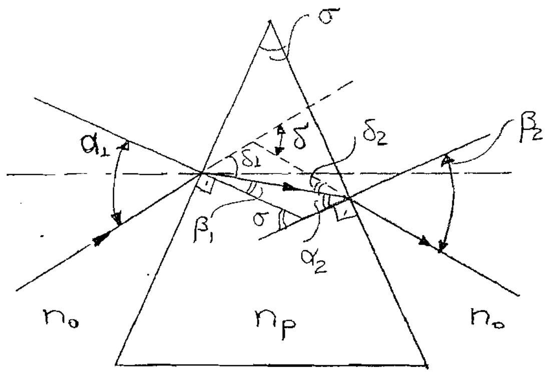
ΟΠΤΙΚΗ ΕΠΙΣΤΗΜΗ & ΤΕΧΝΟΛΟΓΙΑ
ΣΕΙΡΑ ΑΣΚΗΣΕΩΝ No. 2

Problem 1 (Deviation by a Prism):



Angular deviation $\delta = \delta_1 + \delta_2$

$$\delta_1 = \alpha_1 - \beta_1$$

$$\delta_2 = \beta_2 - \alpha_2$$

Thus, $\delta = \alpha_1 - \beta_1 + \beta_2 - \alpha_2 =$

$$= \alpha_1 + \beta_2 - (\alpha_2 + \beta_1)$$

But $\alpha_2 + \beta_1 = \sigma$

and $\delta = \alpha_1 + \beta_2 - \sigma = \alpha_1 + \beta_2(\alpha_1) - \sigma$

To have minimum $\delta \sim \frac{d\delta}{d\alpha_1} = 0 \Rightarrow 1 + \frac{d\beta_2}{d\alpha_1} = 0$

$$\frac{d\beta_2}{d\alpha_1} = -1$$

But from Snell's Law we have:

$$n_0 \sin\alpha_1 = n_p \sin\beta_1 \Rightarrow n_0 \cos\alpha_1 = n_p \cos\beta_1 \frac{d\beta_1}{d\alpha_1} \quad (*)$$

$$n_0 \sin\beta_2 = n_p \sin\alpha_2 = n_p \sin(\sigma - \beta_1) \Rightarrow$$

$$n_0 \cos\beta_2 \frac{d\beta_2}{d\alpha_1} = n_p \cos(\sigma - \beta_1)\,(-1)\,\frac{d\beta_1}{d\alpha_1} \quad (**)$$

From (*) ∧ (**) we have:

$$\cancel{n_0} \cos\beta_2 \frac{d\beta_2}{d\alpha_1} = -\cancel{n_p} \cos(\sigma - \beta_1) \frac{\cancel{n_0} \cos\alpha_1}{\cancel{n_p} \cos\beta_1}$$

$$\frac{d\beta_2}{d\alpha_1} = -\frac{\cos(\sigma - \beta_1)\cos\alpha_1}{\cos\beta_2 \cos\beta_1} = -1 \Rightarrow$$

$$\cos(\sigma - \beta_1)\cos\alpha_1 = \cos\beta_2 \cos\beta_1 \Rightarrow$$

$$\frac{\cos\beta_2}{\cos(\sigma - \beta_1)} = \frac{\cos\alpha_1}{\cos\beta_1} \Rightarrow \frac{\cos\beta_2}{\cos\alpha_2} = \frac{\cos\alpha_1}{\cos\beta_1} \Rightarrow$$

$$\Rightarrow \frac{\cos\beta_2}{\cos\left[\sin^{-1}\left(\frac{n_0 \sin\beta_2}{n_p}\right)\right]} = \frac{\cos\alpha_1}{\cos\left[\sin^{-1}\left(\frac{n_0 \sin\alpha_1}{n_p}\right)\right]} \Rightarrow$$

$$\frac{\cos\beta_2}{\sqrt{1-\frac{n_o^2\sin^2\beta_2}{n_p^2}}} = \frac{\cos\alpha_1}{\sqrt{1-\frac{n_o^2\sin^2\alpha_1}{n_p^2}}} \Rightarrow$$

$$\frac{\cos^2\beta_2}{n_p^2-n_o^2\sin^2\beta_2} = \frac{\cos^2\alpha_1}{n_p^2-n_o^2\sin^2\alpha_1} \Rightarrow \qquad \cos^2\beta_2 = \cos^2\alpha_1 \Rightarrow$$

$\boxed{\alpha_1 = \beta_2}$ for minimum δ

But when $\alpha_1 = \beta_2 \Rightarrow \beta_1 = \alpha_2 \Rightarrow$

$\alpha_2 = \beta_1 = \sigma/2$

Then, from Snell's Law :

$$n_o \sin\alpha_1 = n_p \sin\beta_1 = n_p \sin\frac{\sigma}{2} \Rightarrow$$

$$\boxed{\alpha_1 = \sin^{-1}\left(\frac{n_p}{n_o}\sin\frac{\sigma}{2}\right) = \beta_2 \quad \text{for } \delta = \text{minimum}}$$

In general β_2 can be computed as function of α_1 as follows:

$$\beta_2 = \sin^{-1}\left\{\frac{n_p}{n_o}\sin\alpha_2\right\} = \sin^{-1}\left\{\frac{n_p}{n_o}\sin(\sigma-\beta_1)\right\} =$$

$$= \sin^{-1}\left\{\frac{n_p}{n_o}\left[\sin\sigma\cos\beta_1 - \cos\sigma\sin\beta_1\right]\right\} =$$

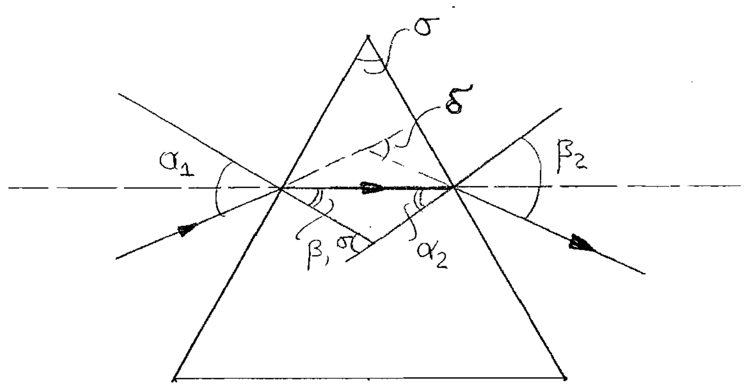
$$= \sin^{-1}\left\{\frac{n_p}{n_o}\left[\sin\sigma\sqrt{1-\frac{n_o^2}{n_p^2}\sin^2\alpha_1} - \cos\sigma\,\frac{n_o\sin\alpha_1}{n_p}\right]\right\} \Rightarrow$$

$$\beta_2 = \sin^{-1}\left\{\frac{1}{n_o}\left[\sin\sigma\sqrt{n_p^2-n_o^2\sin^2\alpha_1} - n_o\cos\sigma\sin\alpha_1\right]\right\} = \beta_2(\alpha_1)$$

And thus,

$$\delta = \alpha_1 + \beta_2(\alpha_1) - \sigma$$

For the minimum deviation case $[\delta = \delta_{min}, \alpha_1 = \beta_2 = \sin^{-1}\left(\frac{n_p}{n_o}\sin\frac{\sigma}{2}\right)]$ and the geometry looks like :



In this minimum deviation case δ is

$$\delta = 2\alpha_{1,min} - \sigma = 2\sin^{-1}\left(\frac{n_p}{n_o}\sin\frac{\sigma}{2}\right) - \sigma$$

$$\boxed{\delta_{min} = 2\sin^{-1}\left(\frac{n_p}{n_o}\sin\frac{\sigma}{2}\right) - \sigma}$$

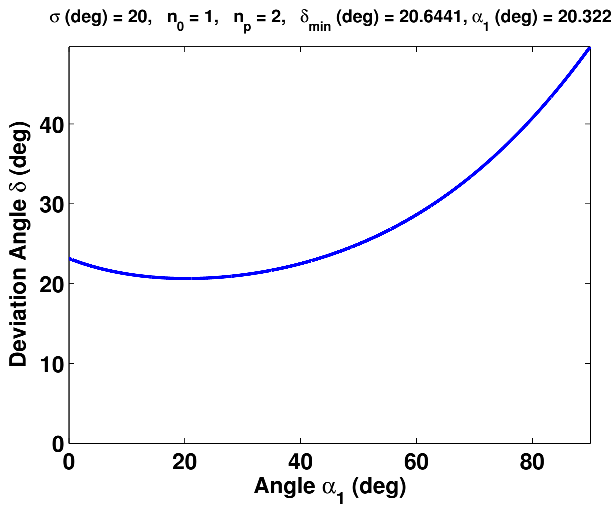
The above equation can also be written as

$$\frac{\delta_{min} + \sigma}{2} = \sin^{-1}\left(\frac{n_p}{n_o}\sin\frac{\sigma}{2}\right) \Rightarrow$$

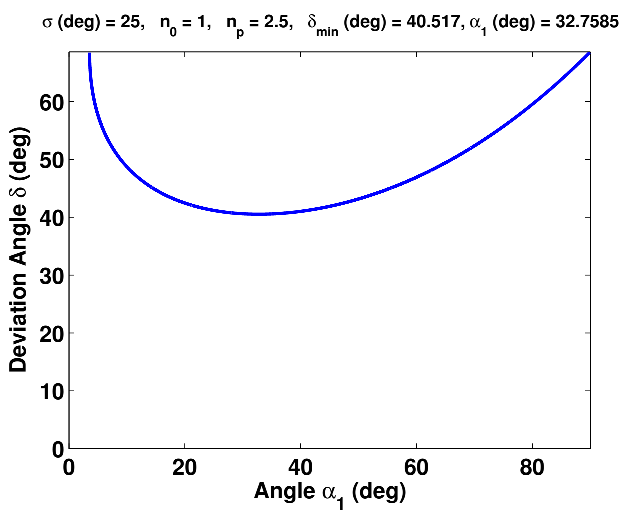
$$\sin\left(\frac{\delta_{min}+\sigma}{2}\right) = \frac{n_p}{n_o}\sin\frac{\sigma}{2} \Rightarrow$$

$$\Rightarrow \boxed{\frac{n_p}{n_o} = \frac{\sin\left(\frac{\delta_{min}+\sigma}{2}\right)}{\sin(\sigma/2)}}$$

For $\sigma = 20\,\text{deg}$, $n_0 = 1.0$, and $n_p = 2.0$ the angle of deviation δ versus the angle of incidence α_1 is shown in the next figure.

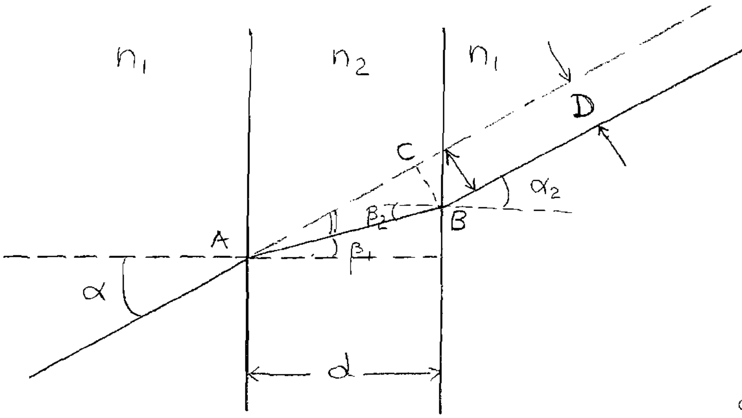


For $\sigma = 25\,\text{deg}$, $n_0 = 1.0$, and $n_p = 2.5$ the angle of deviation δ versus the angle of incidence α_1 is shown in the next figure.



It is observed that in this case a_1 does not start from zero. The reason for this is that α_2 should be less than the critical angle $\theta_{cr} = \sin^{-1}(n_0/n_p)$. Therefore, $\alpha_2 = \sigma - \beta_1 = \sigma - \sin^{-1}(n_0 \sin \alpha_1/n_p) < \theta_{cr}$. From this it can be found that $\alpha_1 > \sin^{-1}((n_p/n_0)\sin(\sigma - \theta_{cr}))$. For the values of the parameters given it can be determined that $\alpha_1 > 3.5565\,\text{deg}$.

Problem 2: (Plane Parallel Plate):



$\beta_1 = \beta_2 = \beta$

Due to symmetry

$\alpha_2 = \alpha$ (since the refractive index in the input and output regions is the same).

From ABC triangle:

$$\left.\begin{aligned} \sin(\alpha-\beta) &= \frac{D}{AB} \\ AB &= \frac{d}{\cos\beta} \end{aligned}\right\} \Longrightarrow$$

$$\sin(\alpha-\beta) = \frac{D}{d/\cos\beta} \Rightarrow \boxed{D = d\left(\frac{\sin(\alpha-\beta)}{\cos\beta}\right)}$$

From Snell's Law: $n_1 \sin\alpha = n_2 \sin\beta \Rightarrow \beta = \sin^{-1}\left(\frac{n_1}{n_2}\sin\alpha\right)$

Thus,

$$\boxed{D = d\,\frac{\sin\left[\alpha - \sin^{-1}\left(\frac{n_1 \sin\alpha}{n_2}\right)\right]}{\left(1 - \frac{n_1^2 \sin^2\alpha}{n_2^2}\right)^{1/2}}}$$

Application: $n_1 = 1.0$, $n_2 = 1.46$, $\alpha = 67.5°$; $d = 0.52$ in

$\beta = 39.2566°$ ~ $D = 0.3208$ in $= 8.148$ mm

Problem 3 (Cartesian Ovoid):

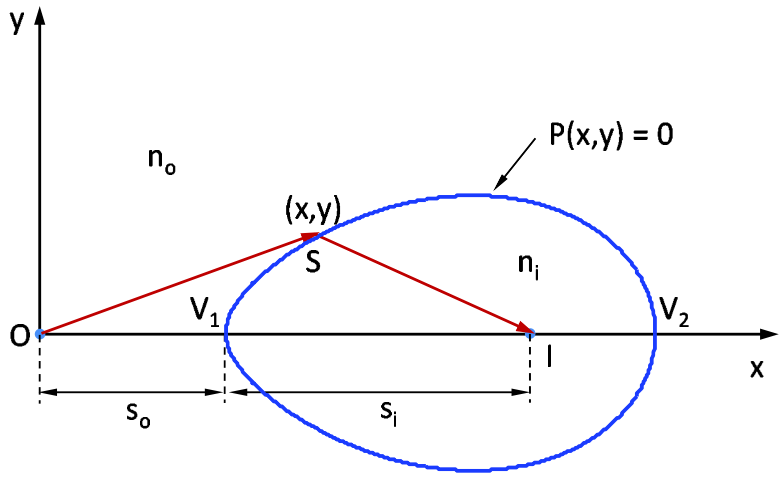


Figure 1: A Cartesian ovoid of refractive index n_i is shown in a surrounding medium of refractive index n_o ($n_o < n_i$). Every ray from the point object O ends up in the point image I. The function of the ovoid is $P(x, y) = 0$. The distances s_o and s_i are the object and image distances from the ovoid's left vertex.

(a) The basic equation of the Cartesian ovoid (as discussed in class) is the following

$$n_o\sqrt{x^2 + y^2} + n_i\sqrt{(s_o + s_i - x)^2 + y^2} = n_o s_o + n_i s_i = A, \tag{1}$$

where s_o and s_i the object and image distances as shown in Fig. 1. In order for the shape shown to be valid the refractive index of the ovoid should be larger than the refractive index of the surrounding region, i.e. $n_i > n_o$. Therefore, the analysis below is valid only if $n_i > n_o$. The vertex points V_1 and V_2 are the points of the ovoid for $y = 0$. These points can be determined by setting $y = 0$ in the previous equation which results in the following

$$n_o|x| + n_i|s_o + s_i - x| = A, \tag{2}$$

which has the following solutions

$$x = s_0, \qquad \text{for} \quad 0 < x < s_o + s_i \tag{3}$$

$$x = \frac{A + n_i(s_o + s_i)}{n_o + n_i} = s_o + \frac{2n_i}{n_o + n_i}s_i, \qquad \text{for} \quad s_o + s_i < x < \infty. \tag{4}$$

The first of the previous equations [Eq. (3)] shows the position of V_1 and the second [Eq. (4)] shows the position of V_2. For the values of the parameters $s_0 = 5\,\text{cm}$, and $s_i = 10, 15, 20\,\text{cm}$, $n_o = 1.0$, and $n_i = 1.50$ the values of V_2, as measured from $x = 0$, (where the object O is) are 17, 23, and 29 cm, respectively.

(b) In order to find the equation of the ovoid, $P(x, y) = 0$, Eq. (1) must be solved. This equation can be transformed by moving one square root to the right hand side and raising to the second power both sides twice. Then the square roots are eliminated and a polynomial form is obtained

$$\begin{aligned} a_1 y^4 + b_1(x) y^2 + c_1(x) &= 0, \quad \text{with} \qquad (5) \\ a_1 &= (n_o^2 - n_i^2)^2, \\ b_1(x) &= 2\left[A^2 + B(x)\right](n_o^2 - n_i^2) - 4An_o^2, \\ c_1(x) &= \left[A^2 + B(x)\right]^2 - 4A^2 n_o^2 x^2, \\ A &= n_o s_o + n_i s_i, \\ B(x) &= n_o^2 x^2 + n_i^2 (s_o + s_i - x)^2. \end{aligned}$$

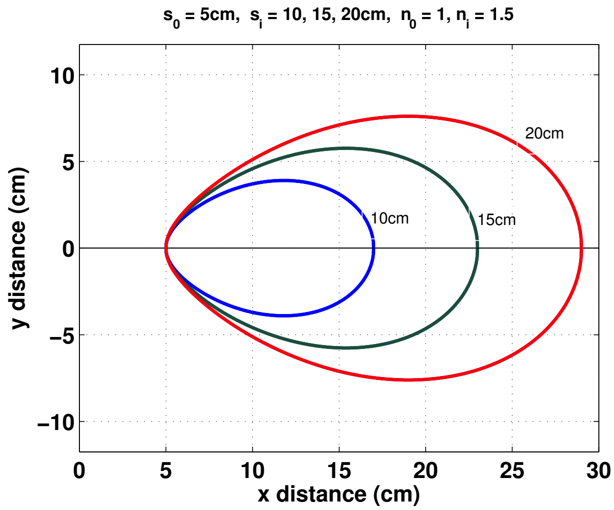
The solutions to Eq. (5) are

$$y_1(x) = \pm \left\{ \frac{-b_1(x) + \sqrt{b_1(x)^2 - 4a_1 c_1(x)}}{2a_1} \right\}^{1/2}, \qquad (6)$$

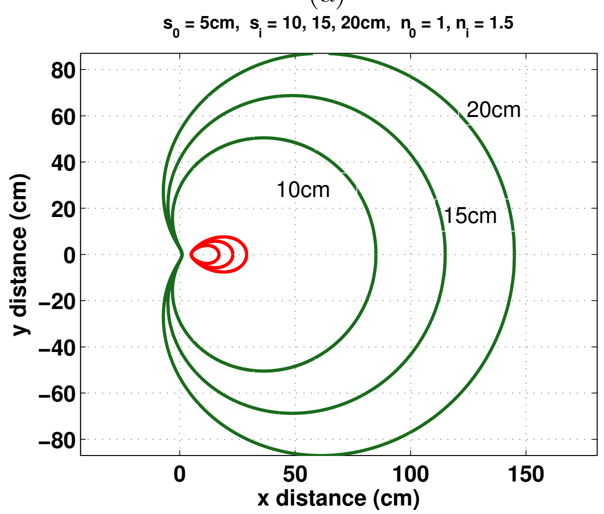
$$y_2(x) = \pm \left\{ \frac{-b_1(x) - \sqrt{b_1(x)^2 - 4a_1 c_1(x)}}{2a_1} \right\}^{1/2}. \qquad (7)$$

The negative solutions show that the ovoid is symmetric with respect to the x-axis. From the two solutions the smaller in absolute value is the one that specifies the ovoid that is seeked ($n_o(OS) + n_i(IS) = n_o s_o + n_i s_i$). I.e. the solution for $y(x)$ that satisfies $P(x, y(x)) = 0$ is the $|y(x)| = \min\{|y_1(x)|, |y_2(x)|\}$. The resulting ovoids for the parameters $s_0 = 5\,\text{cm}$, and $s_i = 10, 15, 20\,\text{cm}$ are shown in Fig. 2a.

(c) In order to find the maximum angle of rays emanating from O and incident on the refractive surface of the ovoid $P(x, y) = 0$, it is necessary to find the tangent from point O, OS, to the ovoid surface. This is schematically shown in Fig. 3. The angle ϕ of the tangent



(a)



(b)

Figure 2: The three generated Cartesian ovoids are shown for the parameters $n_o = 1.0$, $n_i = 1.5$, $s_o = 5\,\text{cm}$ and $s_i = 10, 15,$ and $20\,\text{cm}$. Just for information the other solution that corresponds to the larger root of Eq. (5) is also shown in Fig. 2b along with the smaller solutions in the same scale. The second (larger) solution is not accepted from a physical point of view. It corresponds to the solution of $n_o(OS) - n_i(IS) = -(n_o s_o + n_i s_i)$. These ovoids are known in the literature as Cartesian ovoids.

can be found if the following equation is solved ($\dot{y} = dy/dx$)

$$\tan\phi = \frac{y(x^*)}{x^*} = \frac{dy}{dx}\Big|_{x^*} = \dot{y}(x^*). \tag{8}$$

In order to solve the previous equation it is necessary to find the derivative $\dot{y}(x)$ for the ovoid. This can be found by differentiating Eq. (1) with respect to the variable x keeping $y = y(x)$. After some algebra the equation for determining the derivative $\dot{y}(x)$ at any point

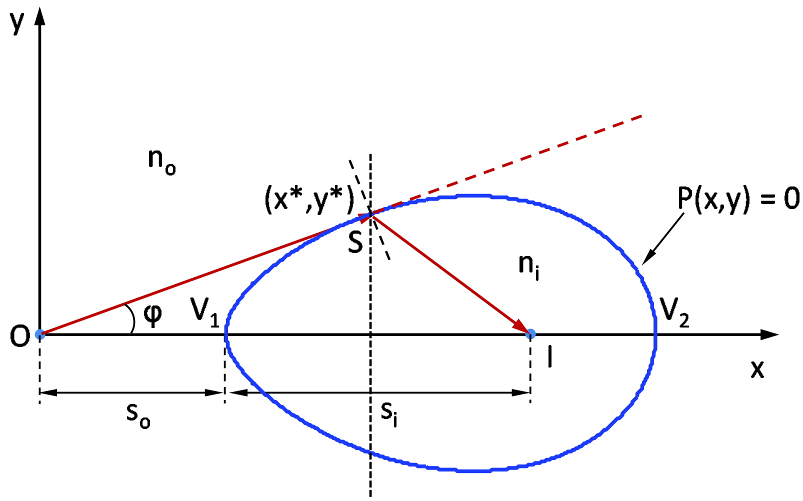


Figure 3: The same Cartesian ovoid with the incident from O beam being tangential to the ovoid. This specifies the maximum angle ϕ that by refraction should pass from point I. The (x^*, y^*) denotes the point on the ovoid that the beam is tangential. Again it is assumed that $n_o < n_i$.

in the ovoid is given by

$$\left[[(A_1(x,y) - A_2(x,y)]y^2\right]\dot{y}^2 + \quad 2[A_1(x,y)xyA_2(x,y)(s_o+s_i-x)y]\dot{y} + \\ A_1(x,y)x^2 - A_2(x,y)(s_o+s_i-x)^2 = 0, \tag{9}$$

$$A_1(x,y) = n_o^2[(s_o+s_i-x)^2+y^2],$$
$$A_2(x,y) = n_i^2(x^2+y^2).$$

Then Eq. (8) can be solved graphically if $f_1(x) = y(x)/x$ and $f_2(x) = \dot{y}(x)$ are plotted as functions of x. The resulting curves and intersections points are shown in Fig. 4. When the numerical solution point (x^*, y^*) is specified the angle ϕ can also be calculated from Eq. (8). The results are tabulated in Table 1.

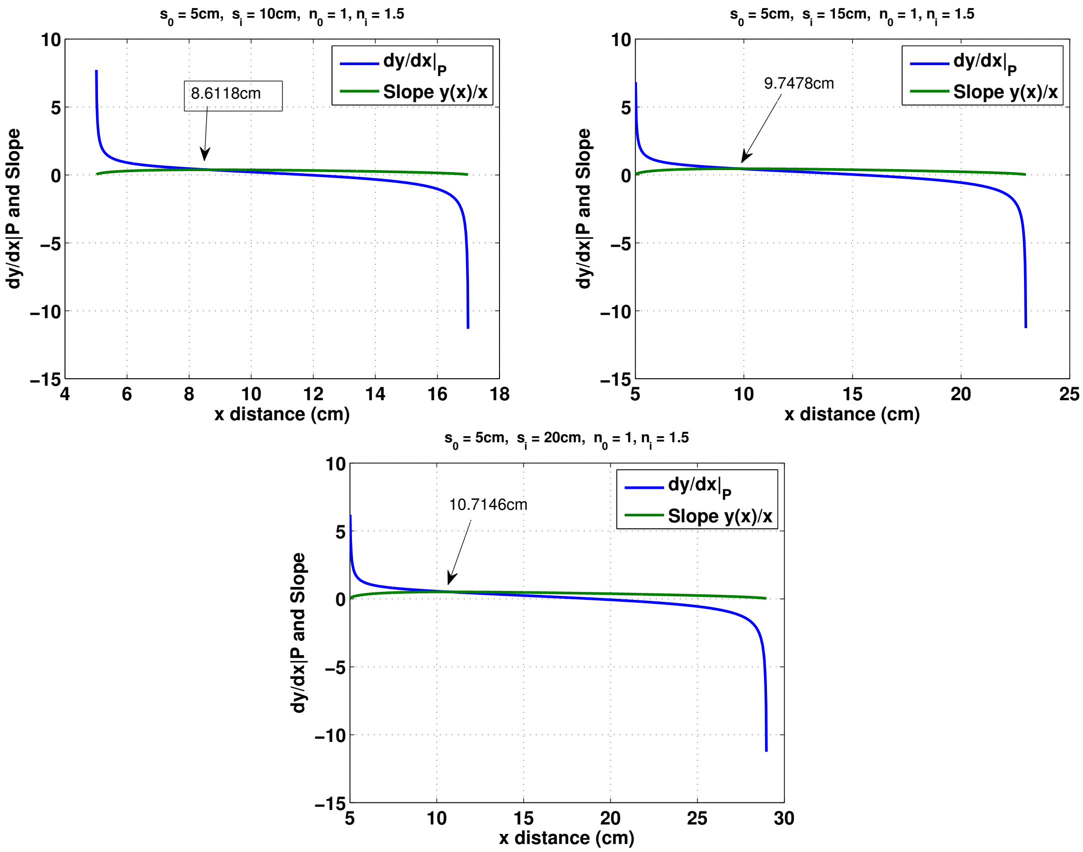


Figure 4: Graphical solution for determing the point for which the incident beam from O is tangential to the ovoid. The intersection of the two curves gives the point x^* and then y^* can be determined from the equation of the ovoid $P(x^*, y^*) = 0$.

Table 1: Results for question (c) of Cartesian Ovoid Problem

s_0 (cm)	s_i (cm)	x^* (cm)	y^* (cm)	ϕ (deg)
5	10	8.6116	3.2925	20.9235
5	15	9.7478	4.4697	24.6331
5	20	10.7146	5.4948	27.1502

Analytical solution for tangent:

Proposed by **Mr. Orfeas Voutiras** (Optical Science & Engineering Class - Spring 2015)

From Eq. (8) $tan\phi = \alpha = y^*/x^*$. Therefore $y = \alpha x$ for the point (x^*, y^*) that is tangential to ovoid. Inserting this relation into Eq. (1) the following equation is derived (setting $s_0 + s_i = D$):

$$\sqrt{x^2 + \alpha^2 x^2} + \frac{n_i}{n_0}\sqrt{(D-x)^2 + \alpha^2 x^2} = \frac{n_0 s_0 + n_i s_i}{n_0} = A'. \tag{10}$$

Re-arranging the above equation and squaring both sides it is straightforward to derive the following second order polynomial equation with respect to x (setting $\kappa = n_i/n_0$):

$$(1+\alpha^2)(\kappa^2 - 1)x^2 + \left[2A'(1+\alpha^2)^{1/2} - 2D\kappa^2\right] x + D^2\kappa^2 - A'^2 = 0. \tag{11}$$

The last equation should have a single solution with respect to x in order to correspond to the tangent to the ovoid. Therefore, the discriminant of the polynomial of Eq. (11) should be zero. This can be expressed from the following equation:

$$\mathscr{D} = 4\left[A'(1+\alpha^2)^{1/2} - D\kappa^2\right]^2 - 4(D^2\kappa^2 - A'^2)(1+\alpha^2)(\kappa^2 - 1) = 0. \tag{12}$$

Setting $p = (1+\alpha^2)^{1/2}$ the previous equation is written as

$$\begin{aligned} \left[A'^2 - (D^2\kappa^2 - A'^2)(\kappa^2 - 1)\right] p^2 - \left[2A'D\kappa^2\right] p + D^2\kappa^2 &= 0 \Longrightarrow \\ a_1 p^2 + a_2 p + a_3 &= 0 \Longrightarrow \end{aligned} \tag{13}$$

$$p = (1+\alpha^2)^{1/2} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1} \Longrightarrow \alpha = \pm\sqrt{p^2 - 1}, \tag{14}$$

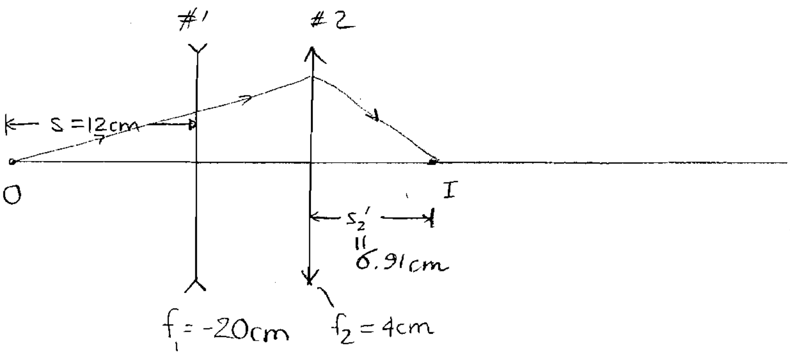
where the root for $|\alpha| < 1$ is the one that corresponds to the ovoid of interest. Using the above analytical solutions the following results can be derived (see Table 2). It is reminded that $\phi = \tan^{-1}(\alpha)$.

Of course the slope does not immediately specify the point (x^*, y^*). However, this can be found solving Eq. (9) for x^* if $y^* = \alpha x^*$ and $\dot{y} = \alpha$.

Table 2: Results for question (c) using the analytical approach:

s_0 (cm)	s_i (cm)	α	ϕ (deg)
5	10	0.3823357	20.92364
5	15	0.4585373	24.63322
5	20	0.5128322	27.15022

Problem 4: (Thin Lenses System):



Lens #1: $\frac{1}{s} + \frac{1}{s_1'} = \frac{1}{f_1} \Rightarrow \frac{1}{0.12} + \frac{1}{s_1'} = -\frac{1}{0.2} \Rightarrow$

$s_1' = -0.075$ m virtual image

Thus distance from the second lens $(0.075 + 0.02)$m $= 0.095$ m

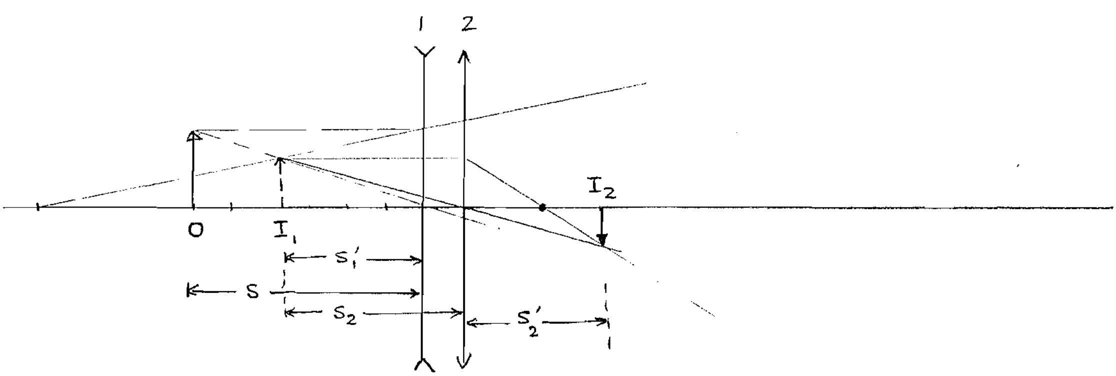
Lens #2:

Object is to the left ~ $s_2 = 0.095$ m

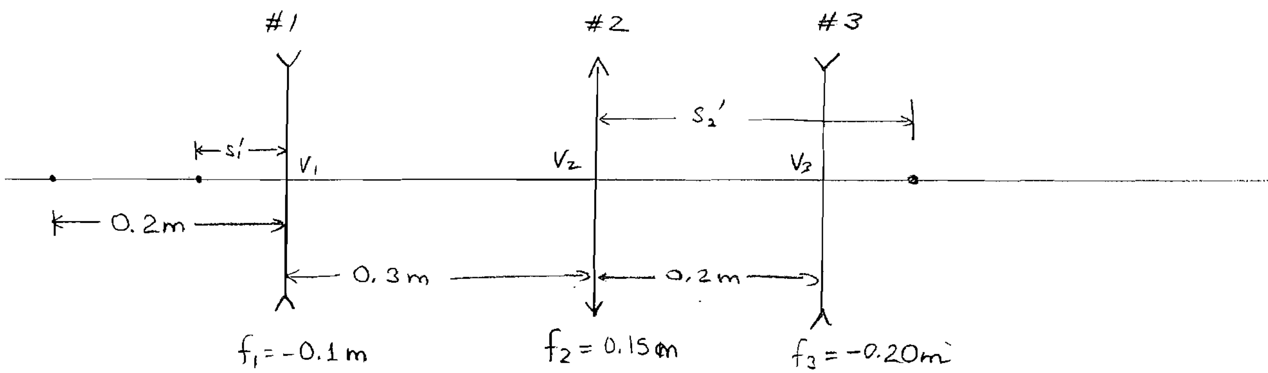
$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \Rightarrow \frac{1}{0.095} + \frac{1}{s_2'} = \frac{1}{0.04} \rightarrow$

$s_2' = 0.0691$ m ~ $s_2' = 6.91$ cm (real image).

m = (-(-0.075)/0.12)(-0.0691/0.095) = -0.455



Problem 5: (Thin Lenses System):



Lens #1:

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \Rightarrow \frac{1}{0.2} + \frac{1}{s_1'} = -\frac{1}{0.1} \Rightarrow s_1' = -0.06667 \text{ m}$$

Lens #2

$s_2 = |s_1'| + 0.3 = 0.36667$ m (real object for #2).

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \rightarrow \frac{1}{0.36667} + \frac{1}{s_2'} = \frac{1}{0.15} \rightarrow s_2' = 0.2538 \text{ m}$$

Lens #3

Distance from V_3 = $0.2538 - 0.2 = 0.0538$ m

Thus real image of #2 is virtual object for #3: $s_3 = -0.0538$ m

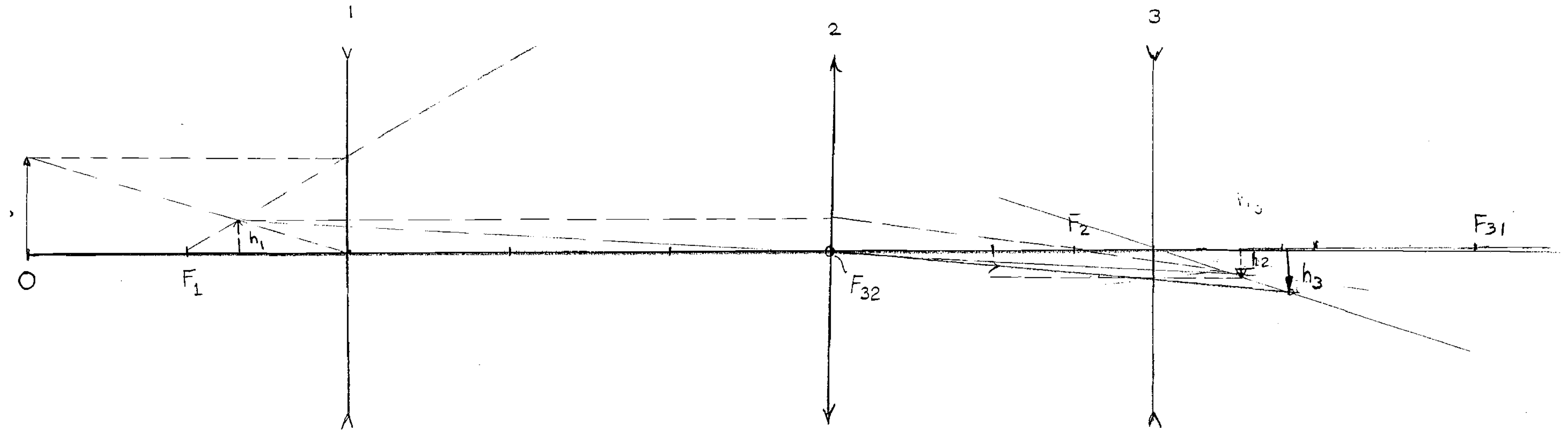
$$\frac{1}{s_3} + \frac{1}{s_3'} = \frac{1}{f_3} \rightarrow -\frac{1}{0.0538} + \frac{1}{s_3'} = -\frac{1}{0.2} \rightarrow s_3' = 0.0736 \text{ m}$$

$\rightarrow$ $s_3' = 7.36$ cm real image.

Magnification: $m = \frac{h_3}{h_0} = \frac{h_3}{h_2}\frac{h_2}{h_1}\frac{h_1}{h_0} = m_3 m_2 m_1$

$$m_1 = -\frac{s_1'}{s_1} = -\frac{-0.06667}{0.2}, \quad m_2 = \frac{-s_2'}{s_2} = -\frac{0.2538}{0.36667}, \quad m_3 = -\frac{0.0736}{-0.0538}$$

Thus $m = -0.316$ (inverted image).


1
2
3
O
F1
h1
F32
F2
h2
h3
F31