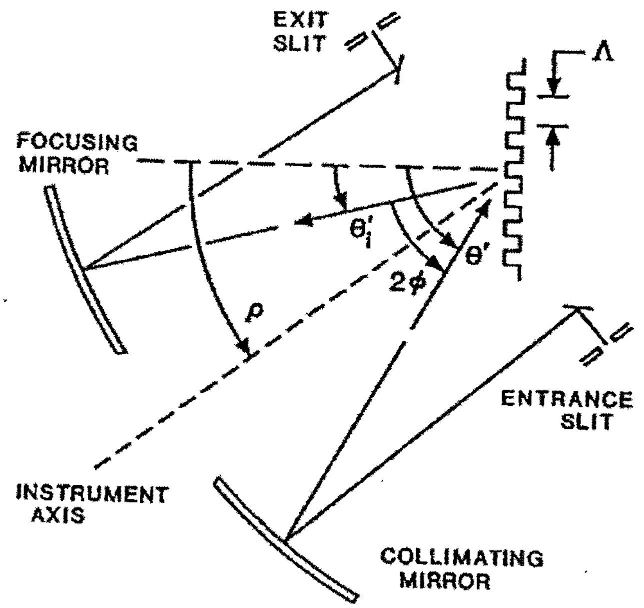
# Optical Engineering
# Problem Set No. 5

## Problem 1: (Czerny-Turner Monochromater)




(a)

Ebert angle $2\phi = 9.2°$, $\Lambda = \frac{1}{1200}$ mm $= 0.8333\,\mu m$

$i = 1$ for the grating. $\rho = 17.5158°$

Grating Equation:

$$\sin\theta' + \sin\theta_i' = i\frac{\lambda_0}{\Lambda n_1} \qquad i = 1 \qquad n_1 = 1.0$$

$$\theta' = \rho + \varphi \qquad \theta_1' = \rho - \varphi \qquad \text{(from figure)}$$

Therefore, $\therefore \lambda_0 = \Lambda\left(\sin\underbrace{(\rho+\varphi)}_{\theta'} + \sin\underbrace{(\rho-\varphi)}_{\theta_1'}\right)$

$\lambda_0 = 500\,nm$

(b) $\left.\frac{d\lambda_0}{d\theta_1'}\right|_{\theta' = \text{constant}} = \Lambda\cos(\rho-\varphi) \rightsquigarrow \frac{d\lambda_0}{d\theta_1'} = 812.2497$ nm/rad

or 14.176 nm/deg

$$\frac{d\lambda_0}{d\rho} = \Lambda\left[\cos(\rho+\varphi) + \cos(\rho-\varphi)\right] = 1584.3 \text{ nm/rad}$$ or 27.651 nm/deg

## Problem 2: (Grating Diffraction)

(a) $\Lambda = 5\mu m$ $n_o = 2.00$ , $d = 250\mu m$

$\lambda_o = 0.6328\ \mu m$ $\theta' = 20°$

The Bragg angle can be specified from the equation

$$m\frac{\lambda_o}{n_o} = 2\Lambda \sin\theta$$ $\theta$ : inside the grating

For $m=1$ we have : $\frac{\lambda_o}{n_o} = 2\Lambda \sin\theta$

$m = 1$

$$\Rightarrow \quad n_o \sin\theta = \frac{\lambda_o}{2\Lambda} \qquad (1)$$

The angle in air $\theta'$ is related with $\theta$ through Snell's Law:

$$n_o \sin\theta = n_I \sin\theta' = \sin\theta' \qquad (n_I = n_{air} = 1.0) \qquad (2)$$

From (1) and (2) ;

$$\boxed{n_o \sin\theta = \sin\theta' = \frac{\lambda_o}{2\Lambda}}$$

$$\theta' = \sin^{-1}\left(\frac{\lambda_o}{2\Lambda}\right) = \sin^{-1}\left(\frac{0.6328}{2.5}\right) = 3.628°$$

$$\theta = \sin^{-1}\left(\frac{\sin\theta'}{n_o}\right) = 1.813°$$

| | | |
|---|---|---|
| $\theta'_B = 3.628°$ | (in air) |
| $\theta_B = 1.813$ | (inside the grating |

(b) The grating equation for the forward-diffracted waves is :

$$i\frac{\lambda_o}{\Lambda} = (n_I \sin\theta' - n_{III} \sin\theta_i'' )$$

$n_I = n_{III} = 1.0$

$$i \frac{\lambda_0}{\Lambda} = \sin 20° - \sin\theta_i'' \Rightarrow$$

$$\sin\theta_i'' = \sin 20° - i \frac{0.6328}{5} \qquad (3)$$

| | | |
|---|---|---|
| $i = -2$ | $\theta''_{-2} =$ | $36.523°$ |
| $i = -1$ | $\theta''_{-1} =$ | $27.942°$ |
| $i = 0$ | $\theta''_{0} =$ | $20°$ |
| $i = +1$ | $\theta''_{+1} =$ | $12.443°$ |
| $i = +2$ | $\theta''_{+2} =$ | $5.100°$ |

All angles are measured CCW from the normal (z-axis).

(c) $$\sin 20° - i \frac{0.6328}{5} \geq -1 \Rightarrow$$

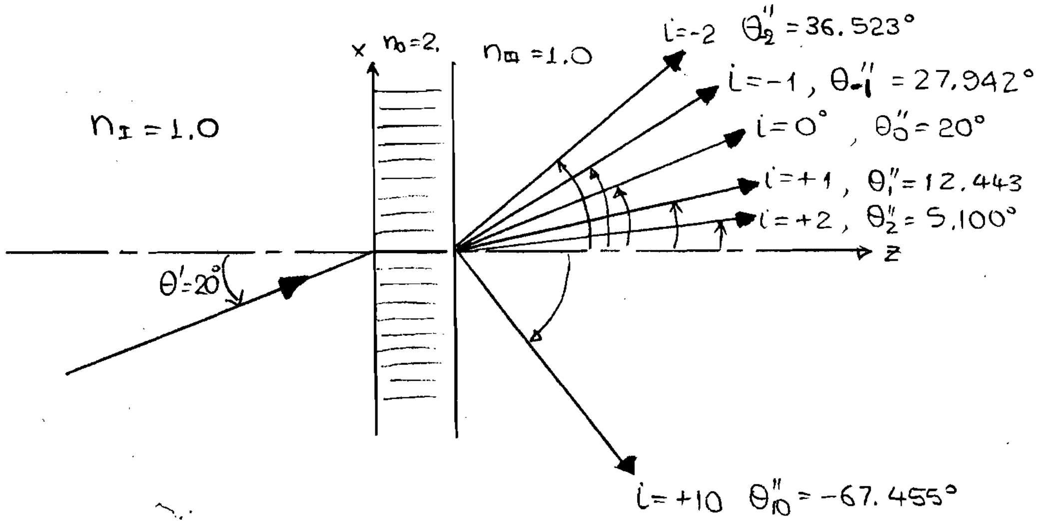
$$i \frac{0.6328}{5} \leq 1 + \sin 20° \Rightarrow i \leq \frac{5}{0.6328}(1 + \sin 20°) \Rightarrow$$

$$i \leq 10.6038 \rightsquigarrow i_{max} = 10$$

$$\theta''_{+10} = -67.455°$$

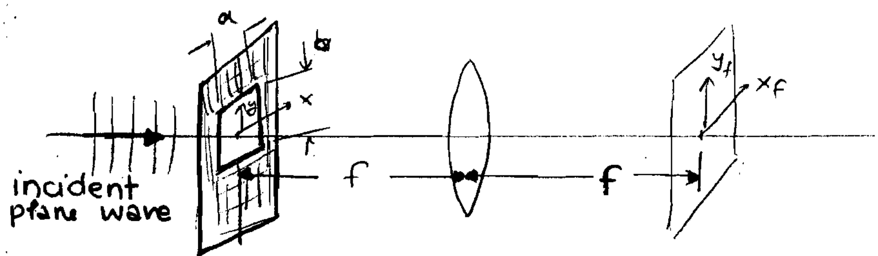
$$\boxed{i = 10, \ \theta''_{+10} = -67.455°}$$

(d)



## Problem 3: (Fraunhofer Diffraction by a Rectangular Aperture)

At the first focal plane use a rectangular aperture.

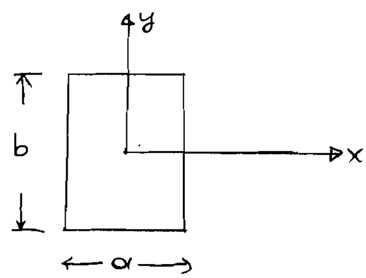



what you see at the second focal plane?

If $f(x,y)$ in the input plane (first focal plane) then at the second focal plane we see:

$$g(x_f, y_f) = \mathcal{F}\{f(x,y)\}_{f_x = \frac{x_f}{\lambda f},\ f_y = \frac{y_f}{\lambda f}}$$

$$f(x,y) = \mathrm{rect}\left(\frac{x}{a}, \frac{y}{b}\right)$$




$$\mathcal{F}\{f(x,y)\} = F(f_x, f_y) = |ab|\,\mathrm{sinc}(a f_x, b f_y) =$$

$$g(x_f, y_f) = ab\ \mathrm{sinc}\left(a\frac{x_f}{\lambda f}, b\frac{y_f}{\lambda f}\right) =$$

$$= ab\ \mathrm{sinc}\left(\frac{x_f}{\lambda f/a}\right)\mathrm{sinc}\left(\frac{y_f}{\lambda f/b}\right) =$$

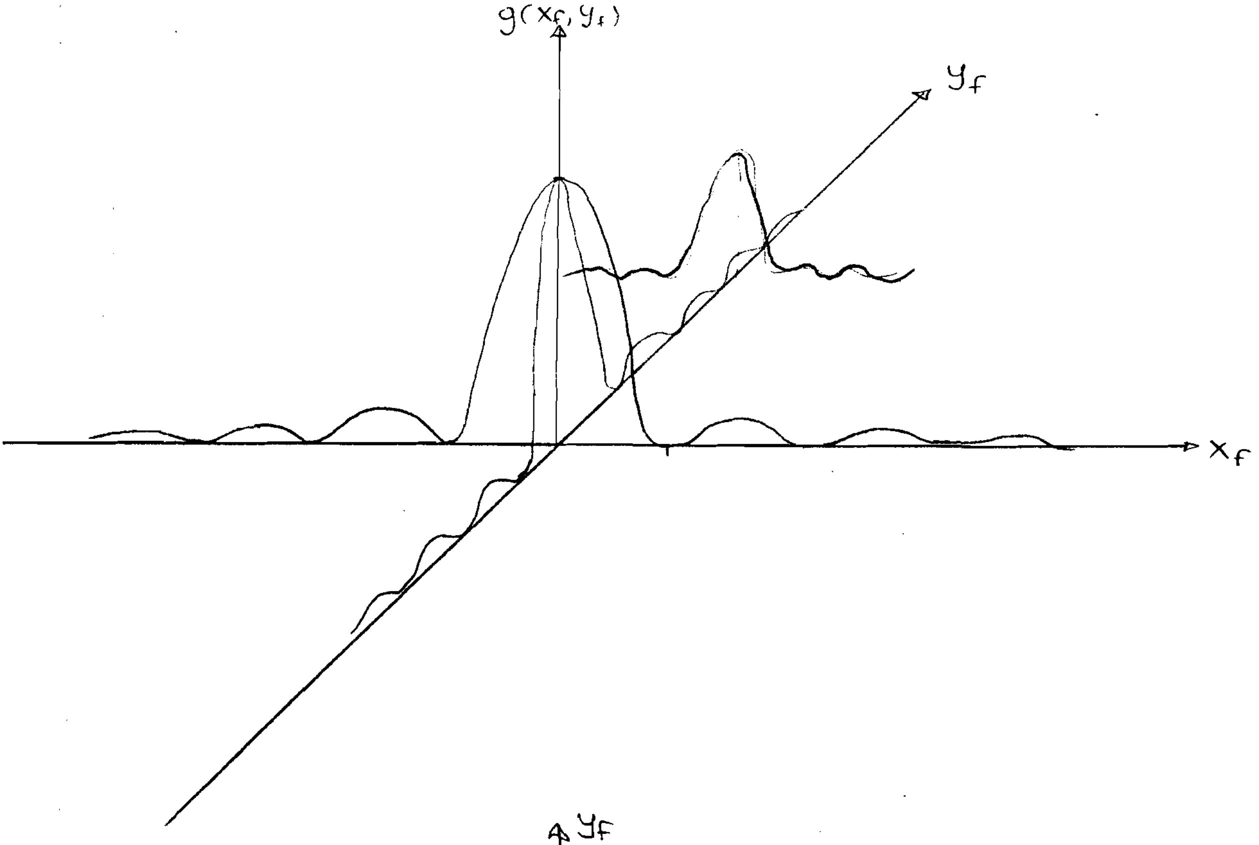
$$= ab\ \frac{\sin\left(\pi\frac{x_f}{\lambda f/a}\right)}{\pi\frac{x_f}{\lambda f/a}}\ \frac{\sin\left(\pi\frac{y_f}{\lambda f/b}\right)}{\pi\frac{y_f}{\lambda f/b}}$$

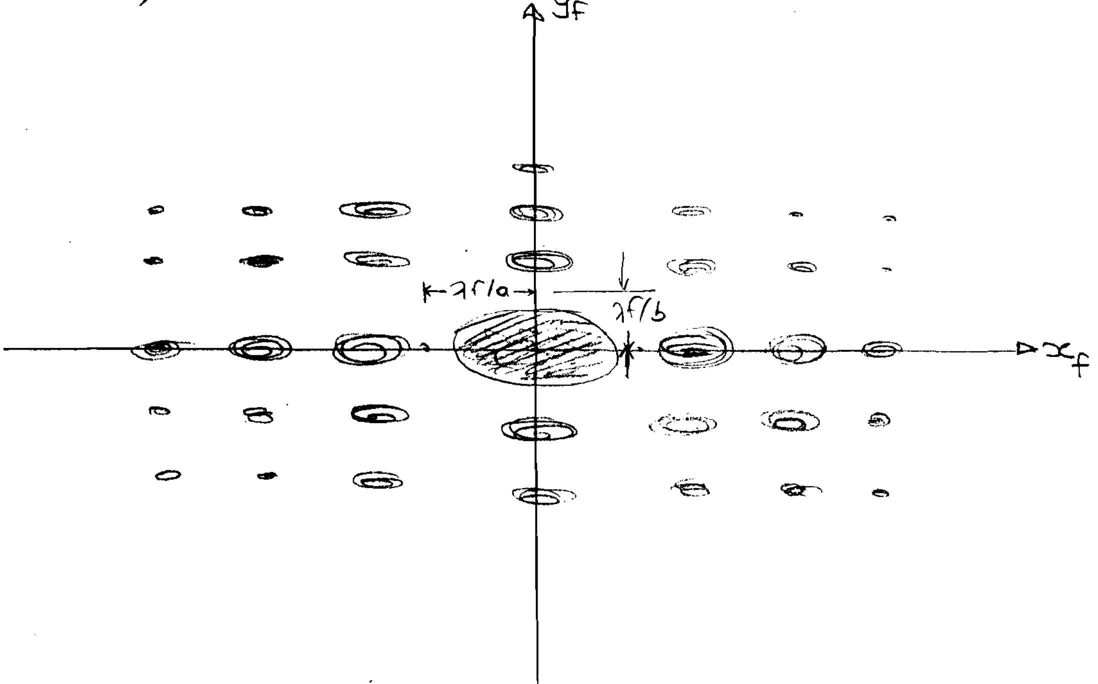
What we see is the intensity

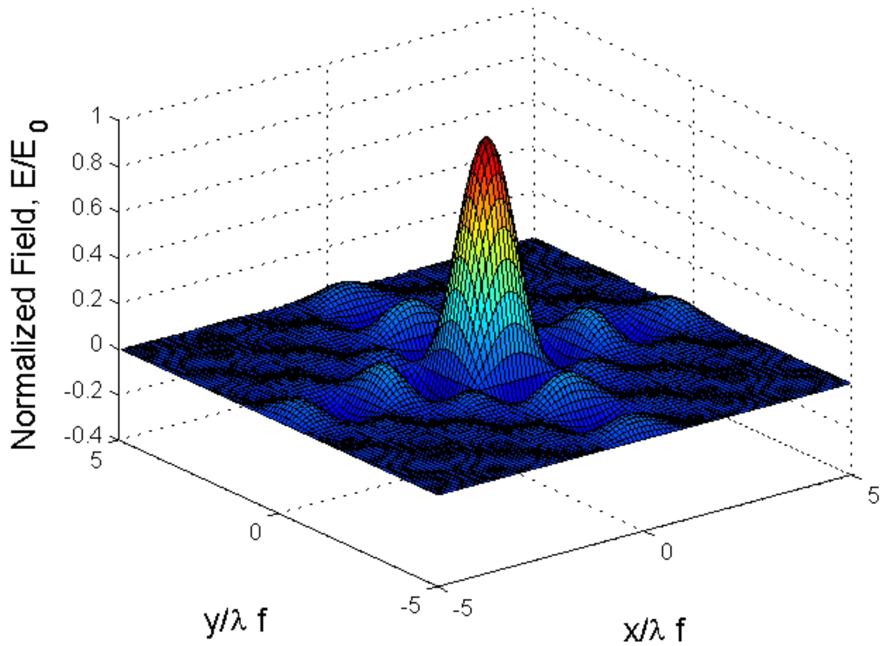
$$|g(x_f, y_f)|^2 = a^2b^2 \cdot \frac{\sin^2\left(\pi\frac{x_f}{\lambda f/a}\right)}{\left[\pi\left(\frac{x_f}{\lambda f/a}\right)\right]^2} \cdot \frac{\sin^2\left(\pi\frac{y_f}{\lambda f/b}\right)}{\left[\pi\left(\frac{y_f}{\lambda f/b}\right)\right]^2}$$

Zeros along $x_f$: $\quad x_f = m\frac{\lambda f}{a}$

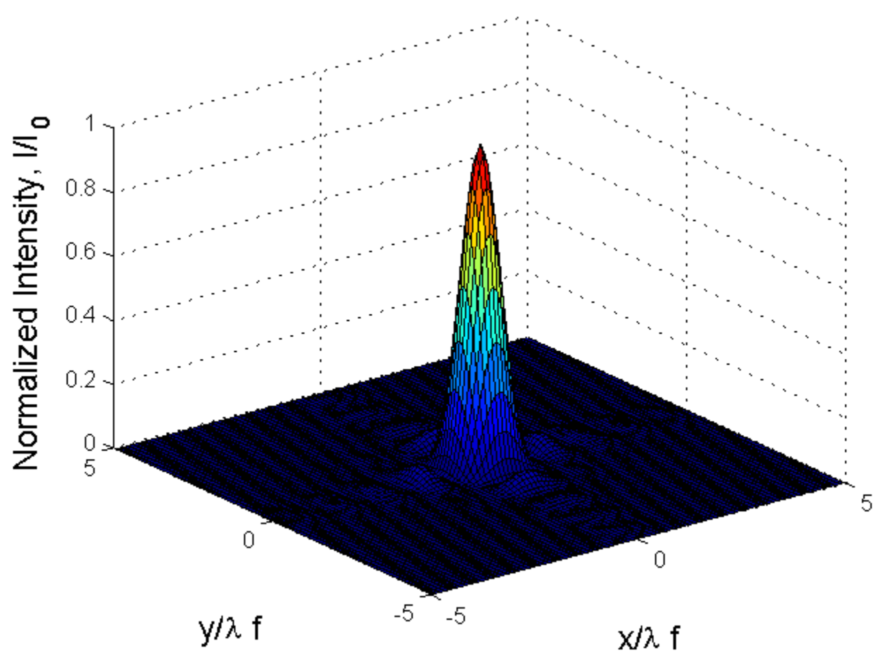
Zeros along $y_f$: $\quad y_f = n\frac{\lambda f}{b}$


g(xf, yf)
yf
xf

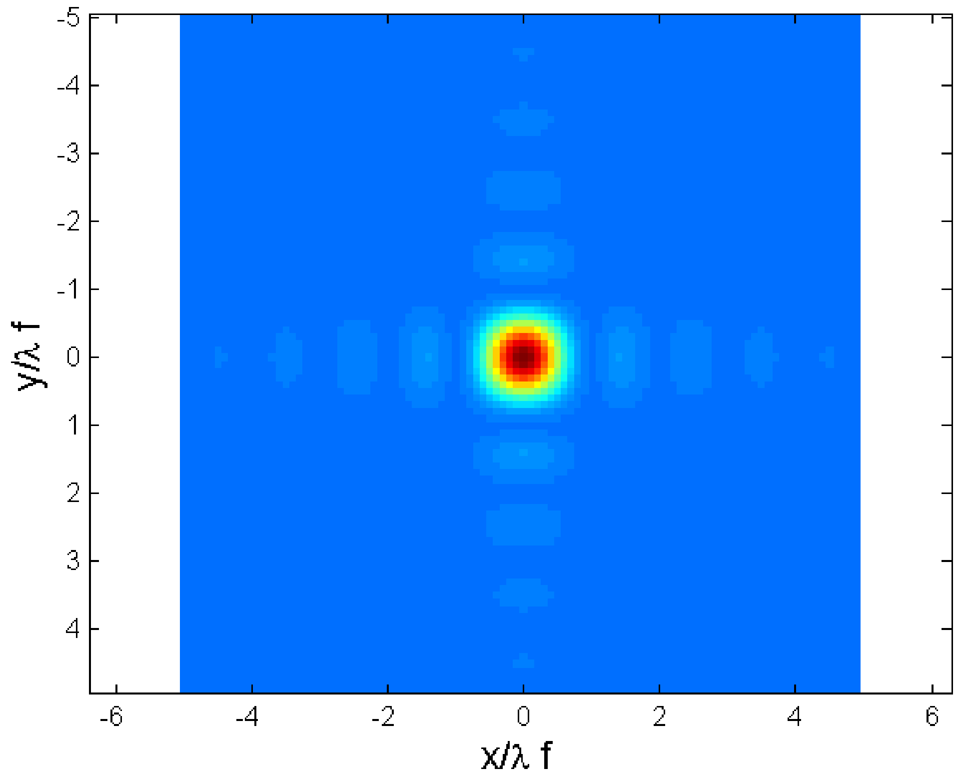

yf
λf/a
λf/b
xf



Normalized Field, E/E0
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
5
0
-5
y/λ f
-5
0
5
x/λ f

Normalized Intensity, I/I0
1
0.8
0.6
0.4
0.2
0
5
0
-5
y/λ f
-5
0
5
x/λ f



-5
-4
-3
-2
-1
0
1
2
3
4
y/λ f
-6
-4
-2
0
2
4
6
x/λ f

## Problem 4: (Edge Diffraction)

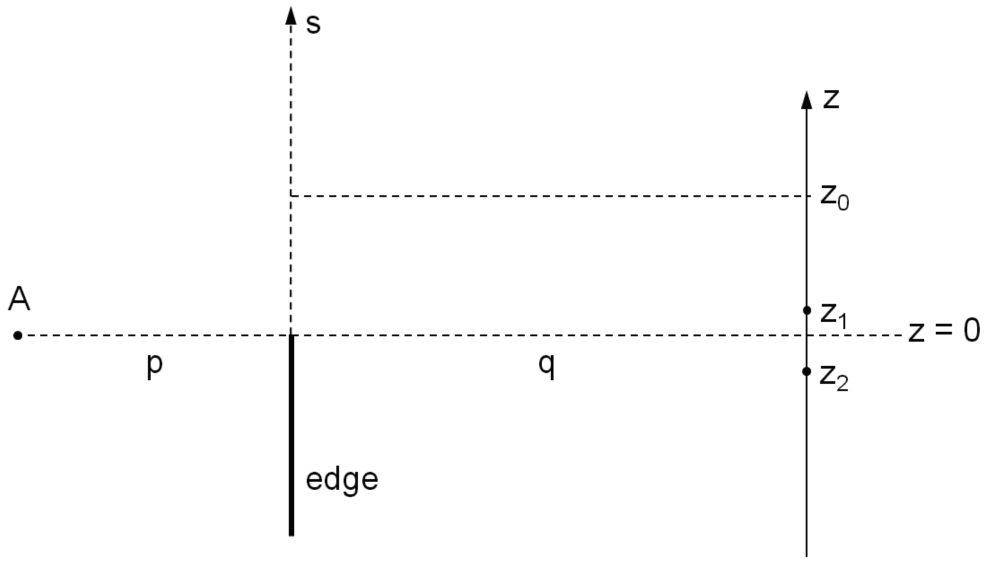



Figure 1: Case (a) with an incident plane wave

(a) The diffraction from the edge will follow the Fresnel diffraction assumptions in order to use the theory of the Cornu spiral with the Fresnel integrals. Since the incident wave is a plane wave $p = \infty$ and $q = 60\,\text{cm}$. The wavelength of the incident plane wave is $\lambda_0 = 589\,\text{nm}$. Let's assume the random point $z_0$ along the $z$-axis in Fig. 1. Then, assume the $s$-axis such as $s = z - z_0$. Therefore the origin of $s$ is the point $z = z_0$. From the Fresnel integral the field at $z_0$ is given by

$$\begin{aligned} E(z_0) &= C\int_0^{\infty} \exp\,[-j\frac{k}{2}(z-z_0)^2\frac{1}{q}]dz, \\ &= C\int_{-z_0}^{\infty} \exp\,[-j\frac{k}{2}s^2\frac{1}{q}]ds, \end{aligned}$$

where $C$ is a constant. Defining the normalized variable $u = s(2/\lambda_0 q)^{1/2}$ as in the book we can write the field at point $z_0$ in the form

$$E(z_0) = \frac{E_u}{\sqrt{2}}\Big[[0.5 - C(-u_0)] - j[0.5 - S(-u_0)]\Big],$$

$$\frac{I(z_0)}{I_u} = \frac{1}{2}\Big[[0.5 - C(-u_0)]^2 + [0.5 - S(-u_0)]^2\Big],$$

where $E_u$ and $I_u$ is the unobstructed amplitude and intensity while $C(u)$ and $S(u)$ the standard Fresnel integrals and $u_0 = z_0(2/\lambda_0 q)^{1/2}$.

Now for $z_0 = z_1 = 1\,\text{mm} \Rightarrow u_1 = z_1(2/\lambda_0 q)^{1/2} = 1.6822$ and $C(-u_1) = -0.3277$ and $S(-u_1) = -0.5666$ which results in $I(z_1)/I_u = 0.9114$.

Similarly, for $z_0 = z_2 = -2\,\text{mm} \Rightarrow u_2 = z_2(2/\lambda_0 q)^{1/2} = -3.3643$ and $C(-u_2) = +0.4159$ and $S(-u_2) = +0.4570$ which results in $I(z_2)/I_u = 0.0045$.

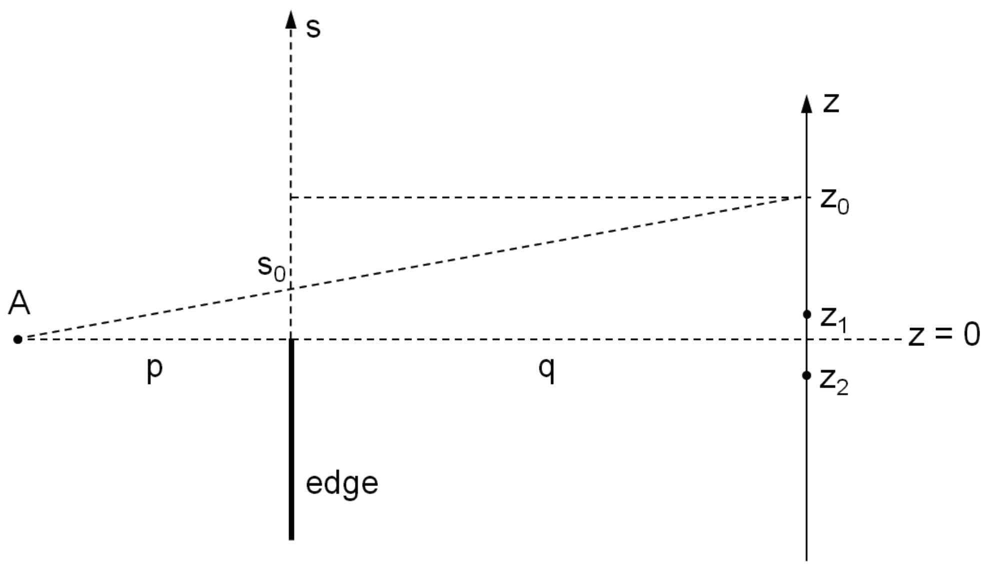



Figure 2: Case (b) with an incident spherical (actually cylindrical) wave emerging from point A

(b) The case of the icident light coming through a slit at point A the main difference is that we have to take into account the relative wavefront that arrives at he edge's plane. According to the theory, the $s_0 = z_0'$ distance on the edge plane corresponding to $z_0$ at the screen plane are related (from similar triangles) by the following equation

$$\frac{z_0'}{z_0} = \frac{p}{p+q},$$

where it is evident that for $p \to \infty$, $z_0' \to z_0$. Then, assume the $s$-axis such as $s = z - z_0'$ as before. Therefore the origin of $s$ is the point $z = z_0'$. From the Fresnel integral the field at $z_0$ is given by

$$E(z_0) \;=\; C \int_0^\infty \exp\Big[-j\frac{k}{2}(z-z_0')^2\Big(\frac{1}{p}+\frac{1}{q}\Big)\Big]dz,$$

$$= \quad C\int_{-z_0'}^{\infty} \exp\Big[-j\frac{k}{2}s^2\Big(\frac{1}{p}+\frac{1}{q}\Big)\Big]ds,$$

where $C$ is a constant. Defining the normalized variable $u = s[(2/\lambda_0)(1/p + 1/q)]^{1/2}$ as in the book we can write the field at point $z_0$ in the form

$$E(z_0) = \frac{E_u}{\sqrt{2}}\Big[[0.5 - C(-u_0)] - j[0.5 - S(-u_0)]\Big],$$

$$\frac{I(z_0)}{I_u} = \frac{1}{2}\Big[[0.5 - C(-u_0)]^2 + [0.5 - S(-u_0)]^2\Big],$$

where $E_u$ and $I_u$ is the unobstructed amplitude and intensity while $C(u)$ and $S(u)$ the standard Fresnel integrals and $u_0 = z_0'[(2/\lambda_0)(1/p + 1/q)]^{1/2}$

Now for $z_0 = z_1 = 1\,\text{mm}$ ($p = 60\,\text{cm}$ and $q = 120\,\text{cm}$), $z_1' = z_1[p/(p+q)] = 1/3\,\text{mm}$ $\Rightarrow u_1 = z_1'[(2/\lambda_0)(1/p + 1/q)]^{1/2} = 0.9712$ and $C(-u_1) = -0.7786$ and $S(-u_1) = -0.4095$ which results in $I(z_1)/I_u = 1.2310$.

Similarly, for $z_0 = z_2 = -2\,\text{mm}$, ($p = 60\,\text{cm}$ and $q = 120\,\text{cm}$), $z_2' = z_2[p/(p+q)] = -2/3\,\text{mm} \Rightarrow u_2 = z_2'(2/\lambda_0 q)^{1/2} = -1.9424$ and $C(-u_2) = +0.4319$ and $S(-u_2) = +0.3536$ which results in $I(z_2)/I_u = 0.0130$.

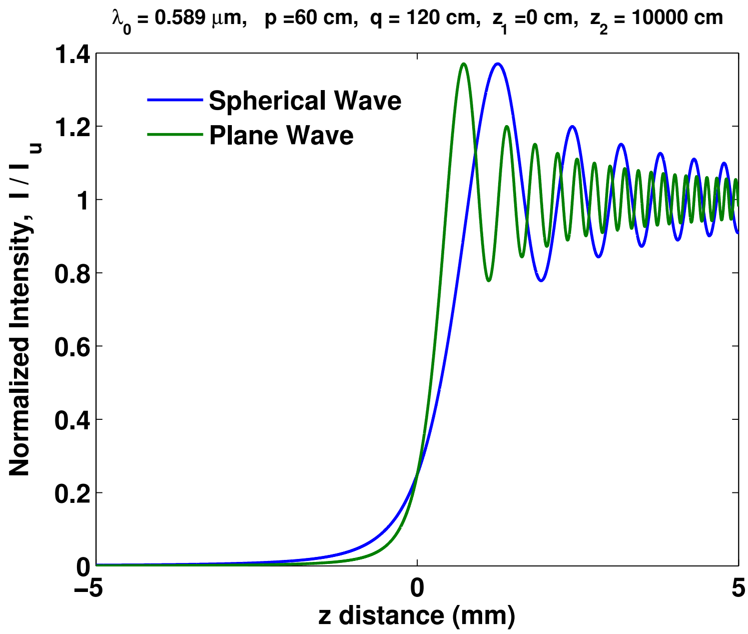



Figure 3: Plot of the intensity pattern $I/I_u$ as a function of the screen distance $z$ for a Spherical wave (blue) and a plane wave (green)