Fundamentals of Gaussian Beams

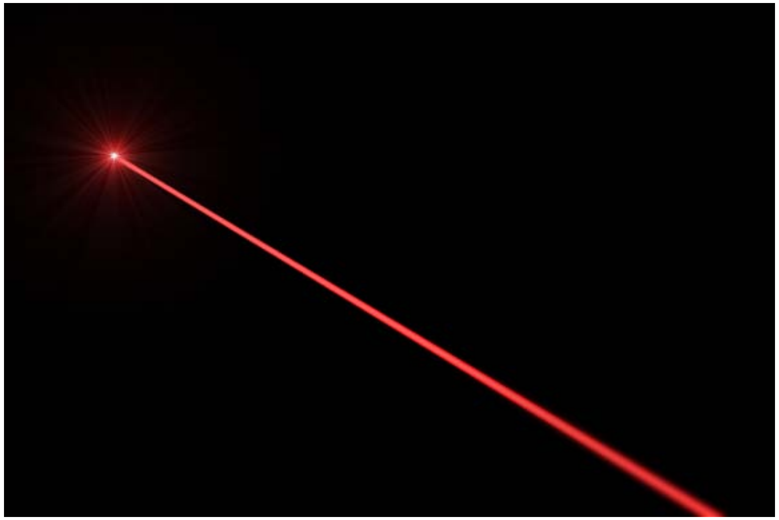
Optical Engineering

Prof. Elias N. Glytsis

School of Electrical & Computer Engineering
National Technical University of Athens

Gaussian Beams

He-Ne Laser



https://wtamu.edu/~cbaird/sq/images/laser_red.png

Green Laser Pointer

https://images-na.ssl-images-amazon.com/images/I/51CVh6BKpUL._SL1000_.jpg

Necessity for an expression of an electromagnetic field with finite cross-sectional area → *Gaussian Beams*

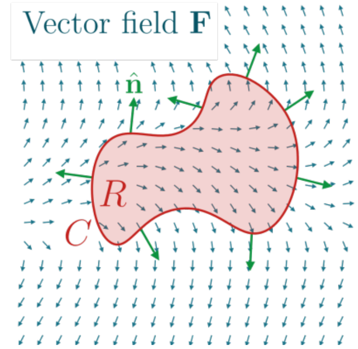
Gaussian Beams – TEM Solutions

Gauss' Law:

$$\vec{\nabla}\cdot\vec{D} = 0 \Longrightarrow \vec{\nabla}\cdot\vec{E} = 0,$$

$$\vec{\nabla}\cdot\vec{E} = 0 \Longrightarrow \vec{\nabla}_t\cdot\vec{E}_t + \frac{\partial E_z}{\partial z} = 0, \quad \vec{E} = \vec{E}_t + E_z\hat{z},$$

2D Divergence



https://cdn.kastatic.org/ka-perseus-images/4054da358c8ceddef75ccf33cd9c6ae47367b003.svg

Approximations:

$$\frac{\partial E_z}{\partial z} \simeq -j\frac{2\pi}{\lambda_0}nE_z,$$

$$\vec{\nabla}_t\cdot\vec{E}_t = \lim_{\Delta S\to 0}\left\{\frac{1}{\Delta S}\oint_{\Delta\ell}\vec{E}_t\cdot\hat{\imath}_{\perp\ell}d\ell\right\} \simeq \lim_{D\to 0}\left\{\frac{4}{\pi D^2}|\vec{E}_t|\pi D\right\} \simeq \frac{|\vec{E}_t|}{D},$$

TEM Approximation:

$$|E_z| \simeq \frac{\lambda_0}{2\pi nD}|\vec{E}_t|$$

$$\frac{|E_z|}{|\vec{E}_t|} \simeq \frac{\lambda_0}{2\pi nD} < 10^{-3}$$

Gaussian Beams – TEM Solutions

TEM Approximation:

$$\vec{E}(x,y,z) = \hat{t}E_0\psi(x,y,z)\exp(-jkz)$$

Helmholtz Scalar Equation

$$\nabla^2 E + k_0^2 n^2 E = 0 \qquad \nabla_t^2\psi - j2k\frac{\partial\psi}{\partial z} + \frac{\partial^2\psi}{\partial z^2} = 0$$

Paraxial Wave Equation

$$\nabla_t^2\psi - j2k\frac{\partial\psi}{\partial z} \simeq 0, \qquad \text{if} \quad \left|\frac{\partial^2\psi}{\partial z^2}\right| \ll 2k\left|\frac{\partial\psi}{\partial z}\right|$$

Fundamental TEM Solution

$$\frac{1}{r_T}\frac{\partial}{\partial r_T}\left(r_T\frac{\partial\psi}{\partial r_T}\right) - j2k\frac{\partial\psi}{\partial z} = 0. \qquad \psi = \exp\left\{-j\left[P(z) + \frac{kr_T^2}{2q(z)}\right]\right\}$$

Gaussian Beams – TEM Solutions

Fundamental TEM Solution

$$\left\{\left[\frac{k^2}{q^2(z)}\left(q'(z)-1\right)\right] r_T^2 \;-\; 2k\left[P'(z)+\frac{j}{q(z)}\right] r_T^0\right\}\psi = 0, \qquad \forall r_T$$

$$\frac{dq}{dz} = q'(z) \;=\; 1,$$

$$\frac{dP}{dz} = P'(z) \;=\; -\frac{j}{q(z)}.$$

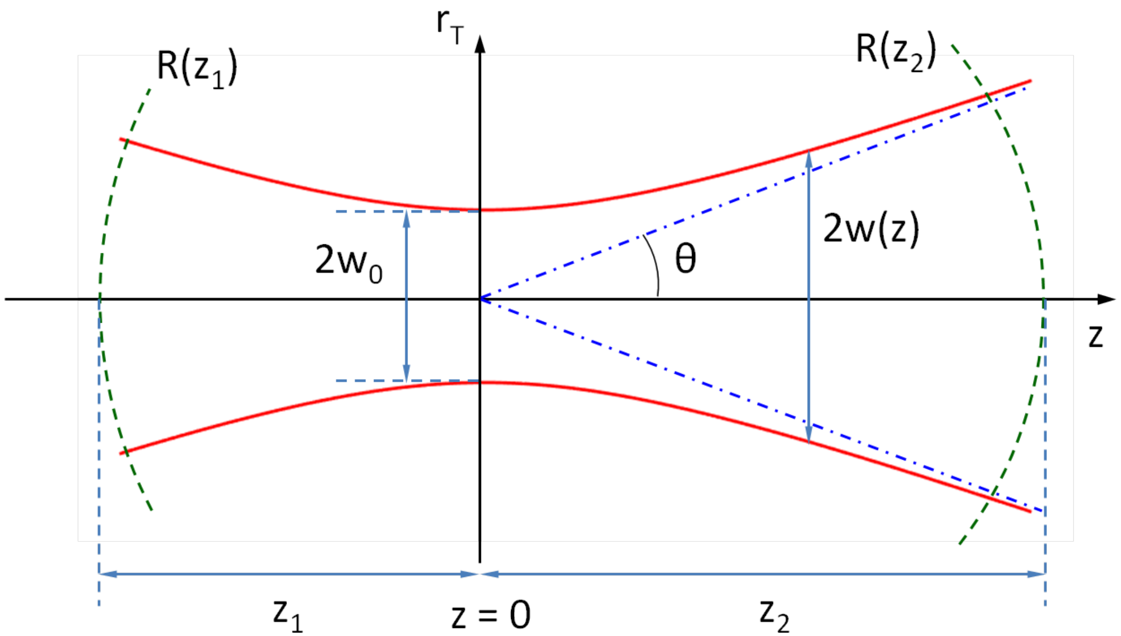
$$\boxed{q(z) = z + jz_0} \qquad |\psi(r_T, z=0)|^2 = \exp(-kr_T^2/z_0)$$

Minimum Beam Waist and Rayleigh Distance

$$\boxed{w_0^2 = \frac{2z_0}{k} = \frac{\lambda_0 z_0}{n\pi} \Longrightarrow z_0 = \frac{\pi n w_0^2}{\lambda_0}} \qquad \psi = \exp\left\{-j\left[P(z)+\frac{kr_T^2}{2q(z)}\right]\right\}$$

$$\boxed{jP(z) = \ln\left[1 - j\left(\frac{z}{z_0}\right)\right] = \ln\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} - j\tan^{-1}\left(\frac{z}{z_0}\right)}$$

Gaussian Beams



$$E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)} \exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}$$

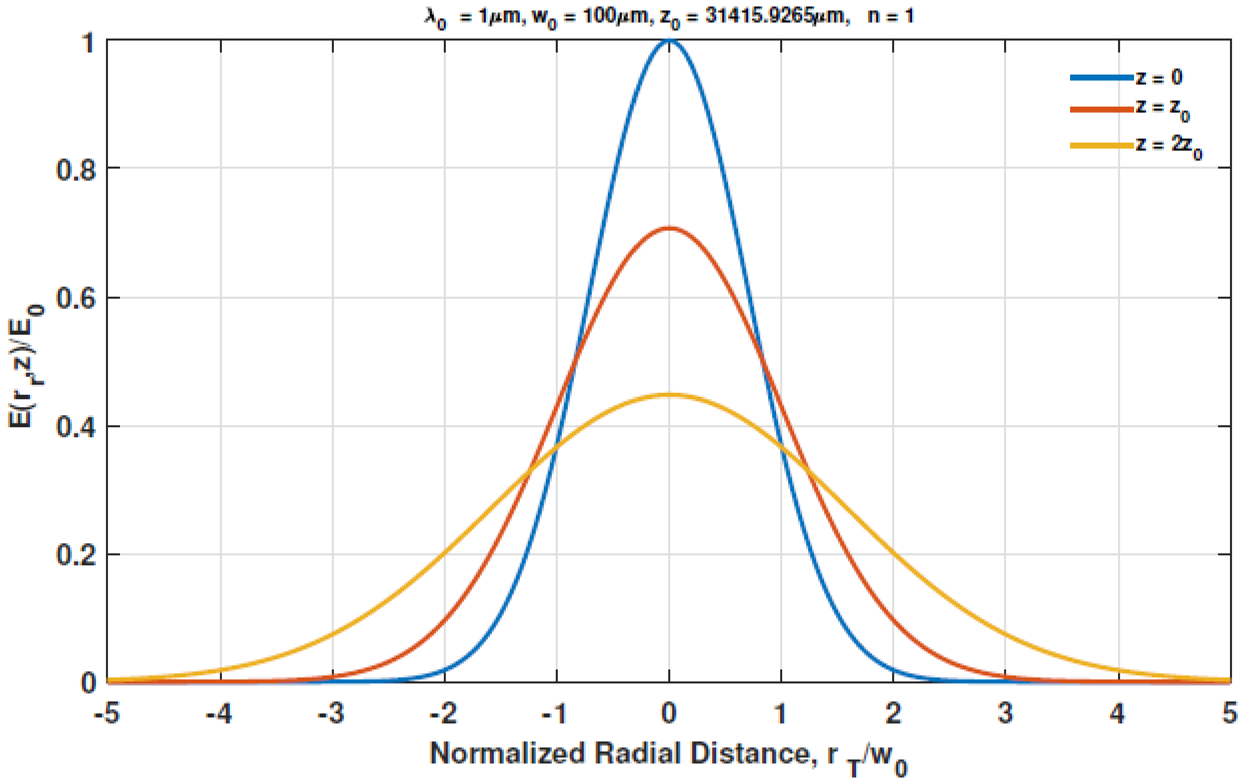
$$\frac{1}{q(z)} = \frac{z}{z^2+z_0^2} - j\frac{z_0}{z^2+z_0^2} = \frac{1}{R(z)} - j\frac{\lambda_0}{\pi n w^2(z)},$$

$$z_0 = \frac{\pi n w_0^2}{\lambda_0} \qquad w^2(z) = w_0^2\left[1+\left(\frac{z}{z_0}\right)^2\right],$$

$$R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right].$$

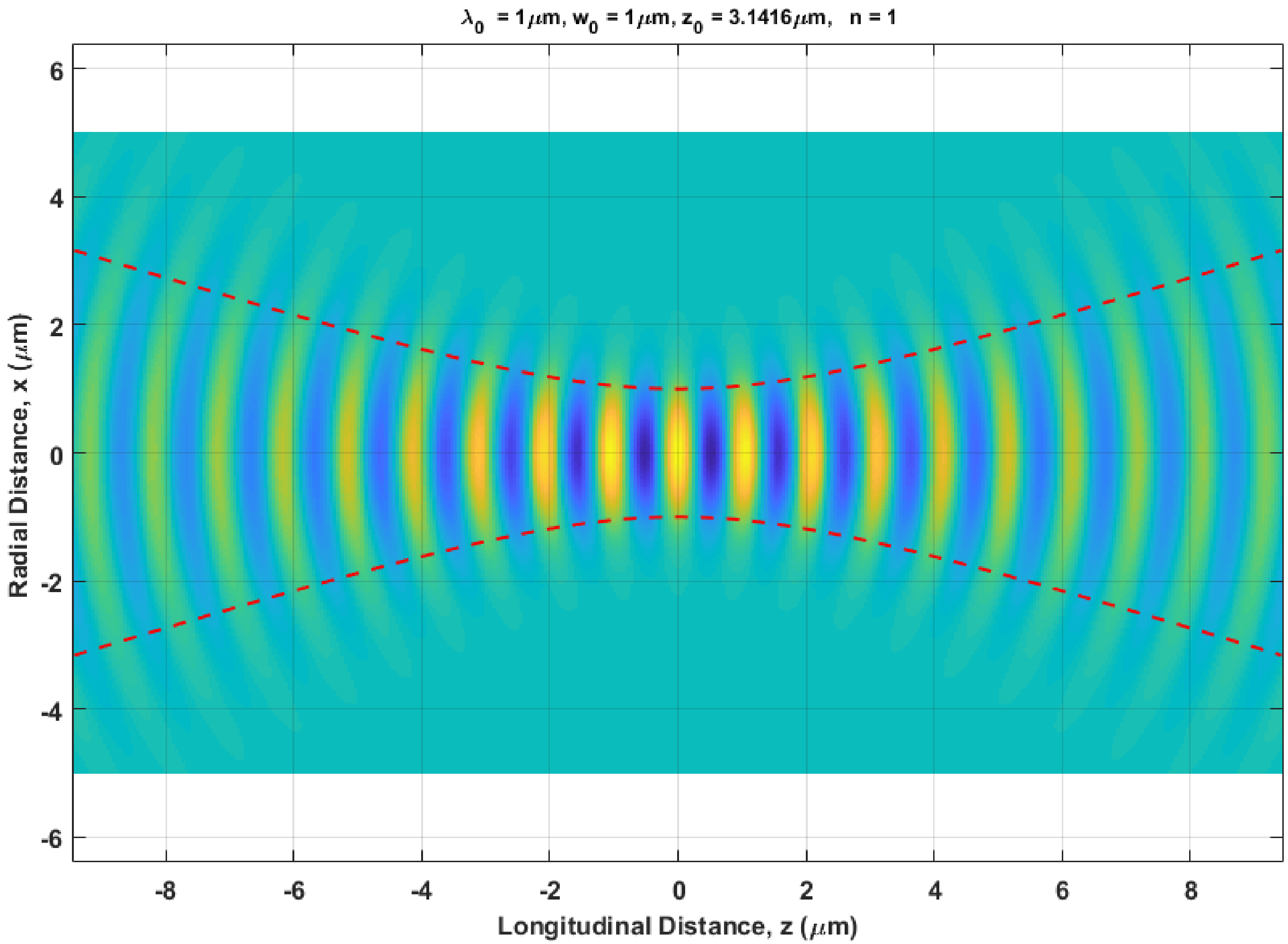
Gaussian Beams

Radial Profiles at various z Distances from Focal Plane



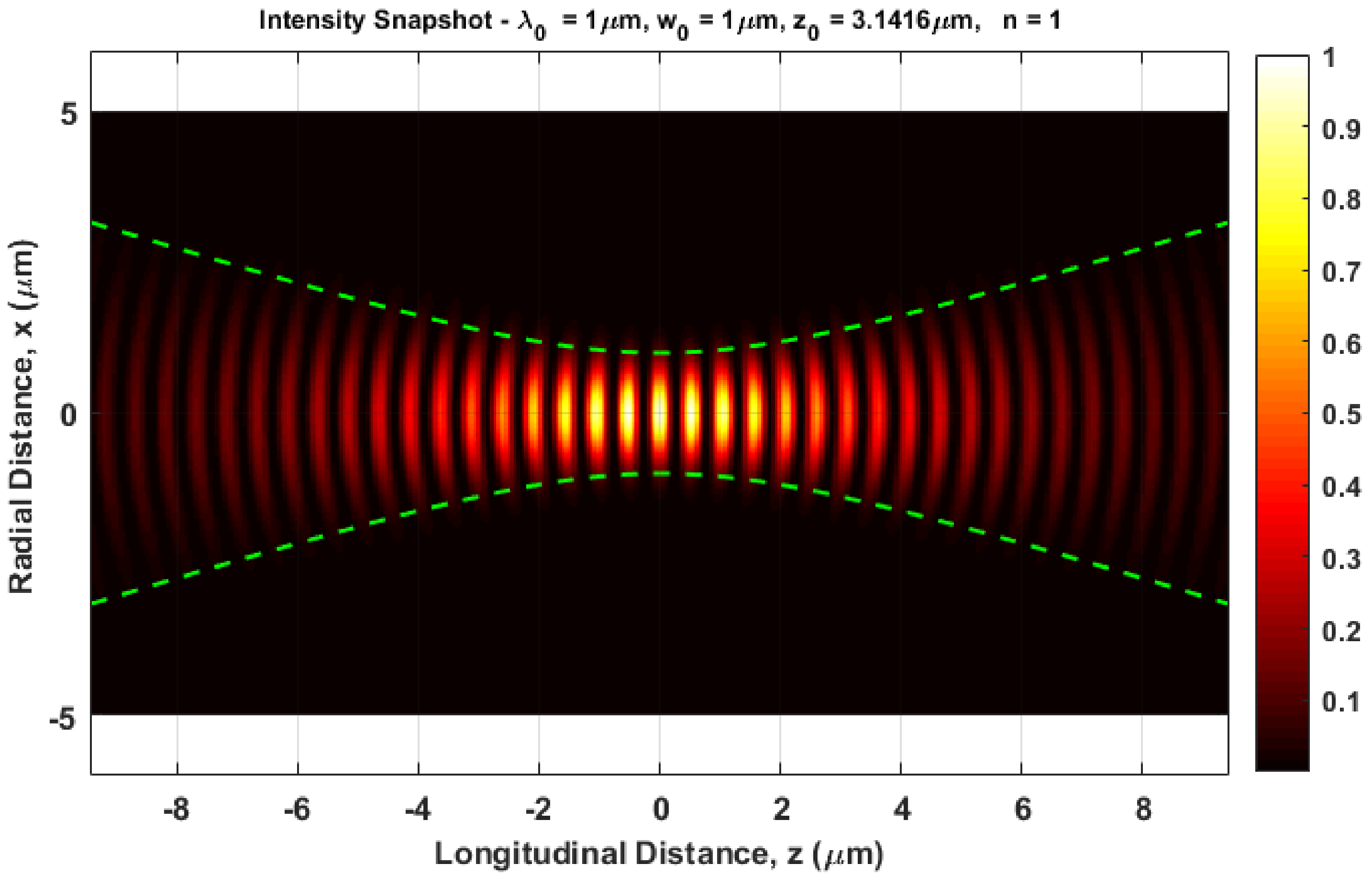
Gaussian Beams

Electric Field Snapshot in Time

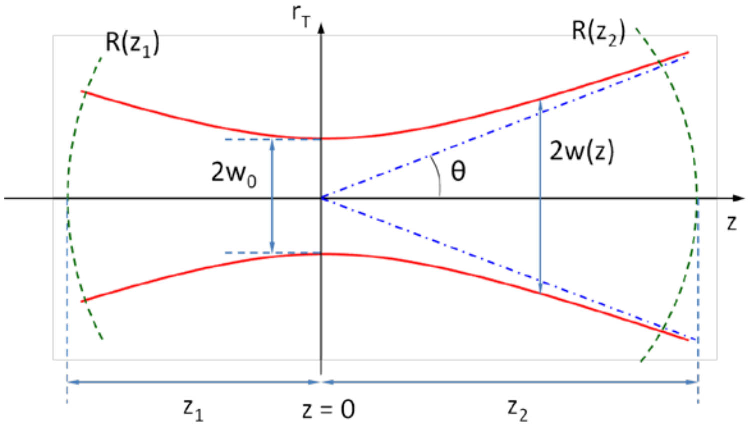


Gaussian Beams

Intensity Snapshot in Time



Gaussian Beams



Gaussian Beam Divergence

$$w(z) \simeq \frac{w_0 z}{z_0} = \frac{\lambda_0 z}{\pi n w_0},$$

$$\tan\theta = \frac{dw(z)}{dz} = \frac{\lambda_0}{\pi n w_0} \Longrightarrow \boxed{\Delta\theta = 2\theta \simeq \frac{2\lambda_0}{\pi n w_0} = \frac{4}{\pi}\frac{\lambda_0/n}{D_0}}$$

Gaussian Beam Power

$$P(z) = \frac{1}{2\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |E_{00}|^2 dx dy = \frac{1}{2\eta}\int_{-\infty}^{+\infty}\int_{0}^{2\pi} |E_{00}(r_T, z)|^2 r_T dr_T d\phi = \frac{1}{2\eta}|\mathcal{E}_0|^2 \left[\frac{\pi w_0^2}{2}\right]$$

Gaussian Beams

Gaussian Beam Radius of Curvature

Spherical Wave

$$
\begin{aligned}
E_{sph} &= \frac{1}{r}\exp(-jkr), \quad \text{where} \\
r &= \sqrt{x^2+y^2+z^2} = \sqrt{r_T^2+z^2} = z\sqrt{1+\frac{r_T^2}{z^2}} \\
r &\simeq z + \frac{1}{2}\frac{r_T^2}{z} \simeq z + \frac{1}{2}\frac{r_T^2}{r}, \quad \text{and} \\
E_{sph} &\simeq \frac{1}{r}\exp(-jkz)\exp\left(-j\frac{kr_T^2}{2r}\right).
\end{aligned}
$$

Fundamental Gaussian Beam

$$
E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)}\exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}
$$

Gaussian Beams

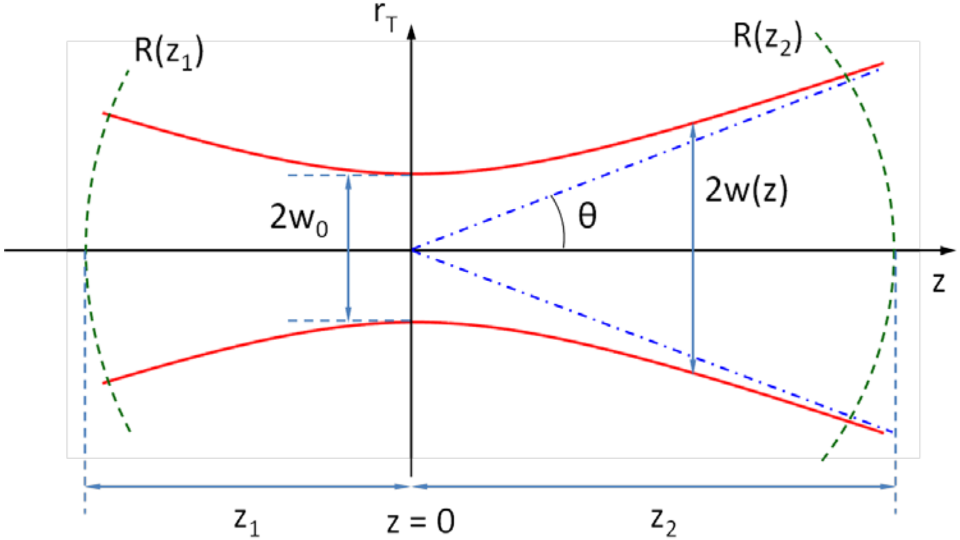
Fundamental Gaussian Beam

$$E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)} \exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}$$

Gaussian Beam Radius of Curvature

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

$$R(z) \simeq \begin{cases} +\infty, & \text{centered at } z_c = -\infty, \text{ for } z \ll z_0, \\ 2z_0, & \text{centered at } z_c = -z_0 \text{ for } z = z_0, \\ z, & \text{centered at } z_c = 0, \text{ for } z \gg z_0. \end{cases}$$



Gaussian Beams

Gaussian Beam Longitudinal Phase Shift – *Gouy's* Shift

$$E_{00}(r_T, z) = E_0 \underbrace{\frac{w_0}{w(z)} \exp\left[-\frac{r_T^2}{w^2(z)}\right]}_{\text{amplitude factor}} \underbrace{\exp\left\{-j\left[kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}}_{\text{longitudinal phase}} \underbrace{\exp\left[-j\frac{kr_T^2}{2R(z)}\right]}_{\text{radial phase}}$$

$$\Phi(z) = kz - \tan^{-1}(z/z_0)$$

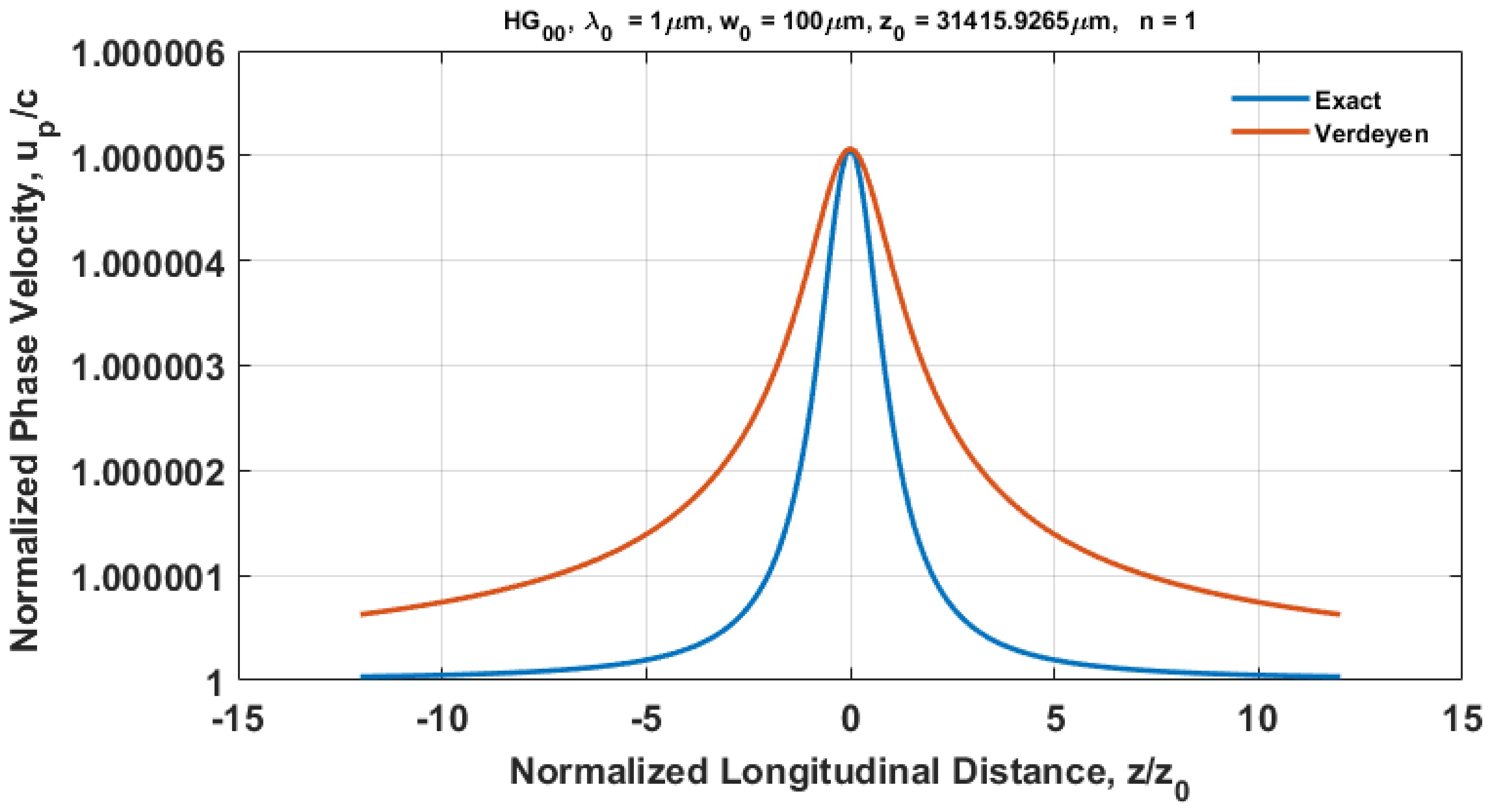
Phase Velocity (exact)

$$u_p = \frac{\omega}{d\Phi/dz} = \frac{\dfrac{c}{n}}{1 - \dfrac{\lambda_0 z_0}{2\pi n}\dfrac{1}{z^2 + z_0^2}}$$

Phase Velocity (approximate)

$$u_p = \frac{\omega}{\Phi/z} = \frac{\dfrac{c}{n}}{1 - \dfrac{\lambda_0}{2\pi n z}\tan^{-1}\left(\dfrac{z}{z_0}\right)}.$$

Gaussian Beam Longitudinal Phase Shift – *Gouy's* Shift



Hermite-Gaussian Beams

$$E_{mp}(x,y,z) = E_{mp}^{HG} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_p\left(\frac{\sqrt{2}y}{w(z)}\right) \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)}\right]$$
$$\exp\left\{-j\left[kz-(1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\} \exp\left[-j\frac{k(x^2+y^2)}{2R(z)}\right]$$

where,

$$w^2(z) = w_0^2\left(1+\frac{z^2}{z_0^2}\right),$$

$$R(z) = z\left(1+\frac{z_0^2}{z^2}\right),$$

$$H_m(u) = (-1)^m e^{u^2}\frac{d^m}{du^m}\left(e^{-u^2}\right),$$

$$H_{m+1}(u) = 2uH_m(u) - 2mH_{m-1}(u), \quad \text{with} \quad H_0(u)=1 \quad \text{and} \quad H_1(u)=2u$$

Some *Hermite* Polynomials

$$H_0(x) = 1$$

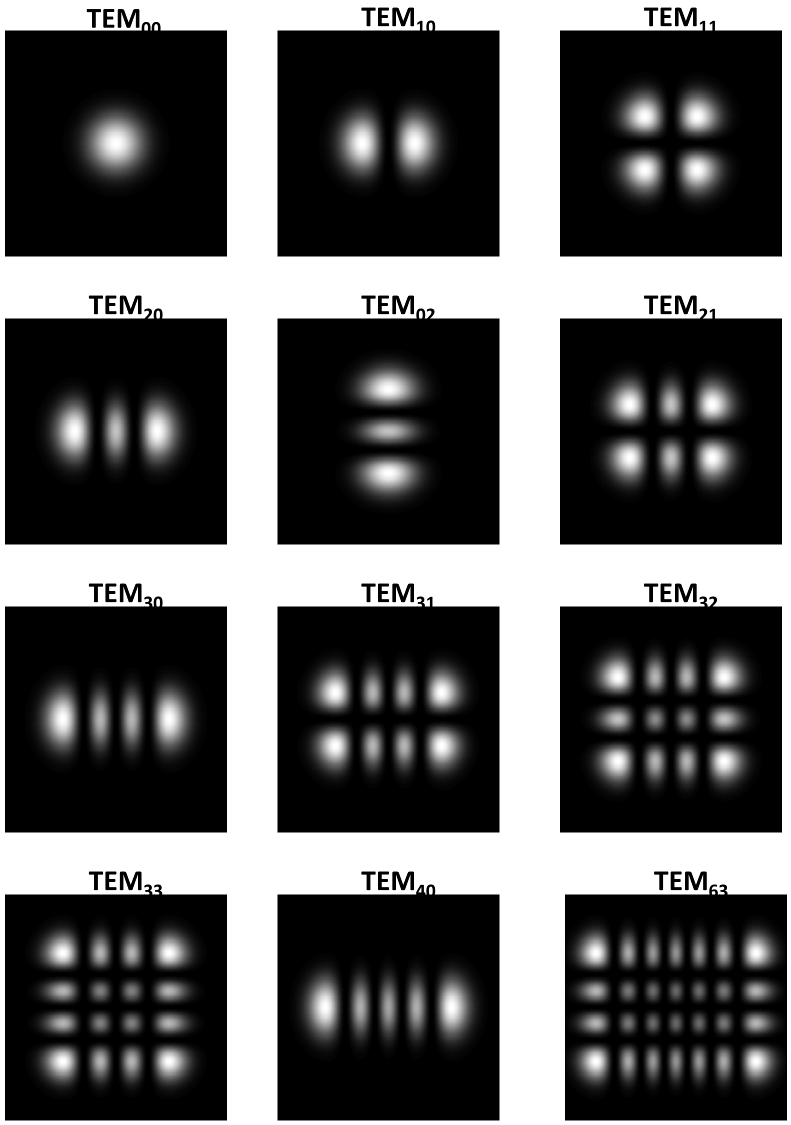
$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

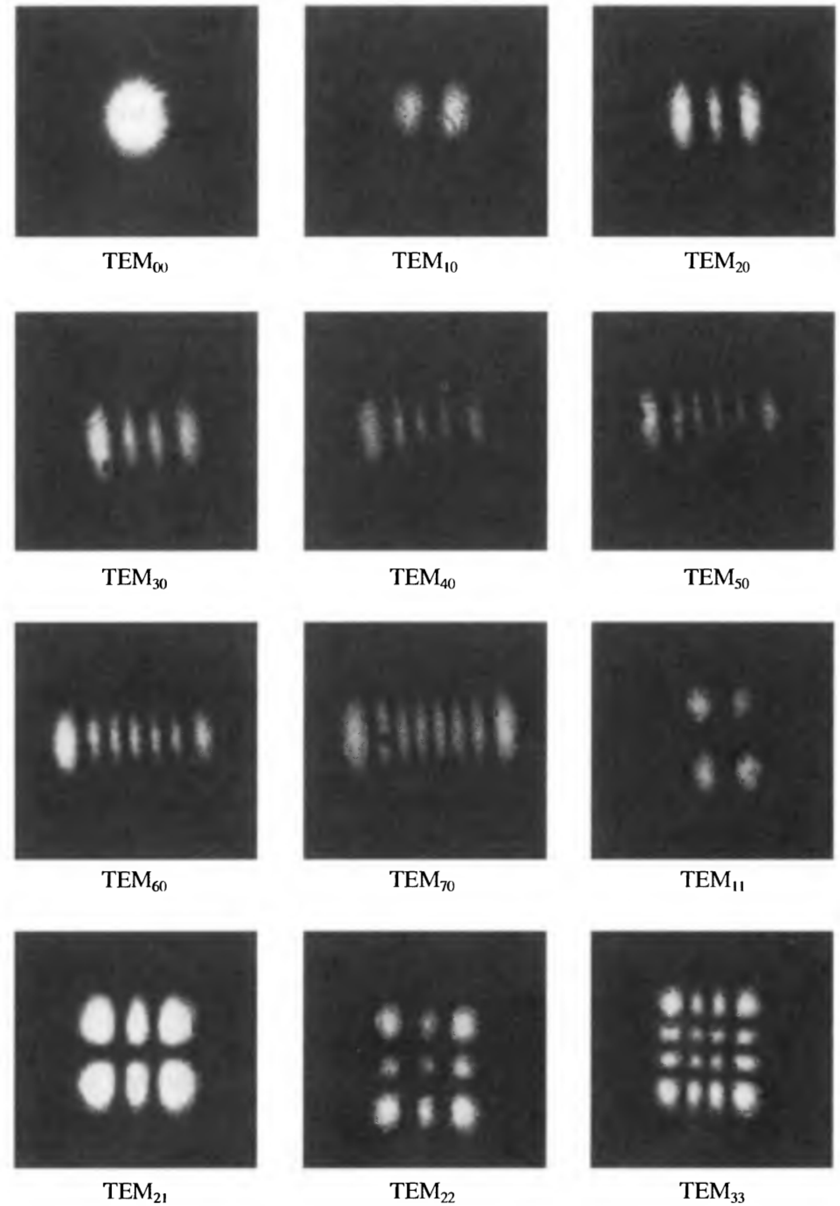
$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

Hermite-Gaussian Beams Patterns



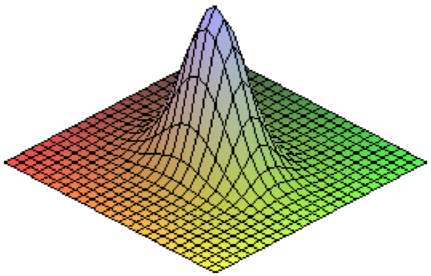
Experimental Patterns of Gaussian Beams



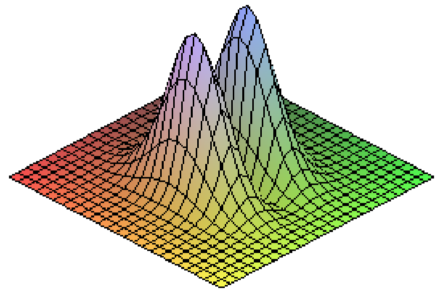
From A. Yariv and P. Yeh, "Photonics" 6th Ed. Oxford University Press, 2007

Hermite-Gaussian Beams Patterns

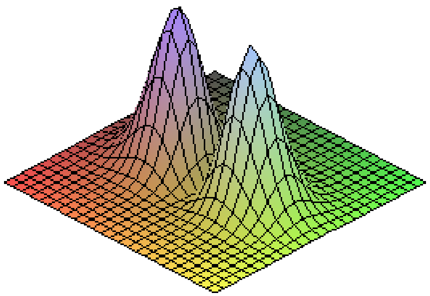
TEM_{00}



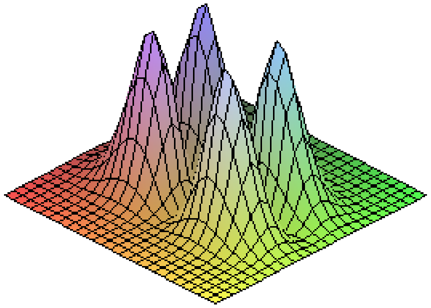
TEM_{10}



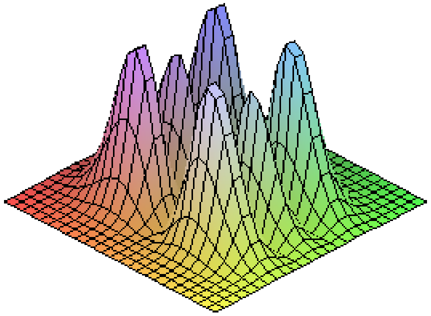
TEM_{01}



TEM_{11}



TEM_{21}



Laguerre-Gaussian Beams

$$E_{pm}(r_T,\phi,z) = E_{pm}^{LG}\frac{w_0}{w(z)}\left(\frac{\sqrt{2}r_T}{w(z)}\right)^{|m|} L_p^{|m|}\left(\frac{2r_T^2}{w^2(z)}\right)\exp\left[-\frac{r_T^2}{w^2(z)}\right]$$
$$\exp\left\{-j\left[kz-(1+|m|+2p)\tan^{-1}\left(\frac{z}{z_0}\right)\right]\right\}\exp(jm\phi)\exp\left[-j\frac{kr_T^2}{2R(z)}\right]$$

where,

$$w^2(z) = w_0^2\left(1+\frac{z^2}{z_0^2}\right),$$

$$R(z) = z\left(1+\frac{z_0^2}{z^2}\right),$$

$$L_p^m(u) = \frac{e^u}{p!}u^{-m}\frac{d^p}{du^p}\left(e^{-u}u^{p+m}\right),$$

$$(k+1)L_{k+1}^p(u) = (2k+1+p-u)L_k^p(u)-(k+p)L_{k-1}^p(u), \quad \text{with}$$

$$L_0^m(u) = 1 \quad \text{and} \quad L_1^m(u) = -u+(m+1),$$

Some *Laguerre* Polynomials

$$L_0^m(x) = 1$$

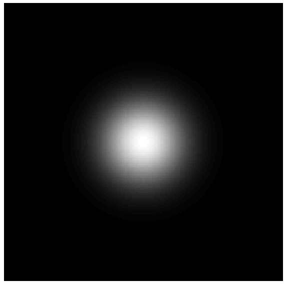
$$L_1^m(x) = -x+(m+1)$$

$$L_2^m(x) = \tfrac{1}{2}[x^2-2(m+2)x+(m+1)(m+2)]$$

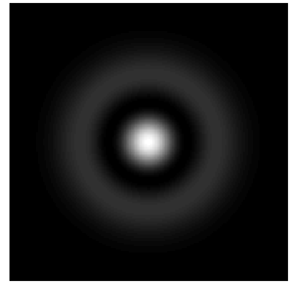
$$L_3^m(x) = \tfrac{1}{6}[-x^3+3(m+3)x^2-3(m+2)(m+3)x+(m+1)(m+2)(m+3)]$$

Laguerre-Gaussian Beams Patterns

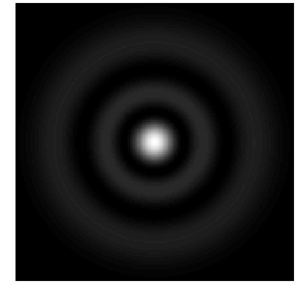
TEM(0,0) - cos(mϕ)



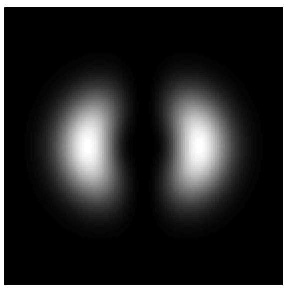
TEM(1,0) - cos(mϕ)



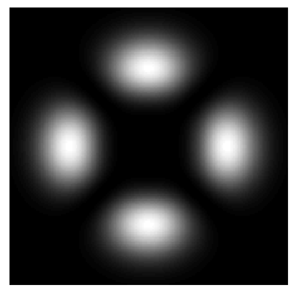
TEM(2,0) - cos(mϕ)



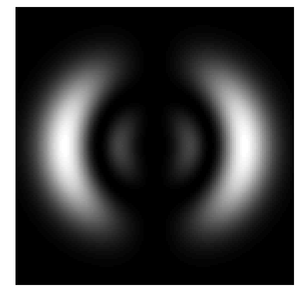
TEM(0,1) - cos(mϕ)



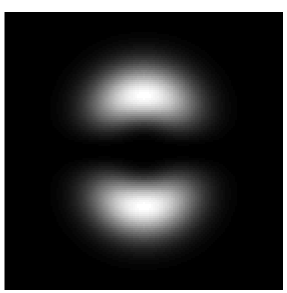
TEM(0,2) - cos(mϕ)



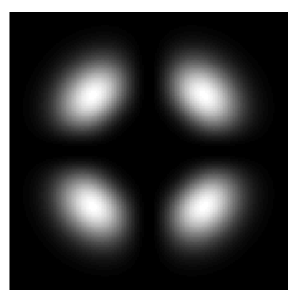
TEM(1,1) - cos(mϕ)



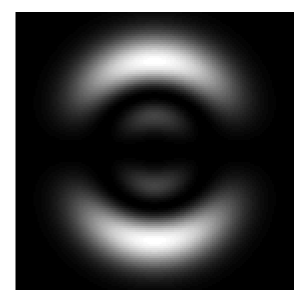
TEM(0,1) - sin(mϕ)



TEM(0,2) - sin(mϕ)



TEM(1,1) - sin(mϕ)



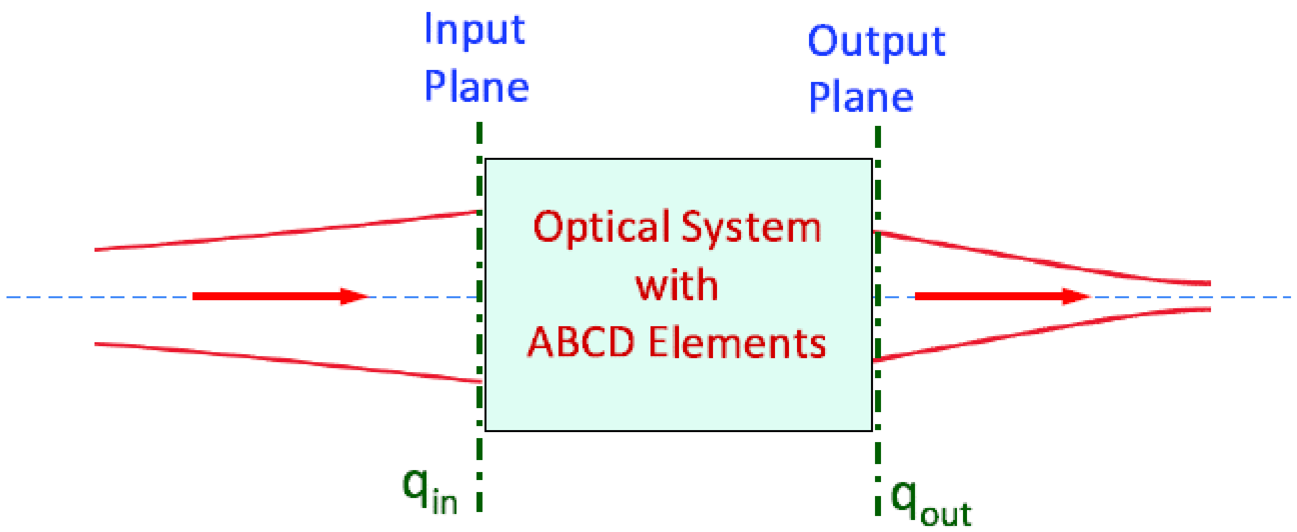
Validity of Paraxial Approximation

$$
\begin{aligned}
\mathcal{P} &= 1 - \frac{N+1}{k_0^2 n^2 w_0^2}, \qquad \text{where} \\
N &= m + p, \qquad \text{for Hermite-Gaussian beams,} \\
N &= 2p + |m|, \qquad \text{for Laguerre-Gaussian beams.}
\end{aligned}
$$

$$
\mathcal{P} \simeq 1 \qquad \text{(the closer to 1 the better the paraxial approximation)}
$$

P. Vaveliuk et al, "Limits of the paraxial approximation in laser beams", Opt. Lett., 32, (927-929), 2007

Gaussian Beams and ABCD Law

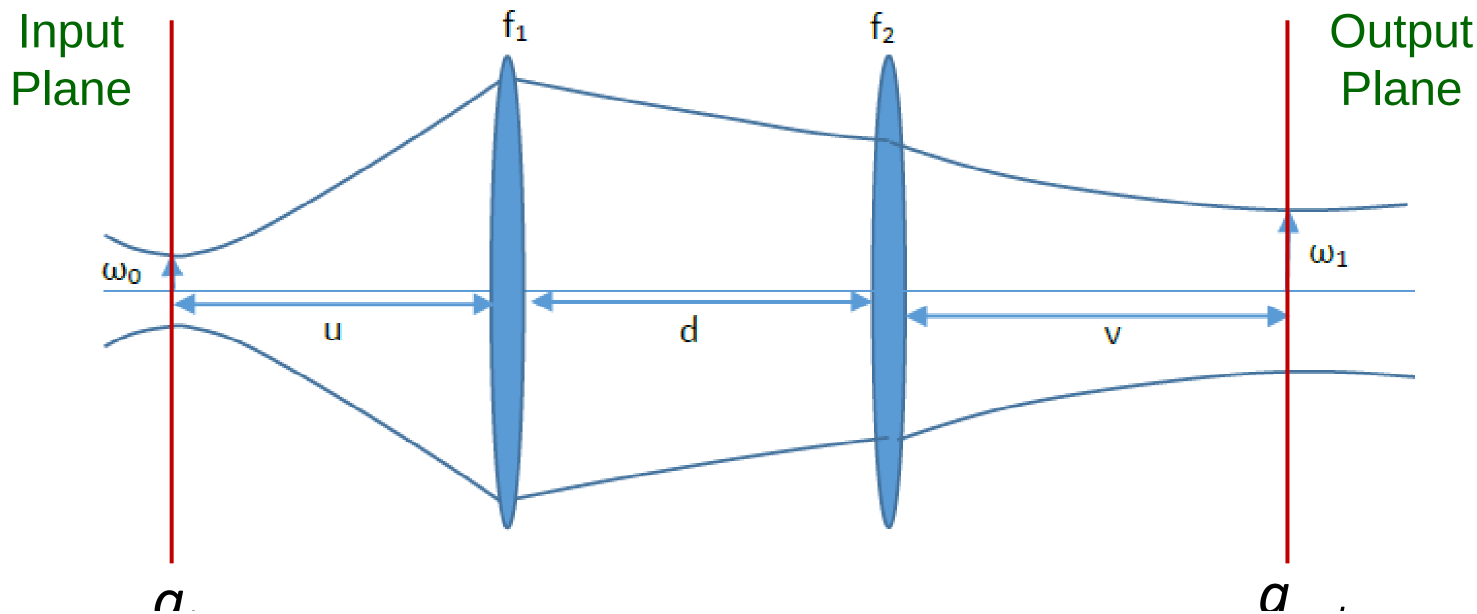


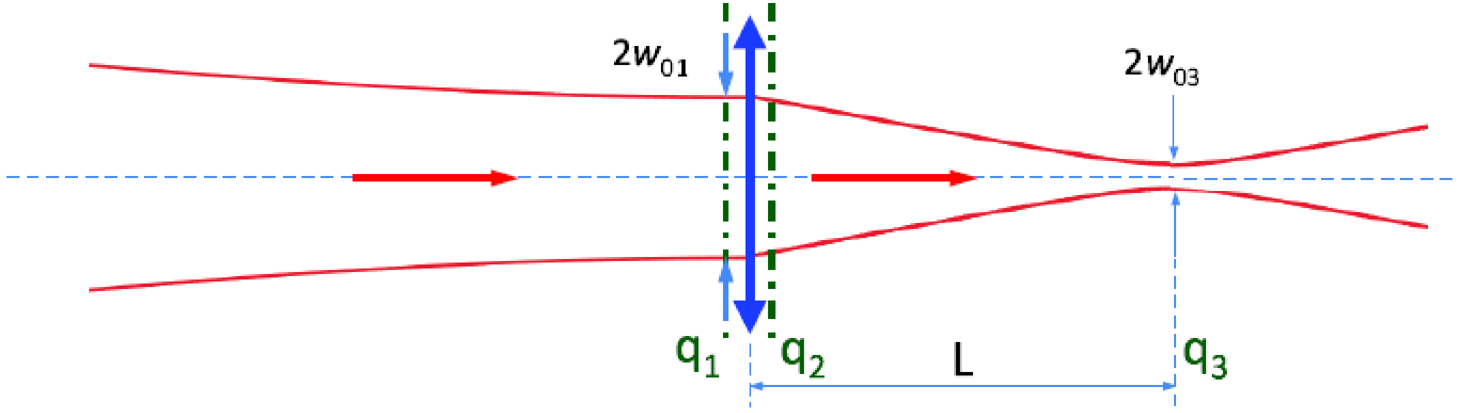
$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

$$\frac{1}{q_{out}} = \frac{C + D(1/q_{in})}{A + B(1/q_{in})}$$

$$\frac{1}{q(z)} = \frac{z}{z^2 + z_0^2} - j\frac{z_0}{z^2 + z_0^2} = \frac{1}{R(z)} - j\frac{\lambda_0}{\pi n w^2(z)}$$

Gaussian Beams and ABCD Law Examples





Gaussian Beams and ABCD Law Examples