_j \times S$, which converges to $(x, g_i) \in \prod_{j \in I} X_j \times S$, and such that $g_i^\nu \in G_i(x^\nu)$. Let $\xi_i \in P_i(x)$. Since P_i has open lowersection, for ν large enough, $\xi_i \in P_i(x^\nu)$. Consequently, $g_i^\nu \cdot (\xi_i - x_i^\nu) \geq 0$. At the limit, one gets $g_i \cdot (\xi_i - x_i) \geq 0$. Since this inequality is true for all $\xi_i \in P_i(x)$, one deduces that $g_i \in -N_{P_i(x)}(x_i) = -N_{\overline{P}_i(x)}(x_i)$ and, thus, g_i belongs to $G_i(x)$.

We finish the proof of the proposition by showing that the preference correspondence P_i satisfies condition P(c). We consider a sequence (x^ν, g_i^ν) of $\prod_{j \in I} X_j \times S$, such that $g_i^\nu \in G_i(x^\nu)$, (x^ν) converges to $x \in \prod_{j \in I} \bar{X}_j$ and $x_i \in \partial X_i$. From condition (c'), for all $\xi_i \in \mathbb{R}^L_{++}$, $\xi_i \in P_i(x^\nu)$ for ν large enough. Consequently, since $g_i^\nu \in -N_{\overline{P}_i(x^\nu)}(x_i^\nu)$, one has $0 \leq g_i^\nu \cdot x_i^\nu \leq g_i^\nu \cdot \xi_i \leq \max_{h \in L}\{\xi_{ih}\}$. Since this inequality is true for all $\xi_i \in \mathbb{R}^L_{++}$, one gets that the sequence $(g_i^\nu \cdot x_i^\nu)$ converges to 0. □

Proof of Proposition 4.1 Note that there is no external effect since each utility function depends only on the consumption bundle of the agent. We recall that condition (c) implies that u_i is strictly quasi-concave. Thus, $P_i(x_i)$ is convex. Furthermore, for all $x_i \in X_i$ and for $r > 0$ small enough, one has $\overline{P}_i(x_i) \cap B(x_i, r) = \{x_i' \in X_i \mid u_i(x_i) \leq u_i(x_i')\} \cap B(x_i, r)$. Thus, since $\nabla u_i(x_i) \neq 0$, one has $N_{\overline{P}_i(x_i) \cap B(x_i, r)}(x_i) = N_{\overline{P}_i(x_i)}(x_i) = \{t \nabla u_i(x_i) \mid t \in \mathbb{R}_+\}$. Consequently, since $\nabla u_i(x_i) \in \mathbb{R}^L_{++}$, there exists a unique element in $N_{\overline{P}_i(x_i)}(x_i) \cap S$, which is $G_i(x_i) = \frac{1}{1_L \cdot \nabla u_i(x_i)} \nabla u_i(x_i)$. Since u_i is twice continuously differentiable, G_i is continuously differentiable. One also has for all $v \in \mathbb{R}^L$,

$$DG_i(x_i)(v) = \frac{1}{1_L \cdot \nabla u_i(x_i)} \left(D^2 u_i(x_i)(v) - \frac{1}{1_L \cdot \nabla u_i(x_i)} (1_L \cdot D^2 u_i(x_i)(v)) \nabla u_i(x_i) \right).$$

One easily remarks that $DG_i(x_i)$ has its range included in $1_L^\perp$. Its restriction to $G_i(x_i)^\perp$ is onto since its kernel is reduced to 0. Indeed, if $v \in G_i(x_i)^\perp$ is in the kernel of $DG_i(x_i)$, one has $D^2 u_i(x_i)(v) = \frac{1}{1_L \cdot \nabla u_i(x_i)} (1_L \cdot D^2 u_i(x_i)(v)) \nabla u_i(x_i)$. If we do the inner product with v, one deduces that $v \cdot D^2 u_i(x_i)(v) = 0$ since $\nabla u_i(x_i) \cdot v = G_i(x_i) \cdot v = 0$. Since the restriction of $D^2 u_i(x_i)$ to $\nabla u_i(x_i)^\perp$ is negative definite, one deduces that $v = 0$.

We end the proof by showing that Assumption P(c) is satisfied. Let (x_i^ν) be a sequence converging to $x_i \in \partial X_i$. Then for all $t > 0$, one has $u_i(x_i^\nu) \leq u_i(t1_L)$ for ν large enough. Indeed, if it is not true, there exists a subsequence of (x_i^ν) which always satisfies $u_i(x_i^\nu) \geq u_i(t1_L)$. Recalling that $\{x_i' \in X_i \mid u_i(t1_L) \leq u_i(x_i')\}$ is closed in $\mathbb{R}^L$, one deduces that the limit x_i of the subsequence belongs to X_i which is impossible since $x_i \in \partial X_i$. For ν large enough, since $G_i(x_i^\nu) \in N_{\overline{P}_i(x_i^\nu)}(x_i^\nu)$, one has $G_i(x_i^\nu) \cdot x_i^\nu \leq G_i(x_i^\nu) \cdot x_i'$ for all $x_i' \in \overline{P}_i(x_i^\nu)$. In particular, since $t1_L \in \bar{P}_i(x_i^\nu)$, $G_i(x_i^\nu) \cdot x_i^\nu \leq G_i(x_i^\nu) \cdot t1_L = t$. Since this inequality is true for every $t > 0$ if ν is large enough, one deduces that the sequence $(G_i(x_i^\nu) \cdot x_i^\nu)$ converges to 0.□

Proof of Proposition 4.2 Let $(\chi, \xi = (\xi_i)_{i \in I}, \eta = (\eta_i)_{i \in I}) \in 1_L^\perp \times (\mathbb{R}^L)^I \times (\mathbb{R}^L)^I$. For all $(q, x = (x_i)_{i \in I}, e = (e_i)_{i \in I}) \in 1_L^\perp \times \prod_{i \in I} X_i \times (\mathbb{R}^L_{++})^I$, one has :

$$DF(q,x,e)(\chi,\xi,\eta) = (\mathrm{proj}_{1_L^\perp} \sum_{i \in I} (\xi_i - \eta_i), (\chi - \sum_{j \in J} D_{x_j} G_i(x)(\xi_j) + [\chi \cdot (x_i - e_i) + p \cdot (\xi_i - \eta_i)] 1_L)_{i \in I}))$$

with $p = q + \frac{1}{\sharp L} 1_L$. Consequently, for all $(\pi, \zeta = (\zeta_i)_{i \in I}) \in (1_L^\perp)^I \times (\mathbb{R}^L)^I$, one has,

$$\begin{aligned}(\pi,\zeta) \cdot DF(q,x,e)(\chi,\xi,\eta) \quad &= \pi \cdot \mathrm{proj}_{1_L^\perp} \textstyle\sum_{i \in I} (\xi_i - \eta_i) \\ &+ \textstyle\sum_{i \in I} \zeta_i \cdot (\chi - \sum_{j \in J} D_{x_j} G_i(x)(\xi_j) + [\chi \cdot (x_i - e_i) + p \cdot (\xi_i - \eta_i)] 1_L) \\ &= \chi \cdot \textstyle\sum_{i \in I} (\zeta_i + (1_L \cdot \zeta_i)(x_i - e_i)) \\ &+ \textstyle\sum_{i \in I} [\xi_i \cdot (\pi - \sum_{j \in I} {}^t D_{x_i} G_j(x)(\zeta_j) + (1_L \cdot \zeta_i) p) + \eta_i \cdot (-\pi - (1_L \cdot \zeta_i) p)]\end{aligned}$$

Thus, the transpose of $DF(q,x,e)$ is defined by :

$$\begin{aligned}{}^t DF(q,x,e)(\pi,\zeta) = \quad &\big(\textstyle\sum_{i \in I} (\zeta_i + (1_L \cdot \zeta_i)(x_i - e_i)), (\pi - \sum_{j \in I} {}^t D_{x_i} G_j(x)(\zeta_j) + (1_L \cdot \zeta_i) p)_{i \in I}, \\ &(-\pi - (1_L \cdot \zeta_i) p)_{i \in I}\big)\end{aligned}$$

$DF(q,x,e)$ is onto if and only if the kernel of its transpose is reduced to 0. If (π, ζ) belongs to the kernel of ${}^t DF(q,x,e)$, one immediately deduces that $\pi = 0$, $1_L \cdot \zeta_i = 0$ for all $i \in I$, and $\sum_{i \in I} \zeta_i = 0$. Consequently, ${}^t DG(x)(\zeta) = 0$. From Assumption DP, there exists $\pi_0 \in 1_L^\perp$ and $w \in (\mathbb{R}^L)^I$ such that $\zeta_i = \pi_0 + DG_i(x)(w)$ for all $i \in I$. Consequently, $\sum_{i \in I} \zeta_i \cdot \zeta_i = \pi_0 \cdot \sum_{i \in I} \zeta_i + \sum_{i \in I} \zeta_i \cdot DG_i(x)(w) = 0 +^t DG(x)(\zeta) \cdot w = 0$. This implies that $\zeta = 0$ and thus the kernel of ${}^t DF(p,x,e)$ is reduced to 0.

Hence F is a submersion which implies that $EM = F^{-1}(0)$ is a differentiable submanifold. Finally, its dimension is the difference between the dimension of $1_L^\perp \times (\mathbb{R}^L)^I \times (\mathbb{R}^L)^I$ and $1_L^\perp \times (\mathbb{R}^L)^I$.□

Proposition 5.1 *Let C be a bounded, closed subset of a finite dimensional Euclidean space E such that $0 \in intC$. Let F be a u.s.c. correspondence with nonempty convex compact values from C to E such that for all $c \in \partial C$, $F(c) \cap -N_C(c) = \emptyset$. Then,* $\deg(F, intC, 0) = 1$.

Proof. From the definition of the degree for a correspondence, one has $\deg(F, \mathrm{int}C, 0) = \deg(f, C, 0)$ for all continuous mapping f from C to E such that the graph of f is in an ε-neighborhood of the graph of F for ε small enough. Since the graph of F is compact and the graph of the normal cone is closed, for ε small enough, one has $f(c) \notin -N_C(c)$ for all $c \in \partial C$.

For all $c \in \partial C$, let $\Gamma(c) = \{u \in \text{int}T_C(c) \mid f(c) \cdot u < 0\}$ and for all $c \in \text{int}C$, $\Gamma(c) = E$. From our assumption, Γ has nonempty values and one easily checks that it has convex values and open inverse values and $0 \notin \Gamma(c)$ for all $c \in \partial C$. Thus, there exists a continuous selection h of Γ. Let us consider the homotopy mapping H from $[0.1] \times C$ to E defined by $H(t,c) = tf(c) + (1-t)(-h(c))$. For all $(t,c) \in [0,1] \times \partial C$, $0 \neq H(t,c)$. Indeed, if it is not true, there exists $(t,c) \in [0,1] \times \partial C$ such that $0 = H(t,c) = tf(c) + (1-t)(-h(c))$. Since f and h do not vanish on ∂C, one has $t \in]0,1[$. From the definition of Γ, $f(c) \cdot h(c) < 0$. Consequently, one obtains $0 = f(c) \cdot (tf(c) + (1-t)(-h(c)) = t\|f(c)\|^2 - (1-t)(f(c) \cdot h(c))$, which implies $f(c) = 0$ and $f(c) \cdot h(c) = 0$, a contradiction.

Since $-h(c) \in -T_C(c)$ for all $c \in \partial C$, it is easy to check that $-h$ is homotopic to the identity mapping on C by a simple "convex" homotopy. So, since the degree is invariant by homotopy, one has $\deg(F, \text{int}C, 0) = \deg(f, \text{int}C, 0) = \deg(-h, \text{int}C, 0) = \deg(\text{id}, \text{int}C, 0) = 1$. □

A numerical example. To state the formula of the mapping g, which represents the preferences of a consumer, we introduce auxiliary mappings and we show some properties on them.

We consider the mappings a, b and α from $]0,1[\times\mathbb{R}$ to $\mathbb{R}$ defined by $a(t,\tau) = \frac{\tau+1}{t}$, $b(t,\tau) = -(2+\tau)$ and $\alpha(t,\tau) = \frac{\tau+1}{t-1}$. We consider the mapping φ from $\mathbb{R}\times]0,1[\times\mathbb{R}$ to $\mathbb{R}$ defined by :

$$\varphi(x,t,\tau) = \begin{cases} a(t,\tau)x^2 + b(t,\tau)x + 1 & \text{if } x \le t \\ \alpha(t,\tau)(x-1)^2 + b(t,\tau)(x-1) & \text{if } x \ge t \end{cases}$$

By a simple computation, one remarks that φ is continuously differentiable. For fixed t and τ, the graph of the mapping $x \to \varphi(x,t,\tau)$ is the union of two pieces of parabola which are connected at the point $(t, 1-t)$. At this point, the derivative is τ. $\varphi(0,t,\tau) = 1$ and $\varphi(1,t,\tau) = 0$. If $\tau < 0$, then $\frac{\partial\varphi}{\partial x}(x,t,\tau) < 0$ and $\varphi(x,t,\tau) \in]0,1[$ for each $x \in]0,1[$. If $\tau = 0$, then $\frac{\partial\varphi}{\partial x}(x,t,0) = 0$ only for $x = t$ and φ is strictly decreasing on $]0,1[$. If $\tau > 0$, then $\frac{\partial\varphi}{\partial x}(x,t,\tau) = 0$ at $x_1(t,\tau) = \frac{t(2+\tau)}{2(1+\tau)}$ and at $x_2(t,\tau) = 1 + \frac{(t-1)(2+\tau)}{2(1+\tau)}$. Furthermore $\varphi(x_1(t,\tau),t,\tau) = 1 - \frac{t(2+\tau)^2}{4(\tau+1)}$ and $\varphi(x_2(t,\tau),t,\tau) = -\frac{(t-1)(2+\tau)^2}{4(\tau+1)}$. φ is strictly decreasing on $]0, x_1(t,\tau)[\cup]x_2(t,\tau), 1[$ and strictly increasing on $]x_1(t,\tau), x_2(t,\tau)[$.

Finally, we define two mappings $\bar{t}$ and $\bar{\tau}$ from $\mathbb{R}_{++}$ to $\mathbb{R}$ as follows:

$$\bar{t}(w) = \begin{cases} \frac{1}{12-8w} & \text{if } w \le 1 \\ \frac{1}{2}w - \frac{1}{4} & \text{if } 1 \le w \le 2 \\ \frac{14w-25}{16w-28} & \text{if } 2 \le w \end{cases}$$

$$\bar{\tau}(w) = \begin{cases} -\frac{1}{2} & \text{if } w \le 1 \\ \frac{1}{2}\sin(2\pi w - \frac{\pi}{2}) & \text{if } 1 \le w \le 2 \\ -\frac{1}{2} & \text{if } 2 \le w \end{cases}$$

One easily checks that $\bar{t}$ and $\bar{\tau}$ are continuously differentiable. We remark that $\bar{\tau} \le \frac{1}{2}$ and $\bar{\tau} \ge 0$ if $w \in [\frac{5}{4}, \frac{7}{4}]$. On this interval, $\bar{t}(w) = \frac{1}{2}w - \frac{1}{4} \in [\frac{3}{8}, \frac{5}{8}]$.

We are now able to give the explicit formula of the mapping g from $\mathbb{R}^2_{++}$ to the simplex of $\mathbb{R}^2$ by its first component :

$$g_1(x,y) = \varphi(\frac{x}{x+y}, \bar{t}(x+y), \bar{\tau}(x+y))$$

We now prove that the mapping g satisfies Assumption P and DP. g is continuously differentiable since φ, $\bar{t}$ and $\bar{\tau}$ are so. We now show that g takes its value in the simplex which means that $g_1(x,y) \in]0,1[$. From the properties of φ, this is true if $\bar{\tau}(x+y) \le 0$. If $\bar{\tau}(x+y) > 0$, then $w \in [\frac{5}{4}, \frac{7}{4}]$. So, one has to check that $\varphi(x_1(\bar{t}(w), \bar{\tau}(w)), \bar{t}(w), \bar{\tau}(w)) = 1 - \frac{\bar{t}(w)(2+\bar{\tau}(w))^2}{4(\bar{\tau}(w)+1)} > 0$ and $\varphi(x_2(\bar{t}(w), \bar{\tau}(w)), \bar{t}(w), \bar{\tau}(w)) = -\frac{(\bar{t}(w)-1)(2+\bar{\tau}(w))^2}{4(\bar{\tau}(w)+1)} < 1$ for $w \in [\frac{5}{4}, \frac{7}{4}]$. This is equivalent to $\bar{t}(w) < \frac{4(\bar{\tau}(w)+1)}{(2+\bar{\tau}(w))^2}$ and $1 - \bar{t}(w) < \frac{4(\bar{\tau}(w)+1)}{(2+\bar{\tau}(w))^2}$.

This is true because $\bar{t}(w) \leq \frac{5}{8}$, $1 - \bar{t}(w) \leq \frac{5}{8}$ and $\frac{4(\tau+1)}{(2+\tau)^2}$ is a decreasing function of τ and its value at $\tau = \frac{1}{2}$ is $\frac{24}{25} > \frac{5}{8}$.

We now prove that the differential of g is onto on $1_2^{\perp}$. Since $1_2^{\perp}$ is of dimension 1, this is equivalent with $\nabla g_1(x, y) \neq (0, 0)$. One has :

$$\frac{\partial g_1}{\partial x}(x, y) = \frac{\partial \varphi}{\partial x}(\xi, t, \tau)\left(\frac{y}{(x+y)^2}\right) + \frac{\partial \varphi}{\partial t}(\xi, t, \tau)\bar{t}'(x+y) + \frac{\partial \varphi}{\partial \tau}(\xi, t, \tau)\bar{\tau}'(x+y)$$

and

$$\frac{\partial g_1}{\partial y}(x, y) = \frac{\partial \varphi}{\partial x}(\xi, t, \tau)\left(\frac{-x}{(x+y)^2}\right) + \frac{\partial \varphi}{\partial t}(\xi, t, \tau)\bar{t}'(x+y) + \frac{\partial \varphi}{\partial \tau}(\xi, t, \tau)\bar{\tau}'(x+y)$$

with $(\xi, t, \tau) = (\frac{x}{x+y}, \bar{t}(x+y), \bar{\tau}(x+y))$. If $\nabla g_i(x, y) = (0, 0)$, then $\frac{\partial \varphi}{\partial x}(\xi, t, \tau) = 0$. Thus, since $\frac{\partial \varphi}{\partial x}(\xi, t, \tau) = 0$ only if $\xi = x_1(t, \tau)$ or $x_2(t, \tau)$ and $\tau \geq 0$, to prove that $\nabla g_i(x, y)$ never vanishes, it suffices to show that

$$\frac{\partial \varphi}{\partial t}(x_1(\bar{t}(w), \bar{\tau}(w)), \bar{t}(w), \bar{\tau}(w))\bar{t}'(w) + \frac{\partial \varphi}{\partial \tau}(x_1(\bar{t}(w), \bar{\tau}(w)), \bar{t}(w), \bar{\tau}(w))\bar{\tau}'(w) \neq 0$$

and

$$\frac{\partial \varphi}{\partial t}(x_2(\bar{t}(w), \bar{\tau}(w)), \bar{t}(w), \bar{\tau}(w))\bar{t}'(w) + \frac{\partial \varphi}{\partial \tau}(x_2(\bar{t}(w), \bar{\tau}(w)), \bar{t}(w), \bar{\tau}(w))\bar{\tau}'(w) \neq 0$$

for all $w \in [\frac{5}{4}, \frac{7}{4}]$.

Since $\frac{\partial \varphi}{\partial t}(x, t, \tau) = -\frac{\tau+1}{t^2}x^2$ and $\frac{\partial \varphi}{\partial \tau}(x, t, \tau) = \frac{1}{t}x^2 - x$ if $x < t$ and $\frac{\partial \varphi}{\partial t}(x, t, \tau) = -\frac{\tau+1}{(t-1)^2}(x-1)^2$ and $\frac{\partial \varphi}{\partial \tau}(x, t, \tau) = \frac{1}{t-1}(x-1)^2 - (x-1)$ if $x > t$, one has to prove that

$$-\bar{\tau}'(w)\bar{\tau}(w)\bar{t}(w) \neq \bar{t}'(w)(2 + \bar{\tau}(w))(1 + \bar{\tau}(w))$$

and

$$-\bar{\tau}'(w)\bar{\tau}(w)(\bar{t}(w) - 1) \neq \bar{t}'(w)(2 + \bar{\tau}(w))(1 + \bar{\tau}(w))$$

Since $\bar{\tau}'(w)\bar{\tau}(w) = \frac{1}{2}\sin(2\pi w - \frac{\pi}{2})\pi\cos(2\pi w - \frac{\pi}{2}) = \frac{\pi}{2}\sin(4\pi w - \pi)$ and $\bar{t}(w) \in [\frac{3}{8}, \frac{5}{8}]$, the left side is less than $\frac{5\pi}{16} < 1$. Since $\bar{t}'(w) = \frac{1}{2}$ and $\bar{\tau}(w) \geq 0$, the right side is greater than 1. Thus they are never equal.

We now show that g satisfies the boundary condition. Let (x^ν, y^ν) a sequence of R^2_{++} which converges to $(x, y) \in \partial R^2_{++}$. Since $g(x^\nu, y^\nu)$ remains in the simplex, if $(x, y) = (0, 0)$, then $\lim_\nu g(x^\nu, y^\nu) \cdot (x^\nu, y^\nu) = 0$. If $(x, y) \neq (0, 0)$, then $(\frac{x^\nu}{x^\nu + y^\nu})$ converges to $\frac{x}{x+y}$, which is equal either to 0 or 1, and $(g_1(x^\nu, y^\nu))$ converges to $\varphi(\frac{x}{x+y}, \bar{t}(x+y), \bar{\tau}(x+y))$ which is equal to 1 if $\frac{x}{x+y} = 0$ or 0 if $\frac{x}{x+y} = 1$. Consequently, $\lim_\nu g(x^\nu, y^\nu) \cdot (x^\nu, y^\nu) = 0$.

We now have a look to the demand when the prices are $(\frac{1}{2}, \frac{1}{2})$ and the income is $w > 0$, that is the element (x, y) which satisfies $\frac{1}{2}x + \frac{1}{2}y = w$ and $g(x, y) = (\frac{1}{2}, \frac{1}{2})$. If $w \leq \frac{5}{4}$ or $w \geq \frac{7}{4}$, then the demand is a singleton because $g_1(x, 2w - x)$ is strictly decreasing with respect to x on $]0, 2w[$ and thus, it takes the value $\frac{1}{2}$ only one time. If we consider the income $w = \frac{3}{2}$, then the demand contains three isolated points. One easily checks that the point $(\frac{3}{4}, \frac{3}{4})$ belongs to the demand since $\bar{t}(\frac{3}{2}) = \frac{1}{2}$. But, since $\bar{\tau}(\frac{3}{2}) > 0$, one deduces from the properties of φ that there exist exactly two elements $x_1 < \frac{3}{2}$ and $x_2 > \frac{3}{2}$ such that $g_1(x_i, 3 - x_i) = \varphi(\frac{2x_i}{3}, \frac{1}{2}, \bar{\tau}(\frac{3}{2})) = \frac{1}{2}$ for $i = 1, 2$. Consequently, the points $(x_1, 3 - x_1)$ and $(x_2, 3 - x_2)$ are also in the demand.

This shows that when the preferences satisfy Assumption P, the demand may be multivalued with a finite number of isolated points.

References

[1] Balasko, Y., (1988), *Foundations of the theory of general equilibrium*, Academic Press.

[2] Bonnisseau, J.M., Florig, M. and A. jofré, (2001a), Continuity and uniqueness of equilibria for linear exchange economies, *Journal of Optimization Theory and Applications*, 109, 237-263.

[3] Bonnisseau, J.M., Florig, M. and A. jofré, (2001b), Differentiability of equilibria for linear exchange economies, *Journal of Optimization Theory and Applications*, 109, 263-288.

[4] Debreu, G., (1970), Economies with finite set of equilibria, *Econometrica*, 38, 387-392.

[5] Debreu, G., (1976), Regular differentiable economies, *American Economic Review*, 66, 280-287.

[6] Dierker, E., (1974), *Topological methods in Walrasian economics*, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag.

[7] Dierker, E., (1982), Regular economies, Chap. 17 in *Handbook of Mathematical Economics*, vol. II, ed. K. Arrow and M. Intriligator, p. 795-830, North-Holland.

[8] Florenzano, M., (1981), *L'équilibre économique général transitif et intransitif : problèmes d'existence*, Editions du CNRS, Paris.

[9] Jouini, E., (1992), "Structure de l'ensemble des équilibres d'une économie non convexe", *Annales de l'Institut Henri Poincaré, Analyse non-linéaire* , 9, 321-336.

[10] Jouini, E., (1993), "The graph of the Walras correspondence", *Journal of Mathematical Economics* , 22, 139-147.

[11] Mantel, R., (1974), On the characterization of aggregate excess demand, *Journal of Economic Theory*, 7, 348-353.

[12] Mas-Colell, A., (1985), *The theory of general economic equilibrium. A differentiable approach*, Cambridge University Press.

[13] Rader, T., (1973), Nice demand functions, *Econometrica*, 41, 913-935.

[14] Rockafellar, T.T. (1970), *Convex Analysis*, Princeton University Press.

[15] Sard, A., (1958), Images of critical sets, *Annals of Mathematics*, 68, 247-259.

[16] Shannon, C., (1994), Regular nonsmooth equations, *Journal of Mathematical Economics*, 23, 147-166.

[17] Smale, S., (1974a), Global Analysis and Economics IIA : Extension of a Theorem of Debreu, *Journal of Mathematical Economics*, 1, 1-14.

[18] Smale, S., (1974b), Global analysis and economics IV : Finiteness and stability of equilibria, *Journal of Mathematical Economics*, 1, 119-127.

[19] Smale, S., (1981), Global analysis and economics, Chap. 8 in *Handbook of Mathematical Economics*, vol. I, ed. K. Arrow and M. Intriligator, p. 331-370, North-Holland.

[20] Sonnenschein, H., (1972), Market excess demand functions, *Econometrica*, 40, 549-563.

[21] Sonnenschein, H., (1973), Do Walras'identity and continuity characterized the class of community excess demand functions?, *Journal of Economic Theory*, 6, 345-354.