Arbitrage with Incomplete Markets and Asymmetric Information

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January 2002

Abstract

This paper deals with the issue of arbitrage with incomplete financial markets and differential information with a focus on information that no-arbitrage asset prices can reveal. Time and uncertainty are represented by two periods and a finite set S of states of nature, one of which will prevail at the second period. Agents may operate limited financial transfers across periods and states via finitely many nominal assets. Each consumer has a private (or idiosyncratic) information about which state will prevail at the second period; this information is represented by a subset S_i of the state space S, associated to each agent i, which defines his set of subjectively realizable states for the second period.

Our analysis is two-fold, namely, we first extend the classical symmetric information analysis to the asymmetric setting, via a concept of no-arbitrage price, and, secondly we study how such no-arbitrage prices convey information to agents. The main difference between the symmetric and the asymmetric settings stems from the fact that a classical no-arbitrage asset price (common to every agent) always exists in the first case, but not in the asymmetric one, thus allowing arbitrage opportunities. This is the main reason why agents will refine their information up to an information structure which precludes arbitrage.

Key words. Asymmetric information, idiosyncratic information, no-arbitrage, incomplete markets, refinement, information revealed by prices.

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1 Introduction

Differences in agents' awareness about tomorrow's possible events, i.e., asymmetric information, may stem from subjective beliefs about the future, a private knowledge of agents regarding their own risk (adverse selection) or decision making processes limiting the set of alternatives (moral hazard). In economies subject to uncertainty and asymmetric information, agents seek to infer relevant information from market indicators, such as prices, to refine their strategies. This paper deals with the issue of arbitrage with incomplete financial markets and differential information with a focus on information that no-arbitrage asset prices can reveal.

Our approach differs from the so called "rational expectations" treatment of asymmetric information, in the sense that we do not assume that agents know the ex ante characteristics of the economy (preferences, endowments of other agents) or a defined relationship between prices and the collection of private information signals in the economy. While full revelation is pervasive in standard models of "rational expectations" [Radner, (1979), Allen (1981), Jordan (1982)], in our approach, agents may keep stable distinct beliefs, i.e., unaffected by changes in the characteristics of the other agents.

In this paper, time and uncertainty are represented by two periods ($t = 0$ and $t = 1$) and a finite set S of states of nature, one of which will prevail at the second period. Agents may operate limited financial transfers across periods and states via finitely many nominal assets. Each consumer has a private (or idiosyncratic) information about which state will prevail at the second period. Asymmetric information is hence represented by a subset S_i of the state space S, associated to each agent i, which defines his set of subjectively realizable states for the second period. Agents receive no wrong information in the sense that the "true state" belongs to $\cap_i S_i$, hence assumed to be nonempty. Similarly, when agents refine their information, i.e., when they infer a smaller set $\Sigma_i \subset S_i$, they also receive no wrong signal, so that $\cap_i \Sigma_i \neq \emptyset$. Such a family $(\Sigma_i)_i$ will be called a refined information structure.

Our analysis is two-fold, namely, we first extend the classical symmetric information non-arbitrage analysis to the asymmetric setting, via a concept of no-arbitrage price, and, secondly, we study how such no-arbitrage prices convey information to agents. The main difference between the symmetric and asymmetric settings stems from the fact that a classical no-arbitrage asset price (common to every agent) always exists in the symmetric case, but not in the asymmetric one, thus allowing arbitrage opportunities. This is the main reason why agents will refine their information up to an information structure precluding arbitrage.

In Section 3, we characterize the absence of future (i.e., at $t = 1$) arbitrage opportunities on the financial market, called the AFAO property, by the existence of a (classical) no-arbitrage price common to every agent [Proposition 3.2]. The failure of the AFAO property in an asymmetric setting has two consequences. First, it leads to define, for every information structure $(S_i)_i$, an extended notion of no-arbitrage asset price [Definition 3.3], which is characterized in the next section. Secondly, we associate to every information structure $(S_i)_i$, the least refined information structure which meets condition AFAO [Proposition 3.5].

In Section 4, we first define, for every agent i and every asset price q, the "revealed information set" $S_i(q) \subset S_i$ [Proposition 4.1]. This allows us to characterize no-arbitrage prices q as those which "reveal" an information structure, i.e., such that $\cap_i S_i(q) \neq \emptyset$ [Proposition 4.2]. We then show that the revealed information structure $(S_i(q))_i$ can be equivalently defined via the overall elimination of the so called "arbitrage states" [Proposition 4.6] and then obtained, in a constructive way, via a sequential elimination of these "arbitrage states" [Proposition 4.8].

Section 5 presents an example which illustrates the previous results in a synthetic way and concludes that rational agents observing a no-arbitrage price may always update their beliefs up to the largest information structure which is arbitrage-free at that price. We also draw some consequences of the previous analysis in terms of consumers' behavior in an asymmetric information setting, a subject which will be developed in a companion paper devoted to financial equilibrium with incomplete markets and idiosyncratic information. [1]

2 The model

2.1 The two-period economy and financial markets

We consider the basic model of a two time-period economy with idiosyncratic (private) information. The economy is finite, in the sense that there are finite sets I, S, and J, respectively, of consumers, states of nature, and assets.

In what follows, the first period will also be referred to as $t = 0$ and the second period, as $t = 1$. There is an a priori uncertainty at the first period ($t = 0$) about which of the states of nature $s \in S$ will prevail at the second period ($t = 1$). The non-random state at the first period is denoted by $s = 0$ and S' stands for the set $\{0\} \cup S$. Each consumer has an idiosyncratic information at the first period about the possible states of nature of the second period, that is, he/she knows that the true state will be in a subset S_i of S, or, equivalently, that the true state will not belong to the complementary set (in S) of S_i.

Agents may operate financial transfers across states in S' (i.e., across the two periods and across the states of the second period) by exchanging a finite number of nominal assets $j \in J$, which define the financial structure of the model. The nominal assets are traded at the first period ($t = 0$) and yield payoffs at the second period ($t = 1$), contingent on the realization of the state of nature. The payoff of asset $j \in J$, when state $s \in S$ is realized, is V_s^j, and we denote by V the $S \times J$-return matrix $V = (V_s^j)$, which does not depend upon the commodity prices p and the asset prices $q \in \mathbb{R}^J$. We summarize by $[(I, S, J), V, (S_i)_{i \in I}]$ the financial and information characteristics, which are fixed throughout the paper and referred to as the (financial and information) structure.

2.2 Idiosyncratic information structures

At the first period, each agent $i \in I$ has some private (idiosyncratic) information $S_i \subset S$ about which states of the world may occur at the next period : either he keeps this information, or

[1] We shall use hereafter the following notations. If I and J are finite sets, the space $\mathbb{R}^I$ (identified to $\mathbb{R}^{\#I}$ whenever necessary) of functions $x : I \to \mathbb{R}$ (also denoted $x = (x(i))_{i \in I}$ or $x = (x_i)$) is endowed with the Euclidean product $x \cdot y := \sum_{i \in I} x(i)y(i)$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm. In $\mathbb{R}^I$, the notation $x \geq y$ means that $x(i) \geq y(i)$ for every i and we let $\mathbb{R}^I_+ = \{x \in \mathbb{R}^L \mid x \geq 0\}$. An $I \times J$-matrix $A = (a_i^j)_{i \in I, j \in J}$ (identified with a classical $(\#I) \times (\#J)$-matrix if necessary) is an element of $\mathbb{R}^{I \times J}$ whose rows are denoted $A[i]$ or A_i for $(a_i^j)_{j \in J} \in \mathbb{R}^J$ $(i \in I)$, and columns $A^j = (a_i^j)_{i \in I} \in \mathbb{R}^I$ $(j \in J)$. To the matrix A, we associate the linear mapping, from $\mathbb{R}^J$ to $\mathbb{R}^I$, also denoted by A, defined by $Ax = (A_i \cdot x)_{i \in I}$. The span of the matrix A, also called the image of A, is the set $< A >:= \{Ax \mid x \in \mathbb{R}^J\}$. The transpose matrix of A, denoted by ${}^tA$, is the $J \times I$-matrix whose rows are the columns of A, or equivalently, is the unique linear mapping ${}^tA : \mathbb{R}^I \to \mathbb{R}^J$, satisfying $(Ax) \cdot y = x \cdot ({}^tAy)$ for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^I$.

he is able to infer that the true state will be in a smaller set $\Sigma_i \subset S_i$. In both cases agents are assumed to receive no wrong information signal, that is, the true state always belongs to the set $\cap_{i\in I} S_i$, or $\cap_{i\in I}\Sigma_i$, hence assumed to be non-empty.

The following definition introduces the notion of information structures and the notion of refinement of information as an order relation on the set of information structures. Heuristically, the information of the agents will be finer if their information sets are smaller.

Definition 2.1 *A collection $\{\Sigma_i | \ i \in I\}$, also denoted $(\Sigma_i)_i$, of subsets of S whose intersection is non-empty is called an (idiosyncratic) information structure.*

Given two information structures $(\Sigma_i^1)_i$, and $(\Sigma_i^2)_i$, we say that $(\Sigma_i^1)_i$ is finer than (refines, is a refinement of) $(\Sigma_i^2)_i$ if $\Sigma_i^1 \subset \Sigma_i^2$ for every i, and we denote it by $(\Sigma_i^1)_i \geq (\Sigma_i^2)_i$.

The nonempty subset $\cap_{j\in I}\Sigma_j$ is called the collective (revealable) information held by the information structure $(\Sigma_i)_i$. It can be realized when the agents decide to share their information, hence leading to the pooled information structure $(\underline{\Sigma}_i)_i$ defined by $\underline{\Sigma}_i := \cap_{j\in I}\Sigma_j$ for every i.

The refinement $(\Sigma_i)_i$ is said to preserve the collective information of $(S_i)_i$ if $\cap_{j\in I} S^j \subset \Sigma_i$ for every i, [or equivalently if $\cap_{j\in I} S^j = \cap_{j\in I}\Sigma_j$, or if $(\underline{S}_i)_i \geq (\Sigma_i)_i$, or if $(\underline{S}_i)_i \geq (\underline{\Sigma}_i)_i$].

Refinement of information clearly preserves the collective (revealable) information, when it is performed without the help of an information source outside the given set I of agents (auctioneer, ...). Hereafter, we do not rule out, however, cases where agents will be able to update their beliefs beyond the collective information initially detained by consumers.

Idiosyncratic information conveys the idea that agents have a different awareness about the possible events which will take place tomorrow. A first interpretation is that each agent may not know the total set of states S but only his idiosyncratic set S_i; hence, he does not know the matrix V but only the rows $V[s]$ of V for $s \in S_i$. A second interpretation is that each agent knows the total set of states S but the agents have different subjective probabilities on S. In this case, the set S_i is the set of states in S for which agent i has a positive subjective probability.

Idiosyncratic information is often encountered in contract or insurance models, where agents have a private knowledge regarding their own risk. This problem is formulated in the following example.

Example (An adverse selection economy). Consider an economy where the random state of nature $s = (s_0, (s_i)_{i\in I}))$ is the product of a macro-economic component $s_0 \in \Sigma_0$, whose probability distribution is known and common to all agents, and of idiosyncratic components $s_i \in \Sigma_i$, representing the individual risks of agent i ($i \in I$), and on which agent i has some private information. It is assumed that, for every i, the risk component s_i is realized at the first period and that its realization, denoted by $\overline{s}_i$, is privately known by agent i and by no other agent (see, for example, Bisin and Gottardi (1999)). In that case, the total information set is

$$S := \Sigma_0 \times \Pi_{j\in I}\Sigma_j,$$

and agent i has for idiosyncratic information set, namely:

$$S_i := \{s = (s_0, (s_j)_{j\in I})) \in \Sigma_0 \times \Pi_{j\in I}\Sigma_j \mid s_i = \overline{s}_i\}.$$

Consider, for instance, a two-period economy with two agents ($I = \{1, 2\}$) and no macroeconomic risk, where endowments are dependent on idiosyncratic-risk as follows. The two consumers have two cars in common which will be sold at the second period. They will then share equally the product of the sale and this will define the endowments e_i of consumers $i = 1, 2$ at $t = 1$.

Each agent i knows the state $\bar{s}_i \in \{1, 2\}$ of one only of the two cars (the one he drove), say "bad" ($\overline{s}_i = 0$) or "good" ($\overline{s}_i = 1$), which can, for example, reflect the fact that he had an accident in the past. This information is detained privately by agent i at $t = 0$ until the next period. The total set of states is thus $S = \{0, 1\}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and the knowledge, by agent i, of the state $\bar{s}_i \in \{1, 2\}$ induces idiosyncratic sets S_i, as defined above. For every i, the sale price p_i at the second period of the car driven by agent i, will depend on the state "bad" or "good". To make things clear, we let $p_1 = (8, 20)$ and $p_2 = (6, 8)$, then agents' endowments are $e_1 = e_2 = (7, 8, 13, 14)$. Finally, let us assume that the consumers can insure themselves against the risk on their future endowments with the following nominal asset with returns :

$$V = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

If agents had no information at period $t = 0$ about their idiosyncratic risk $\bar{s}_i$, (or equivalently if $S_1 = S_2 = S$), then they would have no speculative motive for exchanging assets. Since agents know their risk realizations $\bar{s_i}$ at $t = 0$, one of them may have an informational advantage.

In the following, we consider the case $\bar{s_1} = 1$, and $\bar{s_2} = 0$. Denoting for the sake of simpler notations $S = \{1, 2, 3, 4\}$, where state 1 stands for $(0, 0)$, 2 for $(0, 1)$, 3 for $(1, 0)$, and 4 for $(1, 1)$. Then, clearly, $S_1 = \{3, 4\}$ and $S_2 = \{1, 3\}$. This situation is a prototype of what will be studied in this paper. It will be shown that agents will refine their information to the information structure $\bar{S}_1 = \{3, 4\}$ and $\bar{S}_2 = \{3\}$ and that this refinement can be achieved as follows. There exists a unique no-arbitrage price (in a precise sense given hereafter) $q = 0$ which "reveals" the information structure $\bar{S}_1$ and $\bar{S}_2$, without any agent knowing his partner's characteristics (preferences and endowments). This revelation of information can be achieved in a constructive way by successive elimination of "arbitrage states" as defined in Section 4.

2.3 Consumers' behavior with asymmetric information

Throughout the paper we shall use the following basic equilibrium model as a guideline and an illustration of the results we shall present in arbitrage theory. The main problem, with idiosyncratic information, concerns the modelling of consumers' behavior.

We consider the model of a two time-period finite pure exchange economy with idiosyncratic (private) information, where time and uncertainty are described as previously. In addition, we now assume that there is a finite set, H, of commodities which are available at each period ($t \in \{0, 1\}$) and each state $s \in S$ (thus at each state $S' := \{0\} \cup S$). Then, the commodity space is $(\mathbb{R}^H)^{S'}$ and a consumption is a vector $x = (x(s)) \in (\mathbb{R}^H)^{S'}$. Similarly, a price vector $p \in (\mathbb{R}^H)^{S'}$ will also be denoted by $p = (p(s))$.

Each consumer $i \in I$ (a finite set) is characterized by his/her consumption set $X_i = (\mathbb{R}^H_+)^{S'}$, his/her endowment vector $e_i \in (\mathbb{R}^H)^{S'}$, his/her initial idiosyncratic information set $S_i \subset S$, and a conditional utility function u_i (conditional on the information set $\Sigma \subset S_i$ that the consumer will infer). Denoting $\Sigma' := \{0\} \cup \Sigma$ for every $\Sigma \subset S_i$, then the conditional utility, denoted by $u_i(\cdot|\Sigma)$, is a function from $(\mathbb{R}^H_+)^{\Sigma'}$ to $\mathbb{R}$. It thus defines the preferences of agent i when he/she knows that the random state of the second period will be in the set $\Sigma \subset S_i$. To illustrate this model we present the following important example :

Example (Von Neumann-Morgenstern utility). In the case of a V.N.M. utility function, a fixed utility index $v_i : (\mathbb{R}^H_+)^2 \to \mathbb{R}$ is given and we denote by $p_i(s|\Sigma)$ the subjective probability that agent i assigns to the realization of state $s \in S$, conditionally on the event $s \in \Sigma$. Then, the conditional utility is defined as follows :

$u_i(x|\Sigma) = \sum_{s\in\Sigma} \ p_i(s|\Sigma)v_i(x(0), x(s))$ for every $\Sigma \subset S$ and $x \in (\mathbb{R}^H_+)^{\Sigma'}$. ■

The economy that we have described can thus be summarized by the collection :

$$\mathcal{E} = [(I, H, S, J), V, (S_i, X_i, u_i, e_i)_{i\in I}].$$

Given his initial information set $S_i \subset S$, consumer i will (possibly) need to "infer" a better information set $\Sigma_i \ \subset S_i$, before maximizing his utility under his budget constraint, as explained below. For given commodity prices $p = (p(s)) \in (\mathbb{R}^H)^{S'}$ and asset prices $q \in \mathbb{R}^J$, agent i will then maximize his utility (for the known information set Σ_i) $u_i(\cdot|\Sigma_i)$ on his budget set $B_i(p, q, V, \Sigma_i)$, defined as follows:

$$B_i(p, q, V, \Sigma_i) := \left\{ (x, z) \in (\mathbb{R}^H)^{\Sigma'_i} \times \mathbb{R}^J \ \left| \ \begin{array}{c} p(0) \cdot [x(0) - e_i(0)] \leq -q \cdot z \\ \forall s \in \Sigma_i, \ p(s) \cdot [x(s) - e_i(s)] \leq V[s] \cdot z \end{array} \right. \right\}.$$

The next sections will detail this behavior and seek to answer the following relevant questions. (i) When are agents keeping their initial information sets S_i $(i \in I)$, instead of refining their information? Alternatively, why would agents be obliged to update their beliefs? (ii) What kind of communication process between agents is needed to infer new information sets $(\Sigma_i)_i$? Pooling information? Using some kind of auction? The knowledge of the other agents' characteristics? The only knowledge of prices? (iii) For each agent, how does refinement proceed, which leads from the initial set S_i to some better information set Σ_i? (iv) How can we guarantee that the refinement of information will not lead to the inadmissible situation where $\cap_{i\in I}\Sigma_i = \emptyset$? Will the refined information structure $(\Sigma_i)_i$ preserve the collective information of $(S_i)_i$? In other words shall we always have $\cap_{i\in I} \ S_i = \cap_{i\in I}\Sigma_i$ or not? (v) How and who is fixing the asset prices q? Section 5 will provide answers to these questions.

3 No-arbitrage prices with asymmetric information

3.1 The classical concept of no-arbitrage price

We recall the following standard definition. Given $q \in \mathbb{R}^J$, we denote by $W(q, V, S_i)$ the $S'_i \times J$-matrix, defined by $W(q, V, S_i)[0] = -q$, and $W(q, V, S_i)[s] = V[s]$ for every $s \in S_i$.

Definition 3.1 *The price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price for agent i $(i \in I)$ (or the couple (V, S_i) to be q-arbitrage free) if one of the following equivalent assertions is satisfied :*

(i) there is no portfolio $z_i \in \mathbb{R}^J$ such that $-q \cdot z_i \geq 0$ and $V[s] \cdot z_i \geq 0$ for every $s \in S_i$, with at least one strict inequality;

(ii) $< W(q, V, S_i) > \cap \ \mathbb{R}^{S'_i}_+ = \{0\}$;

(iii) There exists $\lambda_i = (\lambda_i[s]) \in \mathbb{R}^{S_i}_{++}, \ q = \sum_{s\in S_i} \lambda_i[s]V[s]$.

We denote by $Q_c[V, (S_i)_i]$ the set of q which are no-arbitrage prices for every agent $i \in I$, called common no-arbitrage prices.

The equivalence between the three assertions above is standard and relies on the following lemma (see, for example, Magill-Quinzii (1996) for the proof) which will also be used hereafter.

Lemma 1 *Let W be a $S' \times J$ matrix, then the following conditions are equivalent:*
(i) $< W > \cap \, \mathbb{R}^{S'}_{+} = \{0\}$*;*
(ii) $\exists \; \lambda \in \mathbb{R}^{S'}_{++}, \; {}^{t}W \; \lambda = 0$*;*
(ii') $\exists \; \lambda = (\lambda[s]) \in \mathbb{R}^{S'}_{++}, \; \sum_{s \in S'} \; \lambda[s] \; W[s] = 0$.

Remark 1 (Symmetric and asymmetric settings). The main difference between the symmetric and the asymmetric settings is that, in the first case, the set $Q_c[V, (S_i)_i]$ is always nonempty, whereas it is no longer true, in general, in the asymmetric one, as shown in the Example of Section 5. When the information structure $(S_i)_i$ is symmetric, i.e., when all the S_i are equal (say to S), for every $\lambda = (\lambda[s]) \in \mathbb{R}^{S}_{++}$, then $q := \sum_{s \in S} \lambda[s]V[s]$ belongs to $Q_c[V, (S_i)_i]$, hence the discount factors $\lambda[s]$ $(s \in S)$ need not depend on i. ■

Remark 2. The condition that q is a common no-arbitrage price is stronger than the following condition :

$$< W(q, V, \cup_{i \in I} S_i) > \cap \, \mathbb{R}^{\cup_i S'_i}_{+} = \{0\},$$

but not equivalent it (see the example in Section 5). ■

We end the section with a standard property of no-arbitrage prices in terms of consumers' behavior. We first introduce the following assumption (which is used below for $\Sigma_i = S_i$, but will be needed later in the general setting).

Assumption (NSS) *(Non-Satiation of Preferences at every State)* $\forall \; i \in I, \forall \Sigma_i \subset S_i$, $\forall \; s_i \in \Sigma'_i, \forall \; x \in (\mathbb{R}^{H}_{+})^{\Sigma'_i}, \; \exists \; x' \in (\mathbb{R}^{H}_{+})^{\Sigma'_i}, \forall \; s \in \Sigma'_i \backslash \{s_i\}, \; x'(s) = x(s), \; u_i(x' \mid \Sigma_i) > u_i(x \mid \Sigma_i)$.

Proposition 3.1 *Under Assumption* **(NSS)***, if, for every agent $i \in I$, the strategy (x^*_i, z^*_i) maximizes the utility $u_i(\cdot \mid S_i)$ on the budget set $B_i(p^*, q^*, V, S_i)$, then $q^* \in Q_c[V, (S_i)_i]$.*

Proof. By contraposition. If $q^* \notin Q_c[V, (S_i)_i]$, there exists $i \in I$, and $z \in \mathbb{R}^J$ such that $w(z)[0] := -q^* \cdot z \geq 0$ and $w(z)[s] := V[s] \cdot z \geq 0$, for every $s \in S_i$, with at least one strict inequality, say for $s_i \in S_i \cup \{0\}$. From Assumption **(NSS)**, there exists $x'_i \in (\mathbb{R}^{H}_{+})^{S'_i}$ such that $x'_i(s) = x^*_i(s)$ for every $s \in S'_i \backslash \{s_i\}$ and $u_i(x'_i \mid S_i) > u_i(x^*_i \mid S_i)$. Let $\lambda = |p^*(s_i) \cdot [x'_i(s_i) - x^*_i(s_i)]| \; / \; w(z)[s_i]$ and $z'_i = z^*_i + \lambda z$. We let the reader check that $(x'_i, z'_i) \in B_i(p^*, q^*, V, S_i)$. But the conditions $(x'_i, z'_i) \in B_i(p^*, q^*, V, S_i)$ and $u_i(x'_i \mid S_i) > u_i(x^*_i \mid S_i)$ contradict the fact that (x^*_i, z^*_i) maximizes the utility of agent i on his budget set $B_i(p^*, q^*, V, S_i)$. ■

3.2 The absence of future arbitrage opportunities

We shall now characterize the existence of a common no-arbitrage price by the following property.

Definition 3.2 *The financial and information structure $[V, (S_i)_i]$ is said to be future arbitrage-free, or simply arbitrage-free if*
(AFAO) *(Absence of Future Arbitrage Opportunity) there is no $(z_i)_i \in (\mathbb{R}^J)^I$ such that $\sum_{i \in I} z_i = 0$ and $V[s_i] \cdot z_i \geq 0$ for every $i \in I$ and every $s_i \in S_i$, with at least one strict inequality.*

It is important to notice that the above Condition **(AFAO)** always holds for symmetric information structures as shown below.

Remark 3 (Symmetric case). Condition **(AFAO)** always holds if the information structure $(S_i)_i$ is symmetric, i.e., when $S_i = S_j$, for all $i, j \in I$. Otherwise, there exists $(z_i)_{i \in I} \in (\mathbb{R}^J)^I$, such that $\sum_{i \in I} z_i = 0$ and $V[s_i] \cdot z_i \geq 0$, for every $i \in I$ and every $s_i \in S_i$, together with $V[s] \cdot z_{i_0} > 0$, for some $i_0 \in I$, and some $s \in S_{i_0} (= S_i$ for every $i)$. Consequently, $V[s] \cdot (\sum_{i \in I} z_i) = \sum_{i \in I} V[s] \cdot z_i > 0$ holds and contradicts $\sum_{i \in I} z_i = 0$. ■

Remark 4 (Complete markets). In complete markets, only symmetric information structures are arbitrage-free. Indeed, let V be a complete financial structure, that is, such that rank $V = \#S$, and let $(S_i)_i$ be an arbitrage-free information structure. Assume it is not symmetric, then there exists i, j in I and $s_i \in S_i \backslash S_j$. Consider the Arrow-security paying one in state s_i and zero in other states, and let $A^{s_i} \in \mathbb{R}^S$ be the return of this asset, i.e., $A^{s_i}[s] = 1$ if $s = s_i$ and $A^{s_i}[s] = 0$ otherwise. Since rank $V = \#S$, we deduce that $A^{s_i} \in < V >$, that is, there exists $z_i \in \mathbb{R}^J$ such that $A^{s_i} = V z_i$. Defining $z_j = -z_i$, we check that $V[s] \cdot z_i = 0$, for every $s \in S_i$, $V[s_i] \cdot z_i = 1$, for $s_i \in S_i$ and $V[s] \cdot z_j = 0$, for all $s \in S_j$ (since $s_i \notin S_j$). This contradicts the fact that $(V, (S_i)_i)$ is arbitrage-free. ■

We now characterize arbitrage-free information structures in terms of the existence of a common no-arbitrage price.

Proposition 3.2 *For a given structure $[V, (S_i)_i]$, the two following statements are equivalent:*
(i) the structure $[V, (S_i)_i]$ is arbitrage-free, i.e., satisfies Condition **(AFAO)***;*
(ii) the structure $[V, (S_i)_i]$ admits a common no-arbitrage price, i.e., $Q_c[V, (S_i)_i] \neq \emptyset$.

Proof. *[(i)⟹(ii)].* We define the linear mapping $W : (\mathbb{R}^J)^I \to \mathbb{R}^J \times \mathbb{R}^J \times \Pi_{i \in I} \mathbb{R}^{S_i}$ by:

$Wz = (\sum_{i \in I} z_i, -\sum_{i \in I} z_i, [(V[s_i] \cdot z_i)_{s_i \in S_i}]_{i \in I})$ for $z = (z_i)_{i \in I} \in (\mathbb{R}^J)^I$.

Condition (i), stating that $[V, (S_i)_i]$ is arbitrage-free, is equivalent to say :

$$< W > \ \cap \ [\mathbb{R}^J \times \mathbb{R}^J \times \Pi_{i \in I} \mathbb{R}^{S_i}]_+ = \{0\}. \tag{1}$$

A characterization of Condition (1) is given by Lemma 1 and, for this purpose, we let the reader check that the transpose of the linear mapping W is the mapping ${}^tW$ from $\mathbb{R}^J \times \mathbb{R}^J \times \Pi_{i \in I} \mathbb{R}^{S_i}$ to $(\mathbb{R}^J)^I$ defined by:

${}^tW(\alpha, \beta, (\lambda_i)_{i \in I}) = (\alpha - \beta + \sum_{s \in S_i} \lambda_i[s] V[s])_{i \in I}$.

Consequently, from Lemma 1, the above Condition (1) is equivalent to the existence of some α, β in $\mathbb{R}^J_{++}$, and some $\lambda_i = (\lambda_i[s]) \in \mathbb{R}^{S_i}_{++}$, such that

for every $i \in I$, $0 = \alpha - \beta + \sum_{s \in S_i} \lambda_i[s] V[s]$.

But this latter condition, by Definition 3.1, is equivalent to saying that $q := \beta - \alpha$ (which is inedependent of i) is a common no-arbitrage price for the structure $[V, (S_i)_i]$, that is, Condition (ii) holds.

[(ii)⟹(i)]. By contraposition. If condition (i) fails, there exists a collection of portfolios $(z_i)_{i \in I} \in (\mathbb{R}^J)^I$ such that $\sum_{i \in I} z_i = 0$ and $V[s_i] \cdot z_i \geq 0$, for all $i \in I$, all $s_i \in S_i$, with at least one strict inequality. By condition (ii) we let $q \in Q_c[V, (S_i)_i]$. By Definition 3.1, for every $i \in I$, there exists $\lambda_i = (\lambda_i[s])_{s \in S_i} \in \mathbb{R}^{S_i}_{++}$ such that $q = \sum_{s \in S_i} \lambda_i[s] V[s]$. Consequently, $q \cdot z_i = (\sum_{s \in S_i} \lambda_i[s] V[s]) \cdot z_i \geq 0$ and one inequality is strict. Hence, $\sum_{i \in I} q \cdot z_i > 0$, which contradicts $\sum_{i \in I} z_i = 0$. ■

Remark 5 (Non equivalence between **(AFAO)** *and the Absence of Bilateral Arbitrage Opportunity).* We may define a weaker concept than **(AFAO)**, namely the "Absence of Bilateral Arbitrage Opportunity", as follows: for every $i, j \in I, i \neq j$, there are no portfolios $z_i, z_j \in \mathbb{R}^J$ satisfying the conditions $z_i + z_j = 0$ and $V[s_i] \cdot z_i \geq 0$, for every $s_i \in S_i$, $V[s_j] \cdot z_j \geq 0$ for every $s_j \in S_j$, with at least one strict inequality.

Then, condition **(AFAO)** implies condition **(ABAO)** but is not equivalent to it, as shown by the following counterexample. Consider an economy with three agents ($I = \{1, 2, 3\}$), seven states ($S = \{1, 2, 3, 4, 5, 6, 7\}$) and assume that $S_1 = \{1, 2, 3\}, S_2 = \{1, 4, 5\}, S_3 = \{1, 6, 7\}$ and that the payoff matrix V is defined as follows :

$$V = \begin{pmatrix} 0 & 0 \\ -2 & 1 \\ 1 & -1 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}$$

The above structure $[V, (S_i)_i]$ yields no bilateral arbitrage opportunity but is not arbitrage-free (take $z_1 = (-1, -1),\ z_2 = (1, 0),\ z_3 = (0, 1)$). ∎

3.3 No-arbitrage prices with asymmetric information structures

When the initial information structure $(S_i)_i$ is not arbitrage-free, the agents may refine their information and reach an arbitrage-free structure. The common no-arbitrage prices associated to all the refined information structures lead to the following broader concept of no-arbitrage price.

Definition 3.3 *The price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price relative to the structure $[V, (S_i)_i]$ if q is a common no-arbitrage price for some information structure $(\Sigma_i)_i$ refining $(S_i)_i$, that is, if there exists an information structure $(\Sigma_i)_i$ refining $(S_i)_i$ such that $q \in Q_c[V, (\Sigma_i)_i]$.*

We denote by $Q[V, (S_i)_i]$ the set of no-arbitrage prices relative to the structure $[V, (S_i)_i]$.

We point out the following simple, but important, result:

Proposition 3.3 *Every structure $[V, (S_i)_i]$ admits a no-arbitrage price, i.e., $Q[V, (S_i)_i] \neq \emptyset$.*

Proof. The pooled information refinement $(\underline{S_i})_i$, defined by $\underline{S_i} = \cap_{j \in J} S^j$ for every $i \in I$, is arbitrage-free, since it is symmetric (see Remark 3). Hence, $\emptyset \neq Q_c[V, (\underline{S_i})_i] \subset Q[V, (S_i)_i]$. ∎

We end the section with a property of no-arbitrage prices in terms of consumers' behavior.

Proposition 3.4 *Under Assumption* **(NSS)**, *if, first, every agent i refines his individual information (from S_i) to Σ_i, in such a way that $\cap_{i \in I} \Sigma_i \neq \emptyset$ [that is, $(\Sigma_i)_i$ is an information structure refining $(S_i)_i$], and, secondly, every agent i chooses a strategy (x_i^*, z_i^*) which maximizes the utility $u_i(\cdot \mid \Sigma_i)$ on the budget set $B_i(p^*, q^*, V, \Sigma_i)$, then $q^* \in Q_c[V, (\Sigma_i)_i]$, hence q^* is a no-arbitrage price for $[V, (S_i)_{i \in I}]$.*

In the above proposition, we did not make explicit how each agent i is refining his individual information. The purpose of the next section is to explain (i) how the information structure $(\Sigma_i)_i$ can be "revealed" by (the only knowledge of) price q^*, and (ii) when the family $(\Sigma_i)_i$ will satisfy the condition $\cap_{i\in I}\Sigma_i \neq \emptyset$ or the stronger one $\cap_{i\in I}\Sigma_i = \cap_{i\in I}S_i$.

Proof. From Proposition 3.1, under Assumption **(NSS)**, the price q^* is a common no-arbitrage price for the structure $[V, (\Sigma_i)_i]$, i.e., $q^* \in Q_c(V, (\Sigma_i)_i)$. Since the information structure $(\Sigma_i)_i$ is finer than $(S_i)_i$, we deduce that $q^* \in Q(V, (S_i)_i)$, that is, q^* is a no-arbitrage price. ∎

3.4 The least refined arbitrage-free information structure

When the structure $[V, (S_i)_i]$ is not arbitrage-free, from Propositions 3.1 and 3.4, agents' problem will admit no solution until they refine their information up to an arbitrage-free information structure. The following proposition shows that there exists a unique *least refined arbitrage-free information structure*, denoted by $(\bar{S}_i)_i$. Formally, we denote by $\mathcal{S}$, the set of arbitrage-free information structures refining $(S_i)_i$.

Proposition 3.5 *Given the structure $[V, (S_i)_i]$, the set $\mathcal{S}$ admits a unique smallest element for the refinement relation $\geq$, denoted by $(\overline{S_i})$, that is:*

$(\overline{S_i})_i \in \mathcal{S}$ and $(\Sigma_i)_i \in \mathcal{S} \implies (\Sigma_i)_i \geq (\overline{S_i})_i$, or, equivalently, $\Sigma_i \subset \overline{S_i}$, for every i.

Furthermore the set, $Q_c(V, (\overline{S_i})_i)$, of common no-arbitrage prices is nonempty.

Proof. We prepare the proof with a claim.

Claim 1 *We define the upper bound of two information structures $(\Sigma_i^1)_i$ and $(\Sigma_i^2)_i$, denoted by $(\Sigma_i)_i := (\Sigma_i^1)_i \vee (\Sigma_i^2)_i$, by the relations $\Sigma_i := \Sigma_i^1 \cup \Sigma_i^2$ for every $i \in I$. Then, $(\Sigma_i^1)_i \vee (\Sigma_i^2)_i$ is an information structure and, if $(\Sigma_i^1)_i$ and $(\Sigma_i^2)_i$ are both arbitrage free, so is $(\Sigma_i^1)_i \vee (\Sigma_i^2)_i$.*

We first prove the Claim and notice that $(\Sigma_i)_i := (\Sigma_i^1)_i \vee (\Sigma_i^2)_i$ is an information structure. Indeed, for all $i \in I$, $\emptyset \neq \cap_{j\in I}\Sigma_j^1 \subset \Sigma_i^1 \subset \Sigma_i \subset S_i$, hence $\emptyset \neq \cap_{i\in I}\Sigma_i$.

Assume now that $(\Sigma_i^1)_i$ and $(\Sigma_i^2)_i$ are both arbitrage free, but not $(\Sigma_i^1)_i \vee (\Sigma_i^2)_i$. Then, a collection of portfolios $(z_i)_{i\in I} \in (\mathbb{R}^J)^I$ exists, which satisfies, for every $i \in I$, $\sum_i z_i = 0$ and $V[s_i] \cdot z_i \geq 0$, for all $s_i \in \Sigma_i^1 \cup \Sigma_i^2$, with at least one strict inequality. That strict inequality may be assumed, non restrictively, to be met for some $j \in I$ and $s \in \Sigma_j^1$. Hence, the conditions $\sum_i z_i = 0$ and $V[s_i] \cdot z_i \geq 0$, for all $i \in I$ and all $s_i \in \Sigma_i^1$ hold, together with $V[s] \cdot z_j > 0$ for $j \in I$ and $s \in \Sigma_j^1$, which contradicts the assumption that $(\Sigma_i^1)_i$ is arbitrage free. This ends the proof of the Claim.

We come back to the proof of the Proposition. The set $\mathcal{S}$ is finite and we can define the information structure $(\overline{S_i})_i$ as the upper bound of all the elements in $\mathcal{S}$, i.e., $(\overline{S_i})_i := \vee_{(\Sigma_i)_i\in\mathcal{S}}(\Sigma_i)_i$. From the above Claim, $(\overline{S_i})_i$ is an information structure, is arbitrage-free, and satisfies $(\Sigma_i)_i \geq (\overline{S_i})_i$ for every $(\Sigma_i)_i \in \mathcal{S}$. The last assertion of the proposition, i.e., $Q_c(V, (\overline{S_i})_i) \neq \emptyset$, is a consequence of Proposition 3.2. ∎

4 Information revealed by prices

This section shows that every no-arbitrage price $q \in \mathbb{R}^J$ "reveals" a (uniquely defined) information structure, denoted by $(S_i(q))_i$, which is the least refined information structure having q as a common no-arbitrage price, i.e., $q \in Q_c[V, (S_i(q))_i]$. The refinement process is decentralized, in the sense that the price q conveys enough information for each agent to update his beliefs up to the refinement $(S_i(q))_i$, without having any information from the other agents. We shall then show that the information structure $(S_i(q))_i$ can be revealed in a constructive and sequential way, each agent eliminating sequentially his "arbitrage states."

4.1 Individual information sets revealed by prices

Given an (arbitrary) price $q \in \mathbb{R}^J$, for every agent i, we define the notion of the greatest individual q-arbitrage-free set of information, denoted $S_i(q)$, a subset of S_i, as follows:

Proposition 4.1 *For every $q \in \mathbb{R}^J$ and every i, there exists a unique (possibly empty) subset of S_i, denoted by $S_i(q)$, which is the greatest element (for the inclusion) among the subsets $\Sigma \subset S_i$ which satisfy one of the two following equivalent conditions:*

(i) $q = \sum_{s\in\Sigma} \lambda[s]V[s]$ for some $\lambda = (\lambda[s])_{s\in\Sigma} \in \mathbb{R}^{\Sigma}_{++}$;

(ii) the set Σ is q-arbitrage-free, i.e., there exists no $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$ and $V[s] \cdot z \geq 0$ for every $s \in \Sigma$, with one strict inequality.

Consequently, if $S_i(q)$ is nonempty, then $q = \sum_{s\in S_i(q)} \lambda[s]V[s]$ for some $\lambda[s] > 0$ ($s \in S_i(q)$).

We point out that the equivalence between (i) and (ii) is clearly a consequence of Lemma 1. We also notice that the set $S_i(q)$ may be empty (see Section 5.1). Moreover, even if each $S_i(q)$ $(i \in I)$ is nonempty, the collective information set $\cap_i S_i(q)$ may be empty (see again Section 5.1), i.e., the family $(S_i(q))_i$ is not, in general, an information structure, a property that we shall characterize in the next section.

Proof. Let $q \in \mathbb{R}^J$ and denote by $\mathcal{S}_i(q)$ the set of subsets $\Sigma \subset S_i$ such that Σ is q-arbitrage-free for agent i. We first show that the set $\mathcal{S}_i(q)$ is stable for the inclusion, i.e., if Σ^1, Σ^2 belong to $\mathcal{S}_i(q)$, then $\Sigma^1 \cup \Sigma^2$ also belongs to $\mathcal{S}_i(q)$. Indeed, if it is not true, there exists $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$ and $V[s] \cdot z \geq 0$, for every $s \in \Sigma^1 \cup \Sigma^2$, with one strict inequality. Then, either $-q \cdot z > 0$ or $V[s] \cdot z > 0$, for some $s \in \Sigma^1 \cup \Sigma^2$, say in Σ^1. In both cases, this contradicts the fact that Σ^1 is q-arbitrage-free for agent i.

Consequently, we define the set $S_i(q)$ as the union of all the sets in $\mathcal{S}_i(q)$ (which is finite). From above, $S_i(q)$ is q-arbitrage-free for agent i and, clearly, is the greatest set in $\mathcal{S}_i(q)$ satisfying this property. ∎

4.2 Information structures revealed by no-arbitrage prices

The first result states that the family of sets $(S_i(q))_i$ defines an information structure if and only if q is a no-arbitrage price. We let $\mathcal{S}(q)$ be the set of information structures $(\Sigma_i)_i$ refining $(S_i)_i$ and satisfying $q \in Q_c(V, (\Sigma_i)_i)$.

Proposition 4.2 *Let $q \in \mathbb{R}^J$, the following conditions are equivalent:*

(i) q is a no-arbitrage price, i.e., $q \in Q[V, (S_i)_i]$;

(ii) $\cap_{i\in I} S_i(q) \neq \emptyset$, i.e., $(S_i(q))_i$ is an information structure;

(iii) $(S_i(q))_i$ belongs to $\mathcal{S}(q)$ and every $(\Sigma_i)_i$ in $\mathcal{S}(q)$ is finer than $(S_i(q))_i$.

Proof. $[(i) \Longrightarrow (ii)]$. If $q \in Q[V, (S_i)_i]$, from the definition of a no-arbitrage price, there exist an information structure $(\Sigma_i)_i$ refining $(S_i)_i$, and a common no-arbitrage price q for the structure $(\Sigma_i)_i$, that is, $q \in Q_c[V, (\Sigma_i)_i]$. From Definition 3.1, for every $i \in I$, there exists vectors $\lambda_i = (\lambda_i[s])_s \in \mathbb{R}^{\Sigma_i}_{++}$, such that $q = \sum_{s \in \Sigma_i} \lambda_i[s]V[s]$. But, for every i, the set $S_i(q)$ is the greatest one satisfying the previous property, hence $\Sigma_i \subset S_i(q)$. Consequently, $\emptyset \neq \cap_i \Sigma_i \subset \Sigma_i \subset S_i(q) \subset S_i$, for every i, which implies that $\cap_i S_i(q) \neq \emptyset$.

$[(ii) \Longrightarrow (iii)]$ Assume that condition (ii) holds. We first show that $(S_i(q))_i$ belongs to $\mathcal{S}(q)$. Indeed, $(S_i(q))_i$ is an information structure (from (ii)) which clearly refines $(S_i)_i$ (since, for every i, $S_i(q) \subset S_i$) and from its definition (cf. Proposition 4.1) one always has $q \in Q_c[V, (S_i(q))_i]$. Now let $(\Sigma_i)_i \in \mathcal{S}(q)$ then, $q \in Q_c[V, (\Sigma)_i]$, which implies that, for every i, Σ_i is q-arbitrage free. Consequently, from Proposition 4.1, for every i, $\Sigma_i \subset S_i(q)$, that is $(\Sigma_i)_i$ is finer than $(S_i(q))_i$.

$[(iii) \Longrightarrow (i)]$ If Condition (iii) holds, we deduce that $(S_i(q))_i \in \mathcal{S}(q)$, i.e., $(S_i(q))_i$ is an information structure refining $(S_i)_i$, and $q \in Q_c[V, (S_i(q))_i]$. Consequently, $q \in Q_c[V, (S_i(q))_i] \subset Q[V, (S_i)_i]$. ∎

Remark 6 (Equivalent no-arbitrage prices). Let q^1 and q^2 be two no-arbitrage prices, we say that $q^1 \preceq q^2$ if $(S_i(q^2))_i \geq (S_i(q^1))_i$ (for the refinement relation). The relation $\preceq$ clearly defines a preorder on the set $Q[V, (S_i)_i]$ and we can associate to it the equivalence relation $\sim$ defined by $q^1 \sim q^2$ if $[q^1 \preceq q^2$ and $q^2 \preceq q^1]$, or equivalently if, for every i, $S_i(q^1) = S_i(q^2)$, i.e., q^1 and q^2 reveal the same information structure. Since S is finite, one shows that there is a finite number of equivalent classes, denoted $\dot{q}$ (for the relation $\sim$) which define a finite partition of the set $Q[V, (S_i)_i]$. ∎

We shall now reformulate the previous proposition to give a characterization of information structures which can be revealed by prices.

Definition 4.1 *We say that an information structure $(\Sigma_i)_i$, refining the (given) structure $(S_i)_i$, can be revealed by prices if there exists $q \in \mathbb{R}^J$ such that, for every i, $\Sigma_i = S_i(q)$.*

Proposition 4.3 *Let $[V, (S_i)_i]$ be a given structure and let $(\Sigma_i)_i$, be an information structure refining $(S_i)_i$, then the three following conditions are equivalent:*

(i) the information structure $(\Sigma_i)_i$ can be revealed by prices;

(ii) $\emptyset \neq \{q \in \mathbb{R}^J \mid \forall i,\ \Sigma_i = S_i(q)\} \subset Q_c[V, (\Sigma_i)_i]$;

(iii) there exists $q \in Q_c(V, (\Sigma_i)_i)$ such that every $(\Sigma'_i)_i$ in $\mathcal{S}(q)$ is finer than (Σ_i).

Furthermore, if $(\Sigma_i)_i$ can be revealed by a price q, then $(\Sigma_i)_i$ is arbitrage-free and q is a common no-arbitrage price for $(\Sigma_i)_i$, hence a no-arbitrage price for $[V, (S_i)_i]$.

Proof. $[(i) \Longrightarrow (ii)]$. If the information structure $(\Sigma_i)_i$ can be revealed by prices, from the above definition, there exists q such that, for every i, $\Sigma_i = S_i(q)$. Consider such a price q, then $q \in Q_c(V, (S_i(q))_i) = Q_c(V, (\Sigma_i)_i)$ (from Proposition 4.1, defining the information sets $S_i(q)$).

$[(ii) \Longrightarrow (iii)]$. From (ii), there exists $q \in \mathbb{R}^J$ such that, for every i, $\Sigma_i = S_i(q)$ and $q \in Q_c[V, (\Sigma_i)_i]$. From Proposition 4.2, we deduce that every $(\Sigma'_i)_i$ in $\mathcal{S}(q)$ is finer than $(S_i(q))_i = (\Sigma_i)$.

$[(iii) \Longrightarrow (i)]$. From (iii), there exists $q \in Q_c(V, (\Sigma_i)_i)$ and, from the definition of $(S_i(q))$ [i.e., Proposition 4.1] one has, for every i, $\emptyset \neq \Sigma_i \subset S_i(q)$ and $q \in Q_c(V, (S_i(q))_i)$. Consequently, $(S_i(q))_i$ belongs to $\mathcal{S}(q)$, hence from Condition (iii), it is finer than (Σ_i), that is, for every i,

$S_i(q) \subset \Sigma_i$. We have thus shown that, for every i, $S_i(q) = \Sigma_i$, that is, the information structure $(\Sigma_i)_i$ can be revealed by the price q.

The last assertion of the proposition is straightforward from (ii). ∎

We now give an important example of an information structure that can always be revealed by prices, namely the least refined information structure $(\overline{S}_i)_i$ defined in Section 3.4.

Proposition 4.4 *(a) Let* $[V, (S_i)_i]$ *be a given structure, then*

$$\emptyset \neq \{q \in \mathbb{R}^J \mid \forall i, \quad \overline{S}_i = S_i(q)\} = Q_c[V, (\overline{S}_i)_i].$$

Hence, the information structure $(\overline{S}_i)_i$ *can be revealed by every price* $q \in Q_c[V, (\overline{S}_i)_i]$.

(b) If the structure $[V, (S_i)_i]$ *is arbitrage-free, then*

$$\emptyset \neq \{q \in \mathbb{R}^J \mid \forall i, \quad S_i = S_i(q)\} = Q_c[V, (S_i)_i].$$

Proof. *Part (a).* We first notice that, from Proposition 3.5, the set $Q_c[V, (\overline{S}_i)_i]$ is nonempty. We now let $q \in \mathbb{R}^J$ be such that $(\overline{S}_i)_i = (S_i(q))_i$, then, from Proposition 4.1, $q \in Q_c[V, (S_i(q))_i] = Q_c[V, (\overline{S}_i)_i]$. Conversely, let $q \in Q_c[V, (\overline{S}_i)_i]$, then from Proposition 4.2 (iii), for every i, $\overline{S}_i \subset S_i(q)$. This implies that $(S_i(q))_i$ is an information structure (i.e., $\cap_{i \in I} S_i(q) \neq \emptyset$) and it is clearly arbitrage-free (since q is a common no-arbitrage price). But $(\overline{S}_i)_i$ is the least refined arbitrage-free information structure, hence for every i, $\overline{S}_i = S_i(q)$. This ends the proof of the equality. The second part of the Proposition is straightforward.

Part (b). If the structure $[V, (S_i)_i]$ is arbitrage-free, then $\overline{S}_i = S_i$ for every i, from the definition of $(\overline{S}_i)_i$. The result then follows from Part (a). ∎

Let $\underline{\mathcal{S}}$ denote the set of information structures $(\Sigma_i)_i$ refining $(S_i)_i$ and preserving the collective information $\cap_{i \in I} S_i$, that is, such that $\cap_{i \in I} \Sigma_i = \cap_{i \in I} S_i$. We now characterize prices q which reveal information structures $(S_i(q))_i$ in $\underline{\mathcal{S}}$.

Proposition 4.5 *Let* $q \in \mathbb{R}^J$*, the following conditions are equivalent:*

(i) $\cap_{i \in I} S_i \subset \cap_{i \in I} S_i(q)$ *, i.e.,* $(S_i(q))_i$ *preserves the collective information* $\cap_{i \in I} S_i$*;*

(ii) $(S_i(q))_i \in \underline{\mathcal{S}}$*;*

(iii) q is a common no-arbitrage price for some information structure in $\underline{\mathcal{S}}$*,*
i.e., $q \in \cup_{(\Sigma_i)_i \in \underline{\mathcal{S}}} \; Q_c(V, (\Sigma_i)_i)$*;*

(iv) $(S_i(q))_i$ *belongs to* $\mathcal{S}(q) \cap \underline{\mathcal{S}}$ *and every* $(\Sigma_i)_i$ *in* $\mathcal{S}(q) \cap \underline{\mathcal{S}}$ *is finer than* $(S_i(q))_i$*.*

Furthermore there exists prices q satisfying one of the above assertion (i) *to* (iv)*.*

Proof. The proof is a direct consequence of Proposition 4.2. We further notice that there clearly exist prices q, satisfying one of the above equivalent assertions (i) to (iv), since both information structures $(\underline{S}_i)_i$ and $(\bar{S}_i)_i$ belong to $\underline{\mathcal{S}}$. ∎

4.3 Elimination of arbitrage states

For every price $q \in \mathbb{R}^J$, for every $i \in I$, and for every $\Sigma \subset S_i$ we define the two sets :

$$A_i(\Sigma, q) = \{\overline{s} \in \Sigma \mid \exists\, z \in \mathbb{R}^J, -q \cdot z \geq 0, V[s] \cdot z \geq 0, \; \forall\, s \in \Sigma, \; V[\overline{s}] \cdot z > 0\},$$

$$S_i(\Sigma, q) = \Sigma \backslash A_i(\Sigma, q).$$

The set $A_i(\Sigma, q)$ consists in states (of the second period $t = 1$) $\overline{s} \in \Sigma \subset S$, called *arbitrage states*, that provide arbitrage opportunity to agent i ($i \in I$). The set $S_i(\Sigma, q)$ is the first stage of elimination of arbitrage states.

Proposition 4.6 *Let $q \in \mathbb{R}^J$, for every i, there exists a unique (possibly empty) subset of S_i, denoted by $\overline{S_i}(q)$, which is the greatest element (for the inclusion) among all subsets $\Sigma \subset S_i$ such that $A_i(\Sigma, q) = \emptyset$ (or equivalently $S_i(\Sigma, q) = \Sigma$).*

Proof. Let $i \in I$, $q \in \mathbb{R}^J$, and define $\overline{\mathcal{S}}_i(q) = \{\Sigma \subset S_i \mid A_i(\Sigma, q) = \emptyset\}$. We first show that the set $\overline{\mathcal{S}}_i(q)$ is stable for the inclusion, i.e., if Σ^1 and Σ^2 belong to $\overline{\mathcal{S}}_i(q)$, then $\Sigma^1 \cup \Sigma^2$ also belong to $\overline{\mathcal{S}}_i(q)$. Indeed, if $\Sigma^1 \cup \Sigma^2 \notin \overline{\mathcal{S}}_i(q)$, then $A_i(\Sigma^1 \cup \Sigma^2, q) \neq \emptyset$. Let $\overline{s} \in A_i(\Sigma^1 \cup \Sigma^2, q) \subset \Sigma^1 \cup \Sigma^2$, then, there exists $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0, V[s] \cdot z \geq 0$, for every $s \in \Sigma^1 \cup \Sigma^2$, and $V[\overline{s}] \cdot z > 0$. Without any loss of generality, we can assume that $\overline{s} \in \Sigma^1$ and, from above, we deduce that $\overline{s} \in A_i(\Sigma^1, q)$, a contradiction with $\Sigma^1 \in \overline{\mathcal{S}}_i(q)$.

We now define the set $\overline{S_i}(q)$ as the union of all the sets in $\overline{\mathcal{S}}_i(q)$. Since the set $\overline{\mathcal{S}}_i(q)$ is finite, from above, we deduce that $\overline{S_i}(q)$ belongs to $\overline{\mathcal{S}}_i(q)$ and is the greatest element in $\overline{\mathcal{S}}_i(q)$ for the inclusion. ∎

By construction, $\overline{S}_i(q)$ is the largest set in S_i containing no state of arbitrage at $t = 1$, while the set $S_i(q)$ satisfies the stronger condition of being q-arbitrage free for agent i. The link between the two sets is given by the following proposition. An example for which the sets $S_i(q)$ and $\overline{S_i}(q)$ may be different is given in Section 5.

Proposition 4.7 *Let $[V, (S_i)_i]$ be a given structure and let $q \in \mathbb{R}^J$.*
(a) For every agent $i \in I$, $S_i(q) \subset \overline{S_i}(q)$.
(b) For every agent i, the following conditions are equivalent:
(i) $S_i(q) = \overline{S_i}(q)$;
(ii) $\overline{S_i}(q)$ is q-arbitrage free for agent i;
(iii) $\overline{S_i}(q)$ is q-arbitrage free for agent i at the first period (i.e., $t = 0$), in the sense that there is no portfolio $z \in \mathbb{R}^J$ such that $-q \cdot z > 0, V[s] \cdot z \geq 0$, for every $s \in \overline{S_i}(q)$.
(c) If $q \in Q[V, (S_i)_i]$, then, $S_i(q) = \overline{S_i}(q)$, for every $i \in I$.

Proof. *Part (a).* Let $q \in \mathbb{R}^J$, and let $i \in I$. From the definition of the set $S_i(q)$ [i.e., Proposition 4.1], one deduces that $A_i(S_i(q), q) = \emptyset$. Consequently, from the definition of the set $\overline{S_i}(q)$ [i.e., Proposition 4.6], one gets $S_i(q) \subset \overline{S_i}(q)$.
Part (b). [$(i) \Longrightarrow (ii)$]. From its definition [i.e., Proposition 4.1], the set $S_i(q)$ is q-arbitrage free. From (i), $S_i(q) = \overline{S_i}(q)$, hence $\overline{S_i}(q)$ is q-arbitrage free.

[$(ii) \Longrightarrow (iii)$]. It is obvious.

[$(iii) \Longrightarrow (i)$]. By definition of $\overline{S_i}(q)$, for every $\overline{s} \in \overline{S_i}(q)$, there is no $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0, V[s] \cdot z \geq 0$, for every $s \in \overline{S_i}(q)$, and $V[\overline{s}] \cdot z > 0$. That condition, together with Condition (iii), implies that $\overline{S_i}(q)$ is q-arbitrage free for agent i. Consequently, from the definition of the set $S_i(q)$, one has $\overline{S_i}(q) \subset S_i(q)$. From Part (a), we then deduce that $S_i(q) = \overline{S_i}(q)$.
Part (c). If $q \in Q[V, (S_i)_i]$, then there exists an information structure $(\Sigma_i)_i$ refining $(S_i)_i$ such that $q \in Q_c[V, (\Sigma_i)_i]$. From the definition of $S_i(q)$ and Part (a), we deduce that $\emptyset \neq \Sigma_i \subset S_i(q) \subset \overline{S_i}(q)$. In view of Part (b), the proof will be complete if we show that, for every i, $\overline{S_i}(q)$

is q-arbitrage free for the first period ($t = 0$). Indeed, if it is not true, there is some agent i and some portfolio $z \in \mathbb{R}^J$ such that $-q \cdot z > 0, V[s] \cdot z \geq 0$, for every $s \in \overline{S_i}(q)$, and the same inequalities holds, in particular, for every $s \in \Sigma_i \subset \overline{S_i}(q)$. This contradicts the fact that Σ_i is q-arbitrage free for agent i. ∎

4.4 Sequential elimination of arbitrage states

In this section, we shall define the set $\overline{S_i}(q)$ in a constructive way, by eliminating the arbitrage states (as defined previously) sequentially. We shall give two different ways of eliminating the arbitrage states. They are slightly different in formulation, but will be shown to be equivalent.

Let $q \in \mathbb{R}^J$, for every i, and every $\Sigma \subset S_i$, the sets $A_i(\Sigma, q)$ and $S_i(\Sigma, q)$ are defined as in the previous section. We define, by induction on $k \in N$, the sets $S_i^k(q)$ as follows:

$S_i^0(q) = S_i$, and for $k \geq 1$

$S_i^{k+1}(q) = S_i(S_i^k(q), q) := S_i^k(q) \backslash A_i(S_i^k(q), q)$.

Similarly, we define by induction on $k \in N$, the sets $S_i'^k(q)$ as follows:

$S_i'^0(q) = S_i$, and for $k \geq 1$

$$S_i'^{k+1}(q) = \begin{cases} S_i'^k(q), & \text{if } A_i(S_i'^k(q), q) = \emptyset, \\ S_i'^k(q)\backslash\{s^k\} \text{ for some } s^k \in A_i(S_i'^k(q), q) & \text{if } A_i(S_i'^k(q), q) \neq \emptyset. \end{cases}$$

The two sequences $(S_i^k(q))_{k \in N}$ and $(S_i'^k(q))_{k \in N}$ are decreasing, that is, $S_i^{k+1}(q) \subset S_i^k(q)$ and $S_i'^{k+1}(q) \subset S_i'^k(q)$ for every k. Since both sequences are contained in the finite set S_i, each sequence must be constant for k large enough. We let

$S_i^*(q) := \cap_{k \in N}\ S_i^k(q) = S_i^{k^*}(q)$ for some k^* large enough;

$S_i^{**}(q) := \cap_{k \in N}\ S_i'^k(q) = S_i'^{k^{**}}(q)$ for some k^{**} large enough.

Proposition 4.8 *(a) For every $q \in \mathbb{R}^J$, then $\overline{S_i}(q) = S_i^*(q) = S_i^{**}(q)$ for every $i \in I$.*
(b) For every $q \in Q[V, (S_i)_i]$, then $S_i(q) = \overline{S_i}(q) = S_i^*(q) = S_i^{**}(q)$ for every $i \in I$.

Proof. *Part (a).* We prepare the proof with two claims.

Claim 2. *Let $q \in \mathbb{R}^J$ and $\Sigma^1 \subset \Sigma^2 \subset S_i$, then $S_i(\Sigma^1, q) \subset S_i(\Sigma^2, q)$.*

We prove Claim 2 by contraposition. Suppose that there exists some $\overline{s} \in S_i(\Sigma^1, q) \subset \Sigma^1 \subset \Sigma^2$ and $\overline{s} \notin S_i(\Sigma^2, q)$. Then, $\overline{s} \in A_i(\Sigma^2, q)$, that is, there exists $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$, $V[s] \cdot z \geq 0$, for every $s \in \Sigma^2$ and $V[\overline{s}] \cdot z > 0$. Since $\overline{s} \in \Sigma^1 \subset \Sigma^2$, we deduce that $\overline{s} \in A_i(\Sigma^1, q)$, which contradicts the fact that $\overline{s} \in S_i(\Sigma^1, q)$. ∎

Claim 3. $\overline{S_i}(q) \subset S_i^k(q) \subset S_i'^k(q)$, for every k.

We prove Claim 3 by induction on k. Indeed, the above inclusions are true for $k = 0$, since $S_i^0(q) = S_i'^0(q) := S_i$. Suppose that they are also true up to rank k. From Claim 2, we deduce that

$S_i(\overline{S}_i(q), q) \subset S_i(S_i^k(q), q) \subset S_i(S_i'^k(q), q)$.

But, from the definition of the set $\overline{S_i}(q)$ [i.e. Proposition 4.6] and the definitions of the sets $S_i^k(q)$ and $S_i'^k$, one gets $\overline{S_i}(q) = S_i(\overline{S}_i(q), q)$, $S_i^{k+1}(q) := S_i(S_i^k(q), q)$, and

$S_i(S_i'^k(q), q) := S_i'^k(q) \backslash A_i(S_i'^k(q), q) \subset S_i'^{k+1}(q)$.

Consequently, $\overline{S_i}(q) \subset S_i^{k+1}(q) \subset S_i'^{k+1}(q)$. ∎

We now come back to the proof of the Proposition. From Claim 3, taking k large enough, we get $\overline{S_i}(q) \subset S_i^*(q) \subset S_i^{**}(q)$ for every i. But, from the definitions of $S_i^*(q)$ and $S_i^{**}(q)$ we deduce that $A_i(S_i^*(q), q) = A_i(S_i^{**}(q), q) = \emptyset$. Recalling that, from Proposition 4.6, $\overline{S}_i(q)$ is the greatest element (for the inclusion) among the subsets Σ of S_i such that $A_i(\Sigma, q) = \emptyset$, we deduce that $\overline{S}_i(q) = S_i^*(q) = S_i^{**}(q)$.

Part (b). It is a consequence of Part (a) and Proposition 4.7 (c). ∎

5 Conclusion

5.1 A synthesizing example

We first give an example which allows us to synthesize many counterexamples given previously. It also helps to understand the links between prices q and the information sets $S_i(q)$ they "reveal".

Consider the economy with two agents ($I = \{1, 2\}$), five states ($S = \{1, 2, 3, 4, 5\}$), idiosyncratic information sets $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 4, 5\}$, and the payoff matrix :

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- We note that V is not arbitrage-free (consider for agent 1 the portfolio $z_1 = (0, 0, 1)$ and, for agent 2, $z_2 = (0, 0, -1)$, then $V[s_1] \cdot z_1 \geq 0$, for $s_1 \in S_1$, $V[s_2] \cdot z_2 \geq 0$, $s_2 \in S_2$, with one strict inequality).
- For $q = (1, 1, 0)$, $S_1(q) = \{2\}$ and $S_2(q) = \emptyset$, whereas $\overline{S_2}(q) = \{5\}$.
- For $q = (0, 1, 0)$, $S_1(q) = \{1, 2\}$ and $S_2(q) = \{4, 5\}$, hence $S_1(q) \cap S_2(q) = \emptyset$.
- For $q = (-1, 0, 0)$, $S_1(q) = \{1\}$ and $S_2(q) = \{1, 5\}$.
- For $q = (-1, 1, 0)$, $S_1(q) = \{1, 2\}$ and $S_2(q) = \{1, 4, 5\}$.
- The least refined information structure is $\overline{S}_1 = \{1, 2\}$ and $\overline{S}_2 = \{1, 4, 5\}$. Indeed, it is arbitrage-free (since, from above, $q = (-1, 1, 0) \in Q_c(V, (\overline{S}_i)_i))$ and the only information structure, contained in (S_1, S_2), which is less refined than $(\overline{S}_1, \overline{S}_2)$ is (S_1, S_2), which is not arbitrage-free, from above.
- The pooled information structure is $\underline{S}_1 = \underline{S}_2 = \{1\}$, which cannot be revealed by prices since, from above, for every common no-arbitrage price $q \in Q_c(V, (\underline{S}_i)_i) = \{\lambda(-1, 0, 0) \mid \lambda > 0\}$, we have $S_2(q) = \{1, 5\} \neq \{1\}$.

5.2 Inferences permitted by no-arbitrage prices

Given a no-arbitrage price, that is $q \in Q[V, (S_i)_i]$, the successive elimination of arbitrage states as defined in the previous section, may be interpreted as a rational behavior. Indeed, agents observing price q and starting from initial information sets S_i, may always refine their beliefs by ruling out successively these arbitrage states. Agents are then said to update their beliefs by the "no-arbitrage principle". Along Proposition 4.8, whether agents rule out the states of arbitrage one by one, or in bundles, will not change the outcome. Neither will the path (chronology) of elimination represented by the sequences $(S_i'^{k+1} \backslash S_i'^{k})_k$ of the previous section will change this outcome. The "no-arbitrage principle" will always lead to the arbitrage-free information structure $(S_i(q))_i$.

The inference behavior consisting in updating beliefs by the "no-arbitrage principle" does not require any specific knowledge on the ex ante characteristics of the economy (endowments and preferences of the other consumers). In particular, agents need not be aware of a relationship between prices and the private information of the other agents to implement inferences based on the no-arbitrage principle. This is the main difference between the model we consider and rational expectations models with differential information.

In the next section, we summarize the consequences of the "no-arbitrage principle" in terms of consumers' behavior.

5.3 The "no-arbitrage principle" and consumer's behavior

We can now provide some answers to the questions raised about consumers' behavior (in Section 2.3). (*i*) When the initial information structure $(S_i)_i$ is arbitrage-free, consumers may keep their initial information sets. Otherwise, they must refine their beliefs up to an arbitrage-free information structure, to be able to perform their maximization problem. [cf. Proposition 3.1 and 3.4]. (*ii*) The refined information sets $(S_i(q))_i$ are then "revealed" by the (only knowledge of) the asset price q. Hence, refinement is achieved in a decentralized manner : neither the presence of another agent, nor the knowledge of the other agents' characteristics is necessary. [cf. Proposition 4.1 defining the set $S_i(q)$ to be chosen as the new information set Σ_i]. (*iii*) Refinement proceeds in a constructive and sequential way, where each agent is eliminating at each stage one or several states (of the second period) [cf. Proposition 4.8]. States are eliminated when they display some arbitrage opportunities which reveal that the states will not prevail tomorrow. (*iv*) The answer will depend upon the choice of the no-arbitrage price q and we have characterized the two situations for which, either $\cap_{i \in I} \Sigma_i \neq \emptyset$, or $\emptyset \neq \cap_{i \in I} S_i = \cap_{i \in I} \Sigma_i$ [cf. Proposition 4.2, and 4.5, respectively]. (*v*) Neither this paper (in the asymmetric information case), nor the classical literature with symmetric information (where indeterminacy prevails) is explaining who is fixing asset prices q and how they are determined. In the present study, we leave open the possibility that asset prices q be explained by a communication process between agents. In this case, agents will not reach a better collective knowledge of information. A companion paper will tackle the existence problem (and the definition) of equilibria stemming from these comments.

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