EXAMPLE OF A SIMPLE OPTICAL SYSTEM†

A system of two thin lenses is given as shown in Fig. 1. The left thin lens has a focal distance of $f_1 = 50\,\text{mm}$ (converging) and the right thin lens has a focal distance of $f_2 = 25\,\text{mm}$ (converging also). The two thin lenses are separated by 40 mm. An object is placed at a distance of 75 mm to the left of the left thin lens. Find the position and magnification of the final image using (a) the method of matrices, (b) the thin lens equation, and (c) the method of the cardinal points.

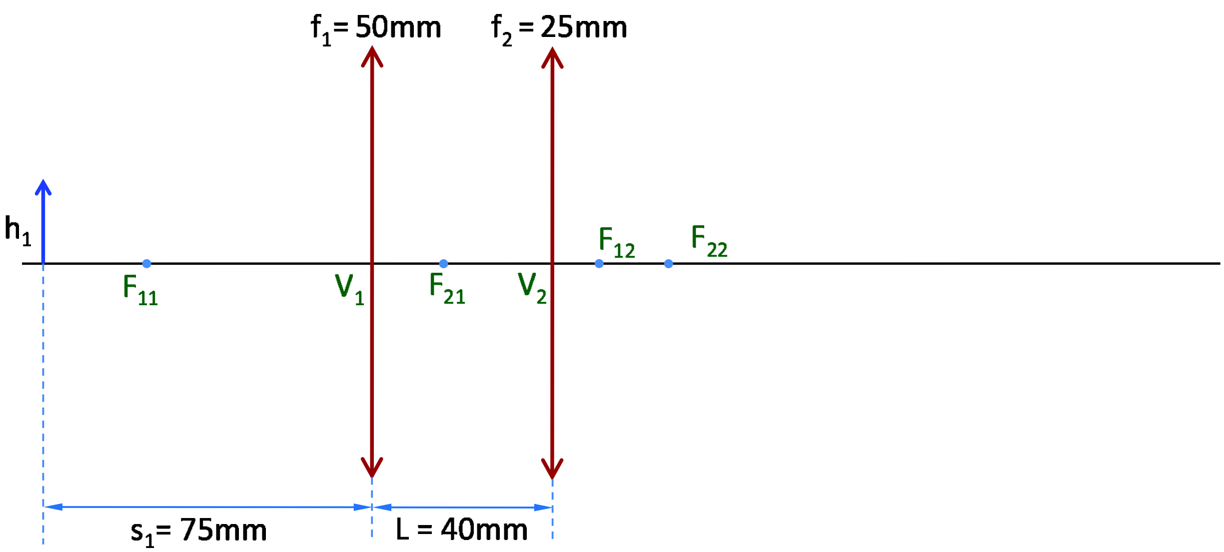

Figure 1: The optical system consisting of two thin-lenses of focal distances of 50mm and 25mm respectively, separated by a distance of 40mm.

A. Method of ABCD Matrices

The matrix ABCD of the two thin lenses separated by a distance $L = 40\,\text{mm}$ can be determined by the multiplication of three basic ABCD matrices, the right thin lens matrix, the propagation by distance L matrix, and the left thin lens matrix. Multiplying the three matrices the ABCD of the optical system comprising of the two thin lenses can be calculated as follows:

$$\begin{bmatrix} A_{TL} & B_{TL} \\ C_{TL} & D_{TL} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{25} & 1 \end{bmatrix} \begin{bmatrix} 1 & 40 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{50} & 1 \end{bmatrix} = \begin{bmatrix} 0.20 & 40\,mm \\ -0.028\,mm^{-1} & -0.6 \end{bmatrix}. \quad (1)$$

Now it is assumed that the image is formed at a distance x to the right of the right thin lens. Then if the object plane is considered as the input plane and the image plane as the output

†written by Prof. Elias N. Glytsis, Last Update: March 30, 2018

plane of the overall optical system, the overall ABCD matrix can be determined as follows (see Figure 2):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{TL} & B_{TL} \\ C_{TL} & D_{TL} \end{bmatrix} \begin{bmatrix} 1 & 75 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.20 - 0.028x & 55 - 2.7x(mm) \\ -0.028\,mm^{-1} & 2.7 \end{bmatrix}. \tag{2}$$

In order to achieve image formation it is necessary for the overall ABCD matrix element B to become zero. When $B = 0$ the value of A gives the overall magnification of the image. Therefore,

$$B = 0 \Longrightarrow 55 - 2.7x = 0 \Longrightarrow x = 20.37\,mm, \tag{3}$$

while the magnification, m, is given by

$$m = A(x = 20.37) = 0.2 - 0.028x = -0.37. \tag{4}$$

Therefore, the image is real and inverted.

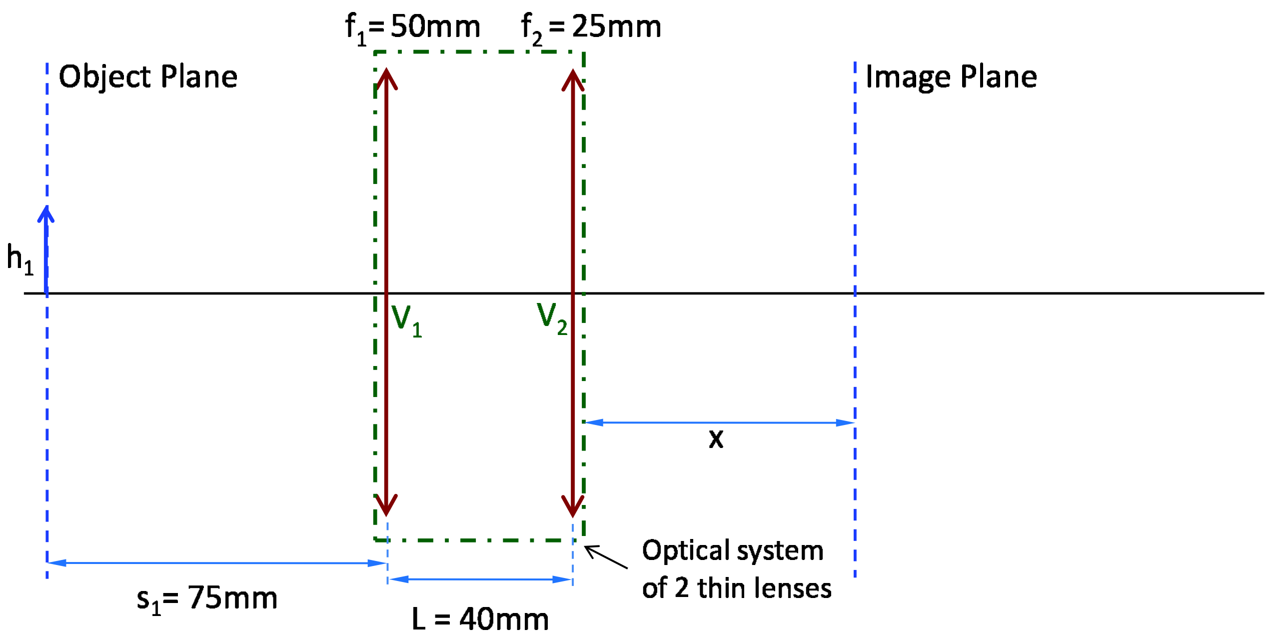

Figure 2: The optical system consisting of two thin-lenses of focal distances of 50mm and 25mm respectively, separated by a distance of 40mm is represented by the dashed rectangle and an equivalent ABCD matrix. The image plane is considered to be at the unknown distance x away from the right thin lens.

B. Method of Thin Lenses

In this approach the thin lens equation is used sequentially for each lens. First the thin lens equation is applied for the left lens. The object distance $s_1 = 75\,\text{mm}$ and the focal distance

$f_1 = +50\,\mathrm{mm}$. Applying the thin lens equation in this case the image distance,s_1' due to the first lens (if the second lens is temporarily neglected)can be determined from the equation

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \Longrightarrow \frac{1}{75} + \frac{1}{s_1'} = \frac{1}{50} \Longrightarrow s_1' = 150\,mm. \tag{5}$$

The image formed by the first lens serves as the object for the second (right) lens. Since the image is to the right of the right lens it will serve as a virtual object for the second lens. Therefore, $s_2 = -(150 - 40) = -110\,\mathrm{mm}$. Applying the thin lens equation for the second lens the final image location can be determined.

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \Longrightarrow \frac{1}{-110} + \frac{1}{s_2'} = \frac{1}{25} \Longrightarrow s_2' = 20.37\,mm. \tag{6}$$

Therefore, the final image is formed 20.37 mm to the right of the right lens and is a real image. The magnification can be found as follows

$$m = \frac{h_2'}{h_1} = \frac{h_1'}{h_1}\frac{h_2'}{h_2 = h_1'} = \Big(-\frac{s_1'}{s_1}\Big)\Big(-\frac{s_2'}{s_2}\Big) = -\frac{150}{75}\Big(-\frac{20.37}{-110}\Big) = -0.37, \tag{7}$$

implying the the final image is inverted (and real as it was determined through s_2'). A ray diagram of the image formation by the two lenses is shown in Fig. 3.

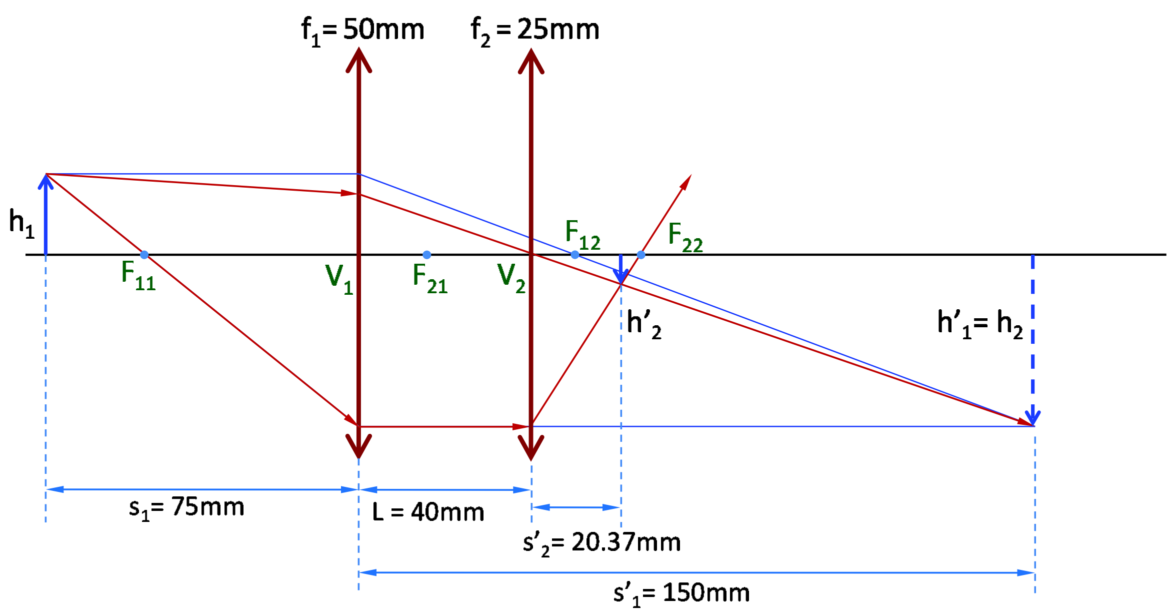

Figure 3: A rays diagram for the two-lenses system.

C. Method of Cardinal Points

In order to determine the cardinal points ($H_1 \equiv N_1$, $H_2 \equiv N_2$, F_1, and F_2), the matric ABCD of the optical system comprised of the two thin lenses must be used. This matrix was computed earlier and is equal to

$$\begin{bmatrix} A_{TL} & B_{TL} \\ C_{TL} & D_{TL} \end{bmatrix} = \begin{bmatrix} 0.20 & 40\,mm \\ -0.028\,mm^{-1} & -0.6 \end{bmatrix}. \tag{8}$$

Using the elements of the matrix the signed distances, f_1, f_2, r, and s (see Fig. 4) can be determined as follows:

$$\begin{aligned} f_1 &= \frac{n_0/n_f}{C_{TL}} = \frac{1}{-0.028} = -35.714\,mm, && (9) \\ f_2 &= -\frac{1}{C_{TL}} = -\frac{1}{-0.028} = +35.714\,mm, && (10) \\ r &= \frac{D_{TL} - (n_0/n_f)}{C_{TL}} = \frac{-0.6 - 1}{-0.028} = +57.1428\,mm, && (11) \\ s &= \frac{1 - A_{TL}}{C_{TL}} = \frac{1 - 0.2}{-0.028} = -28.5714\,mm, && (12) \end{aligned}$$

Now the thin lens equation can be used by suitably chosing the object and image distances s_o and s_i by measuring them from the first and the second principal planes respectively. For this case $s_o = 75 + r = 132.1428$ mm. The focal distance of the system is $f = 35.714$ mm. Then the thin lens equation can be written as follows

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \Longrightarrow \frac{1}{132.1428} + \frac{1}{s_i} = \frac{1}{35.714} \Longrightarrow s_i = 48.9412\,mm. \tag{13}$$

Therefore the position of the real image ($s_i > 0$) from the right thin lens is $s_i - |s| = 48.9412 - 28.5714 = 20.37$ mm. The magnification is $m = -s_i/s_o = -48.9412/132.1428 = -0.37$ as it was computed and with the other methods. A ray diagram using the cardinal points is shown in Fig. 5.

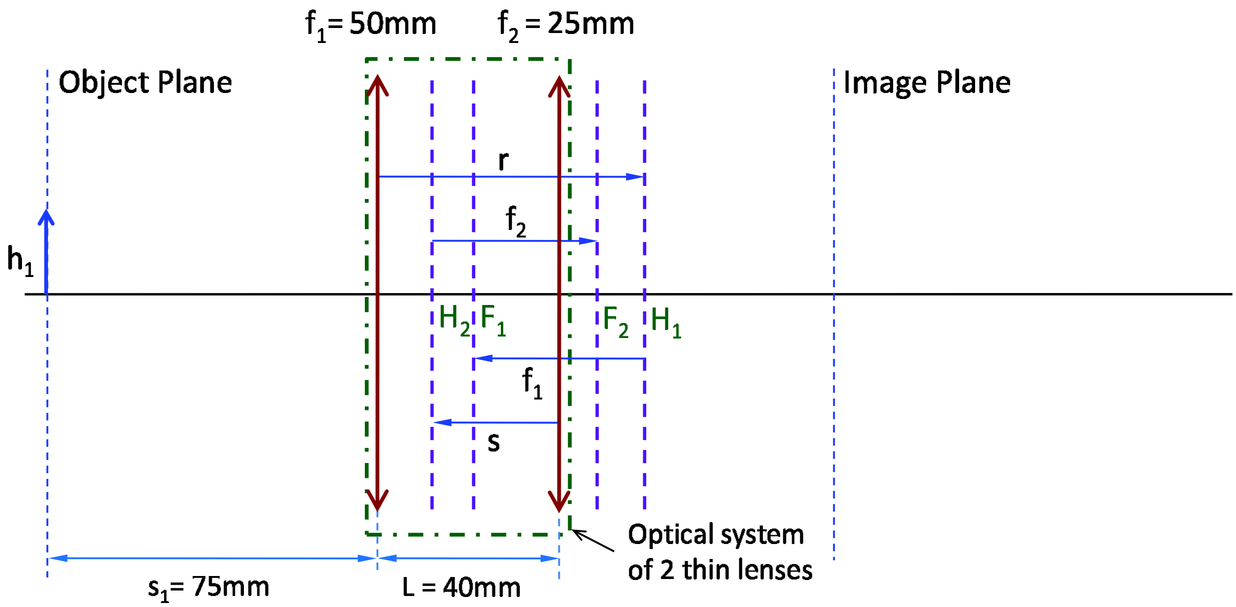

Figure 4: Important distances for the determination of the cardinal points of the two lenses optical system. The nodal points N_1 and N_2 coincide with the principal points H_1 and H_2 respectively.

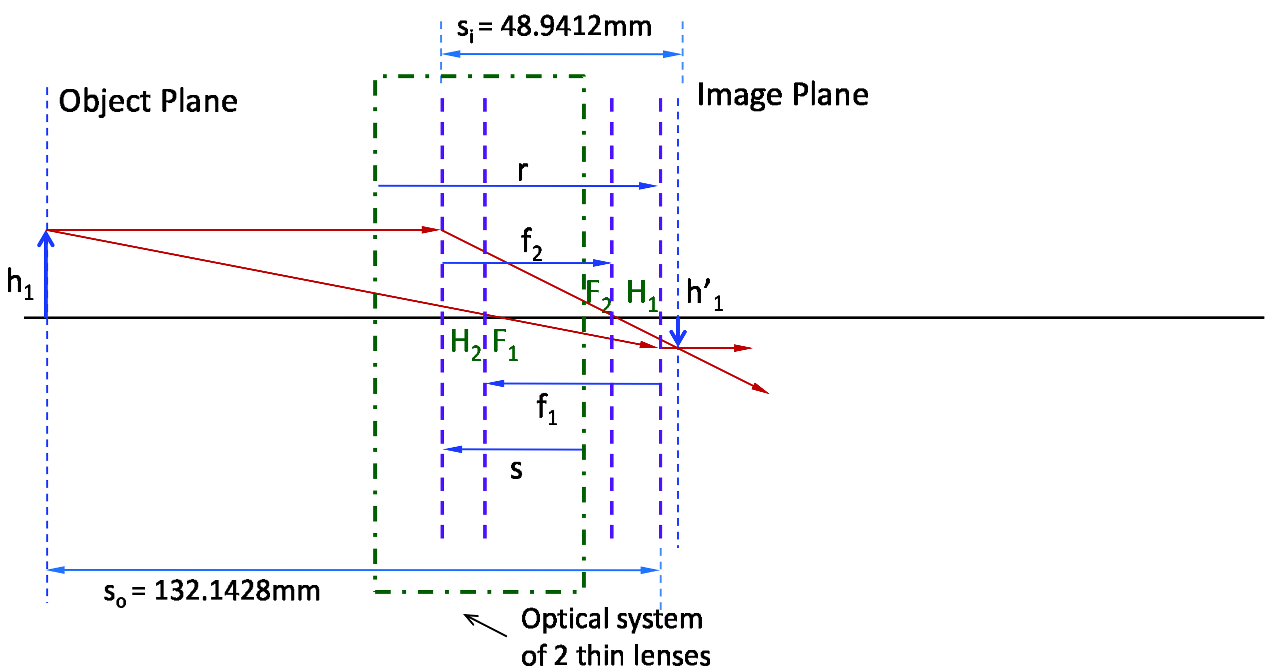

Figure 5: A rays diagram that shows the formation of the image through the use of the cardinal points of the two lenses system.