Social conformity and bounded rationality in games with incomplete information.

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January 2002
Draft version

Abstract

Intepret a set of players all playing the same strategy and all with similar attributes as a society. Is it consistent with self interested behaviour for a population to organise itself into a relatively small number of societies? By introducing the concept of approximate substitutes in non-cooperative games we are able to put a bound on the rationalty of such social conformity for an arbitrary game and arbitrary number of societies. This is then applied, in the context of a non-cooperative pregame, to show that, given $\varepsilon > 0$, there is an integer Q, depending on ε but not on the number of players, such that any sufficiently large game has an ε-equilibrium that induces a partition of the player set into fewer than Q societies. An ε-purification result is also derived.

1 Introduction

Social conformity is an important issue in sociology, economics, and game theory. A culture or society is a group of individuals who have commonalities of language, social and behavioral norms, and customs. Social learning consists, at least in part, in learning the norms and behavior patterns of the society into which one is born and in those other societies which one may join – our professional associations, our workplace, and our community, for example. Social learning may also include learning a set of skills from others that will enable one to fit into a society. The society in question may be broad as "Western civilization" or Canada, or as small as the Econometric Society. A fundamental question is whether this can be consistent with self-interested behavior. Such consistency requires the existence of a Nash equilibrium where individuals within the same society play the same or similar strategies.

In an earlier paper (Wooders, Cartwright and Selten (2001)) we provide conditions under which such an equilibrium will exist. To understand the motivation for the present paper we briefly summarize this work. We take as given a non-cooperative pregame of which a key element is a set Ω of attributes. A component of attribute space is a complete description of the possible characteristics of a player. As such, a player set N and a function α allocating an attribute to each player, induce, through the pregame, a game $\Gamma(N, \alpha)$. We say that a pregame satisfies the large game property if, for sufficiently large games induced by the pregame, (1) payoffs are only slightly altered by a small perturbation of the attributes of players, and (2) each player's payoff is primarily a function of their own strategy and of the numbers of players of each attribute playing each strategy, relative to the size of the population. We interpret a set of players, all with attributes in some convex subset of attribute space and all playing the same pure strategy, as a society. Our main result shows, *that given any $\varepsilon > 0$ and $B \geq 1$, there is an integer $J(\varepsilon, B)$ and real number $\eta(\varepsilon, B)$, such that for any game of complete information which has at least $\eta(\varepsilon, B)$ players, but less than B players of any one attribute, and is induced by a pregame satisfying the large game property, there exists a Nash ε-equilibrium in pure strategies that induces a partition of the set of players into at most $J(\varepsilon, B)$ societies.*

The purpose of this paper is to extend this result which we do in variety of different ways. These can be summarized as (1) to generalize to games of incomplete information, (2) to make social conformity a property of a game rather than a pregame, and (3) to consider alternative notions of conformity. The first of these extensions should lead little explanation. We explain the latter two extensions in turn.

In order to look at social conformity as a property of a game rather than as a property of a pregame we introduce the notion of approximate substitutes of a non-cooperative game. This is a counterpart to the notion of approximate substitutes in cooperative games (Kovalenkov and Wooders (2001)). Informally, two players are approximate substitutes if they, (1) have similar payoff functions, and (2) are such that if they 'exchanged strategies' a third player would be relatively indifferent to this exchange. By putting a bound δ on the similarity of players, in terms of these three criteria, we talk of a δ-substitute partition as a partition of the player set into classes such that any two players with the same class are approximate (δ) substitutes. A game is then defined as a (δ, Q) class game if there exist a δ-substitute partition that partitions the player set into Q classes. It should also be noted that we restrict attention throughout to games with a finite number of strategies.

Within this framework we derive our Theorem 1, namely, *if a game* Γ *is a* (δ, Q) *class game then there exists an* ε*-equilibrium in pure strategies for any* $\varepsilon \geq 2\delta$. One key point to note, is that for any game Γ and any Q there exists some δ for which the game Γ is a (δ, Q) class game. As such, for any game we can put a bound on the ε for which there exists an ε-equilibrium in pure strategies. Alternatively, the value of 2δ could be interpreted as a bound on the rationality of players using pure strategies as opposed to mixed strategies.

The approximate substitute framework allows us to draw conclusions about arbitrary games. It is also useful to have some general examples of (δ, Q) class games for arbitrary values of δ. As such, in the final section, we return to the pregame framework of our earlier paper, extended to the incomplete information case. We are able to connect the concept of games with approximate substitutes to that of games induced by a pregame satisfying the large game property. This is allows us to apply Theorem 1 in deriving an ε-purification result such that, *given any real number* $\varepsilon > 0$ *there exists a real number* $\eta(\varepsilon) > 0$ *such that any game induced by a pregame satisfying the large game property with more that* $\eta(\varepsilon)$ *players has a Nash* ε*-equilibrium in pure strategies.* This result extends the ε-purification result of Wooders, Cartwright and Selten (2001) and contrasts from existing results on the existence of a pure strategy Nash ε-equilibrium in not assuming a continuum of players (cf. Schmeidler 1973, Mas-Colell 1984, Khan 1989, Pascoa 1993,1998, Khan et al. 1997, Araujo and Pascoa 2000).

These purification results illustrate the potential applications of both frameworks of approximate substitutes in non-cooperative games and pregames of non-cooperative games. We apply these frameworks further in addressing the bounded rationality of social conformity. We define a society such that any two players belonging to a society have the same class and play the same strategy. An immediate consequence of Theorem 1 is that *if a game* Γ *is a* (δ, Q) *class game then there exists an* ε*-equilibrium in pure strategies for any* $\varepsilon \geq 2\delta$ *which partitions the player set into no more than* QK *societies*, where K is the number of strategies. This essentially provides a benchmark result which we look to refine.

By way of motivation suppose that a population is playing a form of n person matching pennies, and, as such, in equilibrium half of the time 'heads' should be played and half of the time 'tails' should be played. If players only use pure strategies then in this example we would get two distinct societies between those players who play 'heads' and those who play 'tails'. This would have been the conclusion from our earlier paper and in some instances this would probably seem an appropriate distinction. In some instances, however, it may not be appropriate. For example, the game may be driving and the strategies are to give way or to not give way at road junctions. At any one instance we might expect half of the drivers to give way and half to not give way but, if this is because players are conforming to some highway code, we would clearly not want to think of the player set as being split into two distinct societies. Instead, we would say that players are merely taking different roles within the game and make *actions conditional on their roles*. Making strategy choice conditional on roles to symmetrize a game is standard in the evolutionary game theory literature (e.g. Selten (1980) and Young (2001)).

Clearly, if social conformity is boundedly rational when any two players in a society must use the same action it will be boundedly rational when two players in the same society may potentially use different actions. The converse, however, need not be true. Thus, by relaxing the definition of a society we can hope to demonstrate that it is potentially rational to have a higher level of social conformity. More formally, let the inverse of the number of societies be a measure of the level of social conformity. Our first result shows that it can be (2δ) boundedly rational to have a level of social conformity of $\frac{1}{QK}$. By relaxing the criteria by which we judge a society can we put, for the same criterion of rationality, a larger bound on the level of social conformity?

The extension to incomplete information immediately implies, that despite the fact that every player within a society plays the same strategy, players within the same society may use different actions dependent upon their type. That is, players may take up different roles in their society as determined by the type that 'nature' deals them. In the context of the example, it may be that men play 'heads' and women play 'tails' for example - a persons action being determined by their type which in this case is gender. Allowing imperfect information does not imply, however, that social conformity is any more rational than in games of perfect information. The reasons for this are clear in that the probabilities with which the types of players are drawn by nature may not be appropriate for the equilibrium of the game. In the context of our example, a highway code based on gender may not be entirely satisfactory.

It clearly seems plausible that in some games players will try to find some signalling mechanism by which players can be assigned roles within the society; if gender is not an appropriate signal then try something else. As such we suppose that players can endogenise the set of types and their probability distribution over types. We refer to the set of endogenised types as roles. Thus, instead of being given a type by nature we say that players are assigned to roles within a society. Strategies can be made conditional on a player's role. A society is then essentially such that any two players belonging to the same society have the same approximate type, play the same strategy, and agree on the allocation of players to roles within the society. In the context of our example, drivers would agree on a strategy, say "give way on minor roads", but, furthermore, they would also agree on which roads should be classed as minor roads and which major roads. We are able to show, under such a framework, that *if a game Γ is a (δ, Q) class game then there exists a Nash ε-equilibrium in pure strategies for any $\varepsilon \geq 4\delta$ which partitions the player set into no more than Q societies.* Thus, if players have some way of endogeneising types, that is allocating players to different roles within the society, then we can put a higher bound on the level of social conformity. Indeed, given our constraint that two players in the same society must be of the same class, we get the highest level of social conformity to be expected.

A second way in which we consider refining the notion of a society is to assume that players can play mixed strategies. In most games players probably are restricted to pure strategies (such as driving) but this does not mean there at not games in which mixed strategies can be played (e.g. in sport or gambling). Allowing the use of mixed strategies, we show that *if a game Γ is a (δ, Q) class game then there exists a Nash ε-equilibrium strategy vector for any $\varepsilon \geq 4\delta$ which partitions the player set into less than Q societies.* That is, there exists an ε-equilibrium such that any two players with the same class play the same mixed strategy. In the context of our earlier example this would equate to every player playing 'heads' with probability one half and 'tails' with probability one half.

Together this provides three different ways of looking at social conformity and the notion of a society, (1) all players within a society play the same pure strategy, (2) all players within a society play the same, possibly, mixed strategy, or (3) within a society there is some agreed upon framework within which players are allocated to roles and all players given the same role play the same action. We recall that for any game Γ and any Q there exists some δ for which the game Γ is a (δ, Q) class game. As such, the results we derive in these three contexts apply to any game and for any level of social conformity. One implication of this is that we can put a bound on the rationality of social conformity in an arbitrary game and for an arbitrary number of societies.

We finish by applying these social conformity results to games induced by a pregame satisfying the large game property. One derived result is that, *for any $\varepsilon > 0$ and $B \geq 1$ there exists real numbers $\eta(\varepsilon, B)$ and $Q(\varepsilon, B)$ such that for any game $\Gamma(N, \alpha)$ with at least $\eta(\varepsilon, B)$ players, induced by a pregame satisfying the large game property, there (1) exists a Nash ε equilibrium which partitions the player set into $C \leq Q(\varepsilon, B)$ societies, and (2) if there are less than B players of any one attribute, there exists a Nash ε equilibrium in pure strategies which partitions the player set into $C \leq Q(\varepsilon, B)K$ societies.* It is important to note that the number of societies is bounded independently of the size of the population. Thus, the size of societies become arbitrarily large as the size of the population increases.

We proceed as follows; section 2 introduces the notation and section 3 defines the notion of approximate substitutes. Section 4 looks at approximate purification and section 5 social conformity in games with approximate substitutes. Section 6 looks at large games before we conclude in section 7.

2 A Bayesian Game - definitions and notation

A Bayesian game Γ is given by the tuple (N, A, T, g, u) where N is the player set, A the set of action profiles, T the set of type profiles, g the probability function over type profiles and u the set of utility functions. We define these in turn.

Let $N = \{1, ..., n\}$ be a finite player set. For all $i \in N$ there exists a finite set T_i of *feasible types of player i* and a finite set A_i of *feasible actions of player i* (independent of type). Let $T = \times_i T_i$ be the set of *type profiles* and $A = \times_i A_i$ the set of *actions profiles.* We assume throughout, for convenience, that $T_i = T_\Gamma$ and $A_i = A_\Gamma$ for all $i \in N$ and for some T_Γ

and A_Γ. We will typically index a type as $t_z \in T_\Gamma$ and an action as $a_l \in A_\Gamma$.

A *pure strategy* of a player i is given by a vector $s_i = \{s_i(t_1), ..., s_i(t_{|T_i|})\}$ where $s_i(t_i)$ is interpreted as the action chosen by player i when of type t_i. Denote the set of pure strategies for player i by S_i or equivalently S_Γ. For any player i we allow choice among any feasible pure strategy as allowed by the set of feasible types T_i and actions A_i. Thus, $|S_i| = |A_i|^{|T_i|}$. Let $K = |A_\Gamma|^{|T_\Gamma|}$ be the number of pure strategies.

A *strategy* of a player i is given by a vector $\sigma_i = \{\sigma_{i1}, ..., \sigma_{iK}\}$ where σ_{ik} is interpreted as the probability player i plays pure strategy $s_k \in S_i$. A strategy σ implies a vector $\{\sigma_i(\cdot|t_1), ..., \sigma_i(\cdot|t_{|T_i|})\}$ where $\sigma_i(\cdot|t_i)$ is interpreted as a probability distribution over the set of actions A_i to be used by player i when of type t_i. The value $\sigma_i(a_i|t_i)$ is interpreted as the probability player i uses action a_i given he or she is of type t_i. Let $\Delta(S_i)$ denote the set of strategies for player i. Given strategy σ_i let *support*(σ_i) denote the pure strategies played with strictly positive probability. Let $S = \times_{i \in N} \Delta(S_i)$ denote the set of *strategy vectors*. We refer to a strategy vector σ as *degenerate* if σ_i places unit weight on a unique pure strategy for all $i \in N$. We will typically index a strategy as $s_k \in \Delta(S_\Gamma)$.

Let $C_i = T_i \times A_i$ denote the feasible *compositions of player i*. That is, a composition is a type-action pair. Let $C = \times_{i \in N} C_i$ denote the set of *composition profiles*. For each player $i \in N$ there exists a *utility function* $u_i : C \to \mathbb{R}$. The interpretation is that $u_i(c)$ denotes the payoff of player i if the composition profile is c. We will typically index a composition as $c_r \in C_\Gamma$. Let $u = \{u_1, ..., u_n\}$ denote the set of player utility functions.

For each player $i \in N$ there exists a *prior probability distribution over types* g_i. That is, $g_i(t_i)$ denotes the probability that player i is of type $t_i \in T_i$ if the types of the remaining players $N \backslash \{i\}$ are undetermined. Let g denote a *probability function over the set of type profiles*. Thus, $g(t)$ denotes the probability of type profile $t \in T$. Each player $i \in N$ forms their own beliefs about the types of other players as given by a function p_i mapping T_i into the set of probability distributions over $T_{-i} = \times_{\substack{j \in N \\ j \notin i}} T_j$. The distribution $p_i(t_i)$ is interpreted as the probability function over the type profile of the remaining players in the population conditional on player i knowing his or her own type is t_i. With a slight abuse of notation let $p_i(t_{-i}|t_i)$ denote the probability that player i puts on the type profile being $t = (t_{-i}, t_i) \in T$ given that he or she is of type t_i. We make two assumptions over probability distributions:

1. *independent type allocation:* for all $i \in N$, g_i is independent of the

type profile over the remaining players. That is, $g(t) = \prod_i g_i(t_i)$ where $t = (t_1, ..., t_n)$.

2. *consistent beliefs:* for all $i \in N$ and for all $t_i \in T_i$,

$$p_i(t_{-i}|t_i) = \frac{g(t_{-i}, t_i)}{\sum_{l_{-i} \in T_{-i}} g(l_{-i}, t_i)}$$

We make both these assumptions for simplicity and intuitive appeal. In reality, none of the subsequent results in this paper seem dependent upon these assumptions. To gain some intuition for why this is the case we highlight that our main objective in this paper is to take a Nash equilibrium strategy σ and show that there exists a 'nearby' approximate Bayesian Nash equilibria m that has certain properties. The beliefs of players therefore have little bearing, because we take the initial equilibrium σ as given. That is, if σ was a Nash equilibrium *for some set of beliefs* then we can always find an approximate equilibrium m for *those same set of beliefs.* We say that the probability over type profiles is degenerate if $g(t) = 1$ for some $t \in T$. In this case we say game $\Gamma = (N, T, A, g, u)$ is a *game of perfect information.*

The strategy of a player i, σ_i, and their prior probability distribution over types, g_i, determine a distribution over player i's compositions, γ_i, where $\gamma_i(c_i) = g_i(t_i)\sigma_i(a_i|t_i)$ for composition $c_i = (a_i, t_i)$. We can then derive a *probability distribution over outcomes of the game* γ where $\gamma(c) = \prod_i \gamma_i(c_i)$ for all $c \in C$. Any strategy vector σ and probability function over the set of type profiles g induce a particular probability distribution over outcomes of the game which we index $\gamma_{\sigma,g}$. Thus, given strategy vector σ and a probability function over the set of type profiles g the probability of composition profile $c = ((a_1, t_1), ..., (a_n, t_n))$ is given by

$$\gamma_{\sigma,g}(c) = \prod_i g_i(t_i)\sigma_i(a_i|t_i).$$

Players are assumed to act according to expected payoffs. Thus let E_γ denote the expectations operator where expectations are taken according to the probability distribution over outcomes of the game γ. For each player $i \in N$, let $U_i(\cdot|g)$ denote the *expected utility function of player i conditional on the distribution over player types g*, mapping strategy vectors into the real line, such that

$$U_i(\sigma|g) = E_{\gamma_{\sigma,g}}(u_i(c)).$$

We note how the function U_i accounts for both the uncertainty over player types, and the uncertainty due to mixed strategy vectors.

A strategy vector σ is a *Bayesian Nash ε Equilibrium* if,

$$U_i(\sigma) \geq U_i(s_k, \sigma_{-i}) - \varepsilon$$

for all $s_k \in S_i$ and for all $i \in N$. We say that a Bayesian Nash ε equilibrium m is a *Bayesian Nash ε equilibrium in pure strategies* if m_i is degenerate for all i.

3 Approximate substitutes

We consider approximating games with many players, all of whom may be distinct, by games with a finite set of player classes. In particular, a δ-*substitute partition*, given game $\Gamma = (N, T, A, g, u)$, is a partition of the player set N into subsets such that any two players in the same subset are "within δ of each other". The game Γ is referred to as a (δ, Q)-*class game* if there is a Q-member δ-substitute partition $\{N_1, .., N_Q\}$ of N. Players in the same element of a δ-substitute partition we call δ-substitutes. The set N_q is referred to as a class of player.

This, of course, begs the question of how we measure the distance between two players. In cooperative game theory this distance is typically measured by the difference in value that players can add to coalitions. Informally, a δ-substitute partition is such that, given any coalition structure, 'swapping' around players of the same class, has relatively little effect on the value of the coalitions. The counterpart in a non-cooperative framework would appear to be one in which the distance between players is measured by the effect that players can have on each others payoffs. A δ-substitute partition would now be such that given any strategy profile, 'swapping' around the strategies of players with the same class, has relatively little effect on the payoff to some player who keeps the same strategy. We formalize this below but first introduce two preliminary concepts.

3.1 Representative strategies and similarity of priors

Take as given a partition of the player set N into subsets $N_1, ..., N_Q$. We then introduce the following,

Representative strategy for class q (relative to σ): Given any strategy $\sigma \in S$ and any subset N_q of N let $\sigma(N_q)$ denote the *representative strategy for class q* defined as,

$$\sigma(N_q)(a_l|t_z) = \frac{1}{\sum_{i \in N_q} g_i(t_z)} \sum_{i \in N_q} g_i(t_z) \sigma_i(a_l|t_z)$$

for all $a_l \in A_\Gamma$ and all $t_z \in T_\Gamma$.[1]

We should check that $\sigma(N_q)$ is indeed a strategy for any σ and any N_q. That is, we must show that $\sigma(N_q)$ maps T_Γ into the set of probability distributions over A_Γ. For arbitrary $t_z \in T_\Gamma$,

$$\begin{aligned}
\sum_{a_l \in A_\Gamma} \sigma(N_q)(a_l|t_z) &= \frac{1}{\sum_{i \in N_q} g_i(t_z)} \sum_{a_l \in A_\Gamma} \sum_{i \in N_q} g_i(t) \sigma_i(a_l|t_z) \\
&= \frac{1}{\sum_{i \in N_q} g_i(t_z)} \sum_{i \in N_q} g_i(t_z) \left(\sum_{a_l \in A_\Gamma} \sigma_i(a_l|t_z) \right) \\
&= 1
\end{aligned}$$

and clearly $1 \geq \sigma(N_q)(a_l|t_z) \geq 0$ for all a_l, t_z. Thus, $\sigma(N_q)$ is indeed a strategy.

In interpretation, suppose that $g_i(t_z) = g_j(t_z)$ for all $i, j \in N_q$ and all t_z. We can see that a ‘player’ with strategy $\sigma(N_q)$ truly is a ‘representative’ of the class N_q in the sense that

$$\sigma(N_q)(a_l|t_z) = \frac{1}{|N_q|} \sum_{i \in N_q} \sigma_i(a_l|t_z).$$

That is, the probability the representative has composition (t_z, a_l) equals the average probability players of class N_q have that composition. If for some players $i, j \in N_q$, $g_i(t_z) > g_j(t_z)$ then the representative strategy, given type t_z, is weighted towards the action chosen by player i.

We now introduce our second preliminary concept,

Similarity of prior probabilities for class q (relative to g): Given any strategy any subset N_q of N (and probability function over the set of type profiles g) let $\beta(N_q)$ denote the *similarity of prior probabilities for class q* defined as,

$$\beta(N_q) = \max_{i,j \in N_q} \max_{t_z \in T_\Gamma} |g_i(t_z) - g_j(t_z)| \,.$$

We note that given an assumption of common priors $\beta(N_q) = 0$ for any subset N_q of N.

[1] If $\sum_{i \in \Omega_q} g_i(t_h) = 0$ then let $\sigma_{\omega_q}(\cdot|t_h)$ be any probability distribution over A_Γ.

3.2 δ-substitute partition

With the notion of a representative strategy and similarity of prior probabilities introduced we can now formally define the notion of δ-substitutes. A partition $\{N_1, ..., N_Q\}$ is a *δ-substitute partition* if,

Anonymity: for any two strategy profiles $\sigma^1, \sigma^2 \in S$ if,

$$\max_{a_l \in A_\Gamma} \max_{t_z \in T_\Gamma} \left| \left(\sigma^1(N_q)(a_l|t_z) - \sigma^2(N_q)(a_l|t_z)\right) \sum_{i \in N_q} g_i(t_z) \right| < 1 + \beta(N_q) \left|N_q\right| \tag{1}$$

for all N_q, $q = 1, ..., Q$, then,

$$\left|U_i(s_k, \sigma^1_{-i}) - U_i(s_k, \sigma^2_{-i})\right| < \delta$$

for any player $i \in N$ and any strategy $s_k \in \Delta(S_\Gamma)$.

and,

Similarity of payoffs: for all $q = 1, ..., Q$, for any two players $i, j \in N_q$ and for any strategy profile $\sigma \in S$,

$$|U_i(s_k, \sigma_{-i}) - U_j(s_k, \sigma_{-j})| < \delta$$

for any $s_k \in \Delta(S_\Gamma)$.

The anonymity constraint is a key requirement which we explain in more detail. This condition is essentially composed of two elements. First, the anonymity condition requires that payoffs should depend primarily on just the representative strategies for the Q classes.[2] Thus, there is anonymity in that 'exchanging' the strategies of players with the same class leaves payoffs relatively unaffected. Second, the anonymity condition requires that payoffs should be relatively invariant to bounded changes in the representative strategies of the Q classes. We illustrate by considering two extremes.

Suppose that $|N_q| = 1$ for all $i \in N$. The representative of a class q would then clearly be identical to the actual player of that class. As such, that payoffs should depend only on the representative strategies for the Q classes, is trivial. Note, however, that (1) can now be phrased, $\left|\sigma^1_i(a_l|t_z) - \sigma^2_i(a_l|t_z)\right| g_i(t_z) \leq 1 + \beta$ for all $i \in N$. Thus, any player $i \in N$

[2]They, of course, also depend on the player's own strategy choice.

can change their strategy any way they wish. It seems, therefore, unlikely that anonymity can hold, in this circumstance for any meaningful value of δ. Essentially, we would require that the game is such that each player is indifferent to what their opponents play (which is not much of a game).

By way of contrast, suppose that $|N_1| = n$ and so there is only one class of player. Further, suppose there is common priors so that $\beta(N_1) = 0$. It is now much more restrictive to say that payoffs should depend only on the representative strategies for the Q classes. It would require that payoffs depend only on the 'population average' or the number of players playing each strategy. This is plausible (such an assumption is used in Kalai (2000), for example) but clearly fairly restrictive. Suppose, however, it was the case that payoffs did depended only on the population average. The definition of anonymity now reads $\left|\sigma^1(N_q)(a|t) - \sigma^2(N_q)(a|t)\right| \sum_{i\in N} g_i(t_z) \leq 1$. Thus, the definition of anonymity requires that payoffs be relatively unchanged for only small changes in the representative strategy; this seems reasonable.

Between these two extremes we clearly find a trade off between a small or large number of classes Q. In particular, for an arbitrary game Γ, finding the minimum δ for which there exists a δ-substitute partition would seem to involve a trade-off when varying the size of Q; if Q is large then it seems more plausible that payoffs should depend only on the representative strategies for the Q classes, while, if Q is small, it seems more likely that payoffs should be relatively invariant to bounded changes in the representative strategies of the Q classes.

Let $\mathcal{H}((\delta, Q))$ denote the set of all (δ, Q)-class games. We recall that a game Γ is (δ, Q)-class game if there is a Q-member δ-substitute partition $\{N_1, .., N_Q\}$ of N. Note that for any q and any game Γ, $\Gamma \in \mathcal{H}((\delta, Q))$ for some δ.

4 Purification of mixed strategies

We begin with two technical results and then state and prove out main ε-purification result. First, we introduce some notation. Given a vector $\sigma = (\sigma_1, .., \sigma_n)$ (where $\sigma_i = (\sigma_{i1}, ..., \sigma_{iK}) \in \Delta^K$ for $i = 1, ..., n$) let $\mathcal{M}(\sigma)$ denote the set of vectors $m = (m_1, .., m_n)$ such that for all $i = 1, ..., N$,

1. *support*$(m_i) \subset$*support*(σ_i) for all $i \in N$ and,
2. m_i is degenerate.

Informally, given a strategy vector σ the strategy vector $m \in \mathcal{M}(\sigma)$ if, for all i, strategy m_i is such that player i plays some pure strategy

$s_k \in support(\sigma_i)$ with probability one.

Our main result makes use of the following Lemma from Wooders, Cartwright and Selten (2001),

Lemma 1 (Wooders, Cartwright, Selten): For any vector $\sigma = (\sigma_1, ..., \sigma_n)$ and for any vector $\overline{g} \in \mathbb{Z}_+^K$ such that $\sum_i \sigma_i \geq \overline{g}$, there exists a vector $m = (m_1, ..., m_n) \in \mathcal{M}(\sigma)$ such that:

$$\sum_i m_i \geq \overline{g}.$$

We extend Lemma 1. First, we introduce further notation. Given real number h let $\lfloor h \rfloor$ denote the nearest integer less than or equal to h and $\lceil h \rceil$ the nearest integer greater than or equal to h (i.e. $\lfloor 9.5 \rfloor = 9$ and $\lceil 9.5 \rceil = 10$ etc.). Given vector h denote by $\lfloor h \rfloor$ the vector such that $\lfloor h \rfloor_k = \lfloor h_k \rfloor$ for all k with a similar definition for $\lceil h \rceil$.

Lemma 2: For any vector $\sigma = (\sigma_1, ..., \sigma_n)$ there exists a vector $m = (m_1, ..., m_n) \in \mathcal{M}(\sigma)$ such that:

$$\left\lceil \sum_{i=1}^{n} \sigma_i \right\rceil \geq \sum_{i=1}^{n} m_i \geq \left\lfloor \sum_{i=1}^{n} \sigma_i \right\rfloor.$$

Proof: Denote by $\mathcal{M}^*(\sigma)$ the set of vectors $m = (m_1, ..., m_n) \in \mathcal{M}(\sigma)$ such that $\sum_i m_i \geq \lfloor \sum_i \sigma_i \rfloor$. By Lemma 1 this set is non-empty. Proving the Lemma thus amounts to showing that there exists a vector $m \in \mathcal{M}^*(\sigma)$ such that $\lceil \sum_i \sigma_i \rceil \geq \sum_i m_i$. Suppose not. Then, for every vector $m \in \mathcal{M}^*(\sigma)$ there exists some strategy $s_k \in S$ such that $\sum_i m_{ik} > \lceil \sum_i \sigma_{ik} \rceil$. Choose a vector $m^0 \in \mathcal{M}^*(\sigma)$ such that

$$C \equiv \sum_{s_k : \sum_i m_{ik} > \lceil \sum_i \sigma_{ik} \rceil} \left(\sum_i m_{ik} - \left\lceil \sum_i \sigma_{ik} \right\rceil \right)$$

is minimized. That is, m^0 comes as close as any vector to satisfying the statement of the Lemma. Denote by $s_{\widehat{k}}$ a pure strategy such that

$$\sum_{i=1}^{n} m_{i\widehat{k}} > \left\lceil \sum_{i=1}^{n} \sigma_{i\widehat{k}} \right\rceil.$$

We then introduce the following sets W^t and L^t, $t = 0, 1, 2, ...$,

$$\begin{aligned} W^0 &= \{i : m_{i\widehat{k}} = 1\} \\ L^t &= \{s_k : \sigma_{ik} > 0 \text{ for some } i \in W^t\} \text{ for } t \geq 0 \\ W^t &= \{i : m_{ik} = 1 \text{ for some } s_k \in L^t\} \text{ for } t > 0. \end{aligned}$$

For some t^*, $W^{t^*} = W^{t^*+1} \equiv W$ and $L^{t^*} = L^{t^*+1} \equiv L$. The construction of W^t and L^t imply that for any $s_{k^*} \in L^{t^*}$ there must exist a set of players $\{i_0, i_1, ..., i_t\} \in W^t$ and set of strategies $\{s_{k_1}, ..., s_{k_t}\}$ such that,

$$\begin{aligned} m^0_{i_0\widehat{k}} &= 1 \text{ and } \sigma_{i_0k_1} > 0, \\ m^0_{i_rk_r} &= 1 \text{ and } \sigma_{i_rk_{r+1}} > 0 \text{ for all } r = 1, .., t-1, \\ m^0_{i_tk_t} &= 1 \text{ and } \sigma_{i_tk^*} > 0, \end{aligned}$$

where we allow the possibility that $t = 0, 1$. Suppose there exists a $k^* \in L$ such that

$$\sum_{i=1}^{n} m_{ik^*} \leq \sum_{i=1}^{n} \sigma_{ik^*}.$$

Given the chain of players $\{i_0, i_1, ..., i_t\} \in W$ given above, consider the vector m^* constructed as follows,

$$\begin{aligned} m^*_{i_0\widehat{k}} &= 0 \text{ and } m^*_{i_0k_1} = 1, \\ m^*_{i_rk_r} &= 0 \text{ and } m^*_{i_rk_{r+1}} = 1 \text{ for all } r = 1, ..., t-1, \\ m^*_{i_tk_t} &= 0 \text{ and } m^*_{i_tk^*} = 1, \\ m^*_{ik} &= m^0_{ik} \text{ for all other } s_k \in S \text{ and } i \in N. \end{aligned}$$

It is easily checked that the vector $m^* \in \mathcal{M}(\sigma)$ leads to the desired contradiction by reducing by one the value C. We note, however, that

$$\sum_{i=1}^{n} \sum_{k \in L} m_{ik} = |W| = \sum_{i \in W} \sum_{s_k \in L} \sigma_{ik}.$$

Thus, if

$$\sum_{i=1}^{n} m_{i\widehat{k}} > \sum_{i=1}^{n} \sigma_{i\widehat{k}} \geq \sum_{i \in W} \sigma_{i\widehat{k}}$$

there must exist some $k^* \in L$ such that

$$\sum_{i=1}^{n} m_{ik^*} \leq \sum_{i \in W} \sigma_{ik^*} \leq \sum_{i=1}^{n} \sigma_{ik^*}$$

giving the desired contradiction■

With this we can now state and prove our first main result:

Theorem 1: For any Bayesian game $\Gamma \in \mathcal{H}((\delta, Q))$. Let ε be a positive real number. If, $\varepsilon \geq 2\delta$ then there exists a Bayesian Nash ε-equilibrium in pure strategies.[3]

Proof: By definition there exists a δ-substitute partition of N into non-empty subsets $N_1, ..., N_Q$. Furthermore, using Nash's Theorem there must exist a Nash Equilibrium strategy $\sigma^* \in \Sigma$. This implies, for all $i \in N$, that,

$$U_i(\sigma_i, \sigma^*_{-i}) \geq U_i(s_k, \sigma^*_{-i}) \tag{2}$$

for all σ_i where *support*(σ_i) $\subset$*support*(σ_i^*) and for all $s_k \in \Delta(S)$.

We apply Lemma 2 in turn to each N_q. Doing so implies that there exists a strategy vector $m \in S$ where *support*(m_i) $\subset$*support*(σ_i^*), m_i is degenerate for all $i \in N$ and where,

$$\left\lceil \sum_{i \in N_q} \sigma_i^* \right\rceil \geq \sum_{i \in N_q} m_i \geq \left\lfloor \sum_{i \in N_q} \sigma_i^* \right\rfloor .$$

for all $q = 1, ..., Q$. Thus,

$$\left| \sum_{i \in N_q} m_{ik} - \sum_{i \in N_q} \sigma_i^* \right| \leq 1$$

for all $s_k \in S$ and all q. This implies that,

$$\max_{a_l \in A_\Gamma} \max_{t_z \in T_\Gamma} \left| \sum_{i \in N_q} m_i(a_l|t_z) - \sum_{i \in N_q} \sigma_i^*(a_l|t_z) \right| \leq 1. \tag{3}$$

[3] We remark that Theorem 1 merely requires that the anonymity property hold with regard to the definition of a (δ, Q) class game. That is, similarity of payoffs need not apply.

For each q pick a player $j_q \in N_q$. Then for all q, using the identity $g_i(t_z) = g_{j_q}(t_z) - (g_{j_q}(t_z) - g_i(t_z))$,

$$\begin{aligned}
& \max_{a_l \in A_\Gamma} \max_{t_z \in T_\Gamma} \left| \sum_{i \in N_q} g_i(t_z) m_i(a_l|t_z) - \sum_{i \in N_q} g_i(t_z) \sigma_i^*(a_l|t_z) \right| \\
\leq \; & \max_{a_l \in A_\Gamma} \max_{t_z \in T_\Gamma} \left| g_{j_q}(t_z) \left(\sum_{i \in N_q} m_i(a_l|t_z) - \sum_{i \in N_q} \sigma_i^*(a_l|t_z) \right) \right| + \\
& \max_{a_l \in A_\Gamma} \max_{t_z \in T_\Gamma} \left| \sum_{i \in N_q} (g_{j_q}(t_z) - g_i(t_z))(m_i(a_l|t_z) - \sigma_i^*(a_l|t_z)) \right| .
\end{aligned}$$

Given (3), that $g_i(t_z) \leq 1$ and $g_{j_q}(t_z) - g_i(t_z) \leq \beta(N_q)$ for all $j \in N$, all q and all $t_z \in T_\Gamma$, and finally that $m_i(a_l|t_z) - \sigma_i^*(a|t_z) < 1$ we get,

$$\max_{a_l \in A_\Gamma} \max_{t_z \in T_\Gamma} \left| \sum_{i \in N_q} g_i(t_z) m_i(a_l|t_z) - \sum_{i \in N_q} g_i(t_z) \sigma_i^*(a_l|t_z) \right| < 1 + \beta(N_q) \left| N_q \right|$$

for all q. Thus, denoting $\sigma^*(N_q)$ and $m(N_q)$ as the representative strategy for class $q = 1, ..., Q$, we see that,

$$\max_{a_l \in A_\Gamma} \max_{t_z \in T_\Gamma} \left| (\sigma^*(N_q)(a_l|t_z) - m(N_q)(a_l|t_z)) \sum_{i \in N_q} g_i(t_z) \right| < 1 + \beta(N_q) \left| N_q \right| \tag{4a}$$

for all q.

By anonymity and (4a),

$$\left| U_i(s_k, \sigma_{-i}^*) - U_i(s_k, m_{-i}) \right| < \delta$$

for all $s_k \in \Delta(S_i)$ and for all $i \in N$. Given (2),

$$\begin{aligned}
U_i(m_i, m_{-i}) - U_i(s_k, m_{-i}) & \geq - \left| U_i(m_i, \sigma_{-i}^*) - U_i(m_i, m_{-i}) \right| - \left| U_i(s_k, \sigma_{-i}^*) - U_i(s_k, m_{-i}) \right| \\
& > -2\delta \geq -\varepsilon
\end{aligned}$$

for all $i \in N$ and all $s_k \in \Delta(S_i)$. This completes the proof■

As previously remarked, any game Γ is a (δ, Q) class game for some δ. Theorem 1 allows us, therefore, to put a bound on the rationality of using pure strategies as opposed to mixed strategies. It would be interesting to have a class of games which are (δ, Q) substitute games for arbitrarily small δ. We consider this when looking at large games. First, we consider social conformity in more detail.

5 Social conformity

We begin by defining a *society*. Take as give a δ-substitute partition $\{N_1, ..., N_Q\}$ and a strategy vector $\sigma \in S$. For any strategy $s_k \in \Delta(S_\Gamma)$ and any q, consider the subset N_q^k of N such that $i \in N_q^k$ if and only if $i \in N_q$ and $\sigma_i = s_k$. If N_q^k is non-empty then we refer to the set N_q^k as a society. Thus, a society is (a maximal set) such that every player belonging to that society plays the same strategy and has the same class.

Given a δ-substitute partition $\mathcal{N} = \{N_1, ..., N_Q\}$ and a strategy vector $\sigma \in S$ there exists a unique partition $\{N_1, ..., N_C\}$ of the player set N into societies. We say that $\mathcal{N}$ and σ induce the partition into societies $\{N_1, ..., N_C\}$.

Our first result of this section essentially summarizes the immediate implications of Theorem 1 and should need no proof.

Corollary 1: Let $\Gamma \in \mathcal{H}((\delta, Q))$ be any Bayesian game and let $\mathcal{N}$ be any δ-substitute partition with Q classes. Let ε be a positive real number where $\varepsilon \geq 2\delta$. Then there exists a Bayesian Nash ε-equilibrium in pure strategies m such that $\mathcal{N}$ and m induce the partition into societies $\{N_1, ..., N_C\}$ where $C \leq QK$.

This is clearly an immediate consequence of the fact that any partition into societies induced by a δ-substitute partition with Q classes must have no more than QK societies. It still, however, is an interesting result in that the number of societies is fixed independently of the size of the population. Thus, if we can envisage a 'family of games' in which as the population grows the number of classes remains the same, then Corollary 1 is an important result. We consider this point in the latter half of the paper. In the following three sub-sections we consider three contrasting extensions to this initial result. In the first extension we strengthen the definition of a society, in the latter two sections we weaken the definition.

5.1 Social conformity with connected societies

Assume that there is a one-to-one characteristic function y mapping N into $[0, 1]$. We interpret y as ordering the player set in the sense that we can put some significance to the fact that $y(i) > y(j)$. In particular, the characteristics of a player as given by y may be a measure of similarity. For example, we will assume for any δ-substitute partition $\{N_1, ..., N_Q\}$ and three players $i, j, k \in N$ that if $i, k \in N_q$, for some q, and $y(i) > y(j) > y(k)$ then

$j \in N_q$. We say that a set $Y \subset N$ is *connected with respect to* y when, for any $i, k \in N$, if there exists a player $j \in N$ such that $y(i) > y(j) > y(k)$ then $j \in Y$. Thus, the set of players with the same class is connected.

It may be intuitive for societies to be connected with respect to some characteristic function y. An example may be useful. Suppose that the characteristics of a player can be summarized by a number from the unit interval. Then, if societies are connected this would seem to imply that the majority of players belong to the same society as those players with the characteristics most similar to their own. This example is taken further in the final part of this paper leading to corollary 5.

We now state our second social conformity result,

Corollary 2: Let $\Gamma \in \mathcal{H}((\delta, Q))$ be any Bayesian game and let $\mathcal{N}$ be any δ-substitute partition with Q classes. Let ε be a positive real number where $\varepsilon \geq 6\delta$. Let y denote any characteristic function. Then there exists a Bayesian Nash ε-equilibrium in pure strategies m such that $\mathcal{N}$ and m induce the partition into societies $\{N_1, ..., N_C\}$ where N_c is connected with respect to y for all $c = 1, ..., C$ and where $C \leq JK$.

Proof: By Theorem 1 there exists a Bayesian Nash 2δ equilibrium in pure strategies σ. Thus,

$$U_i(\sigma) \geq U_i(s_k, \sigma_{-i}) - 2\delta$$

for all $s_k \in \Delta(S)$ and for all $i \in N$. For all $s_k \in S_\Gamma$ and all $q = 1, .., Q$, let n^{kq} denote the number of players $i \in N_q$ such that $\sigma_{ik} = 1$. Assume, subject to a possible reordering of the player set that, for all $i, j \in N$, if $i > j$ then $y(i) > y(j)$. Suppose, further, that player 1 belongs to N_1. Let m be informally defined as the strategy vector where for players $i \in \{1, ..., n^{11}\}$, $m_i = s_1$, for players $i \in \{n^{11} + 1, ..., n^{11} + n^{21}\}$, $m_i = s_2$ and so on. Strategy vector m is such that $\mathcal{N}$ and δ induce a partition into societies $\{N_1, ..., N_C\}$ where N_c is connected with respect to y for all $c = 1, ..., C$. We also note that,

$$\sum_{i \in N_q} g_i(t_z) m_i(a_l|t_z) - \sum_{i \in N_q} g_i(t_z) \sigma_i(a_l|t_z) \leq \beta(N_q) |N_q|$$

for all q and all a_l, t_z. Thus, by anonymity for any $i \in N$,

$$|U_i(s_k, m_{-i}) - U_i(s_k, \sigma_{-i})| < \delta$$

for any strategy $s_k \in \Delta(S)$. Furthermore, for any two players $i, j \in N_q$,

$$|U_i(s_k, m_{-i}) - U_j(s_k, m_{-j})| < \delta$$

for any $s_k \in \Delta(S_\Gamma)$.

For every player $j \in N_q$ and for all q, if $m_{jr} = 1$ then there exists a player $i^{rq} \in N_q$ such that $\sigma_{ir} = 1$. This implies that $U_{i^{rq}}(\sigma) \geq U_{i^{rq}}(s_k, \sigma_{-i^{rq}}) - 2\delta$. Note it is possible that $i = j$. Thus,

$$
\begin{aligned}
U_j(m) \quad \geq \quad & U_j(s_k, m_{-j}) - 2\delta \\
& - |U_{i^{rq}}(\sigma) - U_{i^{rq}}(m)| - |U_{i^{rq}}(m) - U_j(m)| \\
& - |U_{i^{rq}}(s_k, \sigma_{-i^{rq}}) - U_{i^{rq}}(s_k, m_{-i^{rq}})| - |U_{i^{rq}}(s_k, m_{-i^{rq}}) - U_j(s_k, m_{-j})| \\
> \quad & U_j(s_k, m_{-j}) - 6\delta
\end{aligned}
$$

for all $s_k \in \Delta(S_\Gamma)$. This completes the proof■

For the statement of this result the definition of a society is strengthened in requiring societies be connected. As such, the bound on the rationality of social conformity increases. That is, it appears potentially less rational that players form societies which are connected as opposed to societies that may or may not be connected.

5.2 Social conformity in mixed strategies

Suppose that players can choose mixed strategies. We retain the same notion of a society such that two players belonging to the same society must have the same class and same strategy. If we allow players to use mixed strategies then we can derive the following result

Theorem 2: Let $\Gamma \in \mathcal{H}((\delta, Q))$ be any Bayesian game and let $\mathcal{N}$ be any δ-substitute partition with Q classes. Let ε be a positive real number where $\varepsilon \geq 4\delta$. Then, there exists a Bayesian Nash ε equilibrium m such that $\mathcal{N}$ and m induce the partition into societies $\{N_1, ..., N_C\}$ where $C = Q$.

Proof: By Nash's Theorem there exists a Bayesian Nash equilibrium σ of the game Γ. That is, for any $i \in N$ and any strategy $s_k \in \Delta(S_\Gamma)$,

$$U_i(\sigma) \geq U_i(s_k, \sigma_{-i}).$$

There also exists a δ substitute partition $\mathcal{N} = \{N_1, ..., N_Q\}$. For each q let $\sigma(N_q)$ denote the representative strategy of class q. Consider now the strategy vector m such that, for all $i \in N$,

$$\text{if } i \in N_q \text{ then } m_i = \sigma(N_q).$$

Clearly $\mathcal{N}$ and m induce a partition into societies $\{N_1, ..., N_C\}$ where $C = Q$.

We note that the representative strategy for each class q is now,

$$\begin{aligned} m(N_q)(a_l|t_z) &= \frac{1}{\sum_{i\in N_q} g_i(t_z)} \sum_{i\in N_q} g_i(t_z)\sigma(N_q)(a_l|t_z) \\ &= \sigma(N_q)(a_l|t_z). \end{aligned}$$

Thus, by the assumption of anonymity,

$$|U_i(s_k,\sigma_{-i}) - U_i(s_k, m_{-i})| < \delta$$

for all $i \in N$ and all $s_k \in \Delta(S_\Gamma)$.

For each q let $S^q = \{s_k \in S_\Gamma$: there exists a player $i \in N_q$ such that $\sigma_{ik} > 0\}$. We note that for all q and for each degenerate strategy s_r where $support(s_r) \subset S^q$ there exists a player $j^{rq} \in N_q$ such that

$$U_{j^{rq}}(s_r, m_{-j^{rq}}) \geq U_{j^{rq}}(s_k, m_{-j^{rq}}) - 2\delta$$

for all $s_k \in \Delta(S_\Gamma)$. We also note for all $i \in N$ that $support(m_i) \subset S_q$. For any q, for any two players $i, j \in N_q$ and for any strategy s_r where $support(s_r) \subset S^q$,

$$|U_i(s_r, m_{-i}) - U_j(s_r, m_{-j})| < \delta$$

by similarity of payoffs. Thus, for any $i \in N$ and for any $s_r \in S^q$,

$$U_i(m) \geq U_i(s_k, m_{-i}) - 4\delta.$$

for any $s_k \in \Delta(S_\Gamma)$. This completes the proof■

This result shows that if players can use mixed strategies then the number of societies can be bounded by the number of classes. That is, if Γ is a (δ, Q) class game then for any $\varepsilon \geq 4\delta$ there exists a Bayesian Nash ε-equilibrium such that any two players belonging to the same class play the same strategy. It is interesting to note that despite Theorem 1 being a purification result it can be used to prove a similar result to Theorem 2 although the bound becomes 'for all $\varepsilon \geq 6\delta$'.

5.3 Social conformity with roles

As discussed in the introduction it sometimes appropriate that players belonging to the same society should play different actions. This is possible through games of incomplete information by making actions conditional on a player's type - a player's type being determined by nature. Suppose, by way of extension, that players could endogenously choose a set of roles and a probability distribution over those roles.

Formally, suppose that there exists a set of *roles* $R = \{r_1, ..., r_K\}$. Each player can then choose their own *probability distribution over roles* $f_i : R_i \to [0,1]$. Given a Bayesian game $\Gamma = (N, A, T, g, u)$ we then consider a *two stage Bayesian game with endogenous types* Γ^R defined such that

1. In stage 1 each player independently chooses a probability distribution over roles f_i, the choices of players as given by $f = (f_1, ..., f_n)$ are freely communicated,

2. In stage 2 the Bayesian game $\Gamma^f = (N, A, T^f, g^f, u^f)$ is played where,

   (a) $T_i^f = T_i \times R_i$ for all $i \in N$,

   (b) $g_i^f(t_z, r_k) = g_i(t_z) f_i(r_k)$ for all $i \in N$, all $t_z \in T_\Gamma$ and all $r_k \in R$

   (c) $u_i^f((a_1, t_1, r_1), ..., (a_n, t_n, r_n)) = u_i((a_1, t_1), ..., (a_n, t_n))$ for any composition profile .

We highlight that in the Bayesian game Γ^f players are allowed to make their action choice conditional on their role in the same as action choice can be made conditional on type. The standard definition of a an approximate Nash equilibrium still applies. We refer to a strategy of a two stage Bayesian game with endogenous types Γ^R as a pure strategy if each player $i \in N$ chooses a pure strategy for game Γ^f given any choice of f in stage 1.

The definition of a society remains the same - namely that two players in the same society are of the same class and play the same strategy. This means that they have the same choice of probability distributions over roles and the same strategy for the resulting Bayesian game. As such, the players share some common goal or identity. This is despite the fact that different players within the society may end up playing using different actions.

We can now state our final result of this section.

Corollary 3: Let $\Gamma \in \mathcal{H}((\delta, Q))$ be any Bayesian game and let $\mathcal{N}$ be any δ-substitute partition with Q classes. Let ε be a positive real number where $\varepsilon \geq 4\delta$. Then, in the two stage Bayesian game with endogenous types Γ^R there exists a Bayesian Nash ε equilibrium in pure strategies σ such that $\mathcal{N}$ and σ induce the partition into societies $\{N_1, ..., N_C\}$ where $C = Q$.

Proof: Consider an arbitrary player $i \in N$. We note that any mixed strategy σ_i of the game Γ can be decomposed as a choice of strategy m_i in a game Γ^f and choice of probability distribution over roles f_i. In particular, suppose that player i plays pure strategy s_k (from game Γ) if given role r_k, for all k. Then, put $f_i(r_k) = \sigma_i(s_k)$ for all k. Having noted this the result is immediate give Theorem 2■

This result demonstrates that if players have some endogenous system by which players can be assigned roles then we can conceive of societies in which players play different actions. There actions are determined by the role that they are playing within the society. Within this framework the number of societies is again equal to the number of classes. Thus, any two players of the same class play the same strategy.

6 Large Games

We begin be reaffirming that for any game Γ and any Q there exists some δ such that game Γ is a (δ, Q) class game. Thus, the results of above apply to any game. Clearly, in interpretation, however, we would want that δ is small. So, what characteristics of a game imply that there will exist a δ-substitute partition for small δ? We would expect such a game to have the characteristics that, (1) a player's payoff is not largely dependent upon the actions of any small subset of the population, and (2) there is a natural way of grouping players with similar characteristics. Games induced from a pregame with the large game property satisfy both these requirements. In this section to demonstrate the later claim and thus provide an example of how Theorem 1 can be applied in practice.

6.1 Definitions

6.1.1 Pregames

Let Ω denote a compact metric space of *player attributes*. Let A_Γ denote a finite set of actions and T_Γ a finite set of types. The set of pure strategies is given by S_Γ and the set of strategies by $\Delta(S_\Gamma)$. A function from $\Omega \times A_\Gamma \times T_\Gamma$ into $\mathbb{R}_+$ is said to be a *weight function* if it satisfies $\sum_{a_l \in A_\Gamma} \sum_{t_z \in T_\Gamma} w(\omega, a_l, t_z) \in \mathbb{Z}_+$ for all $\omega \in \Omega$. Let W denote the set of weight functions.

A *pregame* is given by a tuple $\mathcal{G} = (\Omega, A_\Gamma, T_\Gamma, g, h)$, consisting of a compact metric space Ω, finite sets A_Γ and T_Γ, a function $g : \Omega \times T_\Gamma \longrightarrow [0, 1]$ and a function $h : \Omega \times \Delta(S_\Gamma) \times W \longrightarrow \mathbb{R}_+$.

6.1.2 Societies and games

Let N be a finite set and let α be a mapping from N to Ω, called an *attribute function*. The pair (N, α) is a *population*. The *profile* of the population (N, α) is a function *profile*$(N, \alpha) : \Omega \to \mathbb{Z}_+$ given by

$$profile(N, \alpha)(\omega) = \left|\alpha^{-1}(\omega)\right|$$

Thus, the profile of a population tells us the number of players with each attribute in the population.

Given a population (N, α) and a strategy vector σ for the population (N, α) we say that weight function $w_{\alpha,\sigma}$ is *relative to strategy vector σ and attribute function α* if,

$$w_{\alpha,\sigma}(\omega, a_l, t_z) = \sum_{i \in N:\ \alpha(i)=\omega} \sigma_i(a_l|t_z)g_i(t_z)$$

for all $a_l \in A_\Gamma$, $t_z \in T_\Gamma$ and all $\omega \in \Omega$. In interpretation, given the population (N, α) and the strategy vector σ,

$$\frac{w_{\alpha,\sigma}(\omega, a_l, t_z)}{|profile(N, \alpha)(\omega)|}$$

is the expected proportion of times composition $c_r = (a_l, t_z)$ will be seen by a player of attribute ω. We note that,

$$\begin{aligned} \sum_{a_l \in A_\Gamma} \sum_{t_z \in T_\Gamma} w_{\alpha,\sigma}(\omega, a_l, t_z) &= \sum_{a_l \in A_\Gamma} \sum_{t_z \in T_\Gamma} \sum_{i \in N:\ \alpha(i)=\omega} \sigma_i(a_l|t_z)g_i(t_z) \\ &= |profile(N, \alpha)(\omega)|. \end{aligned}$$

Let W_α denote the set of weight functions relative to attribute function α.

Given a population (N, α) and player $i \in N$, define α_{-i} as the restriction of α to $N \backslash \{i\}$. Thus, given an attribute function α and strategy vector σ, for all $\omega \in \Omega$, all $s_k \in S$ and for all $i \in N$,

$$w_{\alpha_{-i},\sigma}(\omega, a_l, t_z) = \begin{cases} w_{\alpha,\sigma}(\omega, a_l, t_z) - \sigma_i(a_l|t_z)g_i(t_z) \text{ if } \alpha(i) = \omega \\ w_{\alpha,\sigma}(\omega, a_l, t_z) \text{ otherwise.} \end{cases}$$

Weight functions modified by the property that one player of some particular attribute is not included will play a role in the definition of games. We will use $W_{\alpha-\omega}$ to denote the set of weight functions relative to α_{-i} where $\omega = \alpha(i)$.

6.1.3 Induced games

Given a population (N, α), a *game*

$$\Gamma(N,\alpha) = \left(\begin{array}{c} (N,\alpha), A_\Gamma, T_\Gamma, \{g_\omega : T_\Gamma \longrightarrow [0,1] | \omega \in \alpha(N)\} \\ \{h_\omega : \Delta(S_\Gamma) \times W_{\alpha-\omega} \longrightarrow \mathbb{R}_+ | \omega \in \alpha(N)\} \end{array} \right)$$

is induced from the pregame $\mathcal{G} = (\Omega, A_\Gamma, T_\Gamma, g, h)$ by defining

$$g_\omega(t_z) \stackrel{\text{def}}{=} g(\omega, t_z)$$

and

$$h_\omega(s_k, w) \stackrel{\text{def}}{=} h(\omega, s_k, w)$$

for all $s_k \in \Delta(S_\Gamma), w \in W_{\alpha-\omega}$ and $\omega \in \alpha(N)$. In interpretation, $h_{\alpha(i)}(s_k, w)$ is the payoff received by a player $i \in N$ of attribute $\alpha(i)$ from playing the strategy s_k when the strategies of other players are summarized by w. Thus, players of the same attribute have the same payoff function, inherited from the pregame. Similarly, $g_\omega(t_z)$ is the probability that a player $i \in N$ of attribute $\alpha(i)$ is of type t_z.

We impose the standard assumptions on the linearity of payoffs with respect to mixed strategies. To explain further it is useful to relate the utility function induced from the pregame $h_\omega : \Delta(S_\Gamma) \times W_{\alpha-\omega} \longrightarrow \mathbb{R}_+$ to the expected utility function $U_i : S \longrightarrow \mathbb{R}_+$ as used in the first half of this paper. Consider a game $\Gamma(N, \alpha)$ induced from the pregame $\mathcal{G}$. We assume that this game $\Gamma(N, \alpha)$ is equivalent to the Bayesian game $\overline{\Gamma}(N, A, T, \overline{g}, \overline{U})$ where $A = \times_{i \in N} A_\Gamma$, $T = \times_{i \in N} T_\Gamma$, $\overline{g}(t) = \prod_{i \in N} g_\omega(t_i)$ for all $t \in T$ and $U_i(\sigma) = h_{\alpha(i)}(\sigma_i, w_{\alpha_{-i},\sigma}) =$ for all σ and all $i \in N$.

6.2 Large game property

We continue by defining the concepts of global interaction and continuity in attributes. These two concepts allow us to introduce the *large game property.* This property constitutes an assumption on a pregame $\mathcal{G} = (\Omega, A_\Gamma, T_\Gamma, g, h)$. In particular, it places restrictions on the payoff function h and distribution over types g of the pregame. As a preliminary step, let $F(\mathcal{G}, n)$ denote the set of games induced by the pregame $\mathcal{G}$ by populations of finite size n. That is game $\Gamma(N, \alpha) \in F(\mathcal{G}, n)$ if and only if $|N| = n$.

Global Interaction: Given positive real numbers $\delta > 0$ and $\tau > 0$ the game $\Gamma(N, \alpha)$ is said to satisfy δ, τ*-global interaction* when for any two weight functions w_α and g_α, both relative to attribute function α, if,

$$\frac{1}{|N|}\sum_{a_l\in A_\Gamma}\sum_{t_z\in T_\Gamma}\sum_{\omega\in\alpha(N)}|w_\alpha(\omega,a_l,t_z)-g_\alpha(\omega,a_l,t_z)|<\tau$$

then,

$$\left|h_{\alpha(i)}(s_k,w_{\alpha_{-i}})-h_{\alpha(i)}(s_k,g_{\alpha_{-i}})\right|<\delta$$

for all $i\in N$ and all $s_k\in\Delta(S_\Gamma)$.

Continuity in attributes: Given positive real numbers $\delta>0$ and $\tau>0$, the set of games $F(\mathcal{G},n)$ is said to satisfy δ,τ-*continuity in attributes* when for any two games $\Gamma(N,\alpha)$ and $\Gamma(N,\overline{\alpha})$ in $F(\mathcal{G},n)$, if, for all $i\in N$,

$$dist(\alpha(i),\overline{\alpha}(i))<\tau$$

then, for any $j\in N$ and for any strategy vector σ,

$$\left|h_{\alpha(j)}(s_k,w_{\alpha_{-j},\sigma})-h_{\overline{\alpha}(j)}(s_k,w_{\overline{\alpha}_{-j},\sigma})\right|<\delta$$

for all $s_k\in\Delta(S_\Gamma)$. where $w_{\alpha,\sigma}$ and $w_{\overline{\alpha},\sigma}$ are the weight functions relative to strategy vector σ and, respectively, attribute functions α and $\overline{\alpha}$.

We can now introduce the main property,

Large game property: The pregame $\mathcal{G}=(\Omega,A_\Gamma,T_\Gamma,g,h)$ satisfies the *large game property* if for any $\delta>0$ and there exists real numbers $\eta_l(\delta)$, $\tau_g(\delta)>0$ and $\tau_c(\delta)>0$ such that for any $n>\eta(\delta)$ the set of games $F(\mathcal{G},n)$ satisfy $\delta,\tau_c(\delta)$-*continuity in attributes* and any game $\Gamma(N,\alpha)\in F(\mathcal{G},n)$ satisfies $\delta,\tau_g(\delta)$-*global interaction.*

Thus, the pregame $\mathcal{G}$ satisfies the large game property if both global interaction and continuity of payoff functions are satisfied by large games. The large game property implies a form of continuity of $h:\Omega\times\Delta(S_\Gamma)\times W\longrightarrow\mathbb{R}_+$ with respect to changes in the weight function w and attribute ω while the strategy s_k remains constant. It also, implies a form of continuity on $g:\Omega\times T_\Gamma\longrightarrow[0,1]$.

A detailed motivation and explanation of the above assumptions, for games of complete information, is provided by Wooders, Cartwright and

Selten (2001) so we give here only a brief discussion. Global interaction says that players payoffs a largely a function of their own strategy and on the numbers of players we each attribute of each type playing each action. As such a player's payoff is not largely dependent on the actions of any small group of individuals. This clearly has a close relationship with the notion of anonymity in the definition of δ-substitutes.

The assumption of continuity in attributes is a more wide ranging assumption. Essentially, given a strategy vector σ, it says that the attributes of all players can be slightly perturbed and the payoff to each player remains largely unaffected. The first thing we should highlight is how two distinct games $\Gamma(N, \alpha)$ and $\Gamma(N, \overline{\alpha})$ are compared. This is possible through the use of the pregame. The assumption thus formalizes the intuition that if a population changes only slightly from (N, α) to $(N, \overline{\alpha})$ then the games induced by these societies should be largely the same.

One element of continuity in attributes that should be emphasized. Namely, even though the strategies of the players remain the same, in both societies, their attributes have changed and thus their prior probability over types may have changed. As such, the same strategy can imply a different probability distribution over compositions. This would imply that implicit in the continuity in attributes assumption is the idea that players with similar attributes should have similar probability distributions over types.

6.3 Preliminary result

Having defined the large game property we can now go on to apply the results from the first half of this paper. To do this we need to find a connection between the games induced by the a pregame $\mathcal{G}$ satisfying the large game property and the set $\mathcal{H}(\delta, Q)$ for some Q. This relationship is not straightforward. We can show, however, that for any pregame $\mathcal{G}$ and any sufficiently large game $\Gamma(N, \alpha)$ induced by that pregame there is 'nearby' a game $\Gamma(N, \overline{\alpha})$ which is also induced by the pregame and is a (δ, Q)-class game for some Q. Formally, we have,

Lemma 3: If the pregame $\mathcal{G}$ satisfies the large game property then given any real numbers $\delta > 0$ and $\tau > 0$ there exists real numbers $\eta(\delta, \tau)$ and $Q(\delta, \tau)$ such that for any population (N, α), where $|N| > \eta(\varepsilon)$, there exists a population $(N, \overline{\alpha})$ such that $\max_{i \in N}\{dist(\alpha(i), \overline{\alpha}(i))\} < \tau$ and the induced game $\Gamma(N, \overline{\alpha})$ belongs to the set $\mathcal{H}(\delta, Q(\delta, \tau))$.

Proof: Suppose that the statement of the lemma is false. Then there is some $\overline{\delta} > 0$ and $\overline{\tau} > 0$, such that for any real number $\overline{Q}$ and for each integer ν there is a population (N^ν, α^ν) where $|N^\nu| < \nu$ and such that for no population $(N^\nu, \overline{\alpha}^\nu)$ where $\max_{i\in N}\{dist(\alpha(i), \overline{\alpha}(i))\} < \overline{\tau}$ does the induced game $\Gamma(N^\nu, \overline{\alpha}^\nu)$ belongs to the set $\mathcal{H}(\delta, \overline{Q})$.

Given that the pregame $\mathcal{G}$ satisfies the large game property we may choose non-negative real numbers $\eta_l(\delta)$, $\tau_g(\delta) > 0$ and $\tau_c(\delta) > 0$ such that for any $n > \eta(\delta)$ the set of games $F(\mathcal{G}, n)$ satisfy $\delta, \tau_c(\delta)$-continuity in attributes and any game $\Gamma(N, \alpha) \in F(\mathcal{G}, n)$ satisfies $\delta, \tau_g(\delta)$-global interaction. Let $\tau = \min\{\tau_c(\delta), \overline{\tau}\}$. Partition Ω into subsets $\Omega_1, ..., \Omega_Q$ each of diameter less than τ, that is, for any $\omega, \omega' \in \Omega_q$ and for any q, $dist(\omega, \omega') < \tau$. To each subset Ω_q choose and fix an attribute ω_q. Define the attribute function $\overline{\alpha}^\nu$ as follows, for all $i \in N^\nu$,

$$\overline{\alpha}^\nu(i) = \omega_q \text{ if and only if } \alpha(i) \in \Omega_q.$$

For each ν consider the game $(N^\nu, \overline{\alpha}^\nu)$. Let $\mathcal{N}^\nu = \{N_1^\nu, ..., N_Q^\nu\}$ denote, for each ν, the partition of the player set such that $i \in N_q^\nu$ if and only if $\overline{\alpha}^\nu(i) = \omega_q$. We propose $\mathcal{N}^\nu$ as a candidate for a δ-substitute partition in the game $(N^\nu, \overline{\alpha}^\nu)$ for each ν.

We begin by noting that $\beta(N_q) = 0$ is similarity of prior probabilities for each class q. With regard to anonymity, for an arbitrary ν, take any two strategy profiles $\sigma^{\nu 1}$ and $\sigma^{\nu 2}$, of the game $\Gamma(N^\nu, \overline{\alpha}^\nu)$ where,

$$\max_{a_l\in A_\Gamma} \max_{t_z\in T_\Gamma} \left| \left(\sigma^{\nu 1}(N_q)(a_l|t_z) - \sigma^{\nu 2}(N_q)(a_l|t_z)\right) \sum_{i\in N_q} g_i(t_z) \right| < 1 \tag{5}$$

for all q, for all $a_l \in A_\Gamma$ and $t_z \in T_\Gamma$. Let $w_{\overline{\alpha}^\nu, \sigma^{\nu 1}}$ denote the weight function relative to attribute function $\overline{\alpha}^\nu$ and strategy vector $\sigma^{\nu 1}$. Let $w_{\overline{\alpha}^\nu, \sigma^{\nu 2}}$ denote the weight function relative to attribute function α^ν and strategy vector $\sigma^{\nu 2}$. By (5) (and that $\beta(N_q) = 0$ for all q) we have that,

$$\frac{1}{|N^\nu|} \sum_{a_l\in A_\Gamma} \sum_{t_z\in T_\Gamma} \sum_{\omega\in\alpha(N^\nu)} \left| w_{\overline{\alpha}^\nu, \sigma^{\nu 1}}(\omega, a_l, t_z) - w_{\overline{\alpha}^\nu, \sigma^{\nu 2}}(\omega, a_l, t_z) \right| < \frac{Q\,|A_\Gamma|\,|T_\Gamma|}{|N^\nu|}. \tag{6}$$

By global interaction and (6) there exists a ν^* such that if $\nu > \nu^*$,

$$\left| h_{\alpha(j)}(s_k, w_{\overline{\alpha}_{-j}, \sigma^{\nu 1}}) - h_{\alpha(j)}(s_k, w_{\overline{\alpha}_{-j}, \sigma^{\nu 2}}) \right| < \delta$$

for all $s_k \in \Delta(S_\Gamma)$.

We now consider similarity of payoff functions. For an arbitrary ν consider any strategy vector σ^ν of the game $\Gamma(N^\nu, \overline{\alpha}^\nu)$. Let $i, j \in N$ be any two players such that $i, j \in N_q$ for some q. Let $\overline{\sigma}^\nu$ be the strategy vector such that $\overline{\sigma}_i^\nu = \sigma_j^\nu$ and for all $l \neq i$, $\overline{\sigma}_l^\nu = \sigma_l^\nu$. We note that $w_{\alpha_{-j},\overline{\sigma}^\nu} = w_{\alpha_{-i},\sigma^\nu}$. Further, by global interaction for any $\nu > \nu^*$,

$$\left| h_{\overline{\alpha}(j)}(s_k, w_{\alpha_{-j},\sigma^\nu}) - h_{\overline{\alpha}(j)}(s_k, w_{\alpha_{-j},\overline{\sigma}^\nu}) \right| < \delta$$

for all $s_k \in \Delta(S_\Gamma)$. Thus, similarity of payoff functions and anonymity hold for all games $\Gamma(N^\nu, \overline{\alpha}^\nu)$ where $\nu > \nu^*$. This completes the proof■

6.4 Approximate purification in Large games

Lemma 3 allows us to apply all of the results obtained for (δ, Q)-class games to sufficiently large games induced by a pregame satisfying the large game property. We demonstrate with three additional results. The first result shows the existence of a pure strategy Bayesian Nash ε-equilibrium.

Corollary 4: Given a real number $\varepsilon > 0$ there exists a real number $\eta(\varepsilon) > 0$ such that if the pregame $\mathcal{G}$ satisfies the large game property, then for any population (N, α) where $|N| > \eta(\varepsilon)$, the induced game $\Gamma(N, \alpha)$ has an ε-equilibrium in pure strategies.

Proof: Let $\delta = \frac{\varepsilon}{4}$. Given that the pregame $\mathcal{G}$ satisfies the large game property we may choose non-negative real numbers $\eta_l(\delta)$, $\tau_g(\delta) > 0$ and $\tau_c(\delta) > 0$ such that for any $n > \eta(\delta)$ the set of games $F(\mathcal{G}, n)$ satisfy $\delta, \tau_c(\delta)$-continuity in attributes and any game $\Gamma(N, \alpha) \in F(\mathcal{G}, n)$ satisfies $\delta, \tau_g(\delta)$-global interaction. Let $\tau = \tau_c(\delta)$.

By Lemma 3 there exists a real number $\eta(\delta, \tau)$ such that for any population (N, α), where $|N| > \eta(\delta, \tau)$, there exists a population $(N, \overline{\alpha})$ such that $\max_{i \in N}\{dist(\alpha(i), \overline{\alpha}(i))\} < \tau$ and the induced game $\Gamma(N, \overline{\alpha})$ belongs to the set $\mathcal{H}(\delta, Q)$ for some finite real number Q. By Theorem 1, for any population (N, α), where $|N| > \eta(\delta, \tau)$, there exists a population $(N, \overline{\alpha})$ such that $\max_{i \in N}\{dist(\alpha(i), \overline{\alpha}(i))\} < \tau$ and the induced game $\Gamma(N, \overline{\alpha})$ has a Bayesian Nash ε-equilibrium in pure strategies m. Thus, for all $i \in N$,

$$h_{\overline{\alpha}(i)}(m_i, w_{\overline{\alpha}_{-i},m}) \geq h_{\overline{\alpha}(i)}(s_k, w_{\overline{\alpha}_{-i},m}) - \frac{\varepsilon}{2}$$

for all $s_k \in \Delta(S_\Gamma)$. By choice of τ and continuity in attributes,

$$\left| h_{\alpha(i)}(s_k, w_{\alpha_{-i},m}) \geq h_{\overline{\alpha}(i)}(s_k, w_{\overline{\alpha}_{-i},m}) \right| < \frac{\varepsilon}{4}$$

for all $s_k \in \Delta(S_\Gamma)$. Thus, m is a Bayesian Nash ε-equilibrium of the game $\Gamma(N, \alpha)$.

6.5 Social conformity in large games

Corollary 4 and its proof show how the results from the framework of approximate substitutes in non-cooperative games can be applied in a straightforward way within the framework of non-cooperative pregames satisfying the large game property. As such, in the section, we state without proof two social conformity results.

Before, stating our next result we define a further term. Given any population (N, α) and any player $i \in N$ we say that player j is player i's *closest neighbor* if $dist(\alpha(i), \alpha(j)) \leq \min_{k \in N}\{dist(\alpha(i), \alpha(k)\}$. A person may have more than one closest neighbor. The following result, applying corollary 2, demonstrates how we can apply the notion of a characteristic function on the set of players.

Corollary 5: Let $\mathcal{G}$ be any pregame satisfying the large game property and assume the space of attributes is given by $\Omega = [0, 1]$. Given any $\varepsilon > 0$ there exists a real number $\eta(\varepsilon)$ such that for any population (N, α) where $|N| > \eta(\varepsilon)$ there is an ε-equilibrium in pure strategies of the induced game $\Gamma(N, \alpha)$ with the property that at least $|N|(1 - \varepsilon)$ players play the same strategy as a closest neighbor.

In contexts where players attributes can be ordered along the unit interval (Greenberg and Weber (1986) for example) this result clearly demonstrates that for large games it can be efficient for similar players to play similar strategies. We highlight, however, that a player may not play the same strategy as *all* of their closest neighbors but the majority of players will play the same strategy as *a* closest neighbor.

Before stating our final result we refine the definition of a society to reflect the existence of a space of attributes. Firstly we assume that the attributes space Ω is a compact subset of a normed, real linear space.[4] The choice of attribute space allows us to treat convex subsets of Ω and to define a society.[5] Given a population (N, α) and strategy vector σ we interpret a

[4] An example of such an attribute space is provided in Wooders, Cartwright and Selten (2001).

[5] Throughout this paper we will treat convexity as a property relative to an attribute space Ω rather than the linear space of which Ω is a subset. Formally, given an attribute space Ω we say that $A \subset \Omega$ is convex when for any two points $a, b \in A$ and for any $\lambda \in [0, 1]$,

$$\begin{aligned} \text{if } \lambda a + (1-\lambda)b &\in \Omega \\ \text{then } \lambda a + (1-\lambda)b &\in A. \end{aligned}$$

set of players D as a society (relative to α and σ) if (i) there exists some strategy $t \in \Delta(S)$ such that $\sigma_i = t$ for all $i \in D$, and (ii) for any player $i \in N$ if $i \in$ *convex hull*$(\alpha(D))$ then $i \in D$. Thus, any two players belonging to a society D must play the same strategy. Furthermore, to any society D we can associate a convex subset Ω_D of attribute space Ω with the properties that any player i belonging to D has attributes in Ω_D while there exists no other player $j \in N \backslash D$ who has attributes in Ω_D.

Corollary 6: Let $\mathcal{G}$ be any pregame satisfying the large game property. Then, for any $\varepsilon > 0$ there exists real numbers $\eta(\varepsilon)$ and $Q(\varepsilon)$ such that for any population (N, α) where $|N| > \eta(\varepsilon)$ the induced game $\Gamma(N, \alpha)$ has a Bayesian Nash ε equilibrium σ such that $\mathcal{N}$ and σ induce the partition into societies $\{N_1, ..., N_C\}$ where $C \leq Q(\varepsilon)$ for some partition $\mathcal{N}$.

From the earlier discussion it is trivial to note that this result also implies the existence of a Bayesian Nash ε equilibrium in pure strategies m for the two stage Bayesian game with endogenous types $\Gamma^R(N, \alpha)$ which partitions the society into less than $Q(\varepsilon)$ societies for all populations (N, α) where $|N| > \eta(\varepsilon)$.

7 Conclusion

Can it be rational for individuals to form societies in which all players within a society play the same strategy? By using the framework of games with classes we are able to put bounds on the rationality of such social conformity for arbitrary games and for arbitrary numbers of societies. We then apply this to large games induced by a pregame satisfying the large game property. Such games are characterized primarily by the fact that no player's payoff is overly dependent on any small subset of the population. For such games we demonstrate that there does exist a Bayesian Nash ε equilibrium, for arbitrarily small ε, which partitions the player set into a relatively small number of societies. It would appear, therefore, that in such games social conformity can be boundedly rational.

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