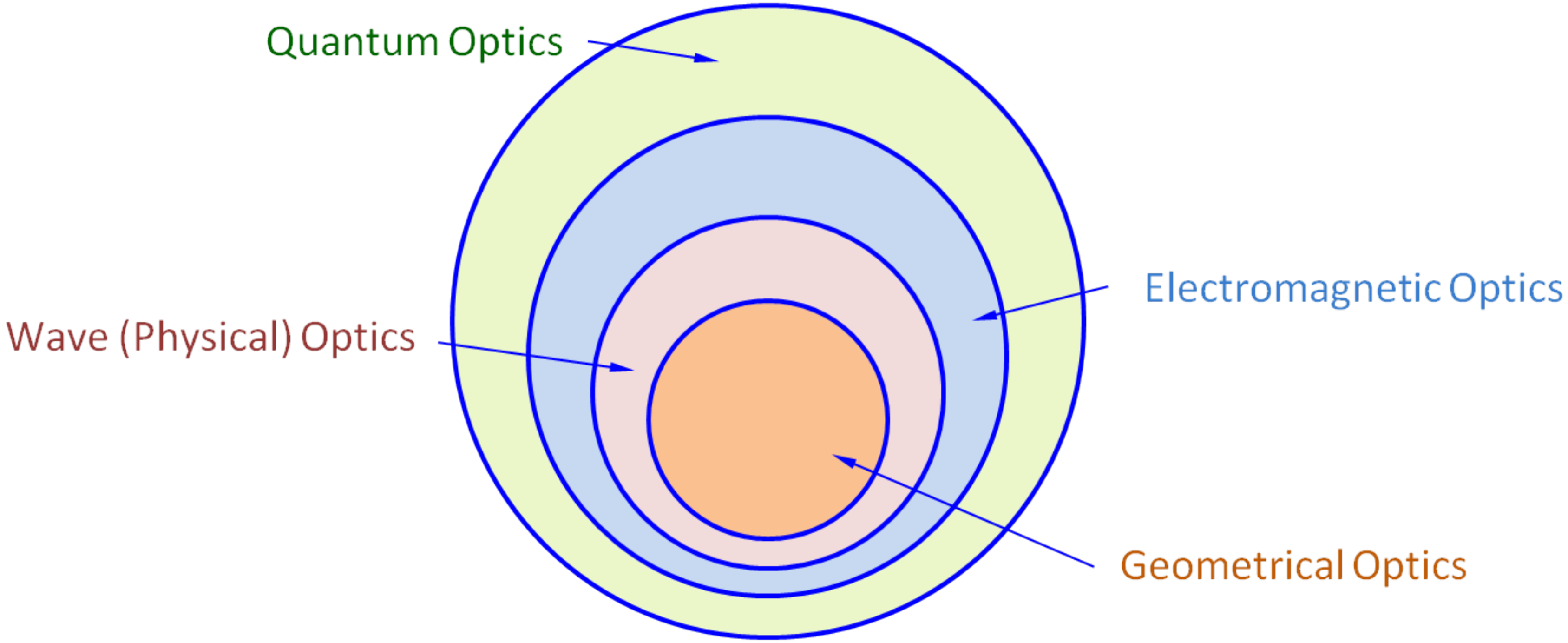
Introduction to Geometrical Optics & Prisms

Optical Engineering

Prof. Elias N. Glytsis

School of Electrical & Computer Engineering
National Technical University of Athens

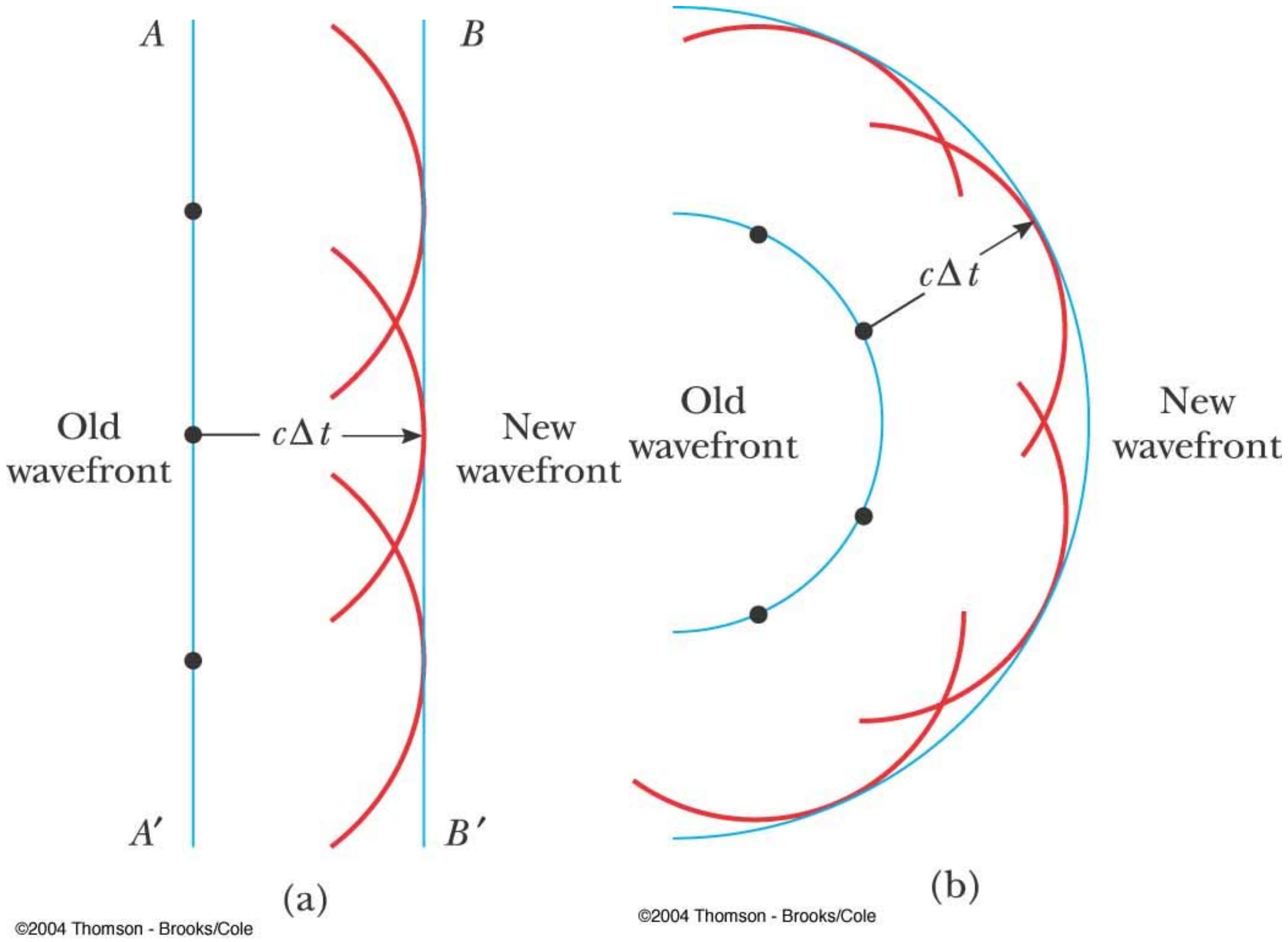
Geometrical Optics



$$\lim_{\lambda \to 0} \{\text{Wave Optics}\} = \text{Geometrical Optics}$$

Originally from the Book: B. E. A. Saleh and M. C. Teich, *Fundamentals of Photonics*, 2nd Ed., J. Wiley 2007

Huygens's Principle (1678 AD)



R. A. Serway and J. W. Jewett, Physics for Scientists & Engineers, 6th Ed.,Thomson_Brooks/Cole, 2004

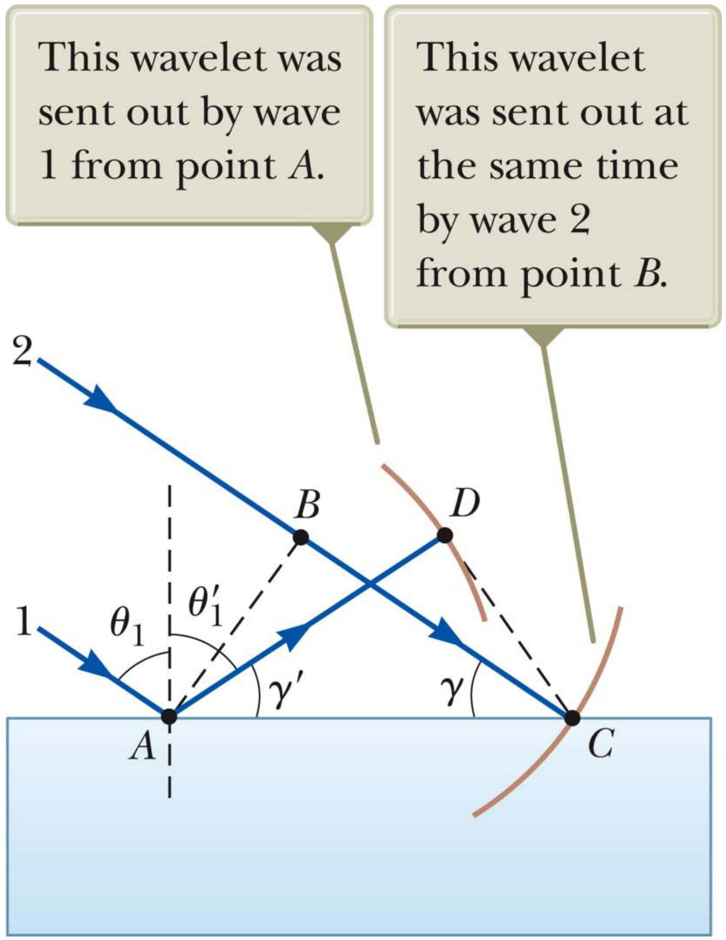
Huygens's Principle and Law of Reflection

AB is a plane wave front of incident light.

The wave at *A* sends out a wavelet centered on *A* toward *D.*
The wave at *B* sends out a wavelet centered on *B* toward *C.*

$$(AD) = (BC) = u\Delta t \Longrightarrow \theta_1 = \theta_1' \Longrightarrow$$

Angle of Incidence = Angle of Reflection



R. A. Serway and J. W. Jewett, Physics for Scientists & Engineers, 6th Ed.,Thomson_Brooks/Cole, 2004

Huygens's Principle and Law of Refraction

Ray 1 strikes the surface and at a time interval *Δt* later, Ray 2 strikes the surface.

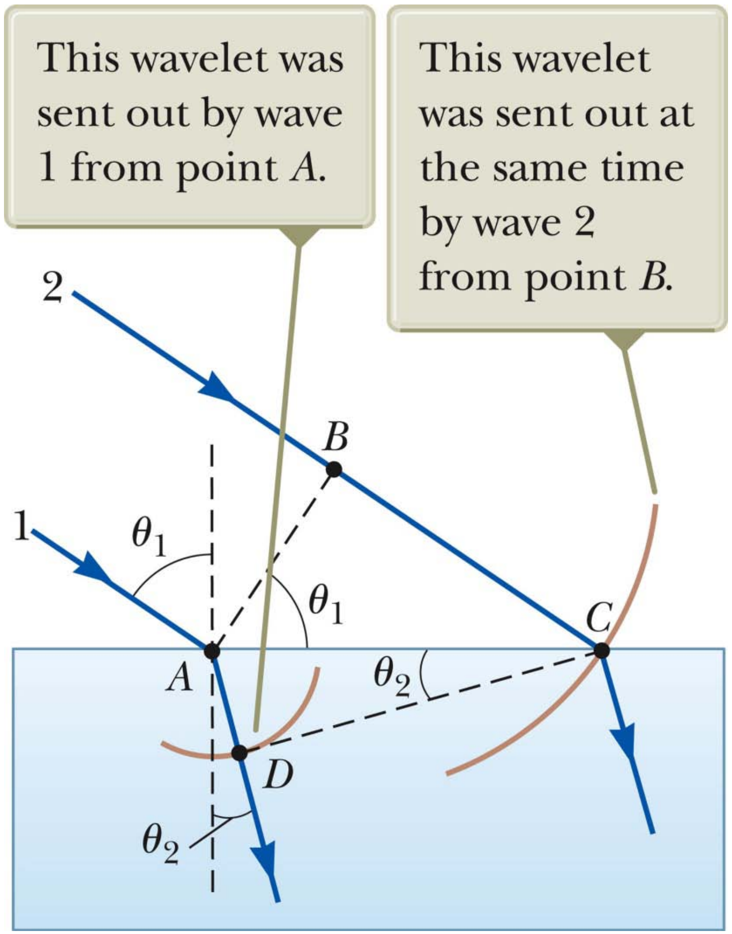
During this time interval, the wave at *A* sends out a wavelet, centered at *A*, toward *D.*

From triangles ABC and ADC, we find

$$\sin\theta_1 \;=\; = \frac{BC}{AC} = \frac{u_1\Delta t}{AC}$$

$$\sin\theta_2 \;=\; = \frac{AD}{AC} = \frac{u_2\Delta t}{AC}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{u_1}{u_2} = \frac{n_2}{n_1} \Longrightarrow n_1\sin\theta_1 = n_2\sin\theta_2$$

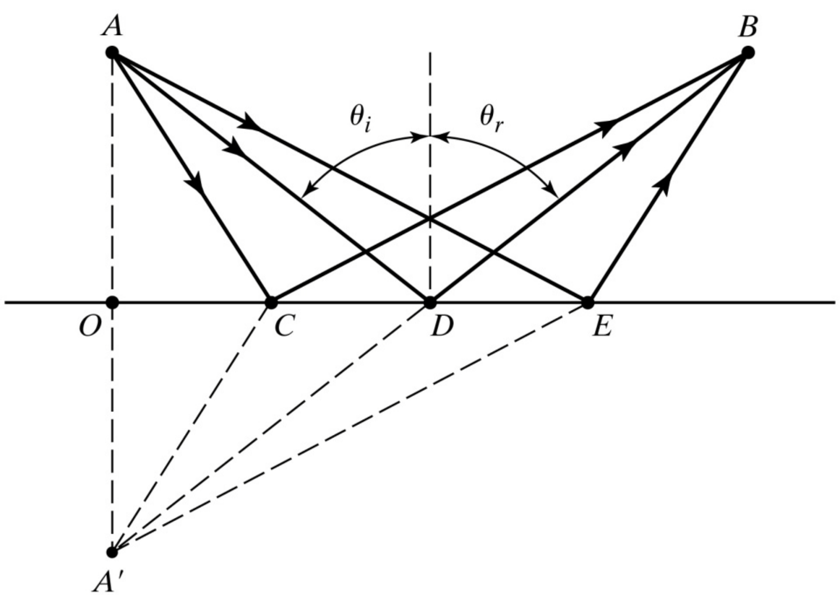


R. A. Serway and J. W. Jewett, Physics for Scientists & Engineers, 6th Ed.,Thomson_Brooks/Cole, 2004

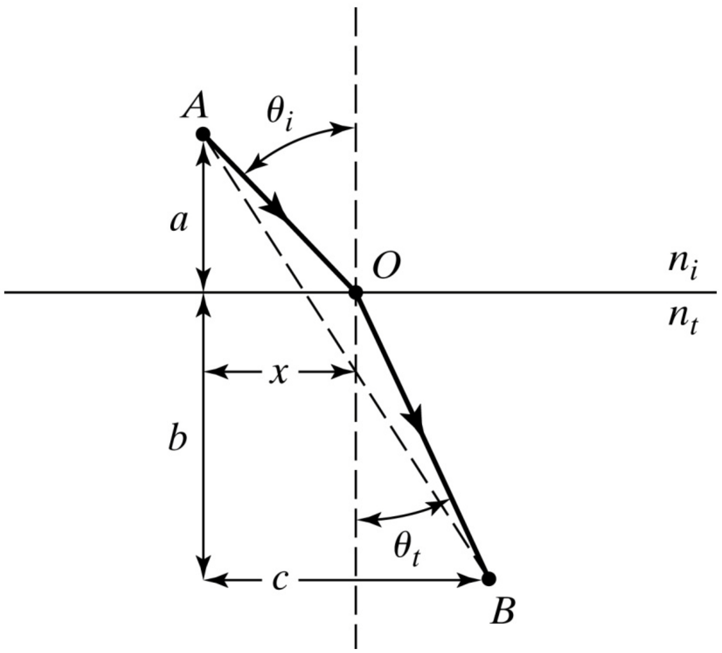
Fermat's Principle (1662 AD)

A ray of light in going from point A to point B will travel an optical path (*OPL*) that minimizes the *OPL*. That is, it is stationary with respect to variations in the *OPL*.

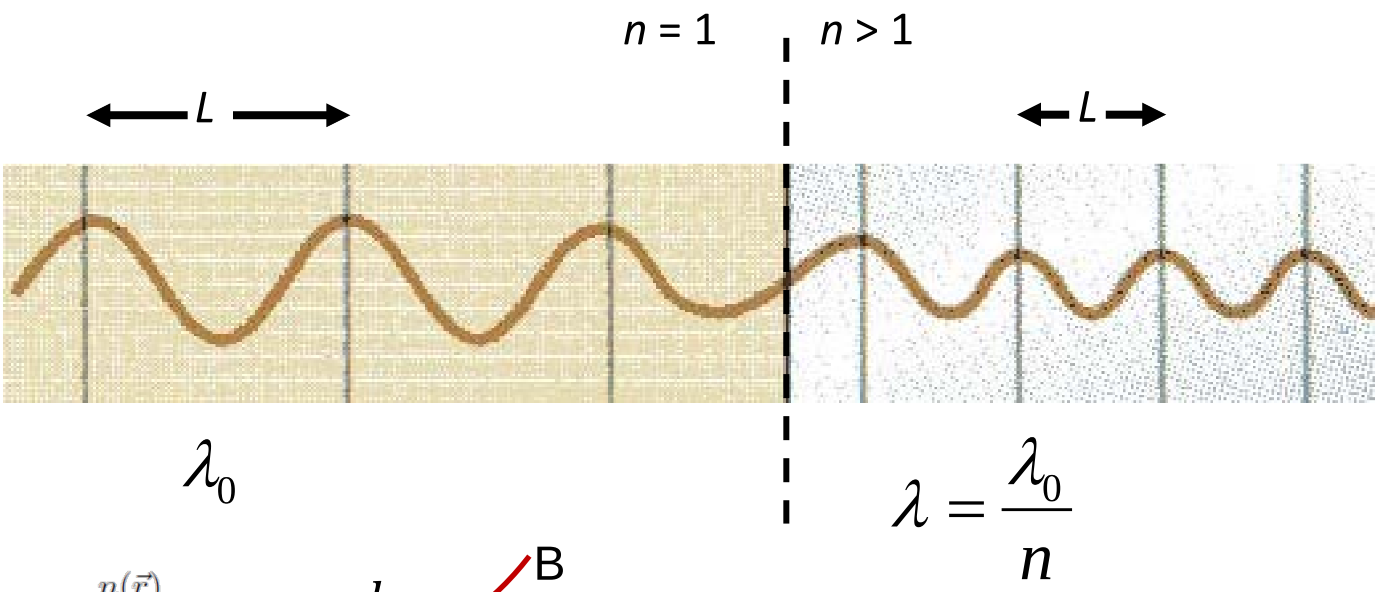
Law of Reflection
(*Hero* - least distance)

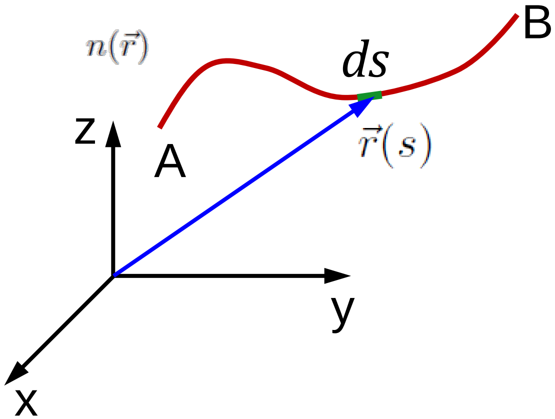


Law of Refraction
(*Fermat* - least time)



Optical Path Length (OPL)

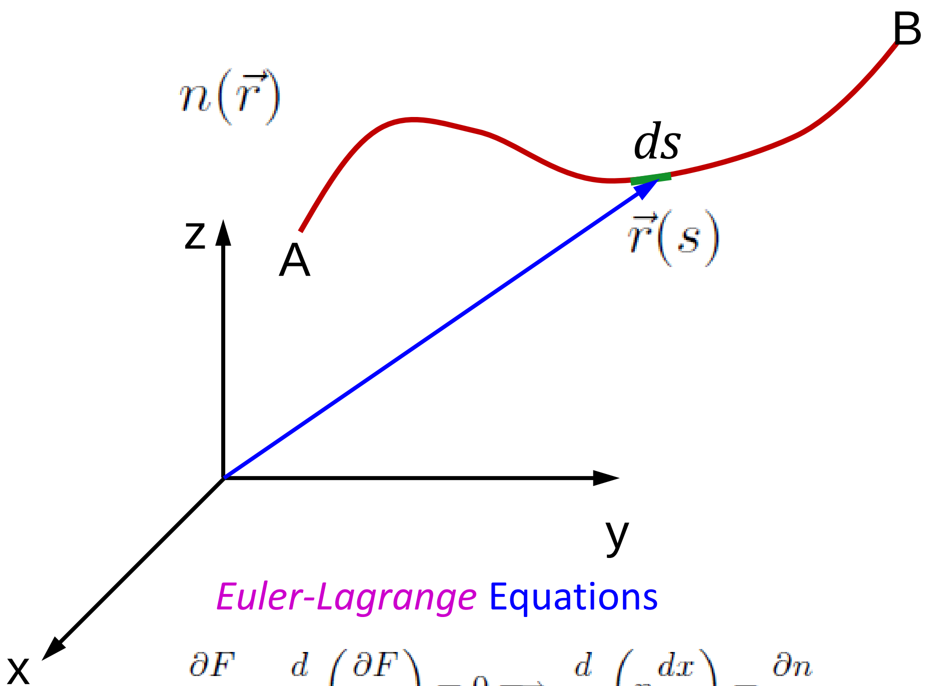




$$OPL = \int_A^B n\left(\vec{r}(s)\right) ds,$$

R. A. Serway and J. W. Jewett, Physics for Scientists & Engineers, 6th Ed.,Thomson_Brooks/Cole, 2004

Fermat's Principle (1662 AD)



Fermat's Principle

$$OPL = \int_A^B n\left(\vec{r}(s)\right) ds$$

$$\vec{r} = x(t)\hat{\imath}_x + y(t)\hat{\imath}_y + z(t)\hat{\imath}_z$$

$$ds = (dx^2 + dy^2 + dz^2)^{1/2} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} dt$$

$$OPL = \int_A^B \underbrace{n(x,y,z)\ \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}_{F(x,y,z,\dot{x},\dot{y},\dot{z})}\ dt$$

Euler-Lagrange Equations

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \Longrightarrow \frac{d}{ds}\left(n\frac{dx}{ds}\right) = \frac{\partial n}{\partial x}$$

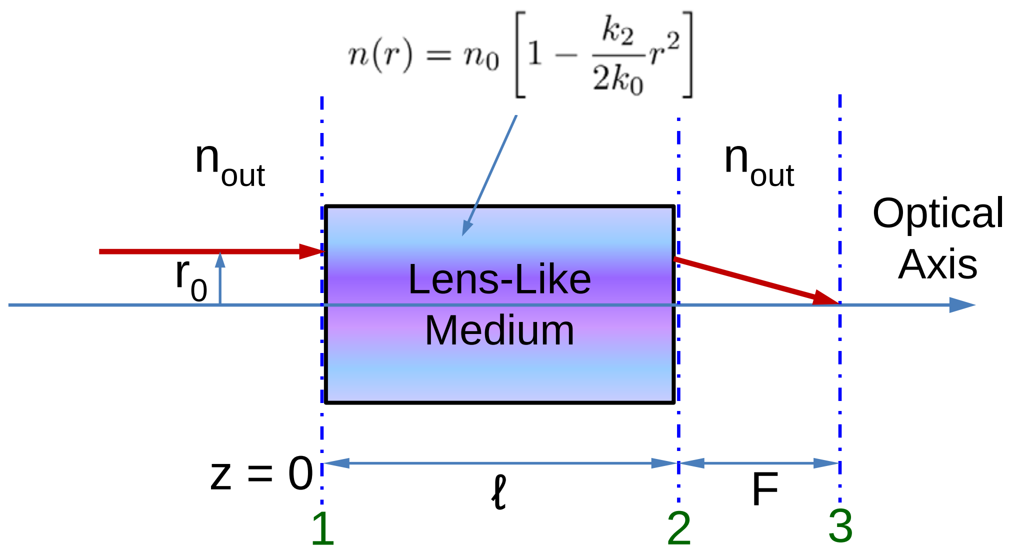
$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0 \Longrightarrow \frac{d}{ds}\left(n\frac{dy}{ds}\right) = \frac{\partial n}{\partial y}$$

$$\frac{\partial F}{\partial z} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}}\right) = 0 \Longrightarrow \frac{d}{ds}\left(n\frac{dz}{ds}\right) = \frac{\partial n}{\partial z}$$

Ray-Path Equation (Eikonal)

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \vec{\nabla}n.$$

Propagation in *Lens-Like Medium*



Ray-Path Equation (Eikonal)

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \vec{\nabla}n$$

Approximate Equation

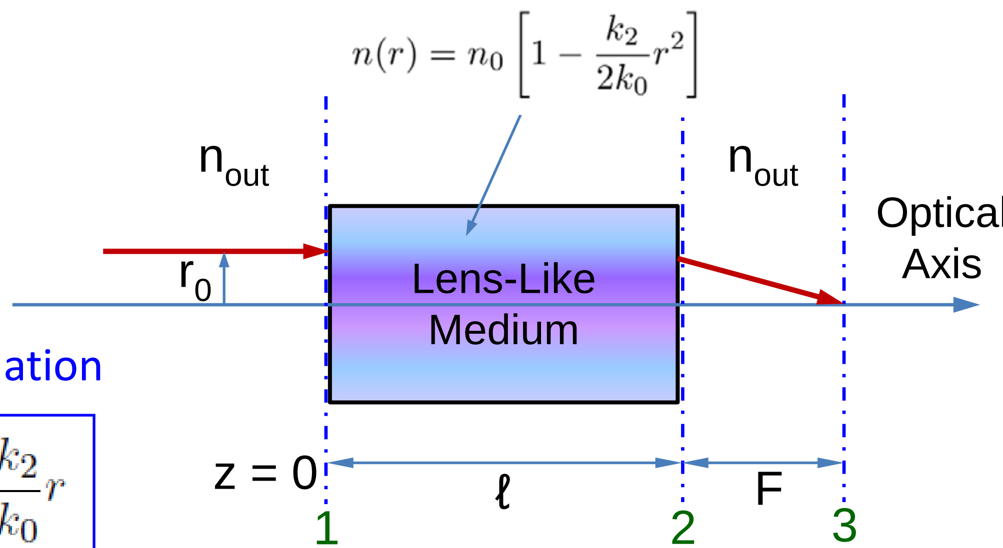
$$n_0\frac{d^2r}{dz^2} \simeq -n_0\frac{k_2}{k_0}r$$

Transformation

$$t = \int ds/n$$

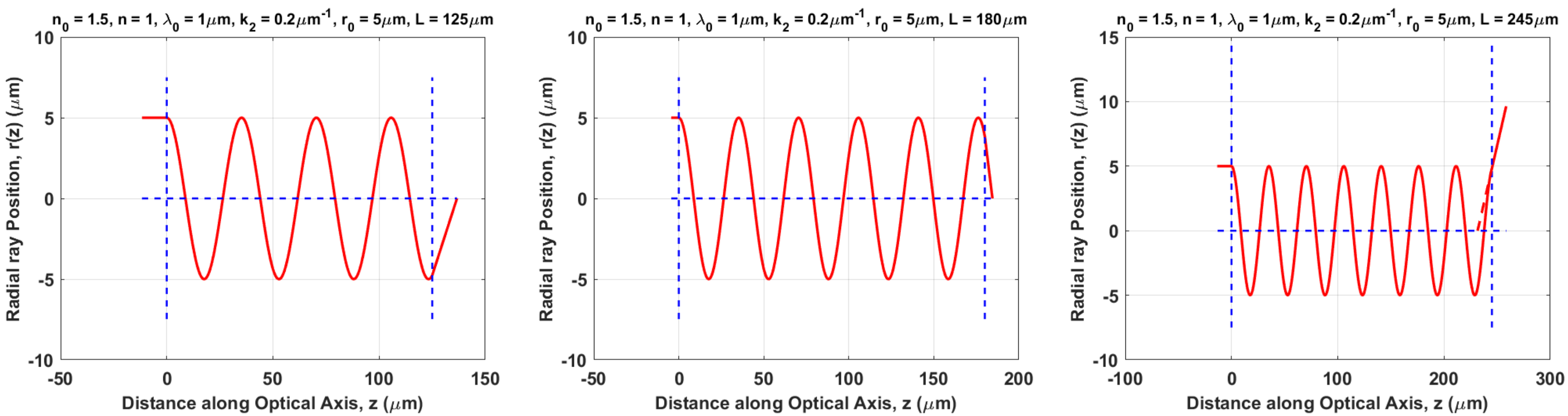
$$\frac{d^2\vec{r}}{dt^2} = n(\vec{r})\vec{\nabla}n$$

Propagation in *Lens-Like Medium*



Approximate Equation

$$n_0 \frac{d^2 r}{dz^2} \simeq -n_0 \frac{k_2}{k_0} r$$



Propagation in *Lens-Like Medium*

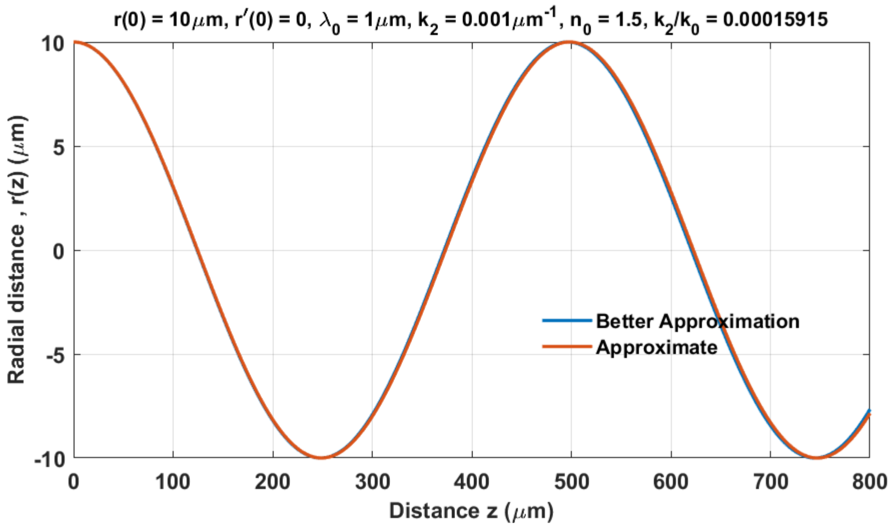
Comparison between Approximate Solutions

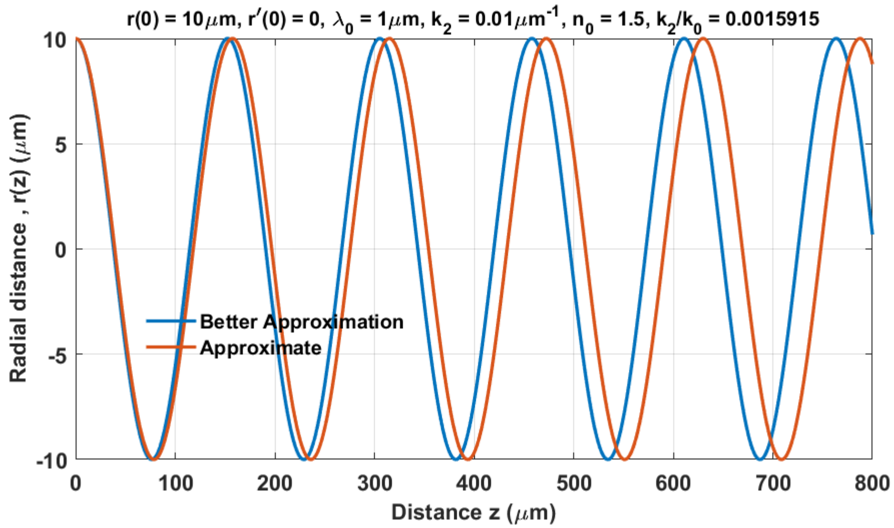
Better Approximation Equation

$$n_0\left(1-\frac{k_2}{2k_0}r^2\right)\frac{d^2r}{dz^2}=-n_0\frac{k_2}{k_0}r$$

Approximate Equation

$$n_0\frac{d^2r}{dz^2}\simeq-n_0\frac{k_2}{k_0}r$$





Propagation in *Lens-Like Medium*

Comparison between Exact Eikonal and other Approximate Solutions

Exact Eikonal Equation

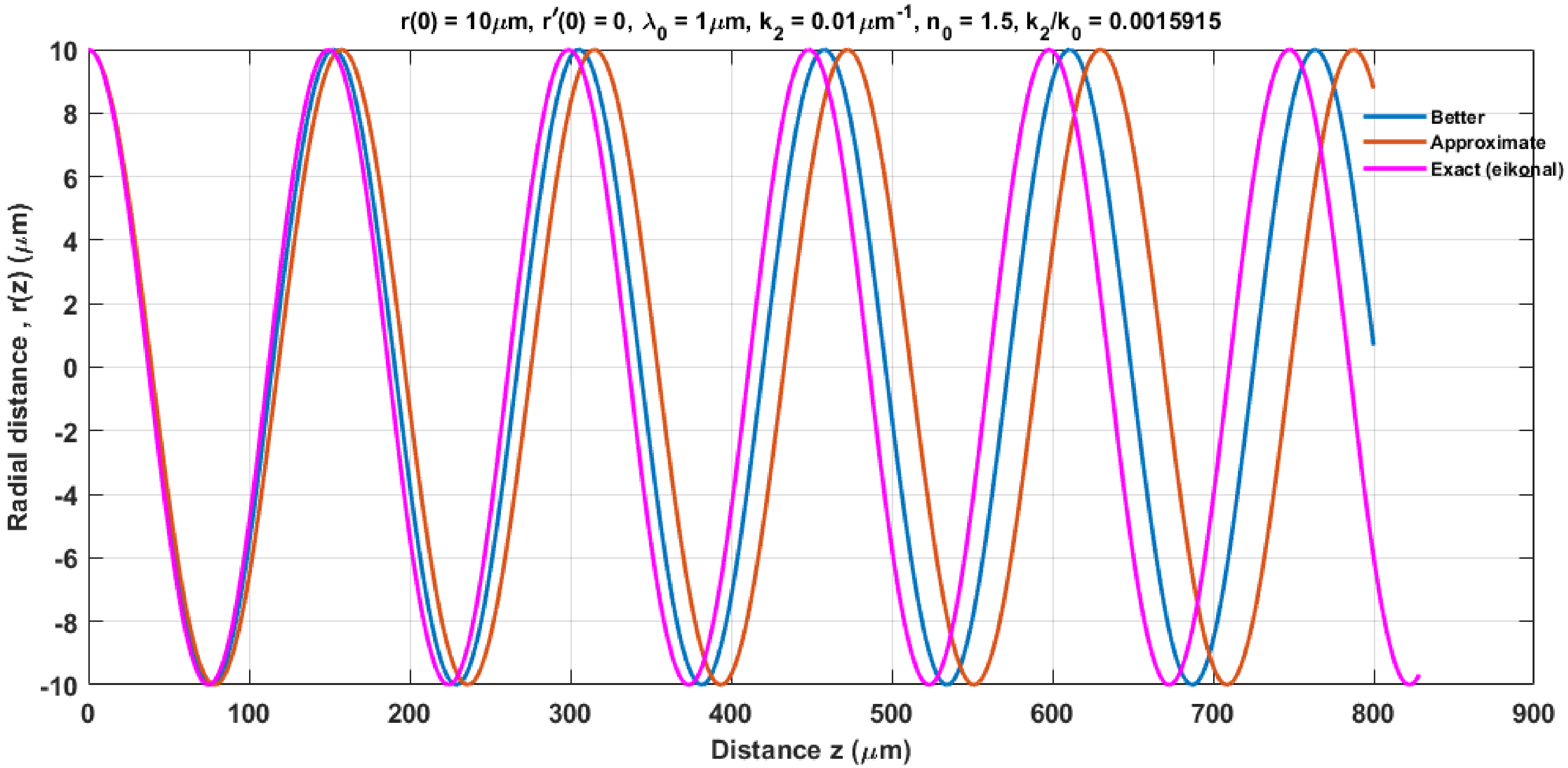
$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \vec{\nabla} n$$

Approximate Equation

$$n_0\frac{d^2 r}{dz^2} \simeq -n_0\frac{k_2}{k_0}r$$

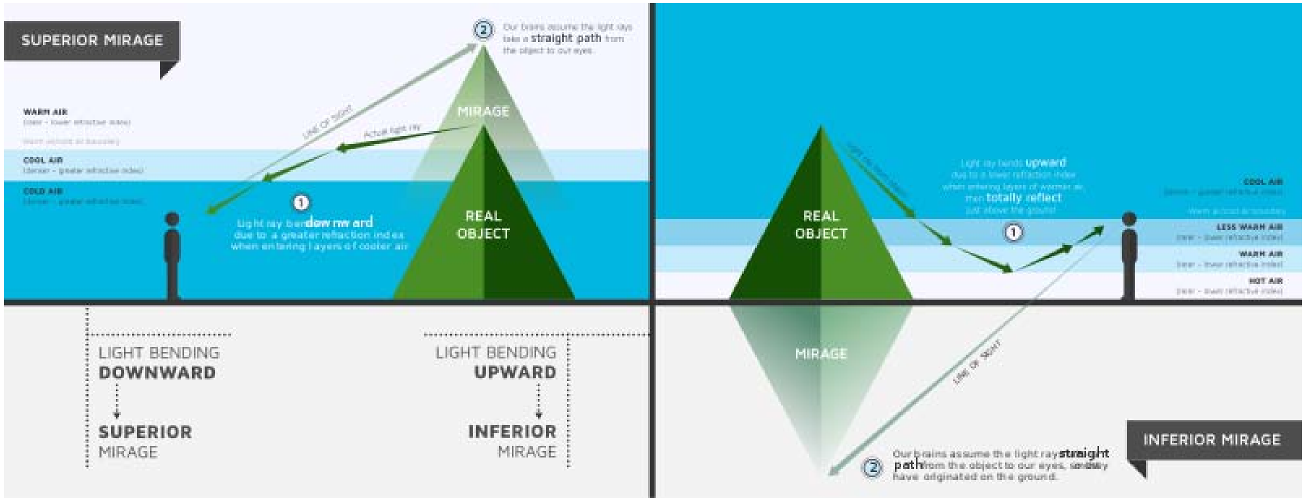
Better Approximation

$$n_0\left(1 - \frac{k_2}{2k_0}r^2\right)\frac{d^2 r}{dz^2} = -n_0\frac{k_2}{k_0}r$$

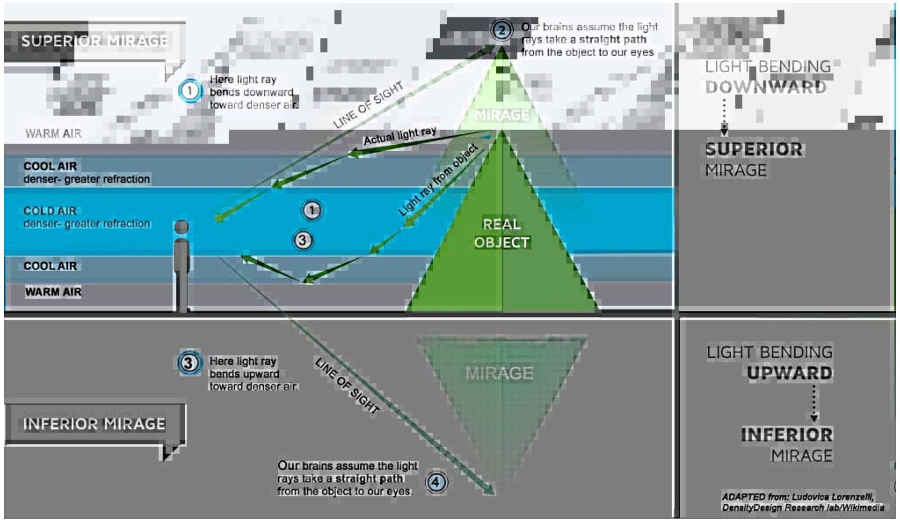


Example of Propagation in *Graded Index Medium*

Inferior and Superior Mirage Phenomenon



https://upload.wikimedia.org/wikipedia/commons/thumb/5/54/Superior_and_inferior_mirage.svg/1200pxSuperior_and_inferior_mirage.svg.png?20141201143416



https://www.friendslakeshorepreserve.com/mirage.html

Example of Propagation in *Graded Index Medium*

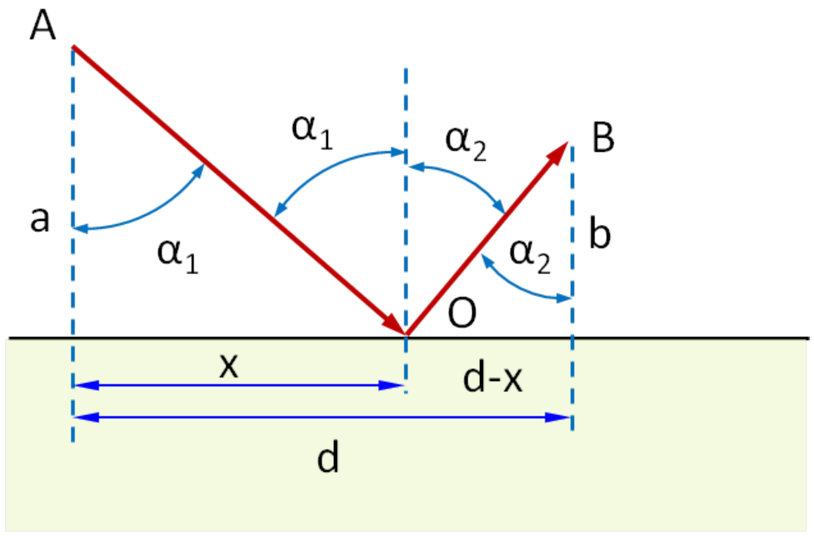
Inferior and Superior Mirage Phenomenon

http://www.astronomycafe.net/weird/lights/mirgal.htm

https://www.eoas.ubc.ca/courses/atsc113/sailing/met_concepts/10-met-local-conditions/10f-optical-phenomena/img-10f/10-superior-mirage.jpg

https://www.eoas.ubc.ca/courses/atsc113/sailing/met_concepts/10-met-local-conditions/10f-optical-phenomena/img-10f/10-inferior-mirage.jpg

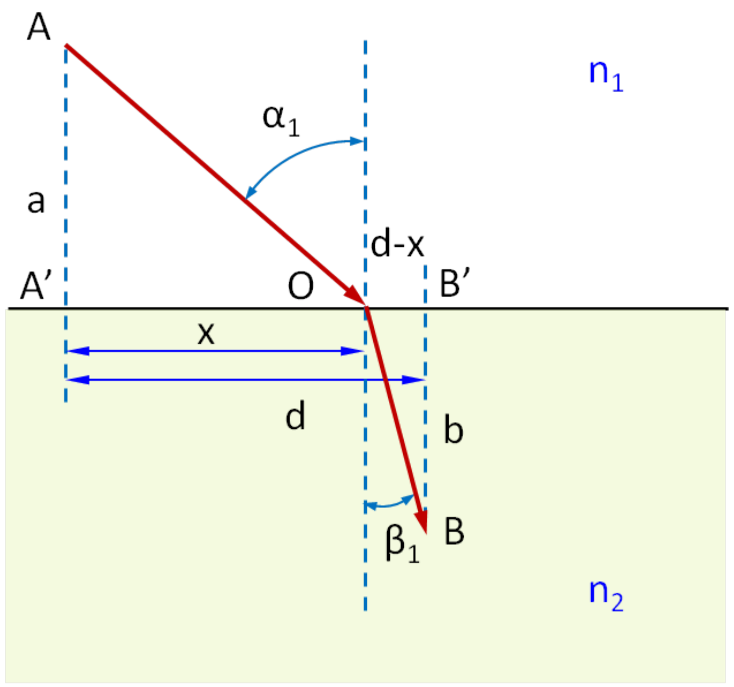
Fermat's Principle and Law of Reflection



$$
\begin{aligned}
OPL &= \int_A^B n\left(\vec{r}(s)\right) ds = n\int_A^O ds + n\int_O^B ds = n(AO) + n(BO) \Longrightarrow \\
\delta[OPL] &= 0 \Longrightarrow n\frac{d(AO)}{dx} + n\frac{d(OB)}{dx} = 0 \Longrightarrow \\
\frac{d}{dx}\left(\sqrt{x^2+a^2}\right) &= -\frac{d}{dx}\left(\sqrt{(d-x)^2+b^2}\right) \Longrightarrow \frac{x}{\sqrt{x^2+a^2}} = \frac{d-x}{\sqrt{(d-x)^2+b^2}} \Longrightarrow \\
\sin\alpha_1 &= \sin\alpha_2 \Longrightarrow \boxed{\alpha_1 = \alpha_2,}
\end{aligned}
$$

Law of reflection

Fermat's Principle and Law of Refraction



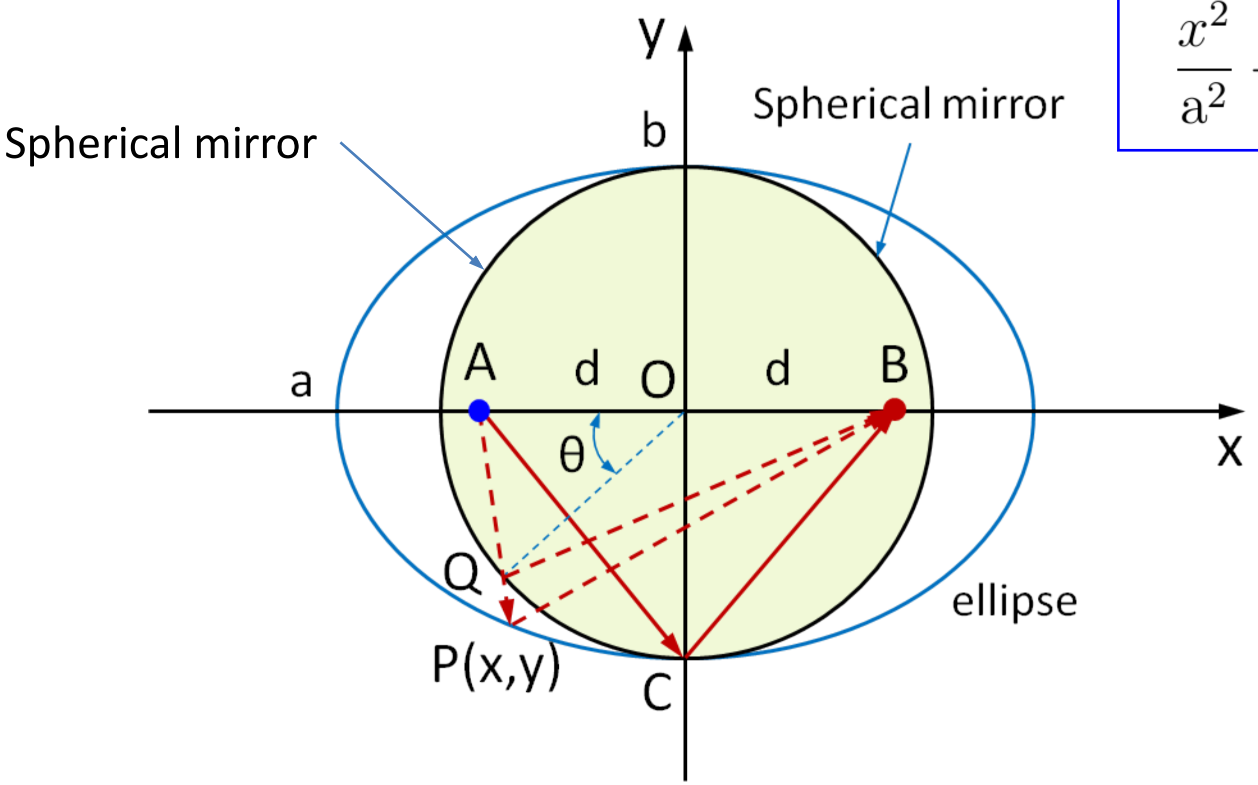
$$
\begin{aligned}
OPL &= \int_A^B n\left(\vec{r}(s)\right) ds = n_1 \int_A^O ds + n_2 \int_O^B ds = n_1(AO) + n_2(BO) \Longrightarrow \\
\delta[OPL] &= 0 \Longrightarrow n_1 \frac{d(AO)}{dx} + n_2 \frac{d(OB)}{dx} = 0 \Longrightarrow \\
n_1 \frac{d}{dx}\left(\sqrt{x^2+a^2}\right) &= -n_2 \frac{d}{dx}\left(\sqrt{(d-x)^2+b^2}\right) \Longrightarrow n_1 \frac{x}{\sqrt{x^2+a^2}} = n_2 \frac{d-x}{\sqrt{(d-x)^2+b^2}} \Longrightarrow \\
n_1 \sin\alpha_1 &= n_2 \sin\beta_1,
\end{aligned}
$$

Snell's Law

Fermat's Principle and Law of Reflection (maximum path)

$$\sqrt{(x-d)^2+y^2}+\sqrt{(x+d)^2+y^2} = 2\sqrt{b^2+d^2} = (\mathrm{a}-d)+(\mathrm{a}+d) = 2\mathrm{a}$$
$$b^2+d^2 = \mathrm{a}^2$$

$$\frac{x^2}{\mathrm{a}^2}+\frac{y^2}{b^2} = 1$$

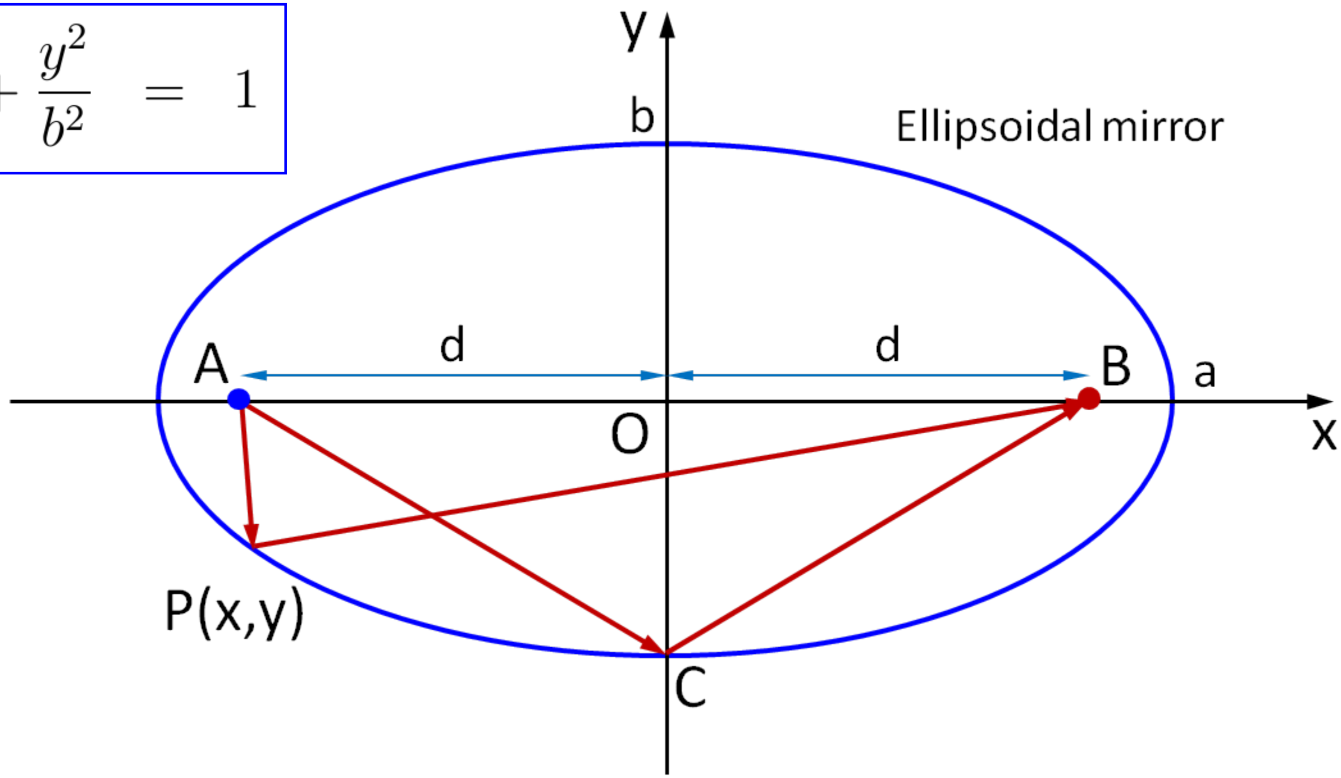


$$(AC)+(CB)=(AP)+(PB)>(AQ)+(QB)$$

Fermat's Principle and Law of Reflection (equal paths)

$$\sqrt{(x-d)^2+y^2}+\sqrt{(x+d)^2+y^2} = 2\sqrt{b^2+d^2}=(\mathrm{a}-d)+(\mathrm{a}+d)=2\mathrm{a}$$
$$b^2+d^2 = \mathrm{a}^2$$

$$\frac{x^2}{\mathrm{a}^2}+\frac{y^2}{b^2} = 1$$

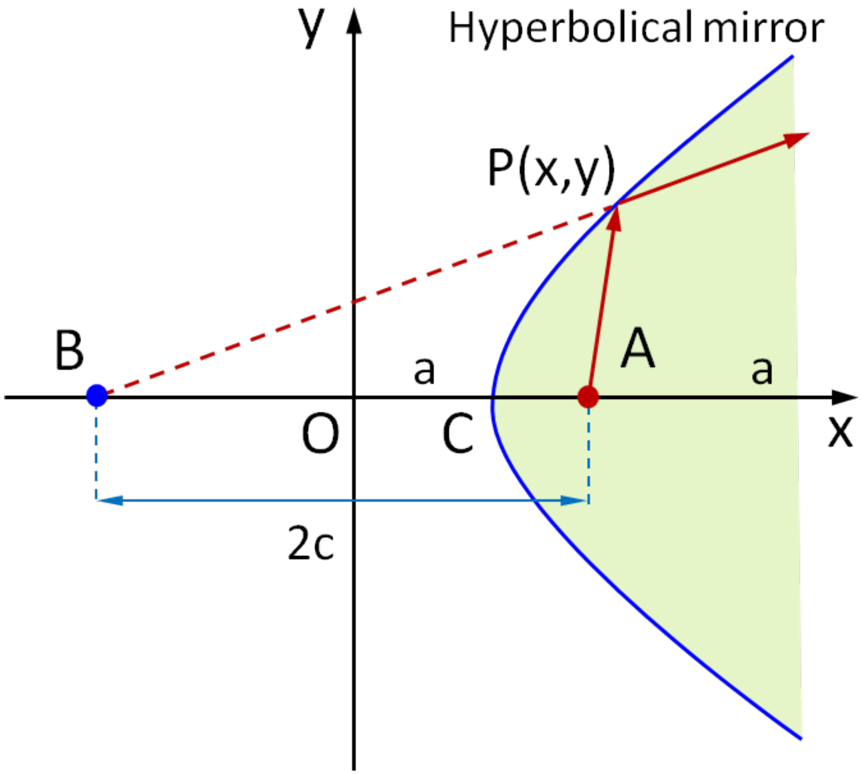


$$(AC)+(CB)=(AP)+(PB)$$

Fermat's Principle and Law of Reflection (Hyperbolical Mirror)

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2} = (a+c)-(c-a) = 2a$$

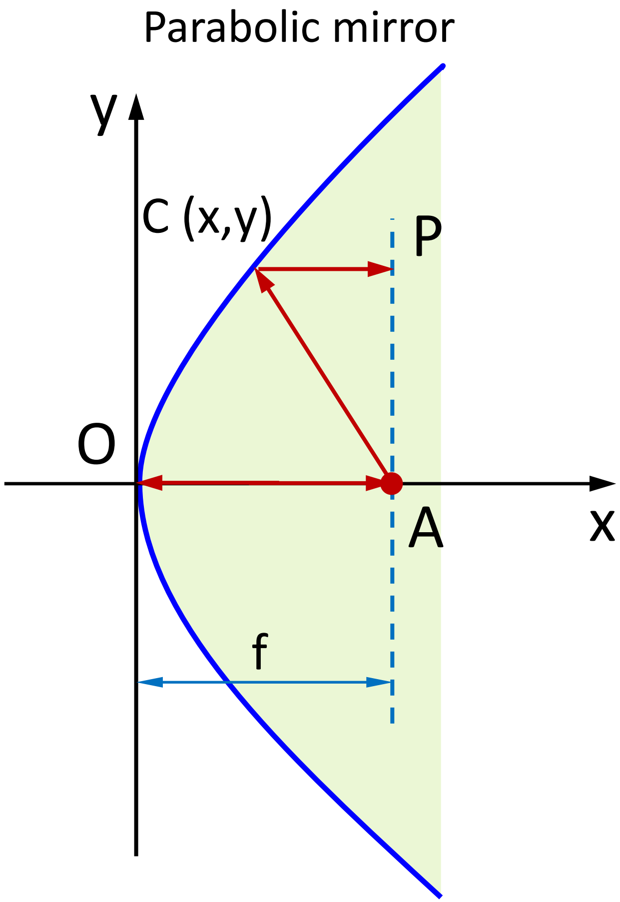
$$a^2+b^2 = c^2$$



$$\frac{x^2}{\mathrm{a}^2}-\frac{y^2}{b^2} = 1$$

$$(PB)-(PA)=(CB)-(CA)=2\mathrm{a}$$

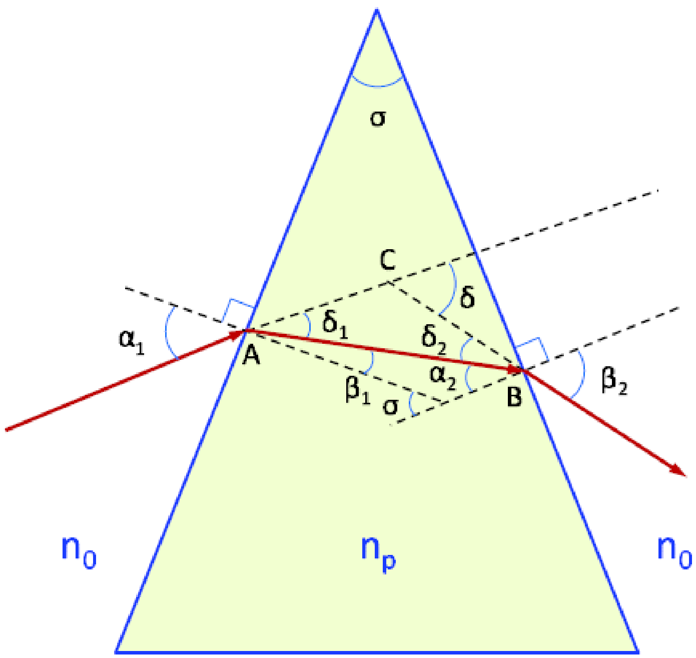
Fermat's Principle and Law of Reflection (Parabolic Mirror)



$$\sqrt{(f-x)^2+y^2}+(f-x)=2f \Longrightarrow y^2=4fx$$

$$(AC)+(CP)=2(AO)$$

Deviation Angle of a *Dispersing Prism*



Geometry & Snell's Law

$$\delta = \delta_1 + \delta_2 = (\alpha_1 - \beta_1) + (\beta_2 - \alpha_2) = \alpha_1 + \beta_2 - (\beta_1 + \alpha_2) =$$
$$= \alpha_1 + \beta_2 - \sigma, \qquad \text{where,}$$
$$\beta_2 = \sin^{-1}\left(\frac{n_p}{n_0}\sin\alpha_2\right),$$
$$\alpha_2 = \sigma - \beta_1 = \sigma - \sin^{-1}\left(\frac{n_0}{n_p}\sin\alpha_1\right),$$

Minimum δ: $\dfrac{d\delta}{\alpha_1} = 0$

$$\frac{n_p}{n_0} = \frac{\sin\left(\dfrac{\sigma+\delta_{min}}{2}\right)}{\sin\left(\dfrac{\sigma}{2}\right)}$$

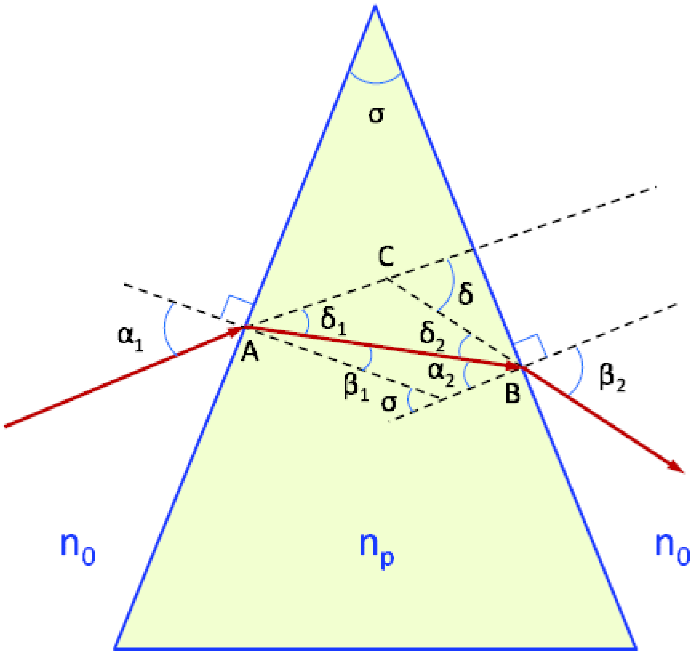
$$\delta_{min} = 2\sin^{-1}\left[\frac{n_p}{n_0}\left(\sin\frac{\sigma}{2}\right)\right] - \sigma$$

Approximation

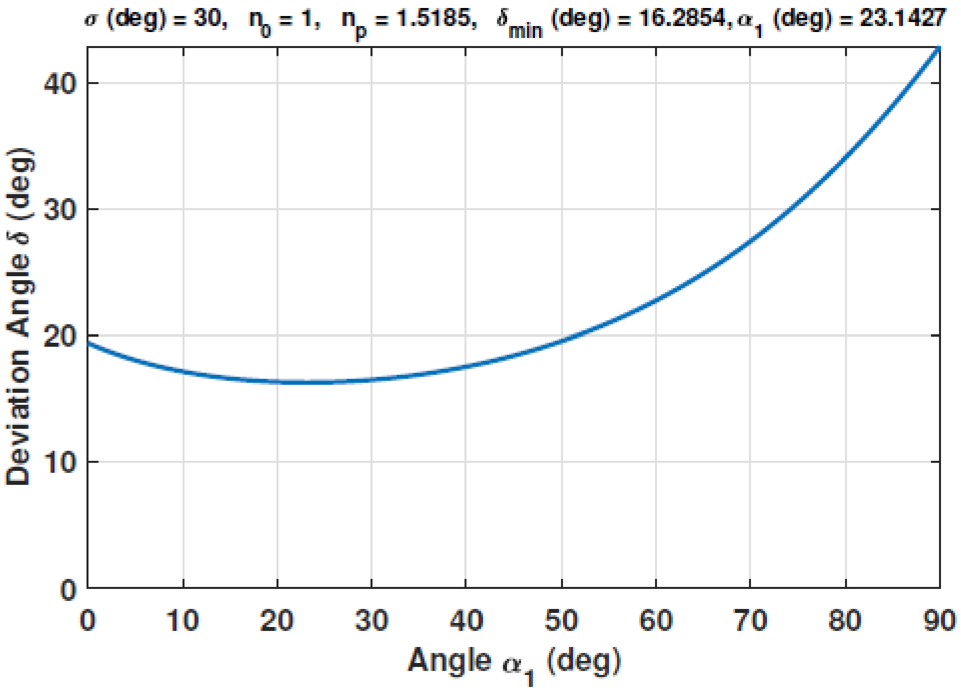
$$\delta_{min} \simeq \left(\frac{n_p}{n_0} - 1\right)\sigma$$

$$\alpha_1 = \beta_2 = \sin^{-1}\left[\frac{n_p}{n_0}\left(\sin\frac{\sigma}{2}\right)\right]$$

Deviation Angle of a *Dispersing Prism*

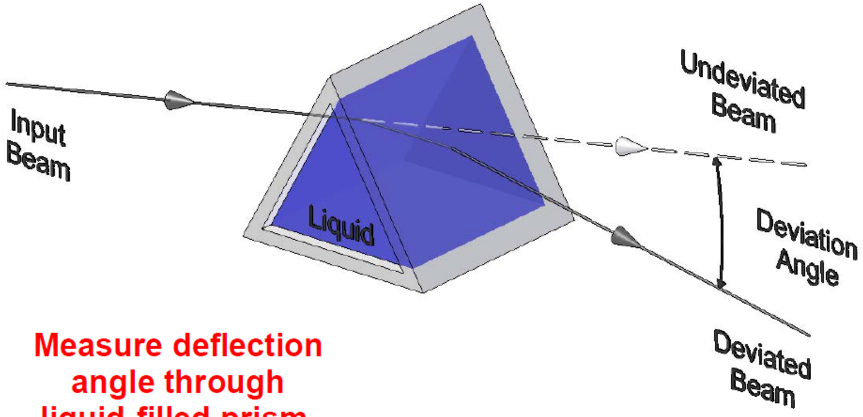


Example for a BK7 Glass Prism σ =30 deg



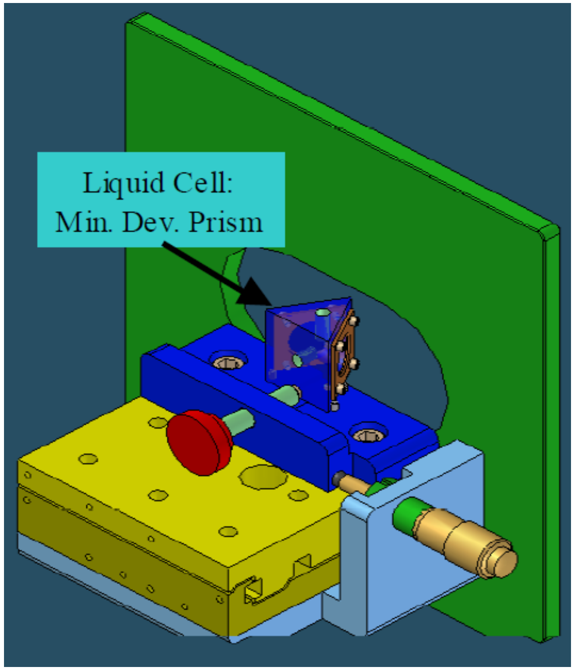
Refractive Index Measurement of Liquids

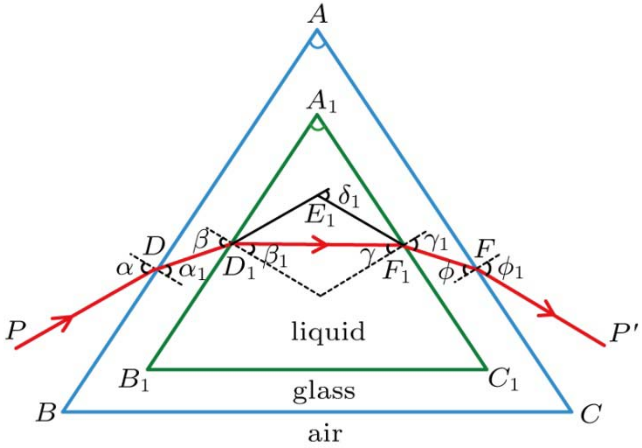
Prism Minimum Deviation Angle Experiment



Measure deflection angle through liquid-filled prism.

R. H. French et al., Proc. SPIE 5377, pp. 1689-1694 (2004)





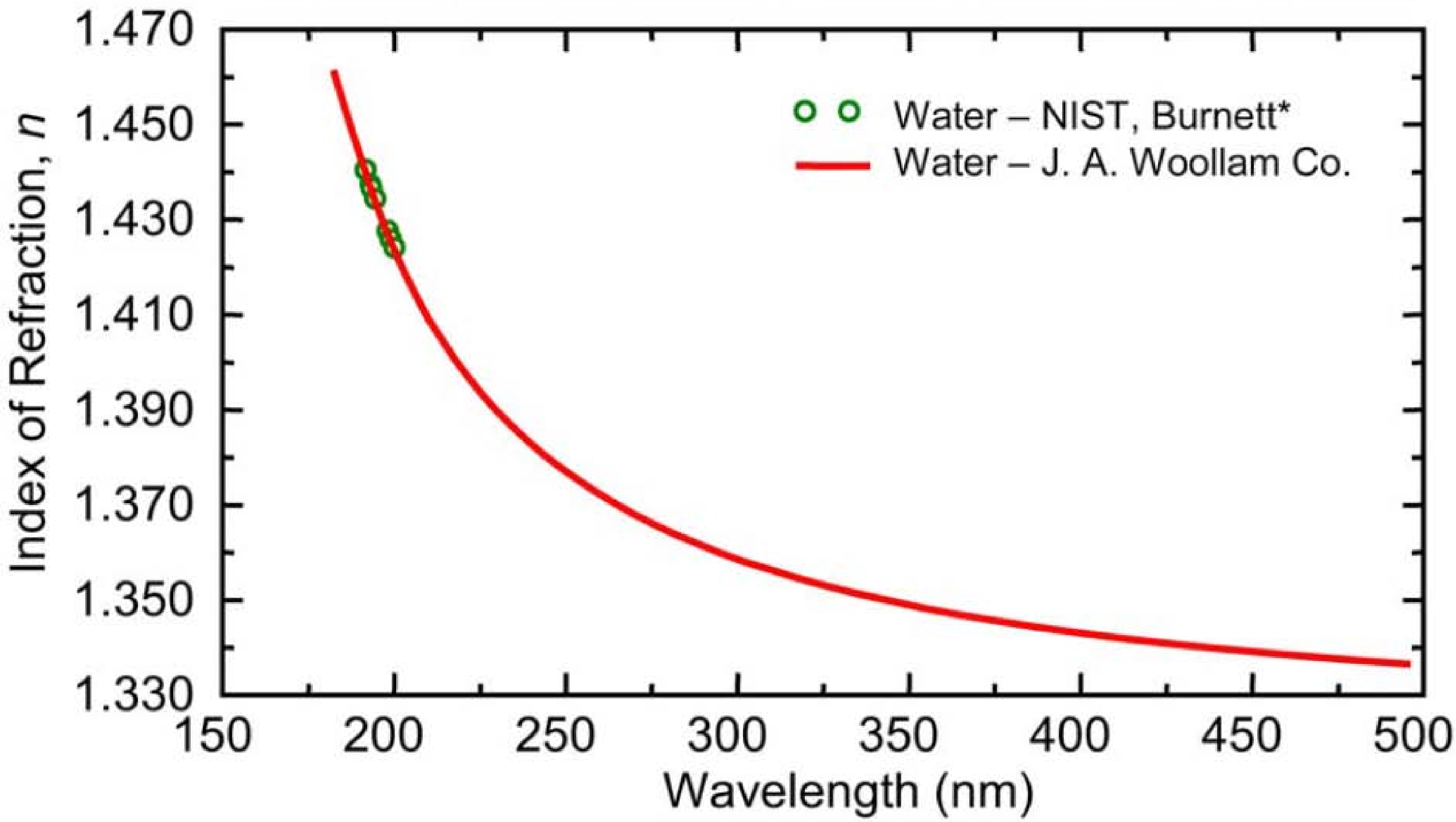
$$\delta_{min} = 2\sin^{-1}\left[\frac{n_p}{n_0}\sin\left(\frac{\sigma}{2}\right)\right] - \sigma$$

$$\delta_{min} \simeq \sigma\left(\frac{n_p}{n_0} - 1\right) \qquad \text{for} \quad \delta, \sigma \ll 1\,rad$$

http://cpb.iphy.ac.cn/article/2020/2027/cpb_29_4_047801/cpb_29_4_047801_f4.jpg

J. Zhou et al., Chinese Physics B, vol. 29, (2020)

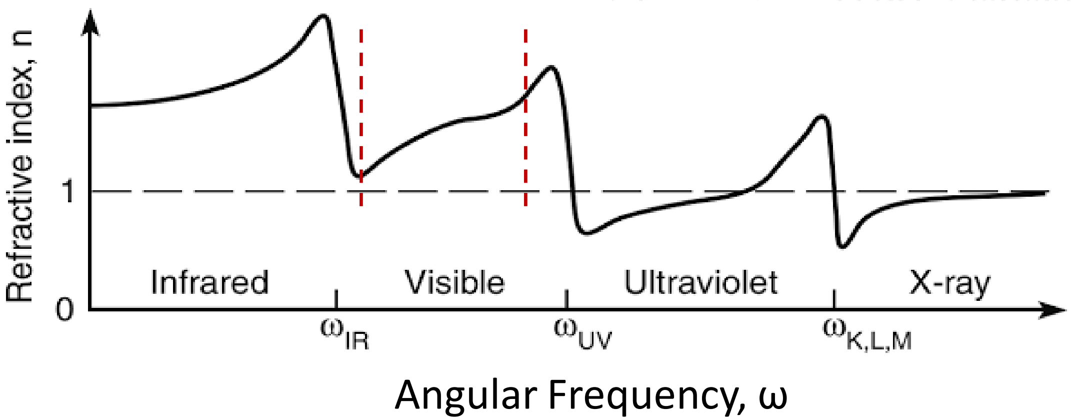
Refractive Index Measurement of Water



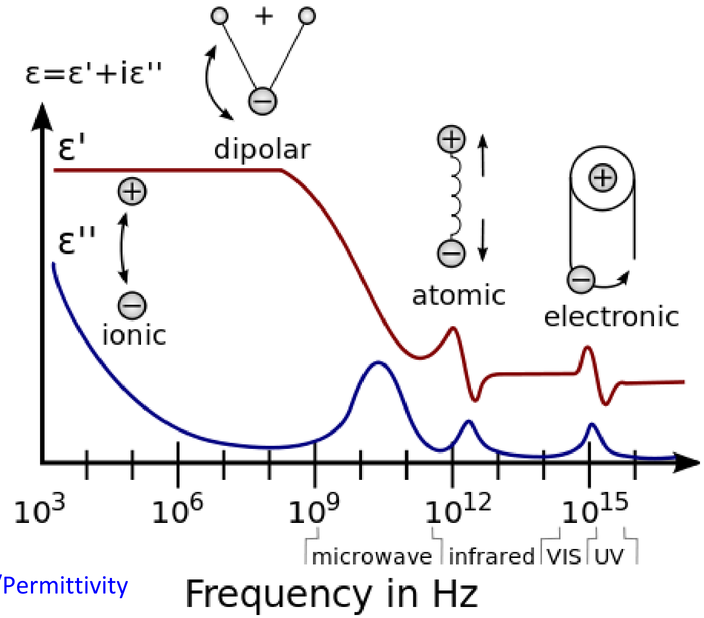
R. H. French et al., Proc. SPIE 5377, pp. 1689-1694 (2004)

Typical Dispersion

The index of refraction vs. wavelength

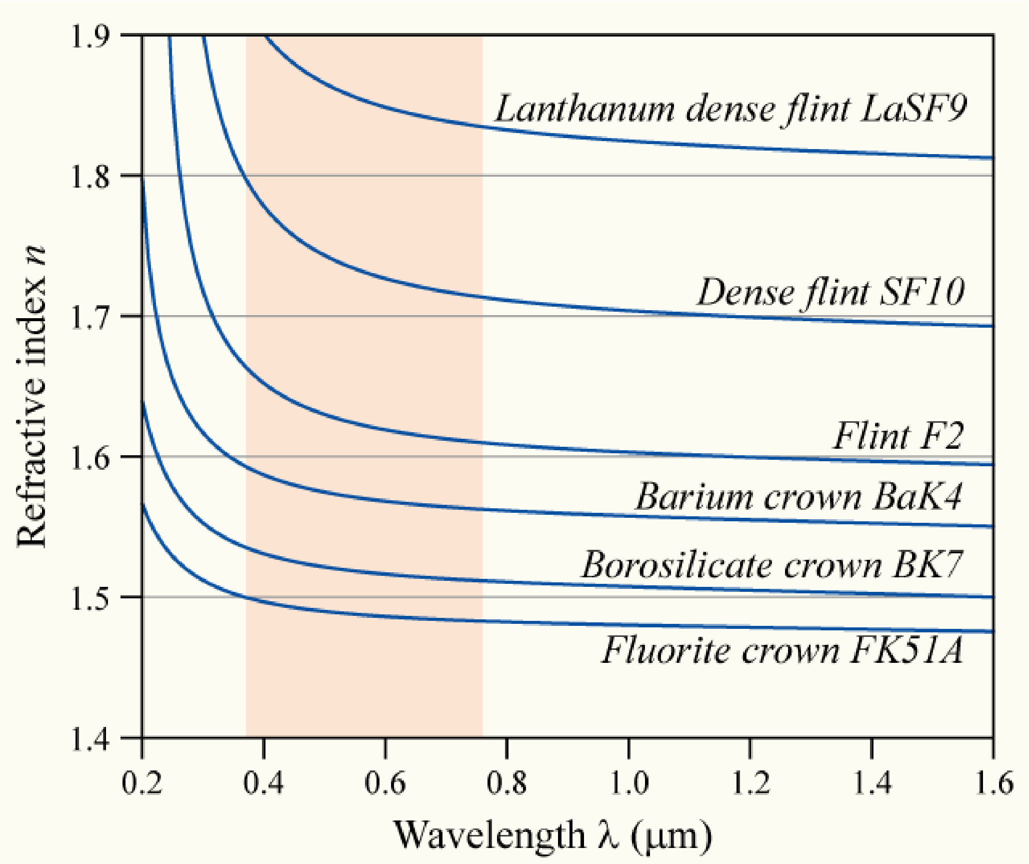


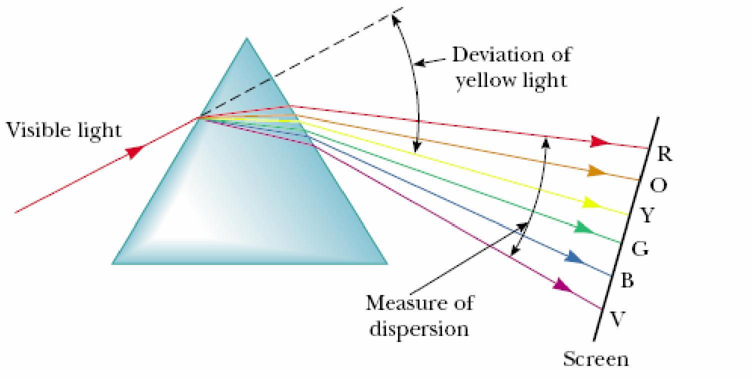
http://1.bp.blogspot.com/-kcHWNW81dI4/UR_qgF5B9dI/AAAAAAAACNg/h5oijsP_gTs/s640/figure+3.png

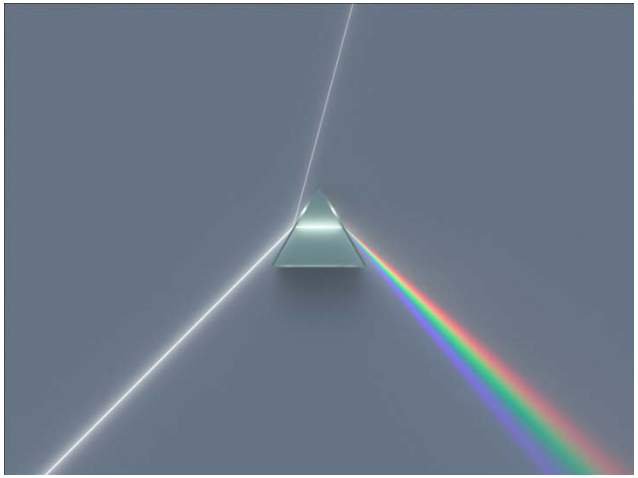


https://en.wikipedia.org/wiki/Permittivity

Normal Dispersion







Normal Dispersion- *Cauchy* Formula

$$n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots$$

Material	**A**	**B** (μm^2)
Fused silica	1.4580	0.00354
Borosilicate glass BK7	1.5046	0.00420
Hard crown glass K5	1.5220	0.00459
Barium crown glass BaK4	1.5690	0.00531
Barium flint glass BaF10	1.6700	0.00743
Dense flint glass SF10	1.7280	0.01342

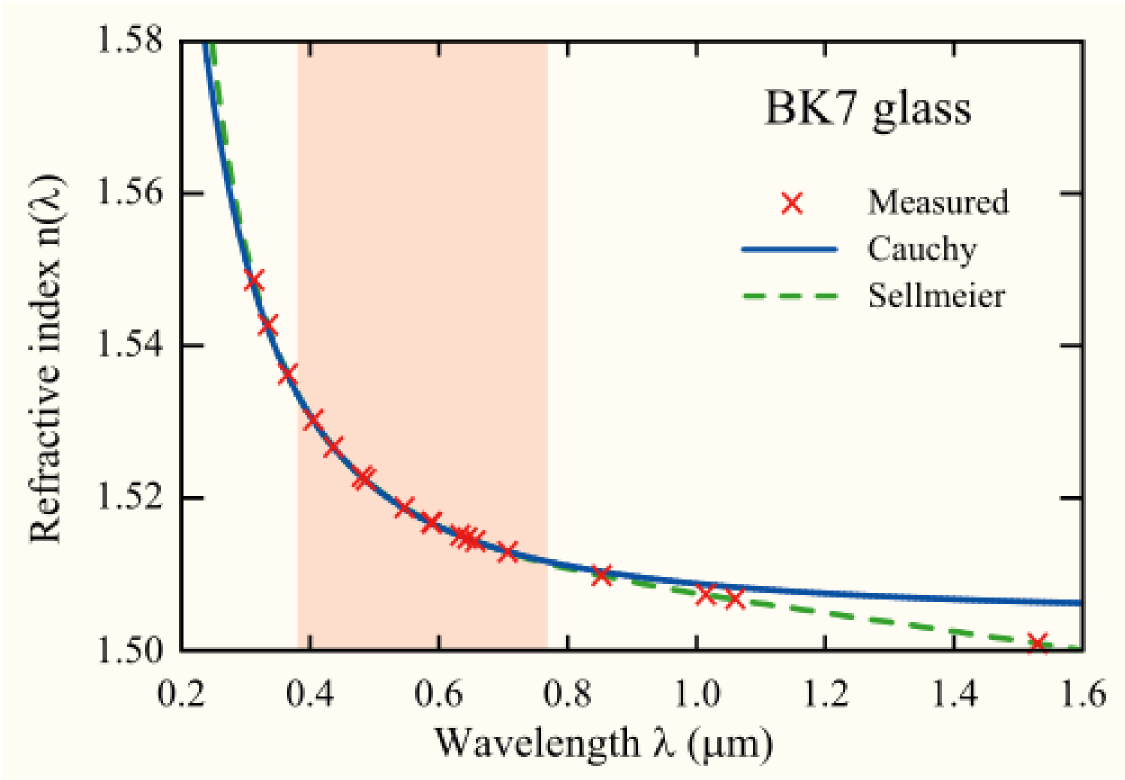
Normal Dispersion - *Sellmeier* Formula

$$n^2(\lambda) = 1 + \sum_i \frac{B_i \lambda^2}{\lambda^2 - C_i}$$

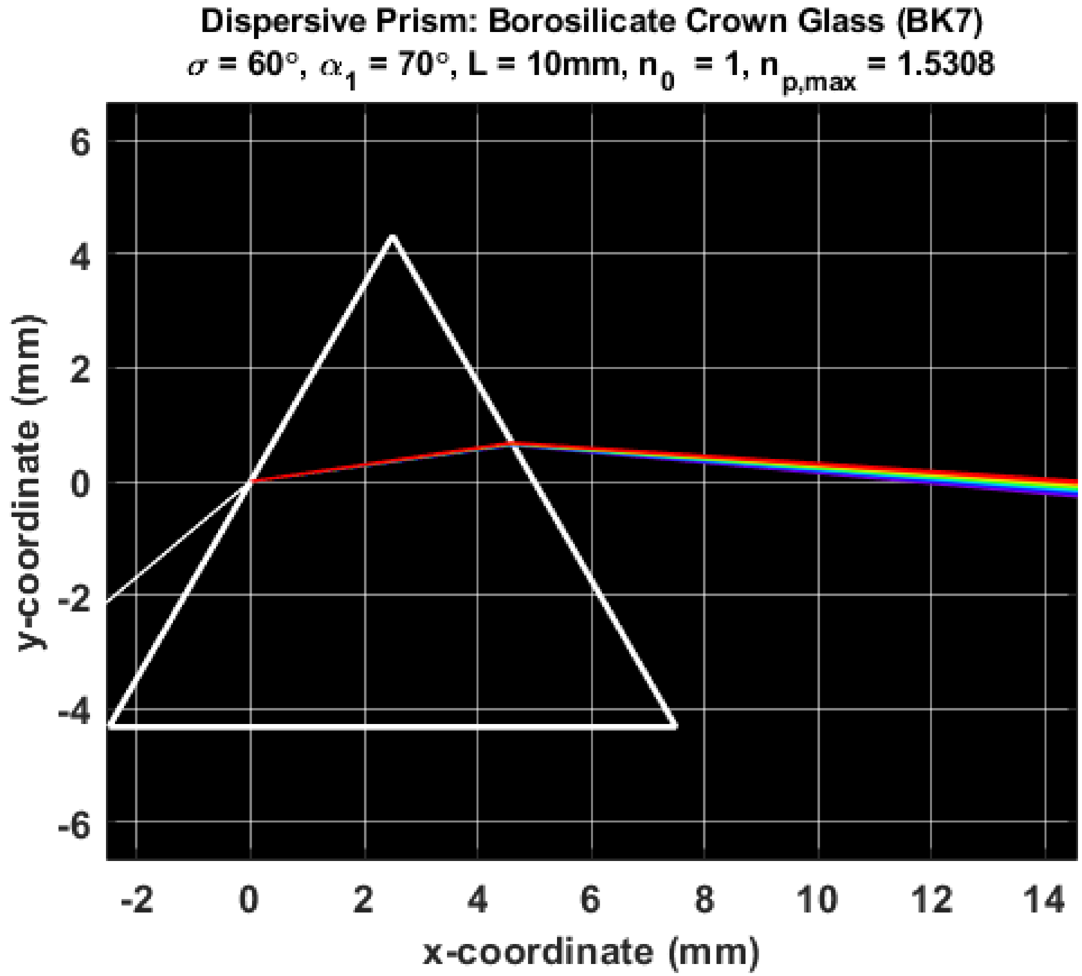
Table of coefficients of Sellmeier equation						
Material	**B_1**	**B_2**	**B_3**	**C_1**	**C_2**	**C_3**
borosilicate crown glass (known as *BK7*)	1.03961212	0.231792344	1.01046945	$6.00069867\times10^{-3}\mu m^2$	$2.00179144\times10^{-2}\mu m^2$	$1.03560653\times10^{2}\mu m^2$
sapphire (for ordinary wave)	1.43134930	0.65054713	5.3414021	$5.2799261\times10^{-3}\mu m^2$	$1.42382647\times10^{-2}\mu m^2$	$3.25017834\times10^{2}\mu m^2$
sapphire (for extraordinary wave)	1.5039759	0.55069141	6.5927379	$5.48041129\times10^{-3}\mu m^2$	$1.47994281\times10^{-2}\mu m^2$	$4.0289514\times10^{2}\mu m^2$
fused silica	0.696166300	0.407942600	0.897479400	$4.67914826\times10^{-3}\mu m^2$	$1.35120631\times10^{-2}\mu m^2$	$97.9340025\ \mu m^2$

Normal Dispersion

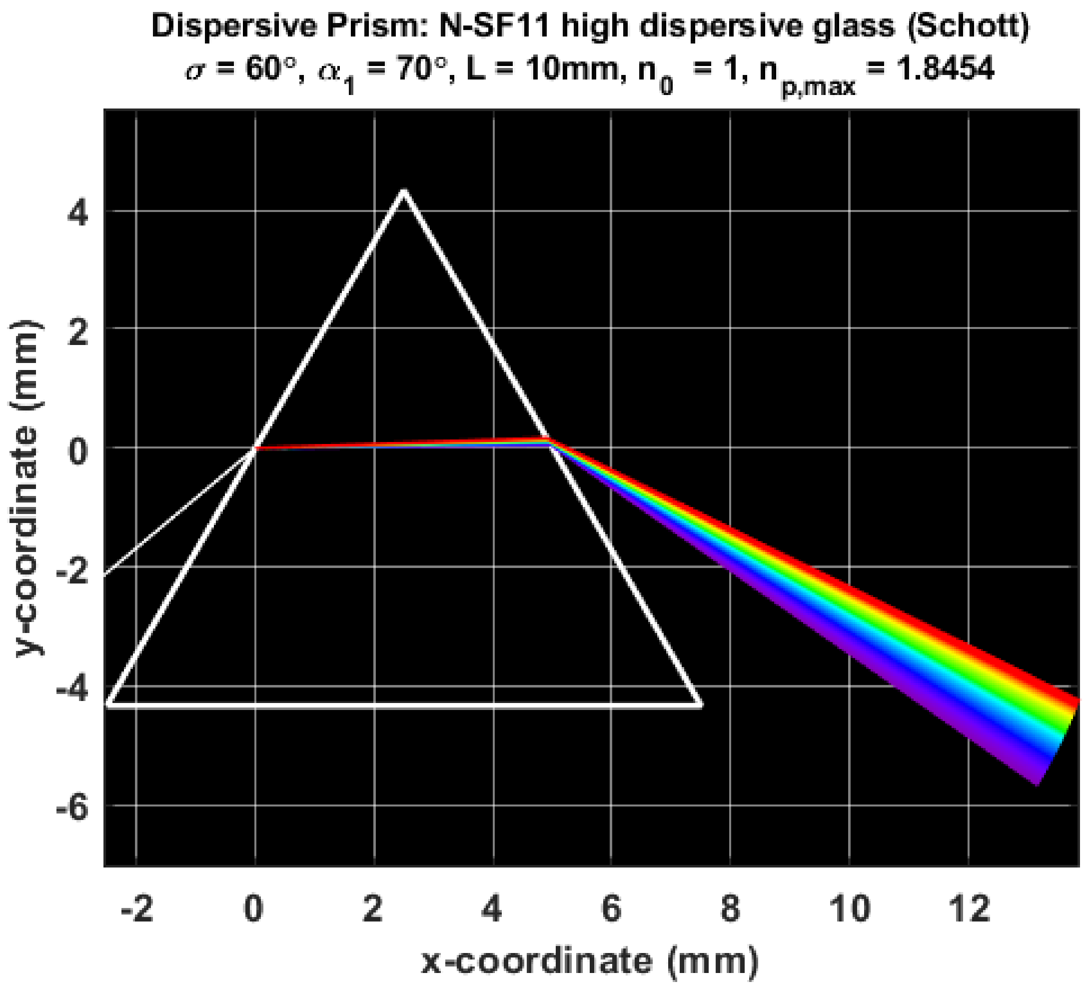
Comparison of *Cauchy* and *Sellmeier* Formulas For BK7 Glass



Prism Dispersion Properties – BK7



Prism Dispersion Properties – Schott N-SF11 Glass



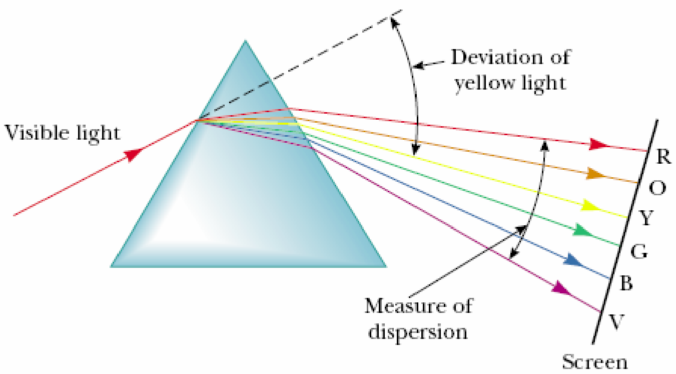
Dispersive Prism – Dispersive Power

$$\delta_{min} = 2\sin^{-1}\left[\frac{n_p}{n_0}\sin\left(\frac{\sigma}{2}\right)\right] - \sigma$$

$$\delta_{min} \simeq \sigma\left(\frac{n_p}{n_0} - 1\right) \qquad \text{for} \quad \delta, \sigma \ll 1\, rad$$

$$\Delta_e = \frac{\delta_{min,F} - \delta_{min,C}}{\delta_{min,D}} \Longrightarrow$$

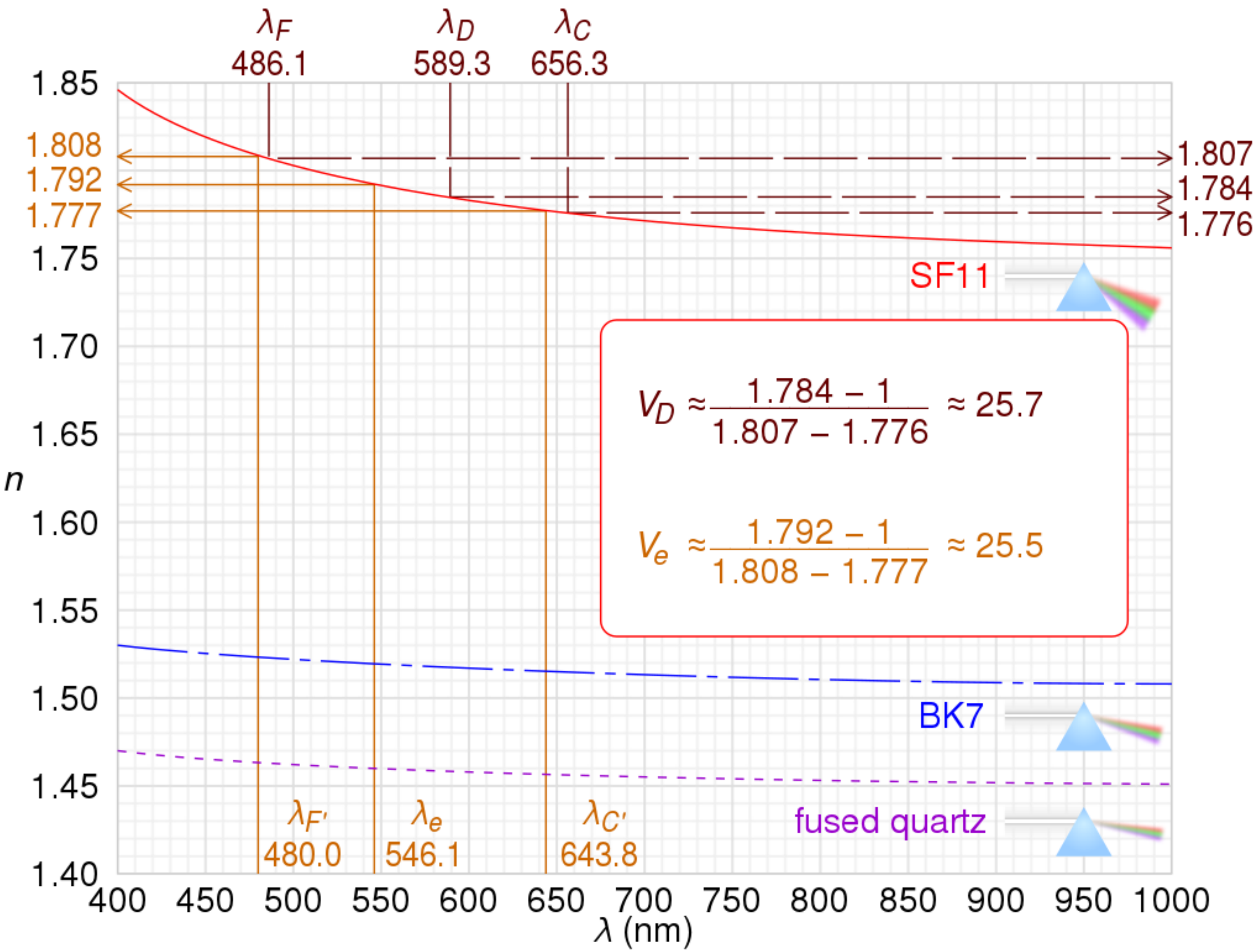
$$\Delta_a \simeq \frac{n_F - n_C}{n_D - n_0},$$



Abbe number: $V = \frac{1}{\Delta}$

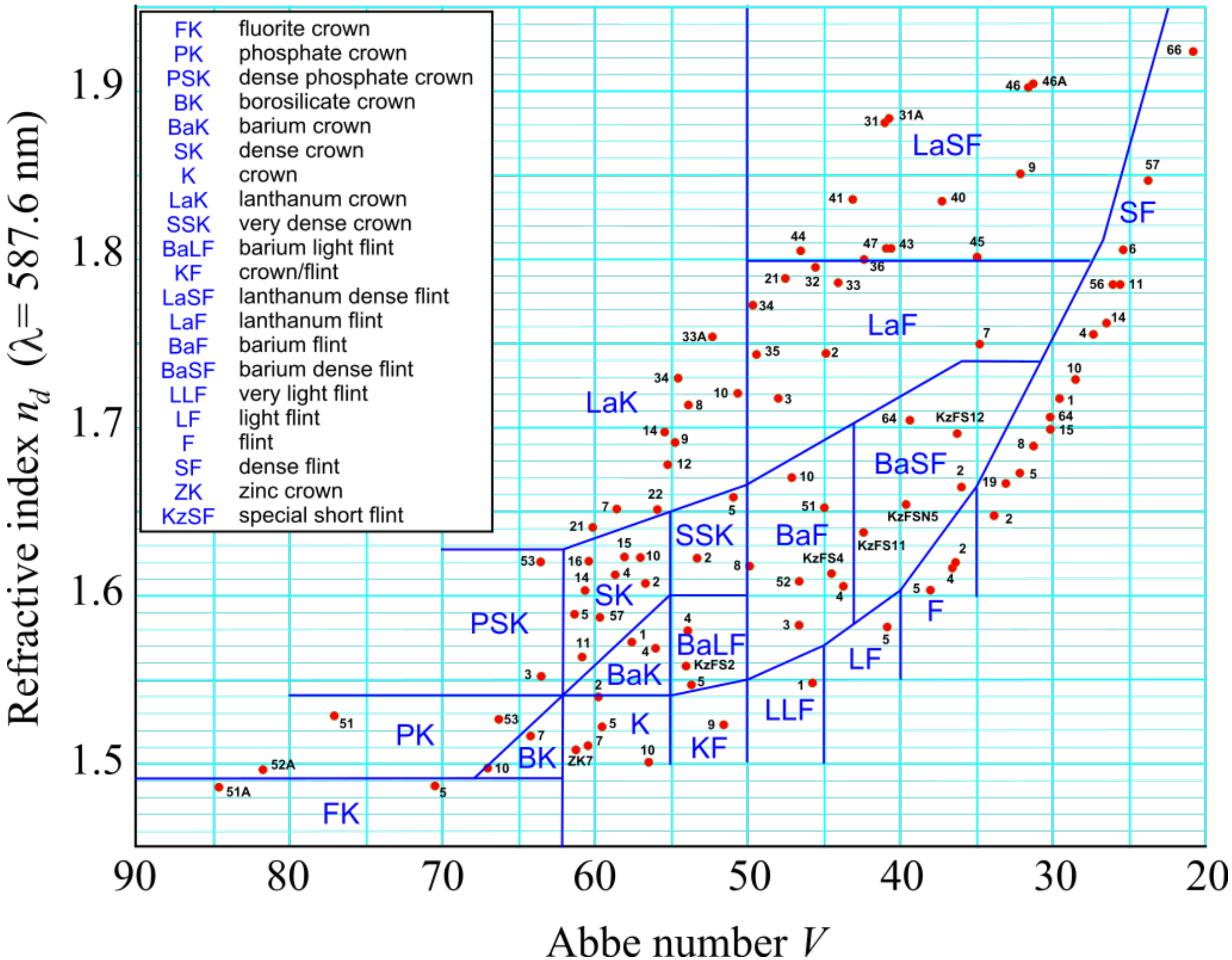
Freespace Wavelength, λ_0 (nm)	Characterization	Crown Glass Refractive Index
486.1	F, blue (H dark line)	n_F = 1.5286
589.2	D, yellow (Na dark line)	n_D = 1.5230
656.3	C, red (H dark line)	n_C = 1.5205

Dispersive Prism – Dispersive Power



https://upload.wikimedia.org/wikipedia/commons/thumb/3/30/Abbe_number_calculation.svg/1024px-Abbe_number_calculation.svg.png

Dispersive Prism – Dispersive Power



An Abbe diagram, also known as 'the glass veil', plots the Abbe number against refractive index for a range of different glasses (red dots). Glasses are classified using the Schott Glass letter-number code to reflect their composition and position on the diagram.

https://en.wikipedia.org/wiki/Abbe_number#:~:text=In%20optics%20and%20lens%20design,of%20V%20indicating%20low%20dispersion.

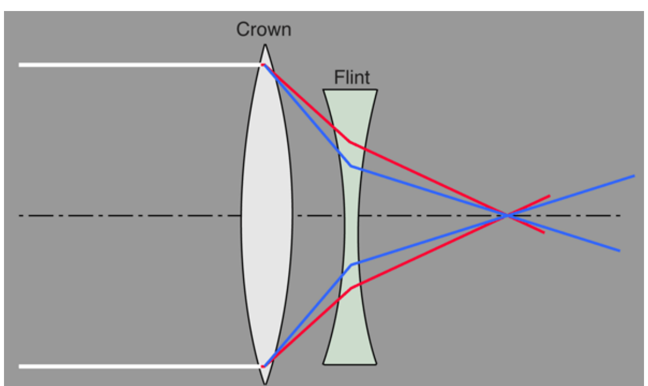
Dispersive Prisms

Dispersive Power / Abbe's number

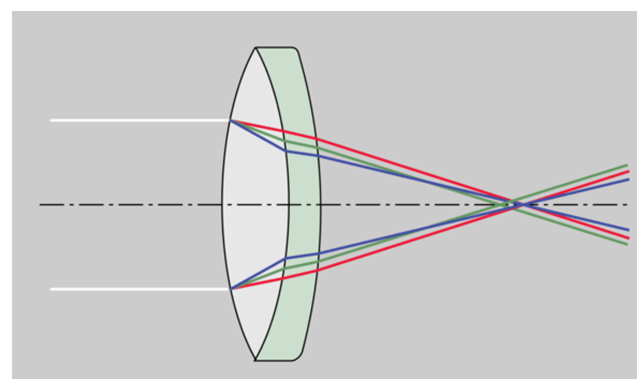
Fraunhofer line	Color	Wavelength (nm)	Spectacle Crown	Extra-dense Flint
			Refractive index	
F	Blue (hydrogen)	486.1	1.5293	1.7378
D	Yellow (sodium)	589.3	1.5230	1.7200
C	Red (hydrogen)	656.3	1.5204	1.7130
$\nu = \dfrac{(n_D - 1)}{(n_F - n_C)}$ = Abbe's number			v value	
			59	29

http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/imggo/disper2.gif

Doublet for Chromatic Aberration

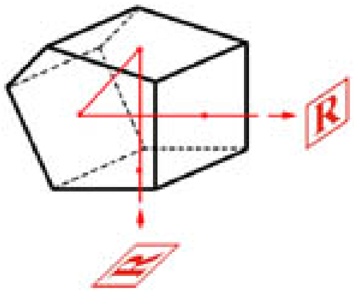


Achromat Doublets

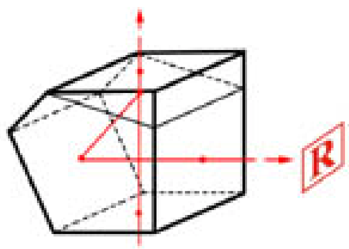


http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/aber2.html#c2

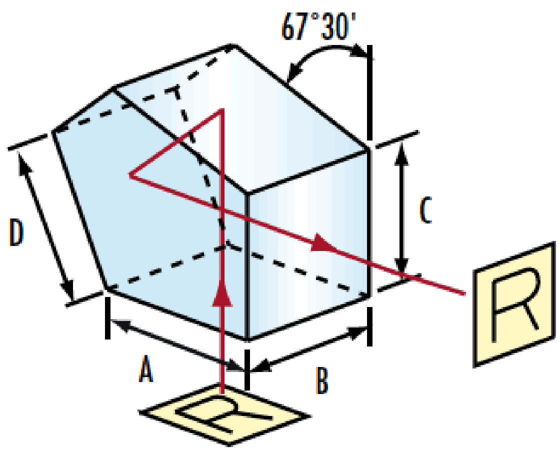
Reflective Prisms



Penta prism can deviate an incident beam without inverting or reversing to 90° . The deviation angle of 90° is independent of any rotation of the prism about an axis parallel to the line of intersection of the two reflecting faces. It is commonly used in Plumb Level, Surveying, Alignment, Rangefinding and Optical Tooling.

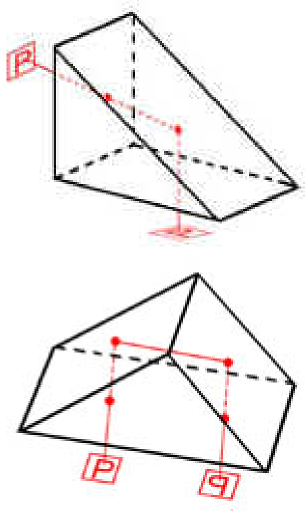


Beamsplitting Penta prism: By adding a wedge and with partial refractive coating on surfaces S1, it can be used as a beamsplitter. It is often used in Plumb Level, Surveying, Alignment, Rangefinding and Optical Tooling.

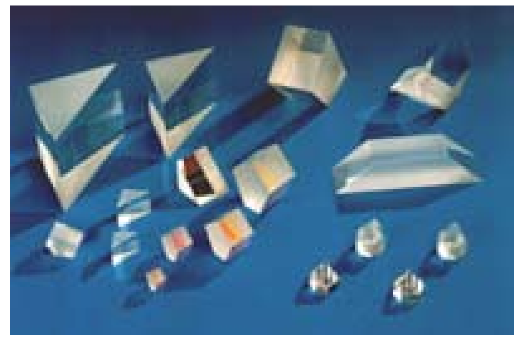


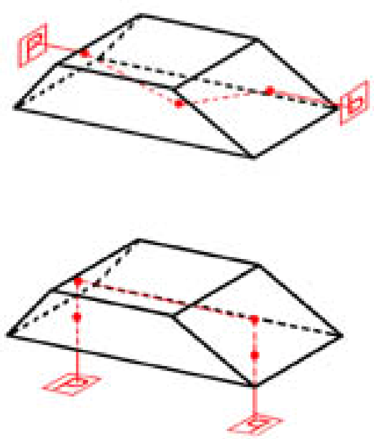
https://www.edmundoptics.eu/resources/application-notes/optics/introduction-to-optical-prisms/

Reflective Prisms



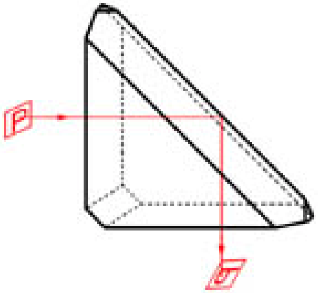
Right angle prism is deviating or deflecting a beam of light with 90 or 180° . It is often used in telescope, periscope and other optical system.



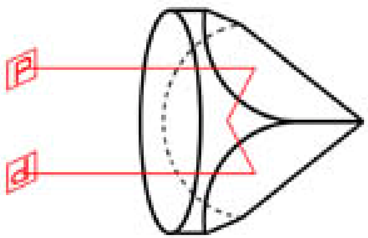


Dove prism has two applications. The main application is used as a rotator. It can rotate an image but without deviating the beam. And when the prism is rotated about the input parallel ray through some angle, the image rotates through twice that angle. It is very important that the application must be used with parallel or collimated beam and the large square reflective surface should be kept very clean. Another application is used as a retroreflector. For this application it perform as a right-angle prism.

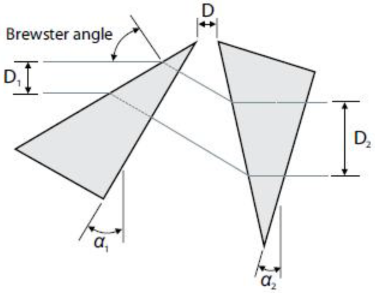
Reflective Prisms

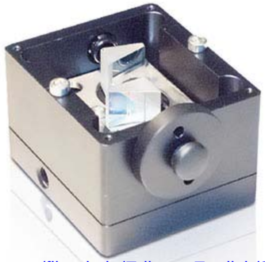


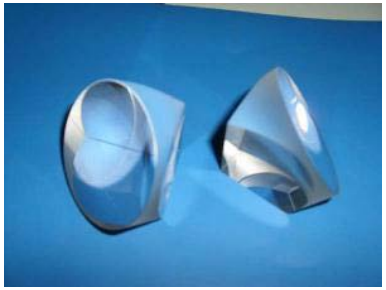
Roof prism (Amici) is combined with a right angle prism and a totally internally reflecting *roof and they* are attached by them largest square surfaces. It can invert and reverse an image, also, deflect the image 90° . Therefore, it is often used in terrestrial telescopes, viewing systems and rangefinders.



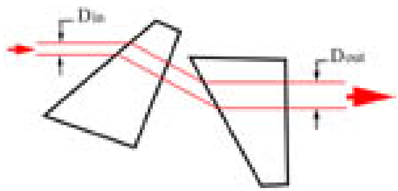
Corner Cube Prism: It has three mutually perpendicular surfaces and a hypotenuse face. Light entering through the hypotenuse is reflected by each of the three surfaces in turn and will emerge through the hypotenuse face parallel to the entering beam regardless of the orientation the incident beam. For its special performance, it is often used to the distance measurement, optical signal process and laser interferometer.





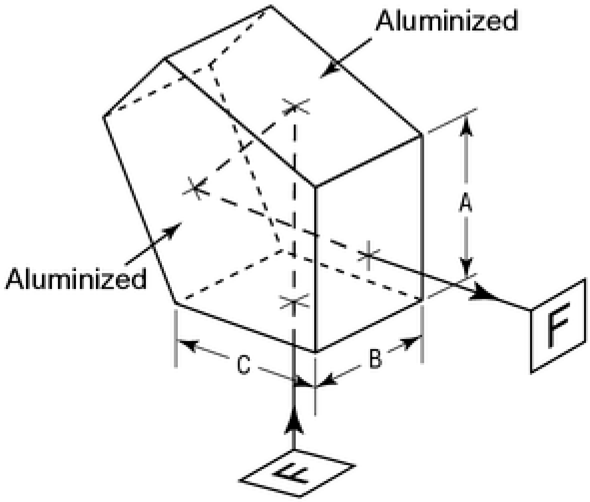


http://www.toptica.com/fileadmin/Editors_English/03_products/09_wavemeters_photonicals/02_photonicals/Anamorphic-Prism-Pair.jpg

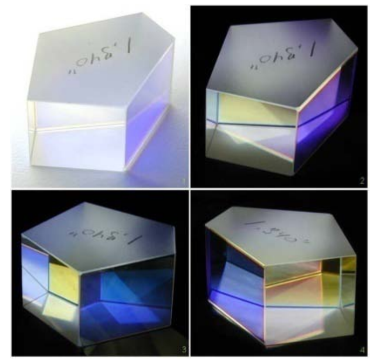


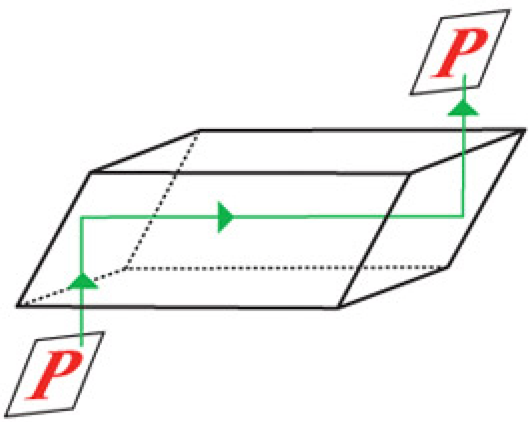
Anamorphic Prisms: These two prisms can expand or contract the beam in one direction without any changes in the other direction. By adjusting the angles among the incident beam and two prisms, the shape of the beam can be changed. It is very easy to turn elliptical bean into circular beam.

Reflective Prisms

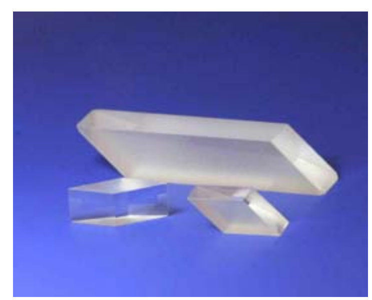


The **Penta Prism** will deviate the beam by 90°without affecting the orientation of the image. It has the valuable property of being a constant-deviation prism, in that it deviates the line-of-sight by 90°regardless of its orientation to the line-of-sight. Note that two of its surfaces must be silvered. These prisms are often used as end reflectors in small rangefinders





The **Rhomboid Prism** displaces the line of sight without producing any angular deviation or changes in the orientation of the image.



Reflective Prisms

Erecting Prisms

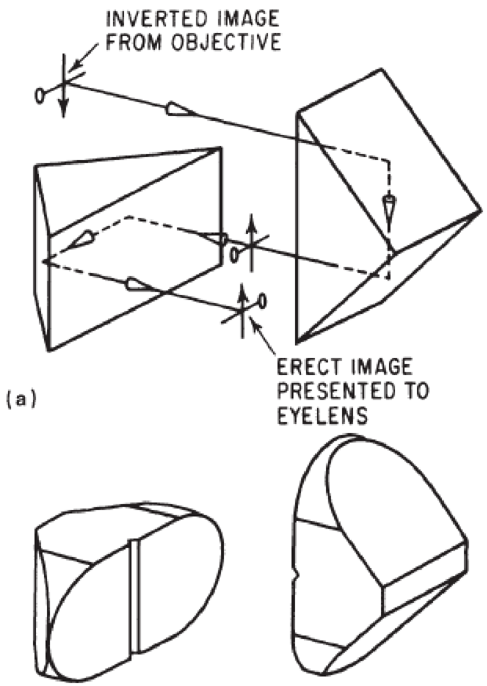


Figure 4.24 Porro prism system (first type) (a) indicating the way the Porro system erects an inverted image. (b) Porro prisms are usually fabricated with rounded ends to save space and weight. Note that the spacing between the prisms has been shown increased for clarity.

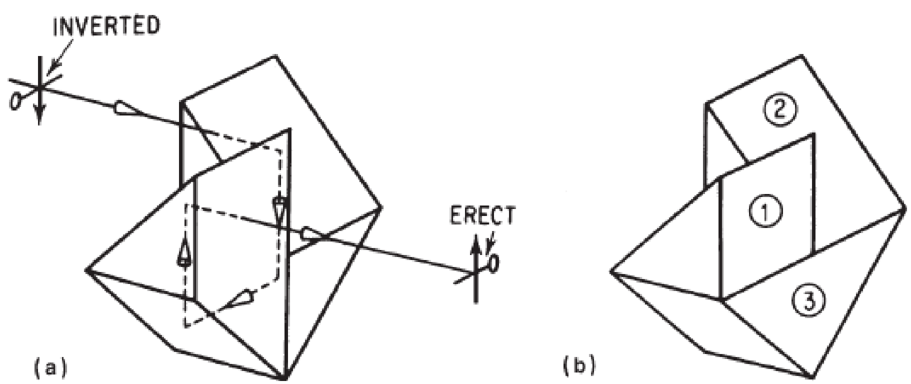
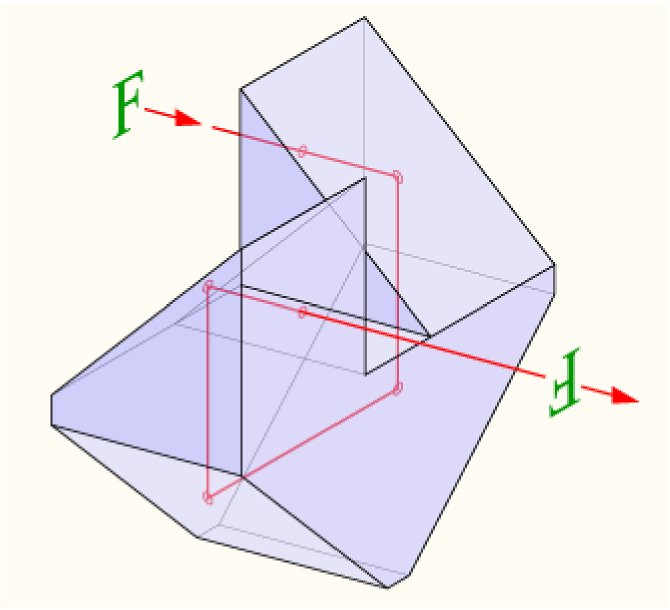
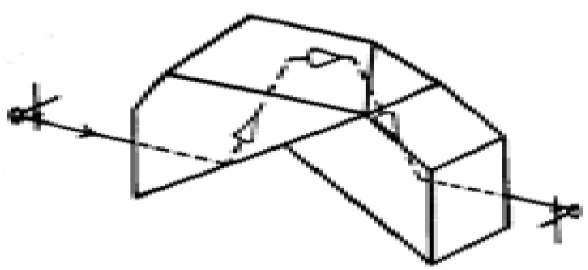


Figure 4.25 Porro prism system (second type) (a) indicating the erection of an inverted image. This system is shown made from two prisms in (a) and from three prisms in (b).

W. J. Smith, Modern Optical Engineering, 3rd Ed. McGraw-Hill, 2000

Porro-Abbe Prism





Porro Prism

- Right angle prism
- Oriented to deviate light by 180 degrees