Introduction to
Geometrical Optics & Prisms

Optical Engineering
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Geometrical Optics

\[ \lim_{\lambda \to 0} \{ \text{Wave Optics} \} = \text{Geometrical Optics} \]

Huygens's Principle (1678AD)

Huygens's Principle and Law of Reflection

\(AB\) is a plane wave front of incident light.

The wave at \(A\) sends out a wavelet centered on \(A\) toward \(D\).
The wave at \(B\) sends out a wavelet centered on \(B\) toward \(C\).

\[
(AD) = (BC') = u\Delta t \implies \theta_1 = \theta'_1 \implies
\]

*Angle of Incidence = Angle of Reflection*

Huygens's Principle and Law of Refraction

**Ray 1** strikes the surface and at a time interval $\Delta t$ later, **Ray 2** strikes the surface.

During this time interval, the wave at $A$ sends out a wavelet, centered at $A$, toward $D$.

From triangles $ABC$ and $ADC$, we find

\[
\frac{\sin \theta_1}{\sin \theta_2} = \frac{u_1}{u_2} = \frac{n_2}{n_1} \implies n_1 \sin \theta_1 = n_2 \sin \theta_2
\]

Fermat's Principle (1662AD)

A ray of light in going from point S to point P will travel an optical path \((OPL)\) that minimizes the \(OPL\). That is, it is \textit{stationary} with respect to variations in the \(OPL\).

Law of Reflection  
(Hero - least distance)  

Law of Refraction  
(Fermat - least time)
Optical Path Length (OPL)

\[ n = 1 \quad \text{or} \quad n > 1 \]

\[ \lambda = \frac{\lambda_0}{n} \]

\[ OPL = \int_A^B n(\vec{r}(s)) \, ds, \]

Fermat's Principle (1662AD)

Fermat’s Principle

\[ I = \min \left\{ \frac{1}{c} \int_{A}^{B} n(\vec{r}) ds \right\} \]

\[ \vec{r} \; = \; x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \]

\[ ds \; = \; \sqrt{x'^2 + y'^2 + z'^2} \, dt \]

\[ I \; = \; \min \left\{ \frac{1}{c} \int_{A}^{B} n(x, y, z) \sqrt{x'^2 + y'^2 + z'^2} \, dt \right\} \]

Ray-Path Equation (Eikonal)

Euler-Lagrange Equations

\[ \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial x'} \right) = 0 \implies \frac{d}{ds} \left( n \frac{dx}{ds} \right) = \frac{\partial n}{\partial x} \]

\[ \frac{\partial F}{\partial y} - \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right) = 0 \implies \frac{d}{ds} \left( n \frac{dy}{ds} \right) = \frac{\partial n}{\partial y} \]

\[ \frac{\partial F}{\partial z} - \frac{d}{dt} \left( \frac{\partial F}{\partial z'} \right) = 0 \implies \frac{d}{ds} \left( n \frac{dz}{ds} \right) = \frac{\partial n}{\partial z} \]
Propagation in Lens-Like Medium

\[ n(r) = n_0 \left[ 1 - \frac{k_2}{2k_0} r^2 \right] \]

Ray-Path Equation (Eikonal)

\[
\frac{d}{ds} \left( n \frac{d\vec{r}}{ds} \right) = \nabla n
\]

Approximate Equation

\[
n_0 \frac{d^2r}{dz^2} \approx -n_0 \frac{k_2}{k_0} r
\]

Transformation

\[
t = \int ds/n
\]

\[
\frac{d^2r}{dt^2} = n(r) \nabla n
\]

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Approximate Equation

\[ n_0 \frac{d^2 r}{dz^2} \simeq -n_0 \frac{k_2}{k_0} r \]

Propagation in Lens-Like Medium

\[ n(r) = n_0 \left[ 1 - \frac{k_2}{2k_0} r^2 \right] \]

Lens-Like Medium

Optical Axis

n_{out}

r_0

z = 0

1

\ell

2

F

3

n_{out}

n_0 = 1.5, n = 1, \lambda_0 = 1\,\mu m, k_2 = 0.2\,\mu m^{-1}, r_0 = 5\,\mu m, L = 125\,\mu m

n_0 = 1.5, n = 1, \lambda_0 = 1\,\mu m, k_2 = 0.2\,\mu m^{-1}, r_0 = 5\,\mu m, L = 180\,\mu m

n_0 = 1.5, n = 1, \lambda_0 = 1\,\mu m, k_2 = 0.2\,\mu m^{-1}, r_0 = 5\,\mu m, L = 245\,\mu m
Propagación en un Medio de Forma Lente

Comparación entre las Soluciones Exactas y Aproximadas

**Ecuación Exacta**

\[
 n_0 \left(1 - \frac{k_2}{2k_0} r^2 \right) \frac{d^2 r}{dz^2} = -n_0 \frac{k_2}{k_0} r
\]

**Ecuación Aproximada**

\[
 n_0 \frac{d^2 r}{dz^2} \approx -n_0 \frac{k_2}{k_0} r
\]
Propagation in Lens-Like Medium

Comparison between Exact Eikonal and other Approximate Solutions

**Exact Eikonal Equation**

\[
\frac{d}{ds} \left( n \frac{dr}{ds} \right) = \nabla n
\]

**Approximate Equation**

\[
n_0 \frac{d^2 r}{dz^2} \approx -n_0 \frac{k_2}{k_0} r
\]

**Better Approximation**

\[
n_0 \left( 1 - \frac{k_2}{2k_0} r^2 \right) \frac{d^2 r}{dz^2} = -n_0 \frac{k_2}{k_0} r
\]

**Figure Legend**

- Better
- Approximate
- Exact (eikonal)

**Parameters**

- \( r(0) = 10 \mu m \)
- \( r'(0) = 0 \)
- \( \lambda_0 = 1 \mu m \)
- \( k_2 = 0.01 \mu m^{-1} \)
- \( n_0 = 1.5 \)
- \( k_2/k_0 = 0.0015915 \)
Fermat's Principle and Law of Reflection

\[ OPL = \int_A^B n(\vec{r}(s)) \, ds = \int_A^O n(\, ds + \int_O^B n(\, ds = n(AO) + n(BO) \]

\[ \delta[OPL] = 0 \implies n \frac{d(AO)}{dx} + n \frac{d(BO)}{dx} = 0 \implies \]

\[ \frac{d}{dx} \left( \sqrt{x^2 + a^2} \right) = -\frac{d}{dx} \left( \sqrt{(d - x)^2 + b^2} \right) \implies \frac{x}{\sqrt{x^2 + a^2}} = \frac{d - x}{\sqrt{(d - x)^2 + b^2}} \]

\[ \sin \alpha_1 = \sin \alpha_2 \implies \alpha_1 = \alpha_2, \quad \text{Law of reflection} \]
Fermat's Principle and Law of Refraction

\[ OPL = \int_{A}^{B} n(\vec{r}(s)) \, ds = n_1 \int_{A}^{O} ds + n_2 \int_{O}^{B} ds = n_1(\text{AO}) + n_2(\text{BO}) \Rightarrow \]

\[ \delta[OPL] = 0 \Rightarrow n_1 \frac{d(\text{AO})}{dx} + n_2 \frac{d(\text{OB})}{dx} = 0 \Rightarrow \]

\[ n_1 \frac{d}{dx} \left( \sqrt{x^2 + a^2} \right) = -n_2 \frac{d}{dx} \left( \sqrt{(d-x)^2 + b^2} \right) \Rightarrow n_1 \frac{x}{\sqrt{x^2 + a^2}} = n_2 \frac{d-x}{\sqrt{(d-x)^2 + b^2}} \Rightarrow \]

\[ n_1 \sin \alpha_1 = n_2 \sin \beta_1, \quad \text{Snell's Law} \]
Fermat's Principle and Law of Reflection (maximum path)

\[
\sqrt{(x - d)^2 + y^2} + \sqrt{(x + d)^2 + y^2} = 2\sqrt{b^2 + d^2} = (a - d) + (a + d) = 2a
\]

\[
b^2 + d^2 = a^2
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

\[
(AC) + (CB) = (AP) + (PB) > (AQ) + (QB)
\]
Fermat's Principle and Law of Reflection (equal paths)

\[ \sqrt{(x - d)^2 + y^2} + \sqrt{(x + d)^2 + y^2} = 2\sqrt{b^2 + d^2} = (a - d) + (a + d) = 2a \]

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

\[ (AC') + (CB) = (AP) + (PB) \]
Fermat's Principle and Law of Reflection (Hyperbolical Mirror)

\[
\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = (a + c) - (c - a) = 2a
\]

\[
a^2 + b^2 = c^2
\]

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

\[
(PB) - (PA) = (CB) - (CA) = 2a
\]
Fermat's Principle and Law of Reflection (Parabolic Mirror)

\[ \sqrt{(f-x)^2 + y^2} + (f-x) = 2f \implies y^2 = 4fx \]

\[ (AC') + (CP) = 2(AO) \]
Deviation Angle of a Dispersing Prism

Geometry & Snell’s Law

\[ \delta = \delta_1 + \delta_2 = (\alpha_1 - \beta_1) + (\beta_2 - \alpha_2) = \alpha_1 + \beta_2 - (\beta_1 + \alpha_2) = \alpha_1 + \beta_2 - \sigma, \]

where,

\[ \beta_2 = \sin^{-1} \left( \frac{n_p}{n_0} \sin \alpha_2 \right), \]

\[ \alpha_2 = \sigma - \beta_1 = \sigma - \sin^{-1} \left( \frac{n_0}{n_p} \sin \alpha_1 \right), \]

Minimum \( \delta \):

\[ \frac{d\delta}{d\alpha_1} = 0 \]

Approximation

\[ \delta_{min} \simeq \left( \frac{n_p}{n_0} - 1 \right) \sigma \]

\[ \alpha_1 = \beta_2 = \sin^{-1} \left[ \frac{n_p}{n_0} \left( \sin \frac{\sigma}{2} \right) \right] \]
Deviation Angle of a Dispersing Prism

Example for a BK7 Glass Prism $\sigma = 30 \text{ deg}$
Refractive Index Measurement of Liquids

Prism Minimum Deviation Angle Experiment

Measure deflection angle through liquid-filled prism.

\[ \delta_{\text{min}} = 2 \sin^{-1} \left[ \frac{n_p}{n_0} \sin \left( \frac{\sigma}{2} \right) \right] - \sigma \]

\[ \delta_{\text{min}} \approx \sigma \left( \frac{n_p}{n_0} - 1 \right) \quad \text{for} \quad \delta, \sigma \ll 1 \text{ rad} \]

Refractive Index Measurement of Water


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Typical Dispersion

The index of refraction vs. wavelength

Normal dispersion

http://1.bp.blogspot.com/-kcHWNW81dl4/UR_qgF5B9dl/AAAAAAACNg/h5oijsP_gTs/s640/figure+3.png
Normal Dispersion

![Graph showing the refractive index of different optical glasses as a function of wavelength.](image)

- **Lanthanum dense flint LaSF9**
- **Dense flint SF10**
- **Flint F2**
- **Barium crown BaK4**
- **Borosilicate crown BK7**
- **Fluorite crown FK51A**

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Normal Dispersion- Cauchy Formula

\[ n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots \]

<table>
<thead>
<tr>
<th>Material</th>
<th>A</th>
<th>B (µm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fused silica</td>
<td>1.4580</td>
<td>0.00354</td>
</tr>
<tr>
<td>Borosilicate glass BK7</td>
<td>1.5046</td>
<td>0.00420</td>
</tr>
<tr>
<td>Hard crown glass K5</td>
<td>1.5220</td>
<td>0.00459</td>
</tr>
<tr>
<td>Barium crown glass BaK4</td>
<td>1.5690</td>
<td>0.00531</td>
</tr>
<tr>
<td>Barium flint glass BaF10</td>
<td>1.6700</td>
<td>0.00743</td>
</tr>
<tr>
<td>Dense flint glass SF10</td>
<td>1.7280</td>
<td>0.01342</td>
</tr>
</tbody>
</table>
Normal Dispersion - Sellmeier Formula

\[ n^2(\lambda) = 1 + \sum_i \frac{B_i \lambda^2}{\lambda^2 - C_i} \]

<table>
<thead>
<tr>
<th>Material</th>
<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>borosilicate crown glass</td>
<td>0.696166300</td>
<td>0.407942600</td>
<td>0.897479400</td>
<td>4.67914826×10⁻³µm²</td>
<td>1.35120631×10⁻²µm²</td>
<td>97.9340025 µm²</td>
</tr>
<tr>
<td>(known as BK7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sapphire (for ordinary wave)</td>
<td>1.5039759</td>
<td>0.55069141</td>
<td>6.5927379</td>
<td>5.48041129×10⁻³µm²</td>
<td>1.47994281×10⁻²µm²</td>
<td>4.0289514×10²µm²</td>
</tr>
<tr>
<td>sapphire (for extraordinary wave)</td>
<td>1.43134930</td>
<td>0.65054713</td>
<td>5.3414021</td>
<td>5.2799261×10⁻³µm²</td>
<td>1.42382647×10⁻²µm²</td>
<td>3.25017834×10²µm²</td>
</tr>
<tr>
<td>fused silica</td>
<td>1.03961212</td>
<td>0.231792344</td>
<td>1.01046945</td>
<td>6.00069867×10⁻³µm²</td>
<td>2.00179144×10⁻²µm²</td>
<td>1.03560653×10²µm²</td>
</tr>
</tbody>
</table>
Normal Dispersion
Comparison of Cauchy and Sellmeier Formulas
For BK7 Glass
Prism Dispersion Properties – BK7

Dispersive Prism: Borosilicate Crown Glass (BK7)

\[ \sigma = 60^\circ, \quad \alpha_1 = 70^\circ, \quad L = 10\text{mm}, \quad n_0 = 1, \quad n_{p,\text{max}} = 1.5308 \]
Prism Dispersion Properties – Schott N-SF11 Glass

Dispersive Prism: N-SF11 high dispersive glass (Schott)
\[ \sigma = 60°, \alpha_1 = 70°, L = 10\,\text{mm}, n_0 = 1, n_{p,\text{max}} = 1.8454 \]
Dispersive Prism – Dispersive Power

\[ \delta_{\text{min}} = 2 \sin^{-1} \left[ \frac{n_p}{n_0} \sin \left( \frac{\sigma}{2} \right) \right] - \sigma \]

\[ \delta_{\text{min}} \approx \sigma \left( \frac{n_p}{n_0} - 1 \right) \quad \text{for} \quad \delta, \sigma \ll 1 \text{ rad} \]

\[ \Delta_e = \frac{\delta_{\text{min},F} - \delta_{\text{min},C}}{\delta_{\text{min},D}} \]

\[ \Delta_a \approx \frac{n_F - n_C}{n_D - n_0} \]

Abbe number: \[ \nu = \frac{1}{\Delta} \]

<table>
<thead>
<tr>
<th>Freespace Wavelength, ( \lambda_0 ) (nm)</th>
<th>Characterization</th>
<th>Crown Glass Refractive Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>486.1</td>
<td>F, blue (H dark line)</td>
<td>( n_F = 1.5286 )</td>
</tr>
<tr>
<td>589.2</td>
<td>D, yellow (Na dark line)</td>
<td>( n_D = 1.5230 )</td>
</tr>
<tr>
<td>656.3</td>
<td>C, red (H dark line)</td>
<td>( n_C = 1.5205 )</td>
</tr>
</tbody>
</table>
Dispersive Prisms

Dispersive Power / Abbe’s number

<table>
<thead>
<tr>
<th>Fraunhofer line</th>
<th>Color</th>
<th>Wavelength (nm)</th>
<th>Spectacle Crown</th>
<th>Extra-dense Flint</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>Blue (hydrogen)</td>
<td>486.1</td>
<td>1.5293</td>
<td>1.7378</td>
</tr>
<tr>
<td>D</td>
<td>Yellow (sodium)</td>
<td>589.3</td>
<td>1.5230</td>
<td>1.7200</td>
</tr>
<tr>
<td>C</td>
<td>Red (hydrogen)</td>
<td>656.3</td>
<td>1.5204</td>
<td>1.7130</td>
</tr>
</tbody>
</table>

\[ \nu = \frac{(n_D - 1)}{(n_F - n_C)} = \text{Abbe’s number} \]

\[ \nu = 59 \quad \text{29} \]

http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/imggo/disper2.gif

Doublet for Chromatic Aberration

Achromat Doublets

http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/aber2.html#c2

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Reflective Prisms

**Penta prism** can deviate an incident beam without inverting or reversing to 90°. The deviation angle of 90° is independent of any rotation of the prism about an axis parallel to the line of intersection of the two reflecting faces. It is commonly used in Plumb Level, Surveying, Alignment, Rangefinding and Optical Tooling.

**Beamsplitting Penta prism:** By adding a wedge and with partial refractive coating on surfaces S1, it can be used as a beamsplitter. It is often used in Plumb Level, Surveying, Alignment, Rangefinding and Optical Tooling.

https://www.edmundoptics.eu/resources/application-notes/optics/introduction-to-optical-prisms/
Reflective Prisms

**Right angle prism** is deviating or deflecting a beam of light with 90 or 180°. It is often used in telescope, periscope and other optical system.

**Dove prism** has two applications. The main application is used as a rotator. It can rotate an image but without deviating the beam. And when the prism is rotated about the input parallel ray through some angle, the image rotates through twice that angle. It is very important that the application must be used with parallel or collimated beam and the large square reflective surface should be kept very clean. Another application is used as a retroreflector. For this application it perform as a right-angle prism.
**Reflective Prisms**

**Roof prism (Amici)** is combined with a right angle prism and a totally internally reflecting roof and they are attached by them largest square surfaces. It can invert and reverse an image, also, deflect the image 90°. Therefore, it is often used in terrestrial telescopes, viewing systems and rangefinders.

**Corner Cube Prism:** It has three mutually perpendicular surfaces and a hypotenuse face. Light entering through the hypotenuse is reflected by each of the three surfaces in turn and will emerge through the hypotenuse face parallel to the entering beam regardless of the orientation the incident beam. For its special performance, it is often used to the distance measurement, optical signal process and laser interferometer.

**Anamorphic Prisms:** These two prisms can expand or contract the beam in one direction without any changes in the other direction. By adjusting the angles among the incident beam and two prisms, the shape of the beam can be changed. It is very easy to turn elliptical bean into circular beam.
Reflective Prisms

The **Penta Prism** will deviate the beam by 90° without affecting the orientation of the image. It has the valuable property of being a constant-deviation prism, in that it deviates the line-of-sight by 90° regardless of its orientation to the line-of-sight. Note that two of its surfaces must be silvered. These prisms are often used as end reflectors in small rangefinders.

The **Rhomboid Prism** displaces the line of sight without producing any angular deviation or changes in the orientation of the image.
Erecting Prisms

Reflective Prisms

Porro-Abbe Prism

Figure 4.24 Porro prism system (first type) (a) indicating the way the Porro system erects an inverted image. (b) Porro prisms are usually fabricated with rounded ends to save space and weight. Note that the spacing between the prisms has been shown increased for clarity.

Figure 4.25 Porro prism system (second type) (a) indicating the erection of an inverted image. This system is shown made from two prisms in (a) and from three prisms in (b).

Porro Prism

• Right angle prism
• Oriented to deviate light by 180 degrees