Physical & Electromagnetic Optics: Basic Principles of Scalar Diffraction

Optical Engineering

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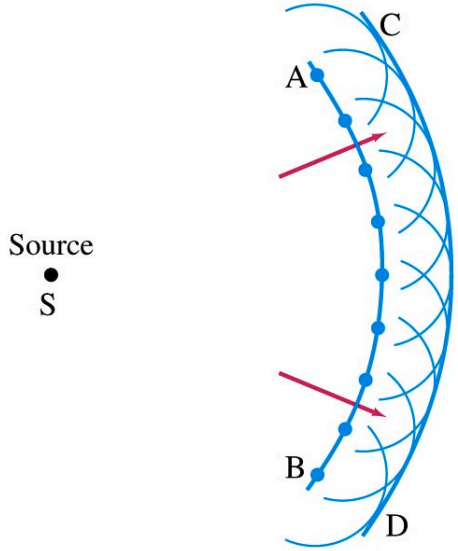
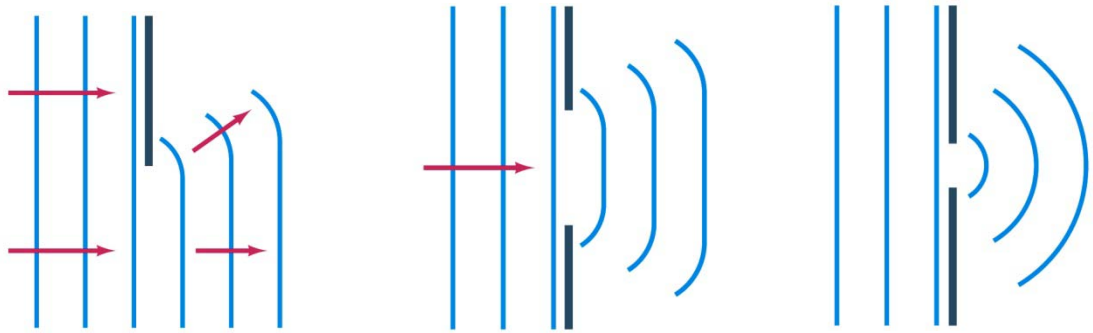
Huygens Principle and Diffraction



Every point on a propagating wavefront becomes a secondary source of spherical wavelets. All these wavelets form the envelope of the next wavefront.

Diffraction – Bending of light into the shadow regions
(wavelength, λ, of the order of obstacles)



https://www.usna.edu/Users/physics/ejtuchol/documents/SP212R/Chapter%20S37.ppt

Diffraction Examples

Circular waves generated by diffraction from the narrow entrance of a flooded coastal quarry

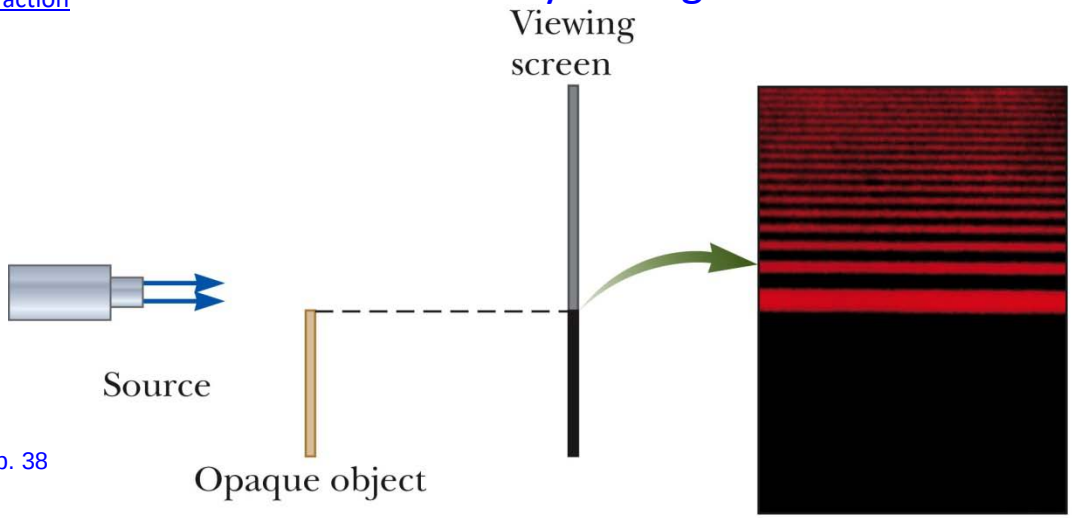
https://en.wikipedia.org/wiki/Diffraction

Diffraction by a Razor

Diffraction by CD

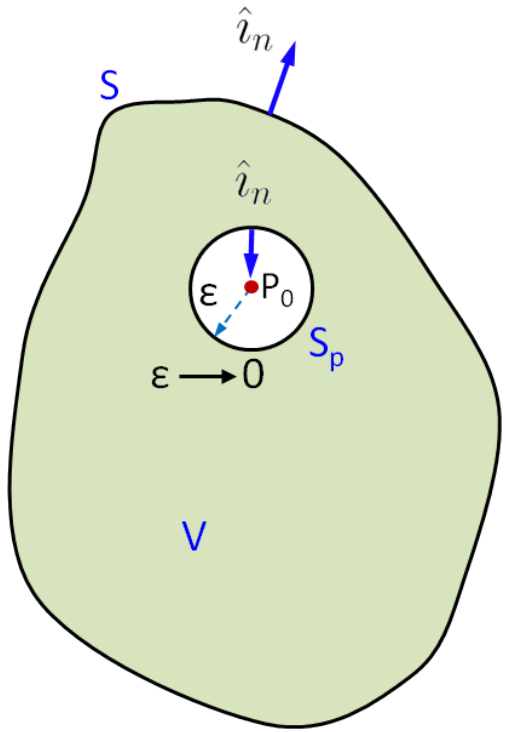
http://hyperphysics.phy-astr.gsu.edu/hbase/phyopt/grating.html

Diffraction by an Edge



Gengage Learning, Chap. 38

Integral Theorem of Helmholtz and Kirchhoff



3D-Green's Function

$$\nabla^2 G(\vec{r},\vec{r'}) + k^2 G(\vec{r},\vec{r'}) = -\delta(\vec{r}-\vec{r'})$$

$$G(\vec{r},\vec{r'}) = \frac{\exp(-jk|\vec{r}-\vec{r'}|)}{4\pi|\vec{r}-\vec{r'}|}$$

Scalar Field at a point as a function of the Fields on the Boundary ***S***

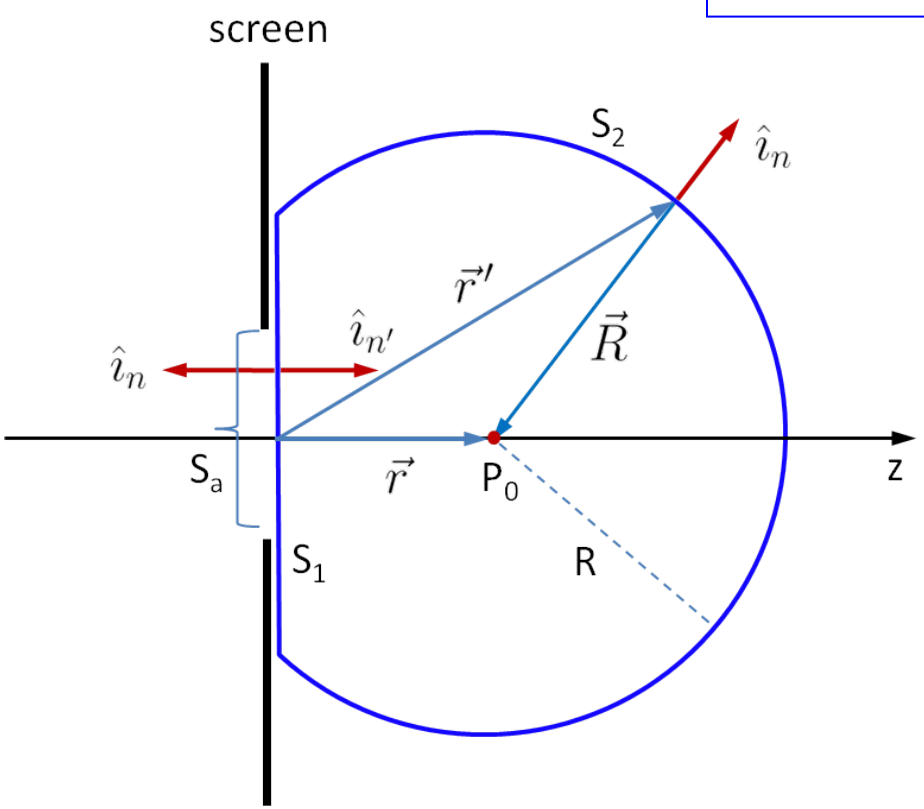
$$U(P_0) = \iint_S \left(G(\vec{r},\vec{r'})\frac{\partial U}{\partial n} - U\frac{\partial G(\vec{r},\vec{r'})}{\partial n} \right) dS$$

A. Ishimarou, "Electromagnetic Wave Propagation, Radiation, and Scattering," Prentice Hall, 1991

J. W. Goodman, "Introduction to Fourier Optics," 2nd Ed., McGraw-Hill, 1996

Kirchhoff's Formulation for a Planar Screen

$$U(P_0) = \iint_{S_1} \left(G(\vec{r},\vec{r'}) \frac{\partial U}{\partial n} - U \frac{\partial G(\vec{r},\vec{r'})}{\partial n} \right) dS$$



Sommerfeld's Radiation Condition

$$\lim_{R\to\infty} \left\{ R\left(\frac{\partial U}{\partial n} + jkU \right) \right\} = 0$$

Sommerfeld's Finiteness Condition

$$\lim_{R\to\infty} \{U\} = 0$$

$$\lim_{R\to\infty} \{R|U|\} < \infty$$

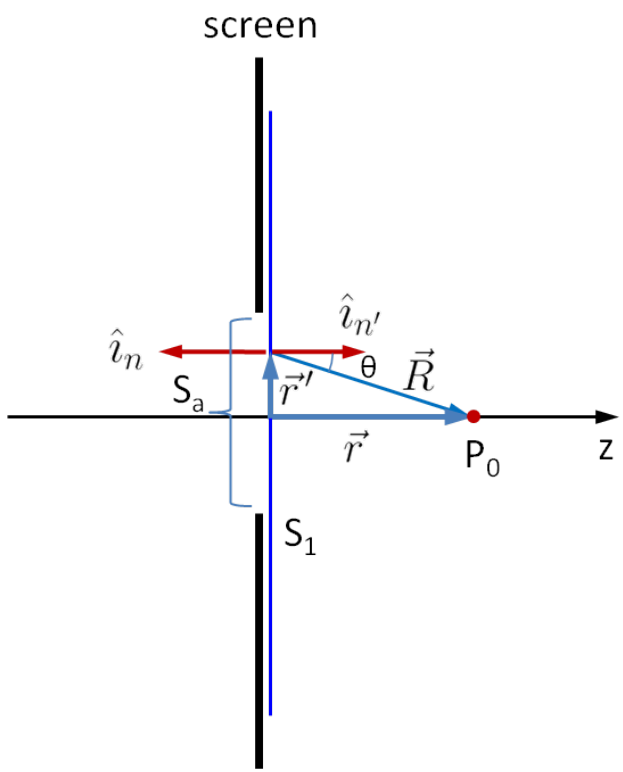
Green's Function Selection

$$G_D(\vec{r},\vec{r'}) = 0 \qquad \vec{r'} \in S_1 \qquad \text{Dirichlet}$$

$$\frac{\partial G_N(\vec{r},\vec{r'})}{\partial n} = 0 \qquad \vec{r'} \in S_1 \qquad \text{Neumann}$$

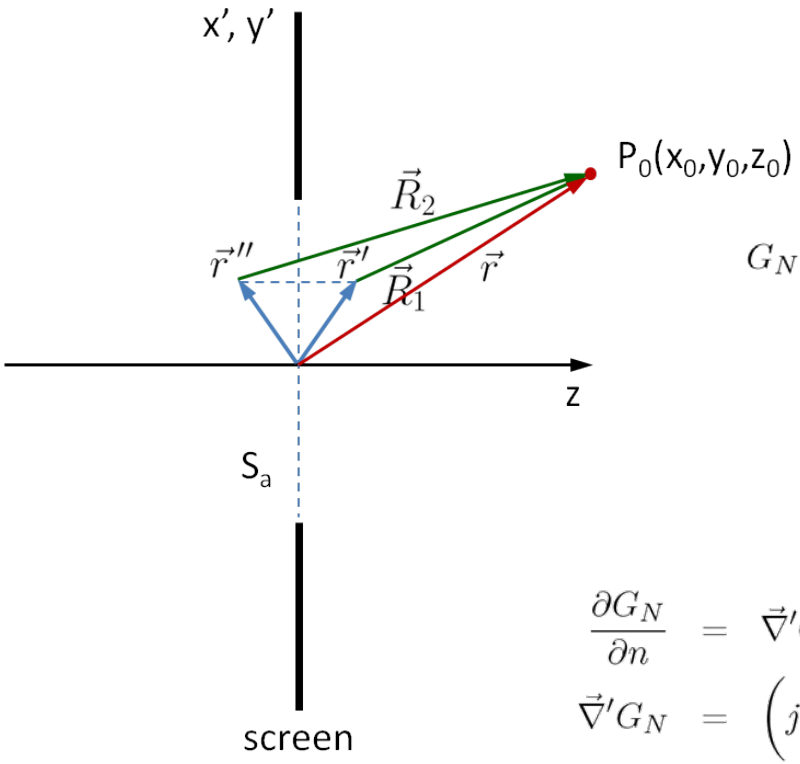
A. Ishimarou, "Electromagnetic Wave Propagation, Radiation, and Scattering," Prentice Hall, 1991

Kirchhoff's Formulation for a Planar Screen



$$
\begin{aligned}
U(P_0) &= \iint_{S_1} \left(G \frac{\partial U}{\partial n} - U \frac{\partial G}{\partial n} \right) ds = \\
&= \iint_{S_1} \left[\frac{e^{-jkR}}{4\pi R} \frac{\partial U}{\partial n} - U \left(jk + \frac{1}{R} \right) \frac{e^{-jkR}}{4\pi R} \left(\hat{\imath}_R \cdot \hat{\imath}_n \right) \right] ds.
\end{aligned}
$$

Rayleigh-Sommerfeld's Formulations



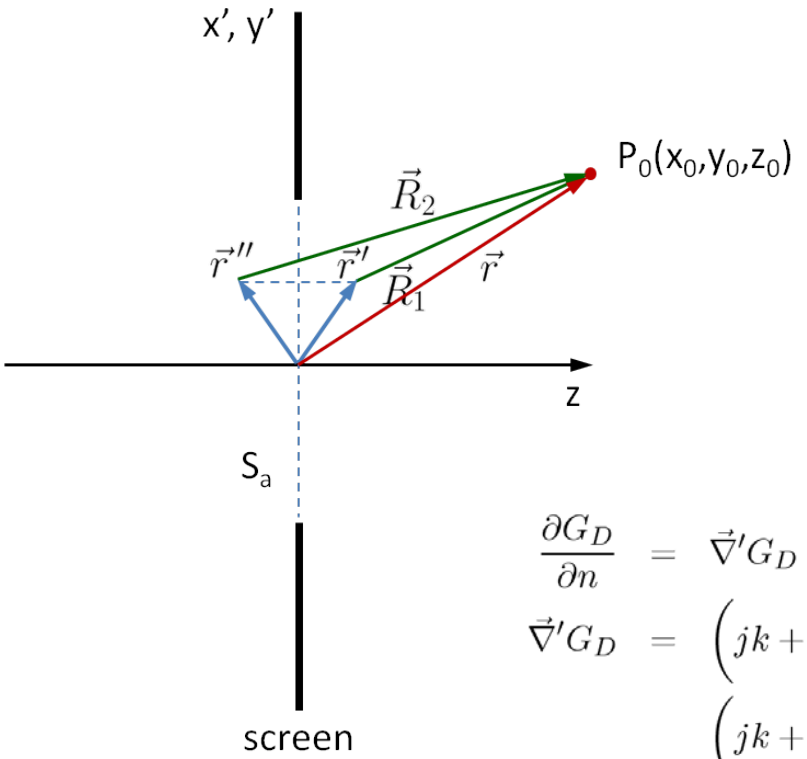
Neumann Green's Function

$$
\begin{aligned}
G_N(\vec{r},\vec{r}';\vec{r}'') &= \frac{\exp(-jk|\vec{r}-\vec{r}'|)}{4\pi|\vec{r}-\vec{r}'|} + \frac{\exp(-jk|\vec{r}-\vec{r}''|)}{4\pi|\vec{r}-\vec{r}''|} = \\
&= \frac{\exp(-jkR_1)}{4\pi R_1} + \frac{\exp(-jkR_2)}{4\pi R_2}, \qquad \text{where,} \\
\vec{R}_1 &= (x_0-x')\hat{\imath}_x + (y_0-y')\hat{\imath}_y + (z_0-z')\hat{\imath}_z, \qquad R_1 = |\vec{R}_1|, \\
\vec{R}_2 &= (x_0-x')\hat{\imath}_x + (y_0-y')\hat{\imath}_y + (z_0+z')\hat{\imath}_z, \qquad R_2 = |\vec{R}_2|,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial G_N}{\partial n} &= \vec{\nabla}' G_N \cdot \hat{\imath}_n, \qquad \hat{\imath}_n = -\hat{\imath}_z, \\
\vec{\nabla}' G_N &= \left(jk + \frac{1}{R_1}\right)\frac{\exp(-jkR_1)}{4\pi R_1}\left(\frac{x_0-x'}{R_1}\hat{\imath}_x + \frac{y_0-y'}{R_1}\hat{\imath}_y + \frac{z_0-z'}{R_1}\hat{\imath}_z\right) + \\
&\quad \left(jk + \frac{1}{R_2}\right)\frac{\exp(-jkR_2)}{4\pi R_2}\left(\frac{x_0-x'}{R_2}\hat{\imath}_x + \frac{y_0-y'}{R_2}\hat{\imath}_y - \frac{z_0+z'}{R_2}\hat{\imath}_z\right),
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial G_N}{\partial n}\right|_{z'=0} &= 0, \qquad \text{while,} \\
G_N|_{z'=0} &= \frac{\exp(-jkR)}{2\pi R}.
\end{aligned}
$$

Rayleigh-Sommerfeld's Formulations



Dirichlet Green's Function

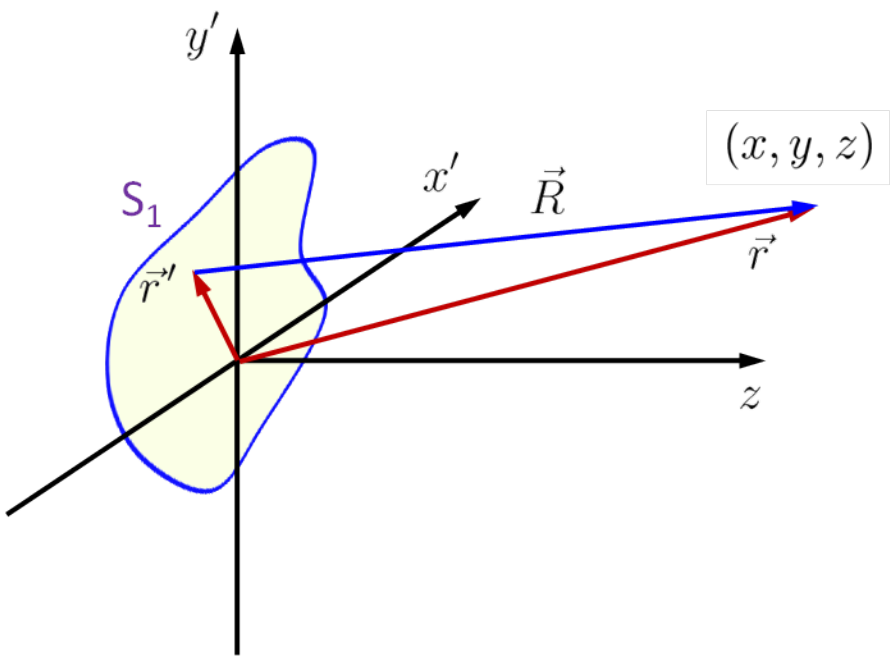
$$
\begin{aligned}
G_D(\vec{r},\vec{r}';\vec{r}'') &= \frac{\exp(-jk|\vec{r}-\vec{r}'|)}{4\pi|\vec{r}-\vec{r}'|} - \frac{\exp(-jk|\vec{r}-\vec{r}''|)}{4\pi|\vec{r}-\vec{r}''|} = \\
&= \frac{\exp(-jkR_1)}{4\pi R_1} - \frac{\exp(-jkR_2)}{4\pi R_2},
\end{aligned}
$$

$$
\begin{aligned}
\vec{R}_1 &= (x_0-x')\hat{\imath}_x + (y_0-y')\hat{\imath}_y + (z_0-z')\hat{\imath}_z, \qquad R_1 = |\vec{R}_1|, \\
\vec{R}_2 &= (x_0-x')\hat{\imath}_x + (y_0-y')\hat{\imath}_y + (z_0+z')\hat{\imath}_z, \qquad R_2 = |\vec{R}_2|,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial G_D}{\partial n} &= \vec{\nabla}' G_D \cdot \hat{\imath}_n, \qquad \hat{\imath}_n = -\hat{\imath}_z, \\
\vec{\nabla}' G_D &= \left(jk + \frac{1}{R_1}\right)\frac{\exp(-jkR_1)}{4\pi R_1}\left(\frac{x_0-x'}{R_1}\hat{\imath}_x + \frac{y_0-y'}{R_1}\hat{\imath}_y + \frac{z_0-z'}{R_1}\hat{\imath}_z\right) - \\
&\quad \left(jk + \frac{1}{R_2}\right)\frac{\exp(-jkR_2)}{4\pi R_2}\left(\frac{x_0-x'}{R_2}\hat{\imath}_x + \frac{y_0-y'}{R_2}\hat{\imath}_y - \frac{z_0+z'}{R_2}\hat{\imath}_z\right), \\
\left.\frac{\partial G_D}{\partial n}\right|_{z'=0} &= -\left(jk + \frac{1}{R}\right)\frac{\exp(-jkR)}{4\pi R}\frac{2z_0}{R}, \qquad \text{where,}
\end{aligned}
$$

$$
\begin{aligned}
R &= [(x_0-x')^2 + (y_0-y')^2 + z_0^2]^{1/2}, \qquad \text{while,} \\
G_D|_{z'=0} &= 0.
\end{aligned}
$$

Rayleigh-Sommerfeld's 1 Formulation for a Planar Screen



$$U(P_0) = \iint_{S_1} \left(G(\vec{r}, \vec{r}') \frac{\partial U}{\partial n} - U \frac{\partial G(\vec{r}, \vec{r}')}{\partial n} \right) dS$$

$$G_D(\vec{r}, \vec{r}') = 0$$

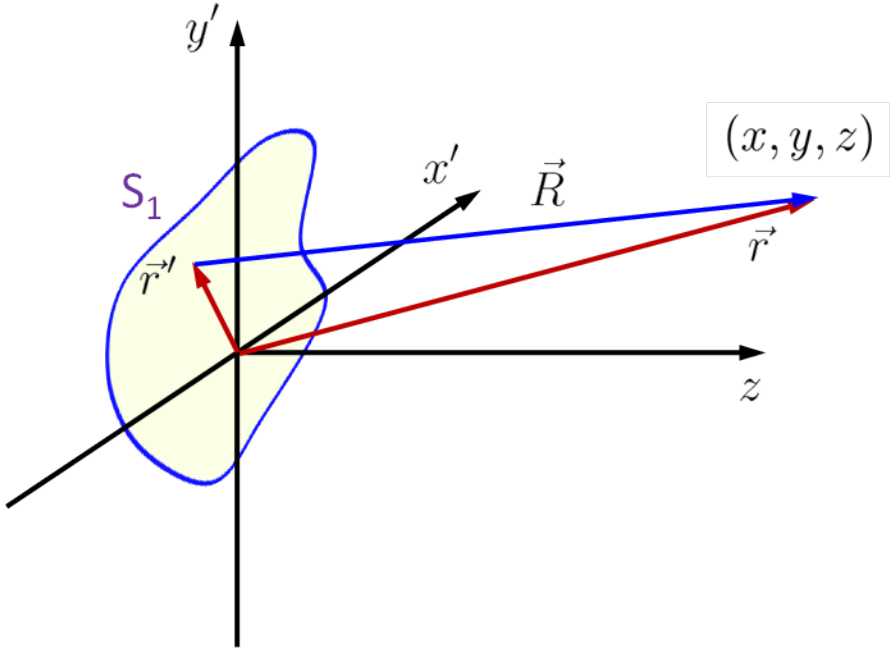
$$\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n} = -\left(jk + \frac{1}{R} \right) \frac{e^{-jkR}}{4\pi R} \frac{2z}{R}$$

$$U(\vec{r}) = \iint_{S_1} \frac{\exp(-jkR)}{2\pi R} \left(jk + \frac{1}{R} \right) \frac{z}{R} U_i(\vec{r}') dS'$$

$$R = [(x - x')^2 + (y - y')^2 + z^2]^{1/2}$$

A. Ishimarou, "Electromagnetic Wave Propagation, Radiation, and Scattering," Prentice Hall, 1991

Rayleigh-Sommerfeld's 2 Formulation for a Planar Screen



$$U(P_0) = \iint_{S_1} \left(G(\vec{r}, \vec{r}') \frac{\partial U}{\partial n} - U \frac{\partial G(\vec{r}, \vec{r}')}{\partial n} \right) dS$$

$$G_N(\vec{r}, \vec{r}') = \frac{e^{-jkR}}{2\pi R}$$

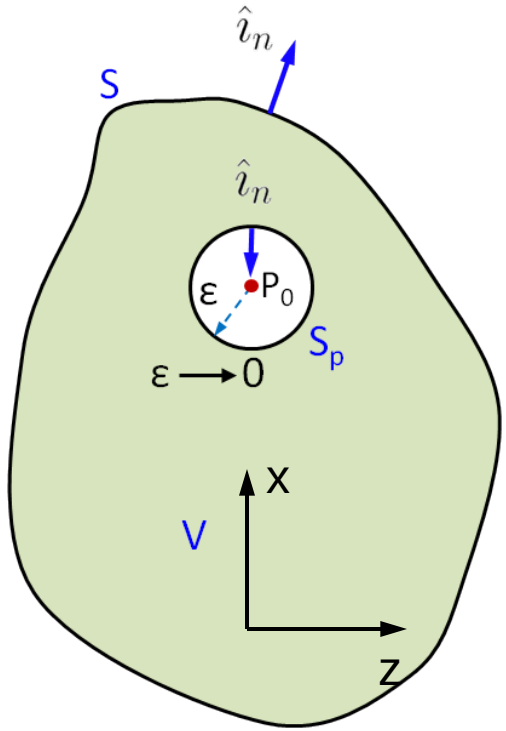
$$\frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n} = 0$$

$$U(\vec{r}) = \iint_{S_1} \frac{\exp(-jkR)}{2\pi R} \frac{\partial U_i(\vec{r}')}{\partial n} dS'$$

$$R = [(x - x')^2 + (y - y')^2 + z^2]^{1/2}$$

A. Ishimarou, "Electromagnetic Wave Propagation, Radiation, and Scattering," Prentice Hall, 1991

Integral Theorem for 2D Diffraction Problems



2D-Green's Function

$$\nabla^2 G^{2d}(\vec{r},\vec{r}\,') + k^2 G^{2d}(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$$

$$G^{2d}(\vec{r},\vec{r}\,') = \frac{j}{4} H_0^{(2)}\left(k|\vec{r}-\vec{r}\,'|\right)$$

$$\vec{r} = x\hat{x} + z\hat{z}$$

$$\vec{r}\,' = x'\hat{x} + z'\hat{z}$$

Scalar Field at a point as a function of the Fields on the Boundary Curve C

$$U(P_0) = \int_C \left(G(\vec{r},\vec{r'})\frac{\partial U}{\partial n} - U\frac{\partial G(\vec{r},\vec{r'})}{\partial n} \right) d\ell$$

A. Ishimarou, "Electromagnetic Wave Propagation, Radiation, and Scattering," Prentice Hall, 1991

J. W. Goodman, "Introduction to Fourier Optics," 2nd Ed., McGraw-Hill, 1996

Summary of Diffraction Formulas

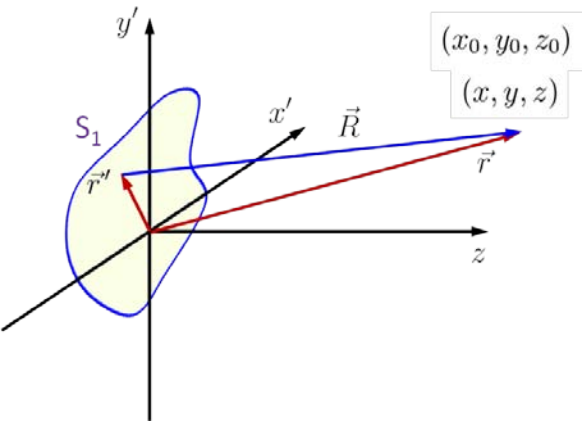
Rayleigh-Sommerfeld 1

$$
\begin{aligned}
U(x,y,z) &= \iint \frac{e^{-jkR}}{2\pi R}\left(jk+\frac{1}{R}\right)\frac{z}{R}U_i(x',y')dx'dy', \qquad \text{3D Green's Function} \\
R &= \sqrt{(x-x')^2+(y-y')^2+z^2}, \\
U(x,z) &= \int \frac{k}{2j}H_1^{(2)}(kR)\frac{z}{R}U_i(x')dx', \qquad \text{2D Green's Function} \\
R &= \sqrt{(x-x')^2+z^2} \\
H_1^{(2)}(z) &= J_1(z)-jY_1(z), \qquad \text{Hankel's Function}
\end{aligned}
$$

Rayleigh-Sommerfeld 2

$$
\begin{aligned}
U(x,y,z) &= \iint \frac{e^{-jkR}}{2\pi R}\frac{\partial U_i(x',y')}{\partial n}dx'dy', \qquad \text{3D Green's Function} \\
R &= \sqrt{(x-x')^2+(y-y')^2+z^2}, \\
U(x,z) &= \int \frac{-j}{2}\frac{k}{2j}H_0^{(2)}(kR)\frac{\partial U_i(x')}{\partial n}dx', \qquad \text{2D Green's Function} \\
R &= \sqrt{(x-x')^2+z^2}
\end{aligned}
$$

Fresnel Approximation



$$U(x_0, y_0, z_0) = \iint_{S_a} \left[U_{inc} \left(jk + \frac{1}{R} \right) \frac{\exp(-jkR)}{2\pi R} \frac{z_0}{R} \right]_{z'=0} dx'dy', \qquad \text{where,}$$

$$R = \left[(x_0 - x')^2 + (y_0 - y')^2 + z_0^2 \right]^{1/2} = z_0 \left[1 + \frac{(x_0 - x')^2}{z_0^2} + \frac{(y_0 - y')^2}{z_0^2} \right]^{1/2} \simeq$$

$$\simeq z_0 \left[1 + \frac{(x_0 - x')^2 + (y_0 - y')^2}{2z_0^2} \right] = z_0 + \frac{(x_0 - x')^2 + (y_0 - y')^2}{2z_0}.$$

$(1+X)^{1/2} \simeq 1 + X/2$ was used when $|X| \ll 1$.

$$\boxed{U(x_0, y_0, z_0) \simeq jk \frac{\exp(-jkz_0)}{2\pi z_0} \iint_{S_a} \left[\exp\left\{ -jk \left[\frac{(x_0 - x')^2}{2z_0} + \frac{(y_0 - y')^2}{2z_0} \right] \right\} U_{inc}(x', y') \right] dx'dy'}$$

$$z_0 \gg \left(\frac{\pi}{4\lambda_0} D^4 \right)^{1/3} \qquad D = \max\{[(x_0 - x')^2 + (y_0 - y')^2]^{1/2}\}$$

Fresnel Number

$$N_F = \frac{D^2}{\lambda_0 z_0}$$

$$\frac{\pi}{4} N_F \theta_{max}^2 \ll 1$$

$$\theta_{max} = D/z_0$$

Fresnel diffraction $N_F \gtrsim 1$ while for Fraunhofer diffraction $N_F \ll 1$

Fresnel Approximation

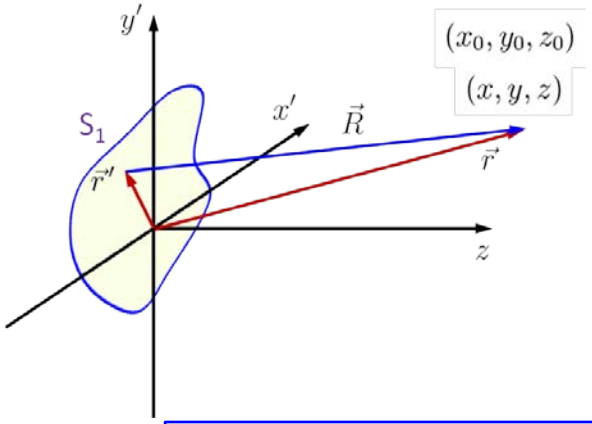
Fresnel Approximation in 2D

$$
\begin{aligned}
U(x_0, z_0) &= \int_{x'} \left[U_{inc} \frac{k}{2j} H_1^{(2)}(kR_T) \frac{z_0}{R_T} \right]_{z'=0} dx' \\
&\simeq \frac{k}{2j} e^{j3\pi/4} \sqrt{\frac{2}{\pi k z_0}} \int_{x'} \exp(-jkR_T) U_{inc}(x') dx' \\
&\simeq j\sqrt{\frac{k}{2\pi z_0}} e^{-jkz_0} \int_{x'} \exp\left[-j\frac{k}{2z_0} (x' - x_0)^2 \right] U_{inc}(x') dx'
\end{aligned}
$$

with the condition $\quad z_0 \gg \left[\left(\frac{\pi}{4\lambda} \right) (x_0 - x')^4 \right]^{1/3}.$

$$
H_\nu^{(2)}(w) \simeq \sqrt{\frac{2}{\pi w}} \exp\left[-j \left(w - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) \right], \qquad \text{for} \quad w \gg 1.
$$

Fraunhofer Approximation



$$k\left[\frac{(x_0-x')^2}{2z_0}+\frac{(y_0-y')^2}{2z_0}\right] = k\left[\frac{x_0^2+y_0^2}{2z_0}+\frac{x'^2+y'^2}{2z_0}-\frac{x_0x'+y_0y'}{z_0}\right] =$$

$$\simeq k\left[\frac{x_0^2+y_0^2}{2z_0}-\frac{x_0x'+y_0y'}{z_0}\right], \qquad \text{if,}$$

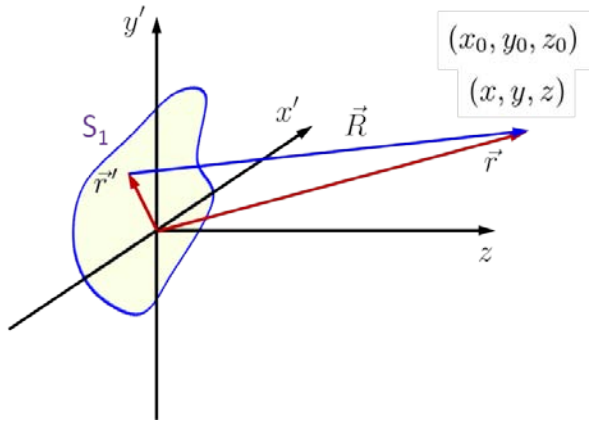
$$k\left(\frac{x'^2+y'^2}{2z_0}\right) \ll 1 \Longrightarrow z \gg \frac{kD_{max}^2}{2} = \frac{\pi}{\lambda}D_{max}^2 \simeq \frac{Area}{\lambda},$$

$$U_{ff}(x_0,y_0,z_0) = jk\frac{\exp(-jkz_0)}{2\pi z_0}\exp\left(-jk\frac{x_0^2+y_0^2}{2z_0}\right) \iint_{S_a}\left[\exp\left\{+jk\left(\frac{x_0x'+y_0y'}{z_0}\right)\right\}U_{inc}(x',y')\right]dx'dy'$$

$$= jk\frac{\exp(-jkz_0)}{2\pi z_0}\exp\left(-jk\frac{x_0^2+y_0^2}{2z_0}\right)\iint_{-\infty}^{\infty}\exp\left\{+jk\left(\frac{x_0x'+y_0y'}{z_0}\right)\right\}\left[t_a(x',y')U_{inc}(x',y')\right]dx'dy'$$

$$= jk\frac{\exp(-jkz_0)}{2\pi z_0}\exp\left(-jk\frac{x_0^2+y_0^2}{2z_0}\right)\mathcal{F}\left\{t_a(x',y')U_{inc}(x',y')\right\}\Big|_{\substack{k_x=kx_0/z_0\\ k_y=ky_0/z_0}}$$

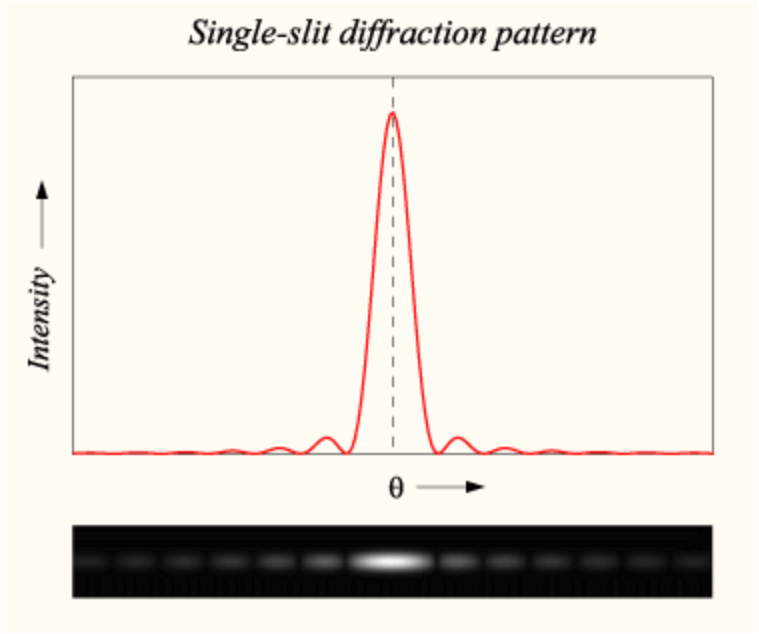
Fraunhofer Approximation in 2D



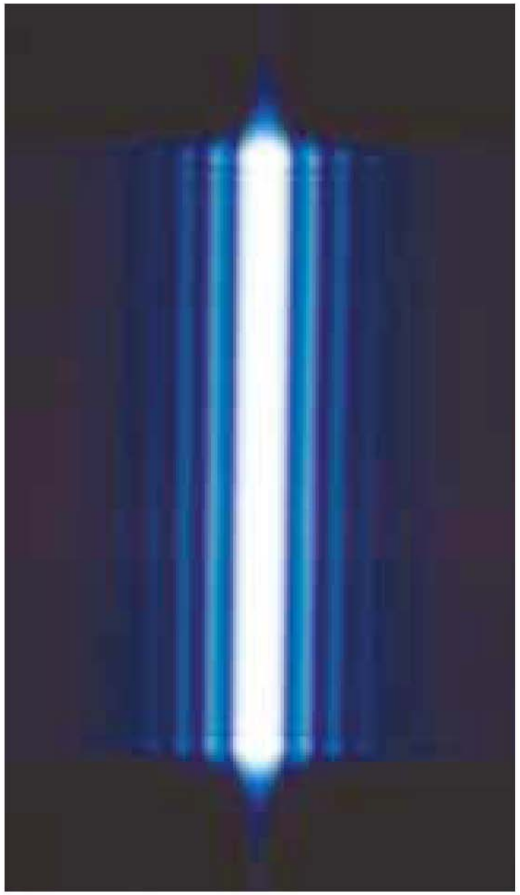
$$
\begin{aligned}
U_{ff}(x_0, z_0) &= j\sqrt{\frac{k}{2\pi z_0}} e^{-jkz_0} \int_{x'} \exp\left[-j\frac{k}{2z_0}\left(x' - x_0\right)^2\right] U_{inc}(x')dx' \\
&\simeq j\sqrt{\frac{k}{2\pi z_0}} e^{-jkz_0} \exp\left(-j\frac{k}{2z_0}x_0^2\right) \int_{x'} \exp\left[+j\frac{kx_0}{z_0}x'\right] U_{inc}(x')dx' \\
&\simeq j\sqrt{\frac{k}{2\pi z_0}} e^{-jkz_0} \exp\left(-j\frac{k}{2z_0}x_0^2\right) \int_{-\infty}^{+\infty} \exp\left[+j\frac{kx_0}{z_0}x'\right] \left[t_a(x')U_{inc}(x')\right] dx' \\
&= j\sqrt{\frac{k}{2\pi z_0}} e^{-jkz_0} \exp\left(-j\frac{k}{2z_0}x_0^2\right) \mathcal{F}\left\{t_a(x')U_{inc}(x')\right\}\Big|_{k_x = kx_0/z_0}.
\end{aligned}
$$

with the condition $\quad z_0 \gg \frac{\pi}{\lambda}x'^2 \simeq \frac{Length^2}{\lambda}.$

Diffraction from a Single Slit

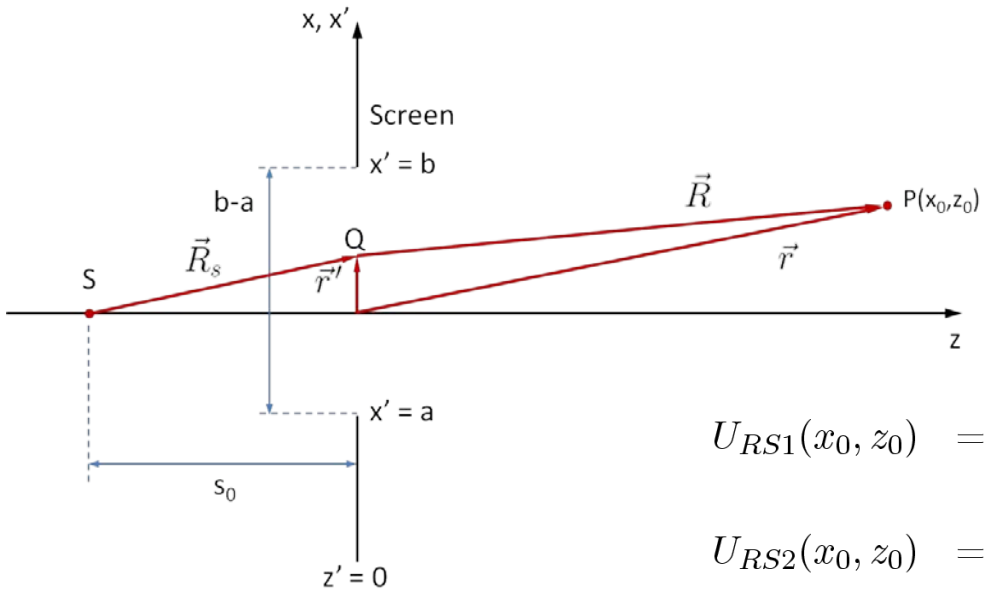


http://onderwijs1.amc.nl/medfysica/doc/LightDiffraction.htm



R.A. Serway and J. W. Jewett, "Physics for Scientists and Engineers", 6th Ed., Thomson Brooks/Cole, 2004

Diffraction from a Single Slit



Rayleigh-Sommerfeld Formulations

$$U_{RS1}(x_0, z_0) = \int_a^b \left[U_{inc} \frac{k}{2j} H_1^{(2)}(kR_T) \frac{z_0}{R_T} \right]_{z'=0} dx', \qquad \text{and}$$

$$U_{RS2}(x_0, z_0) = \int_a^b \left[-\frac{j}{2} H_0^{(2)}(kR_T) \frac{\partial U_{inc}}{\partial n} \right]_{z'=0} dx', \qquad \text{where,}$$

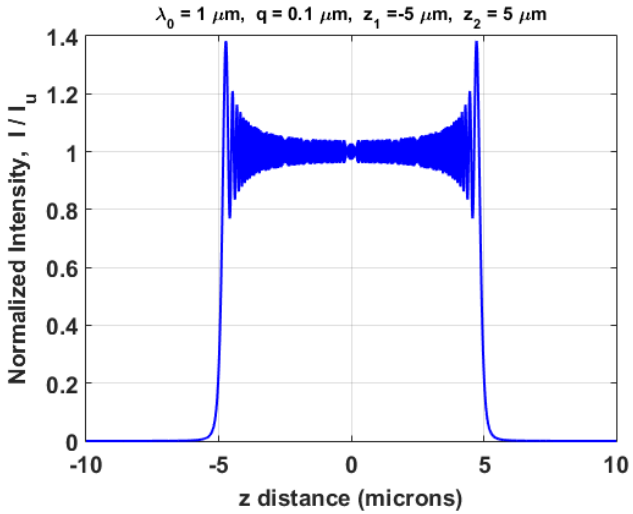
$$R_T = \left[(x_0 - x')^2 + z_0^2 \right]^{1/2}.$$

Incident Wave

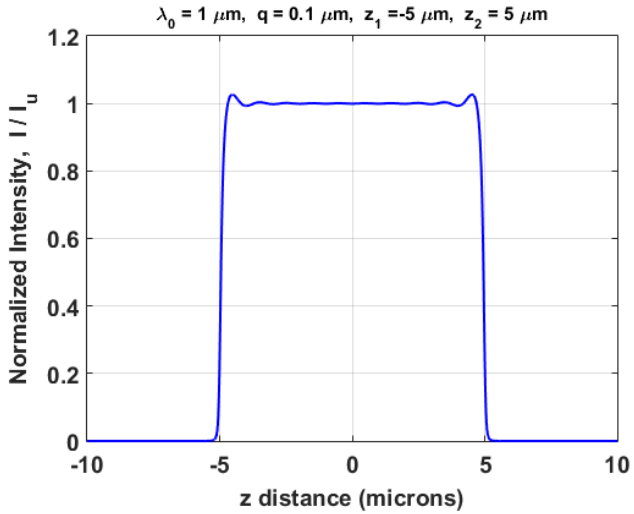
$$U_{inc}(x', z') = \begin{cases} U_0 \exp(-jkz'), & \text{(plane wave)} \\ U_0 \dfrac{\exp(-jkR_s)}{\sqrt{R_s}}, & R_s = [x'^2 + (z' + s_0)^2]^{1/2}, \quad \text{(diverging)}, \\ U_0 \dfrac{\exp(+jkR_s)}{\sqrt{R_s}}, & R_s = [x'^2 + (z' - s_0)^2]^{1/2}, \quad \text{(converging)}, \end{cases}$$

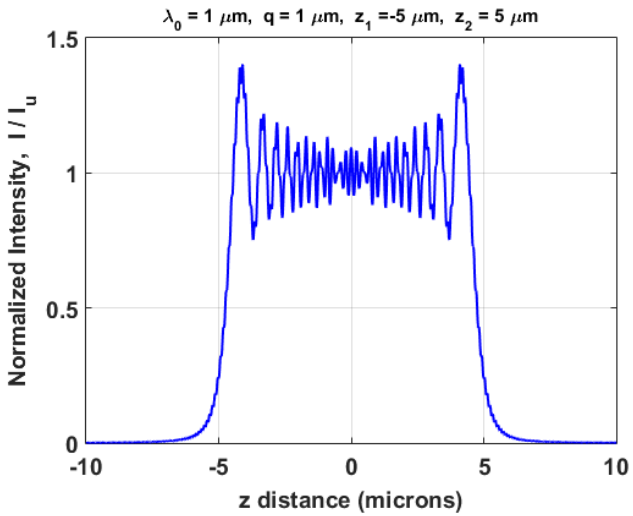
Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm

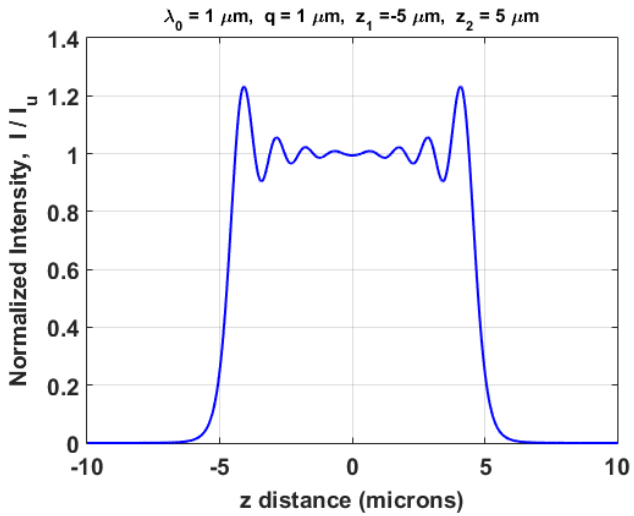
Fresnel Approximation



Rayleigh-Sommerfeld 1

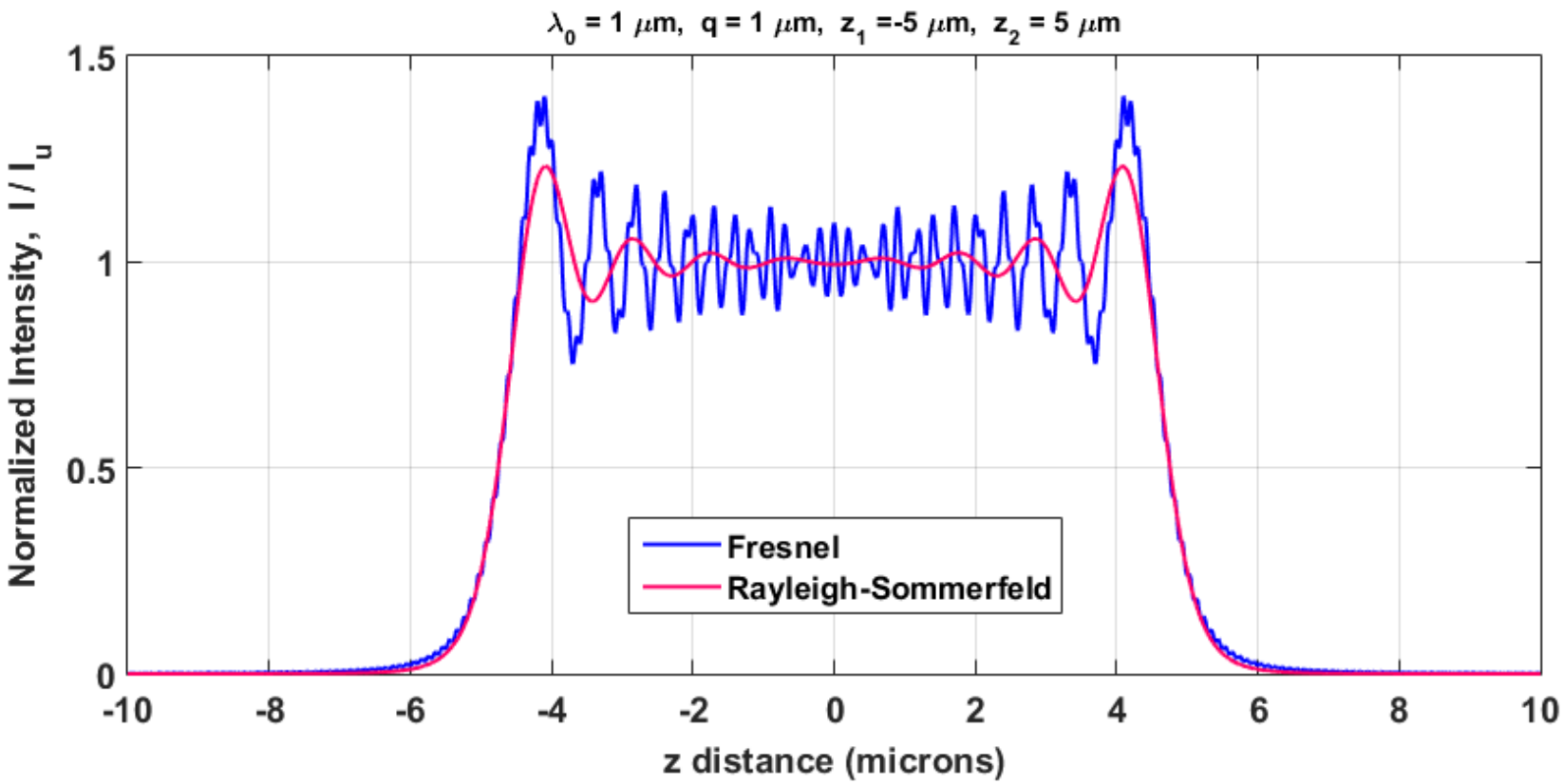




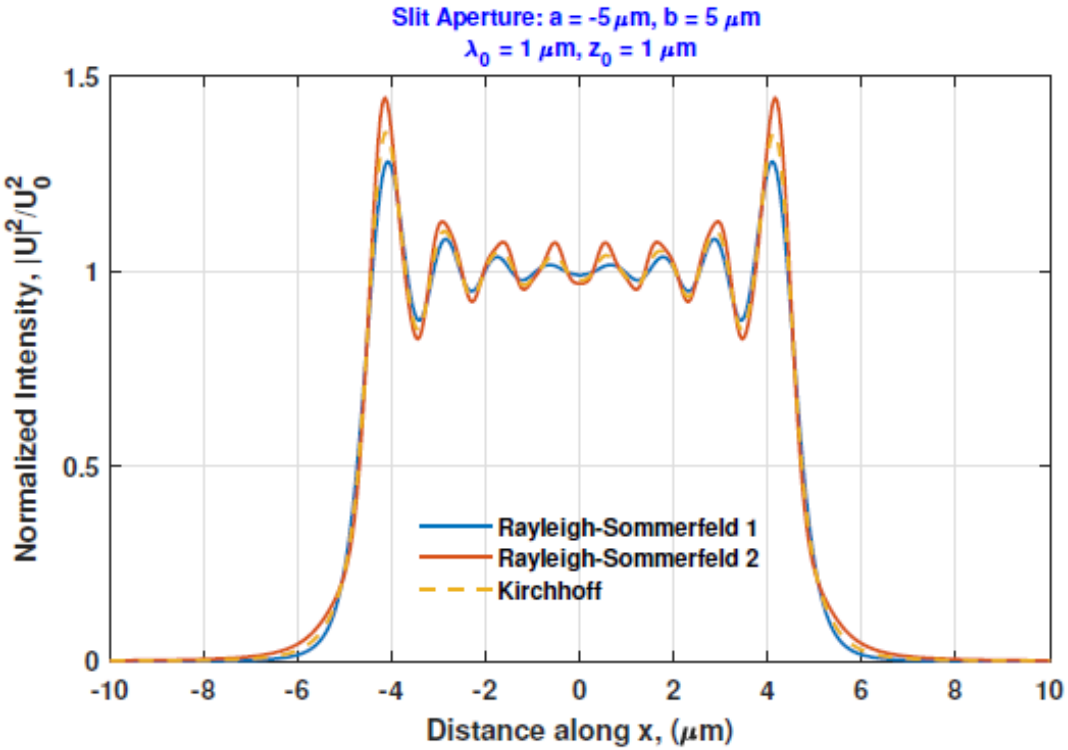


Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm

Comparison of Rayleigh-Sommerfeld (1) and Fresnel Approximation

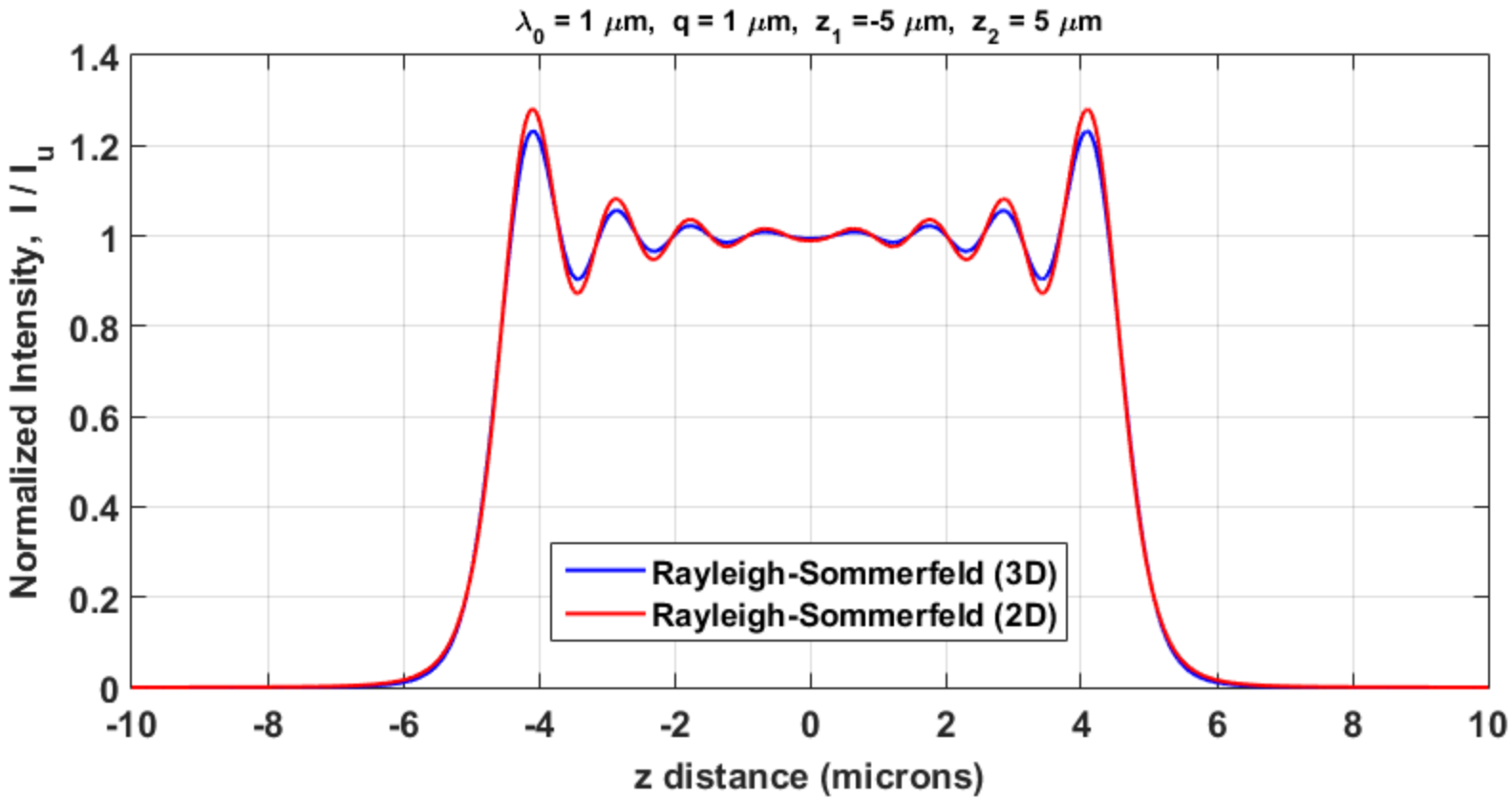


Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm



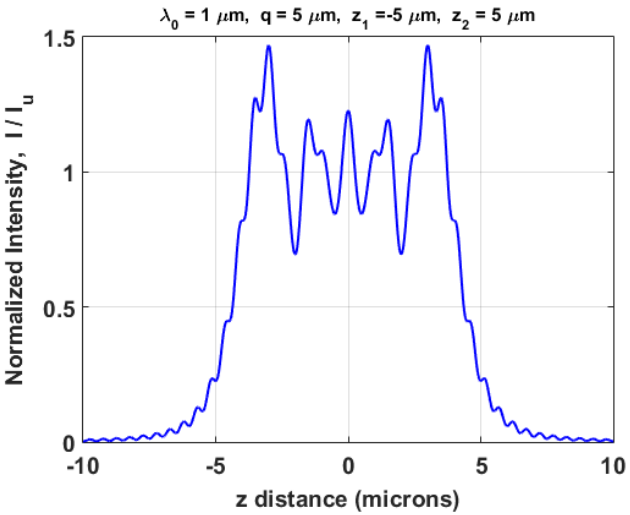
Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm

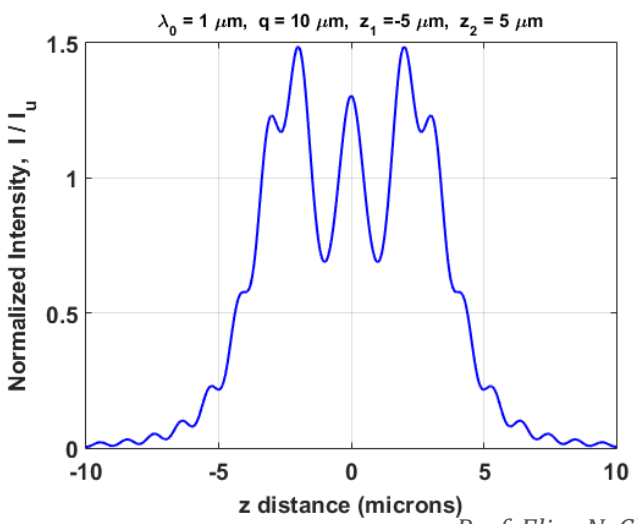
Comparison of Rayleigh-Sommerfeld (1) with 2D and 3D Green's Function



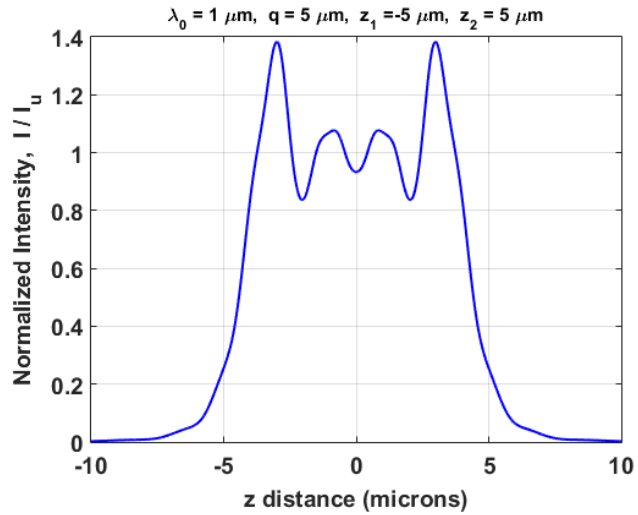
Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm

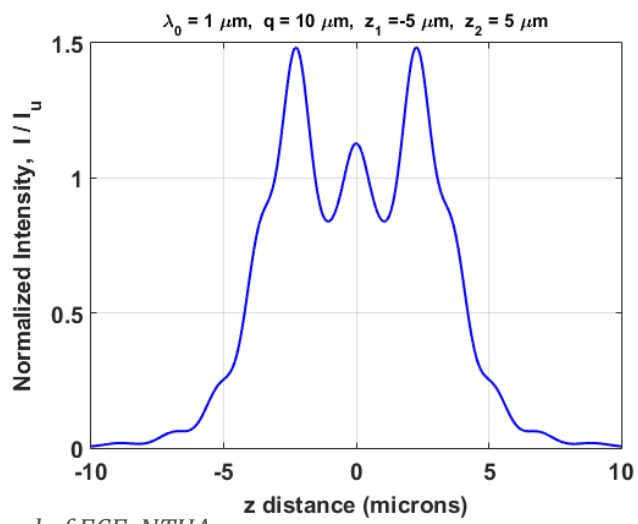
Fresnel Approximation





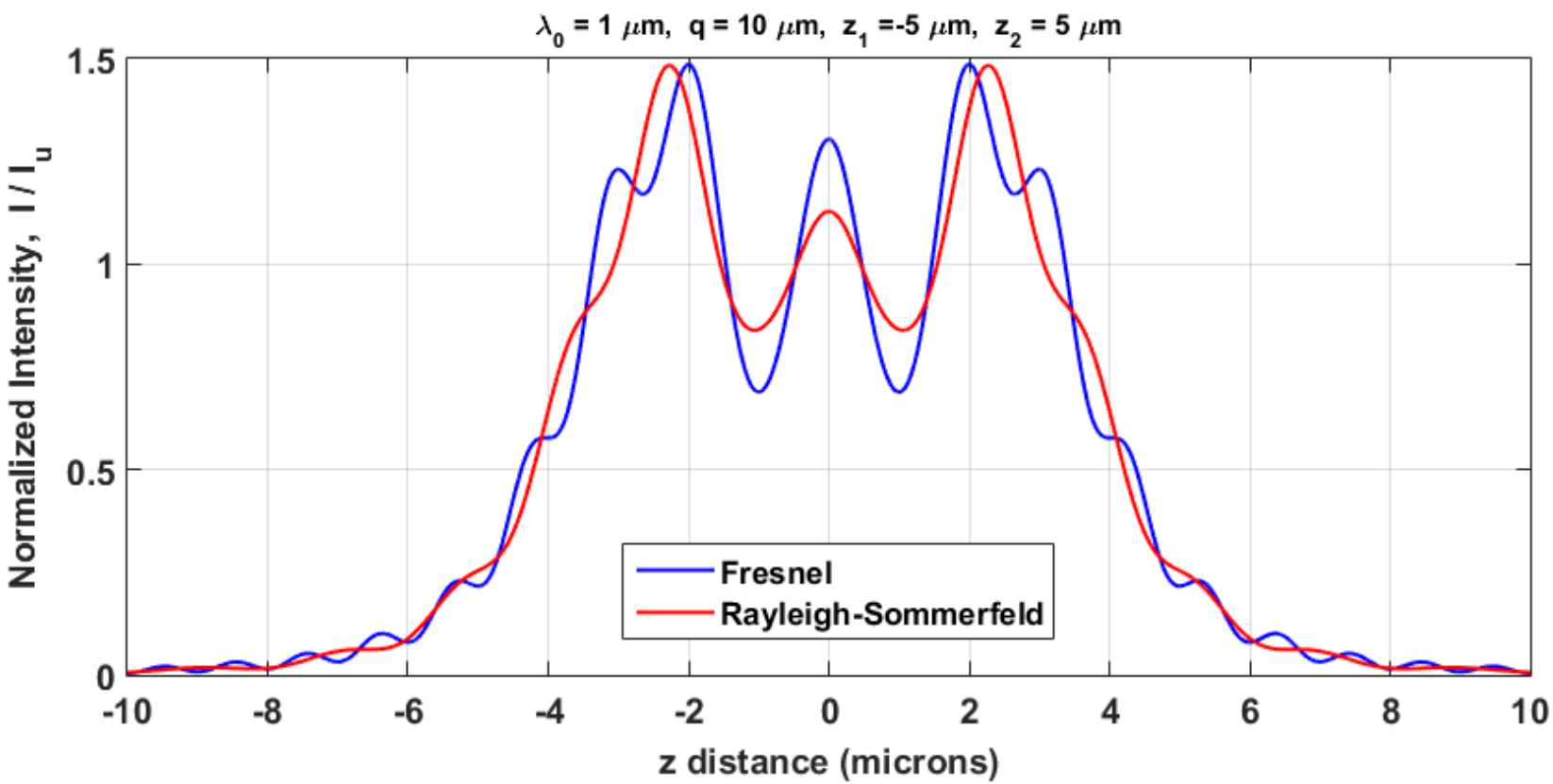
Rayleigh-Sommerfeld 1





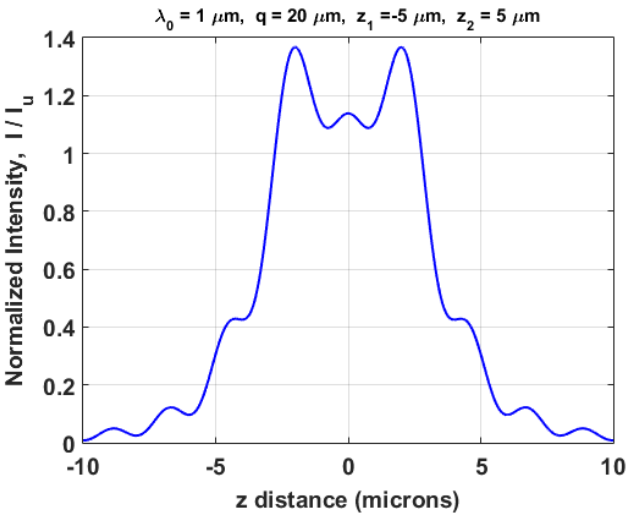
Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm

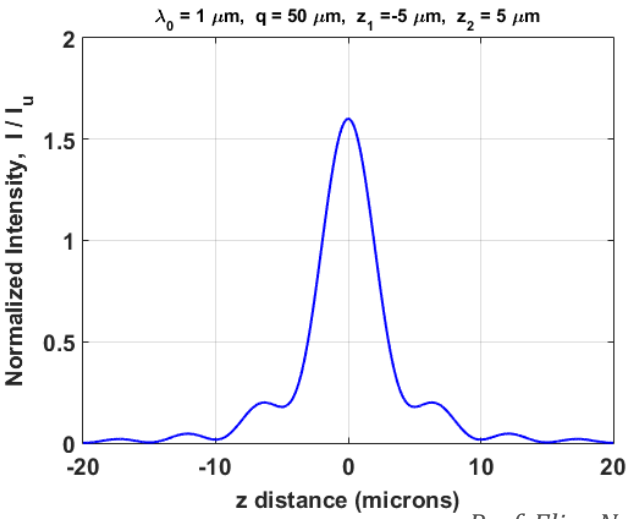
Comparison of Rayleigh-Sommerfeld (1) and Fresnel Approximation



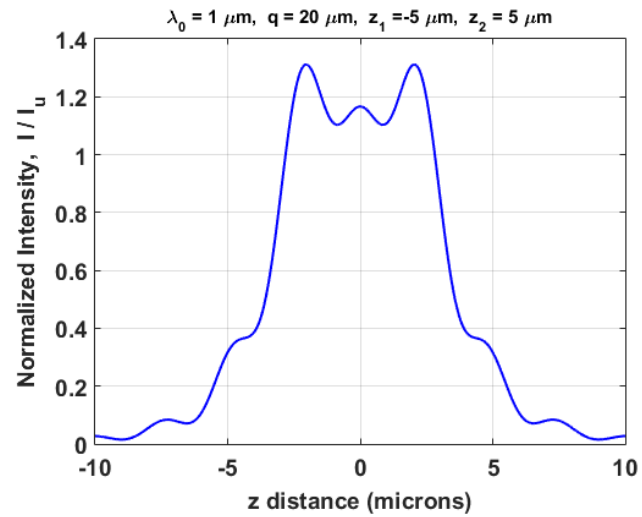
Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm

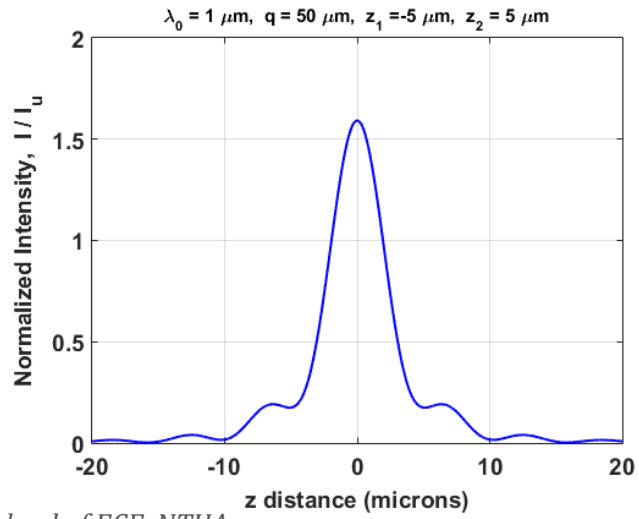
Fresnel Approximation





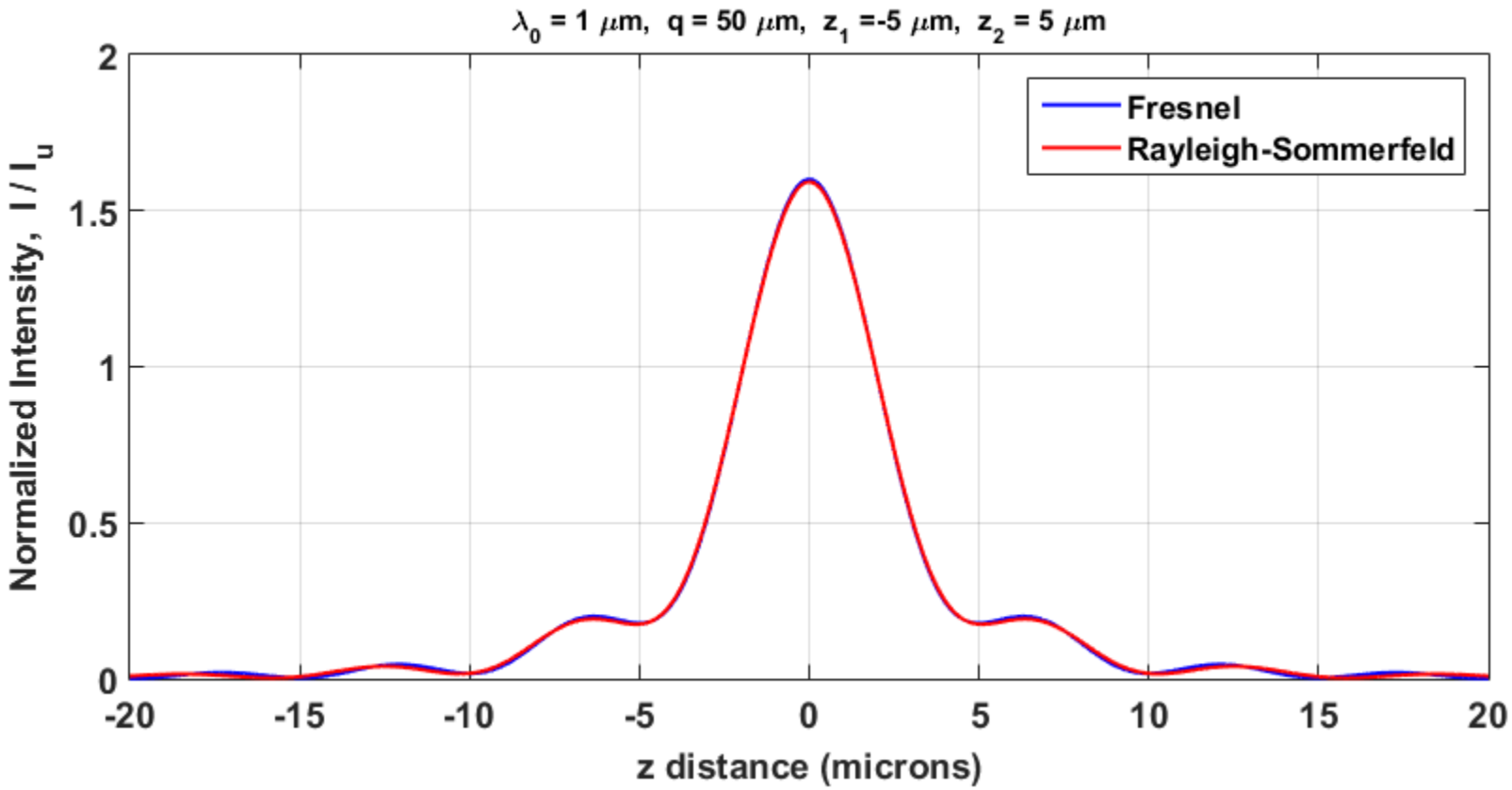
Rayleigh-Sommerfeld 1





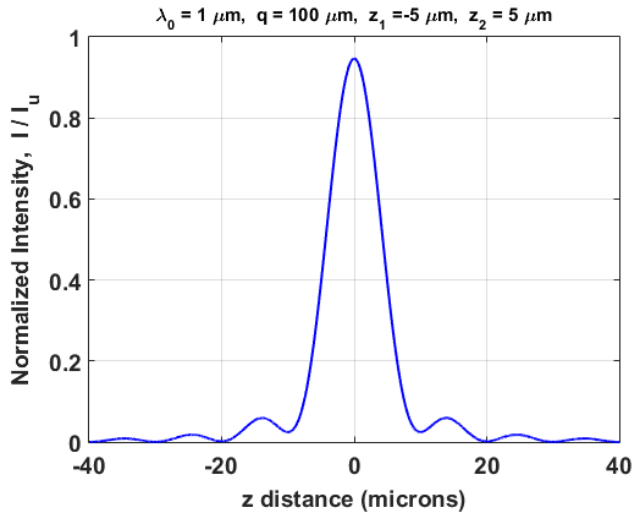
Diffraction from a Single Slit
$d = 10\mu m$, $\lambda_0 = 1\mu m$

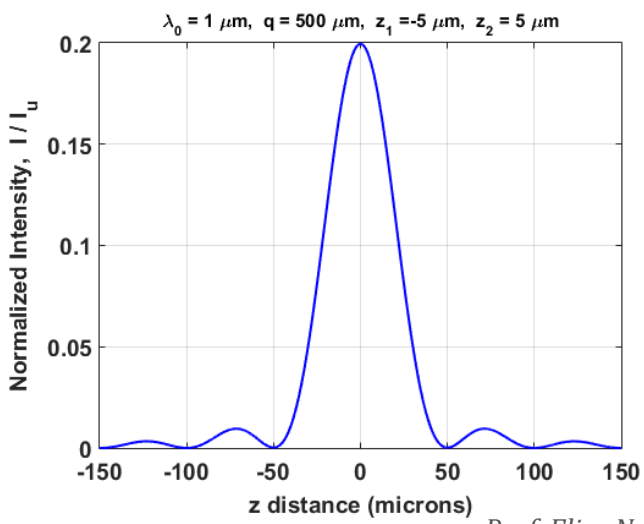
Comparison of Rayleigh-Sommerfeld (1) and Fresnel Approximation



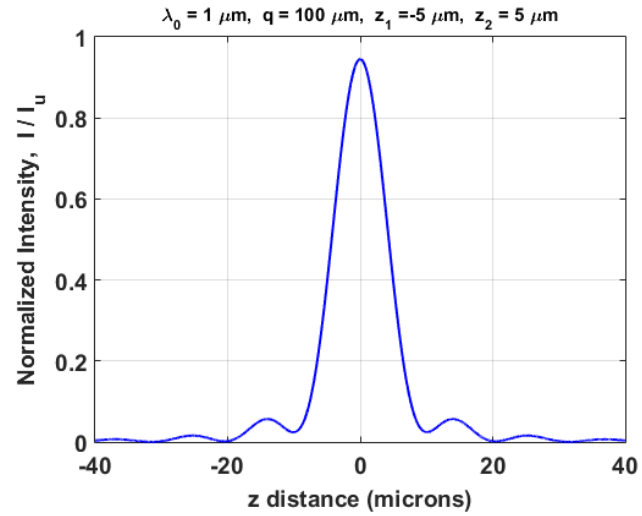
Diffraction from a Single Slit
d = 10μm, λ_0 = 1μm

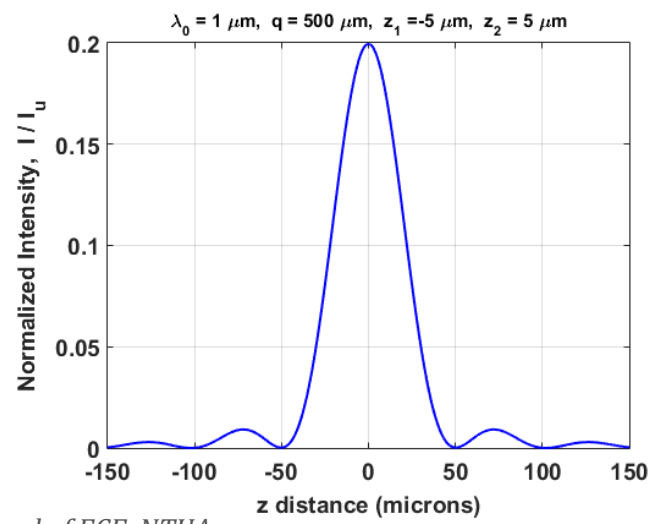
Fresnel Approximation



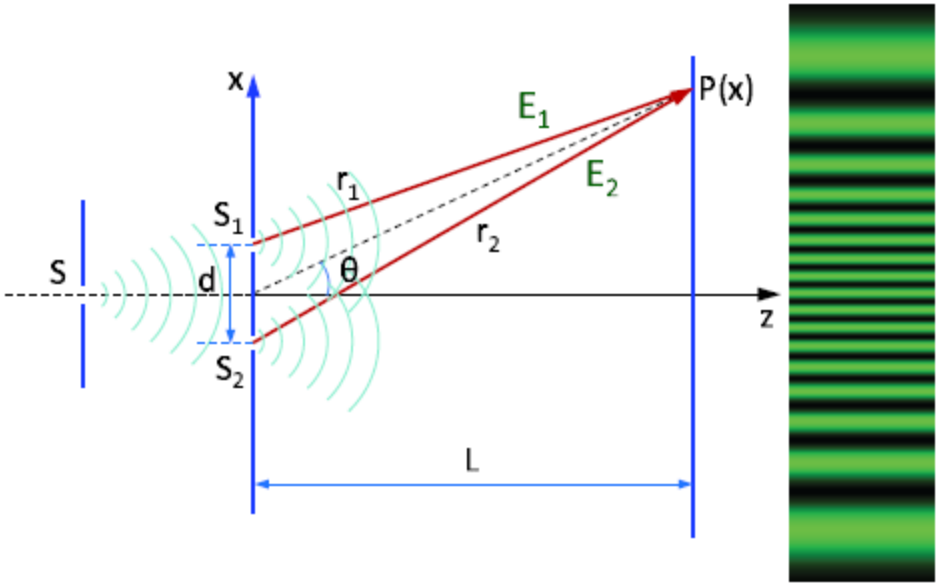


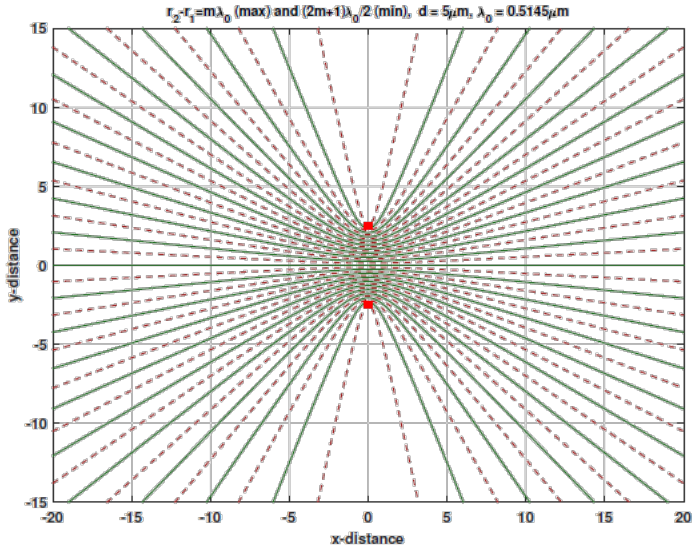
Rayleigh-Sommerfeld 1



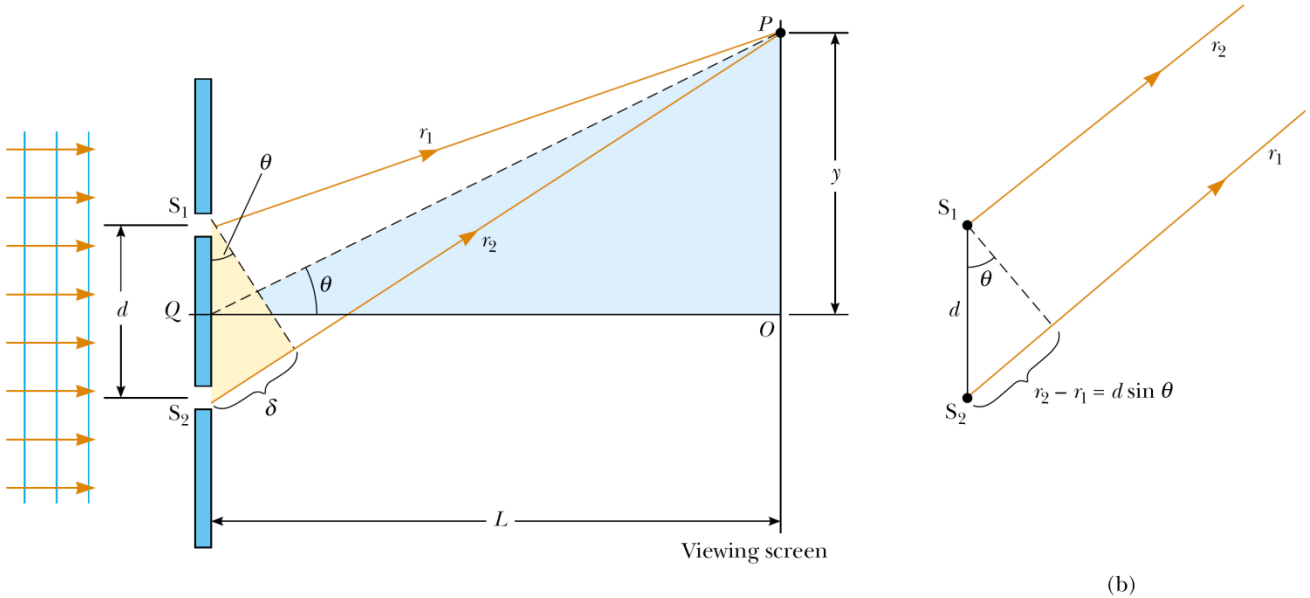


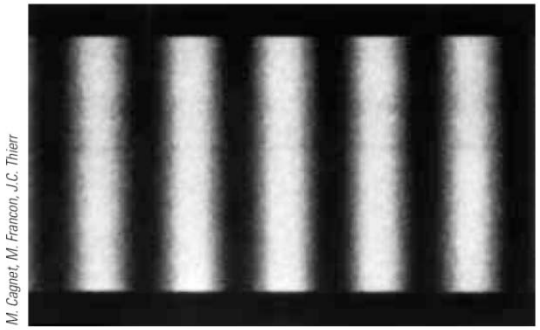
Interference from a Double Slit -Young's Experiment

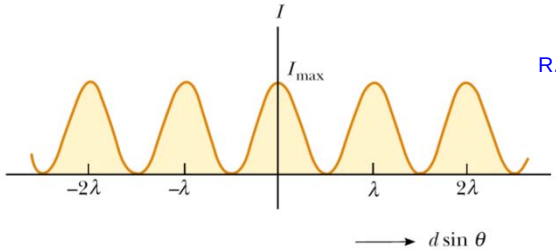




Interference from a Double Slit -Young's Experiment

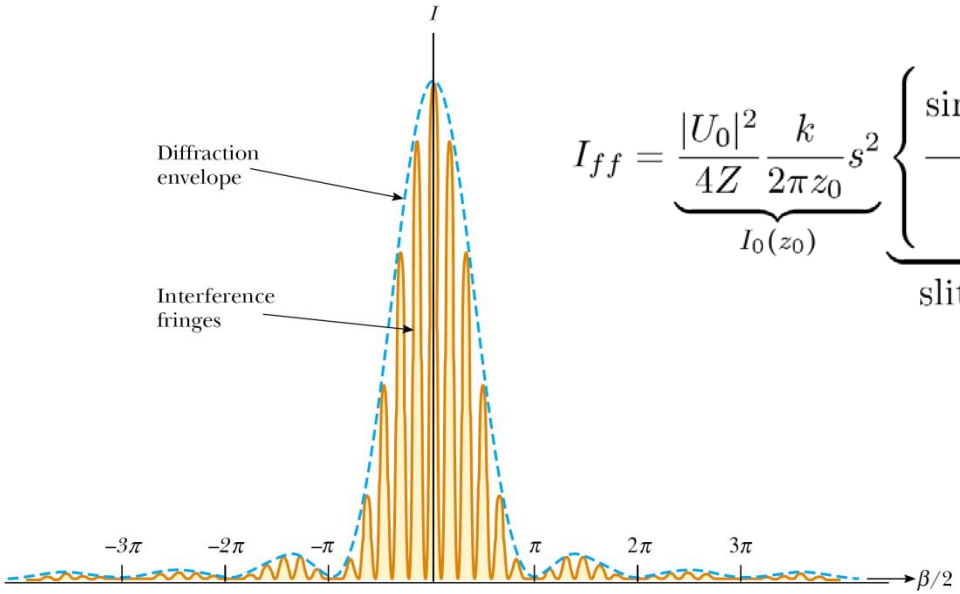






R.A. Serway and J. W. Jewett, "Physics for Scientists and Engineers", 6th Ed., Thomson Brooks/Cole, 2004

Diffraction from a Double Slit



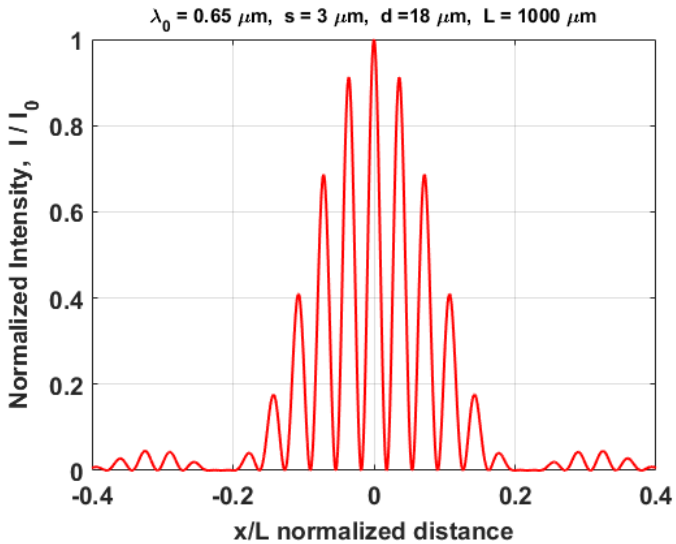
$$I_{ff} = \underbrace{\frac{|U_0|^2}{4Z}\frac{k}{2\pi z_0}s^2}_{I_0(z_0)} \underbrace{\left\{\frac{\sin\left(\frac{kx_0}{z_0}\frac{s}{2}\right)}{\frac{kx_0}{z_0}\frac{s}{2}}\right\}^2}_{\text{slit diffraction}} \underbrace{\left\{\frac{\sin\left(N\frac{kx_0}{z_0}\frac{\Lambda}{2}\right)}{\sin\left(\frac{kx_0}{z_0}\frac{\Lambda}{2}\right)}\right\}^2}_{\text{array factor}}$$

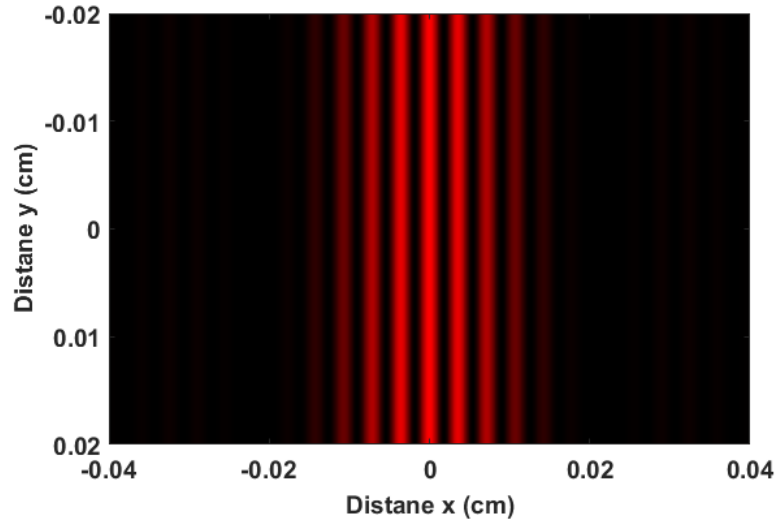
Courtesy of Central Scientific Company

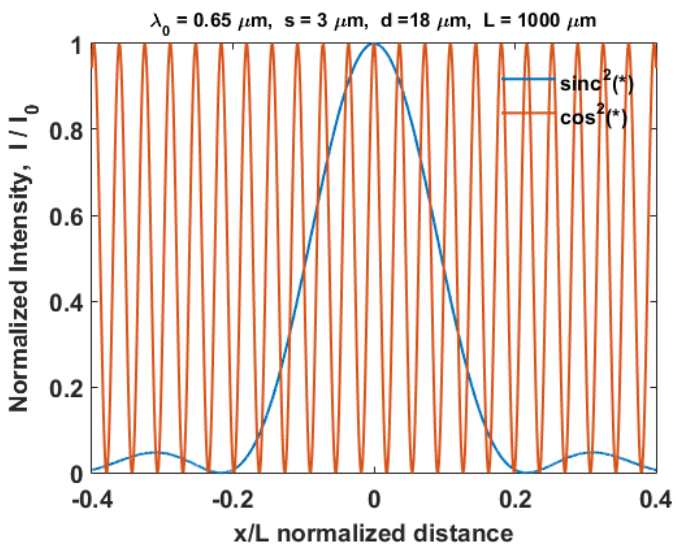
The combined effects of two-slit and single-slit interference. This is the pattern produced when 650nm light waves pass through two 3.0μm slits that are 18μm apart. Notice how the diffraction pattern acts as an "envelope" and controls the intensity of the regularly spaced interference maxima.

R.A. Serway and J. W. Jewett, "Physics for Scientists and Engineers", 6th Ed., Thomson Brooks/Cole, 2004

Diffraction from a Double Slit

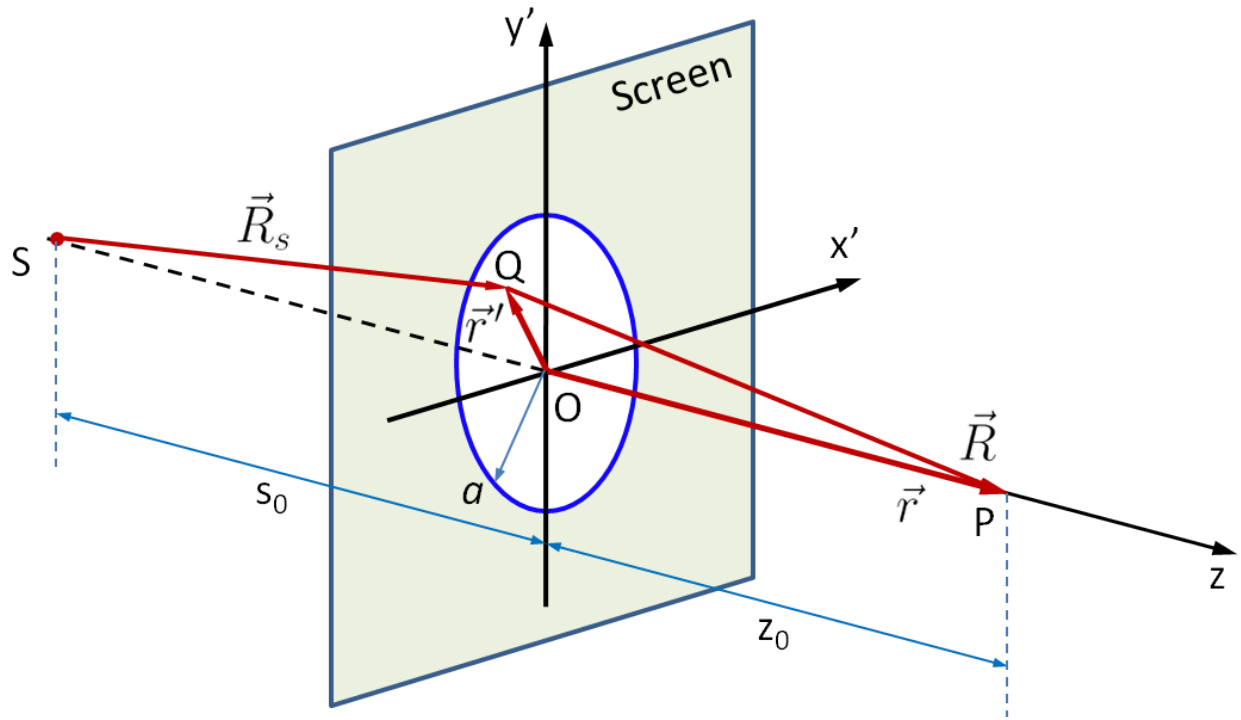






The combined effects of two-slit and single-slit interference. This is the pattern produced when 650-nm light waves pass through two 3.0µm slits that are18µm apart. Notice how the diffraction pattern acts as an "envelope" and controls the intensity of the regularly spaced interference maxima.

Diffraction from a Circular Aperture



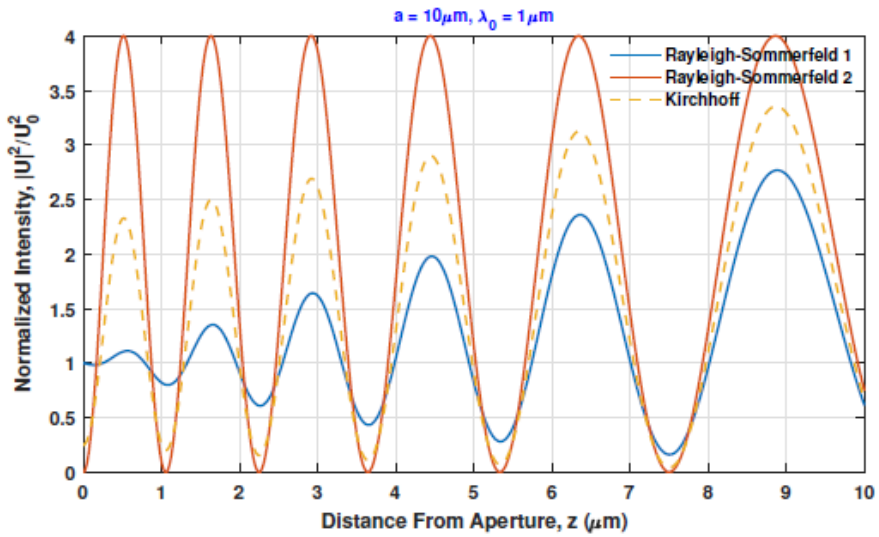
$$U(x,y,z) = \iint \frac{e^{-jkR}}{2\pi R}\left(jk+\frac{1}{R}\right)\frac{z}{R}U_i(x',y')dx'dy', \qquad \text{3D Green's Function}$$

$$R = \sqrt{(x-x')^2+(y-y')^2+z^2},$$

$$U(x,y,z) = \iint \frac{e^{-jkR}}{2\pi R}\frac{\partial U_i(x',y')}{\partial n}dx'dy', \qquad \text{3D Green's Function}$$

$$R = \sqrt{(x-x')^2+(y-y')^2+z^2},$$

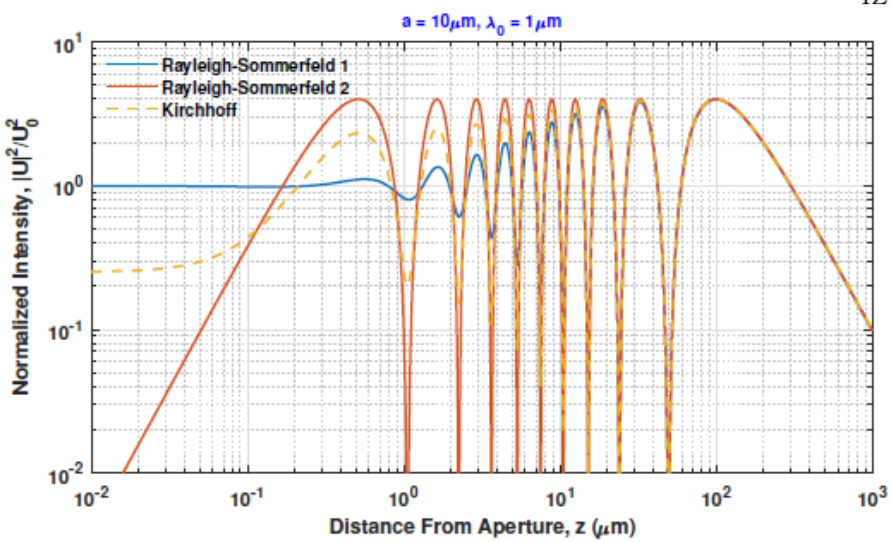
Diffraction from a Circular Aperture



(a)

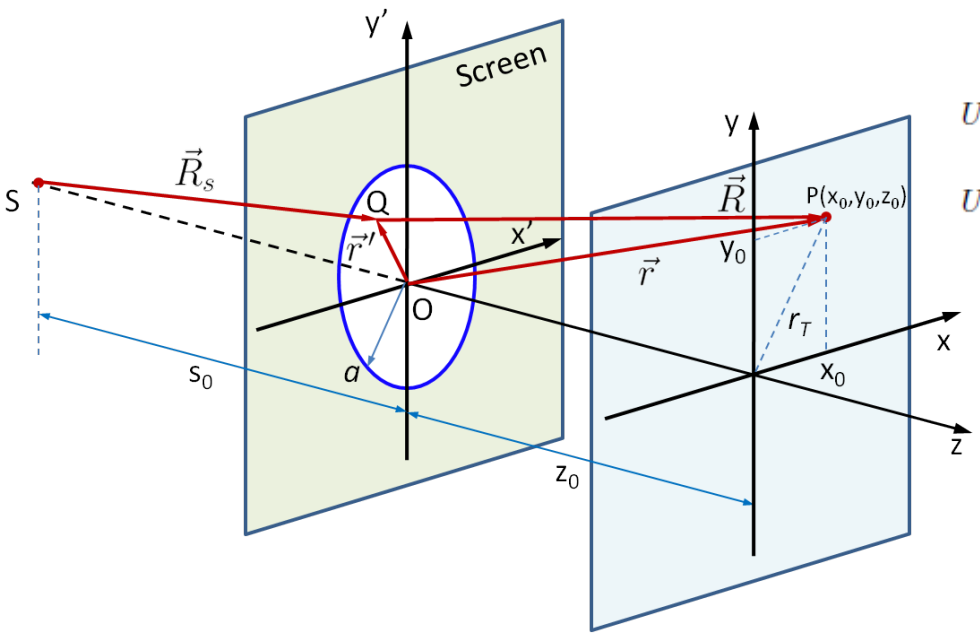
$$I_{RS_1}(z_0) = \frac{|U_0|^2}{4Z}\left[1+\frac{z_0^2}{a^2+z_0^2}-\frac{2z_0}{\sqrt{a^2+z_0^2}}\cos\left(k\sqrt{a^2+z_0^2}-kz_0\right)\right] = I_0\left[1+\frac{z_0^2}{a^2+z_0^2}-\frac{2z_0}{\sqrt{a^2+z_0^2}}\cos\left(k\sqrt{a^2+z_0^2}-kz_0\right)\right],$$

$$I_{RS_2}(z_0) = \frac{|U_0|^2}{4Z}\left[2-2\cos\left(k\sqrt{a^2+z_0^2}-kz_0\right)\right] = I_0\left[2-2\cos\left(k\sqrt{a^2+z_0^2}-kz_0\right)\right]$$



(b)

Diffraction from a Circular Aperture



$$U_{RS_1}(x_0,y_0,z_0) = \int_0^a \int_0^{2\pi} U_0 \left(jk + \frac{1}{R}\right) \frac{\exp(-jkR)}{2\pi R} \frac{z_0}{R} r'_T dr'_T d\phi',$$

$$U_{RS_2}(x_0,y_0,z_0) = \int_0^a \int_0^{2\pi} \frac{\exp(-jkR)}{2\pi R} (jkU_0) r'_T dr'_T d\phi',$$

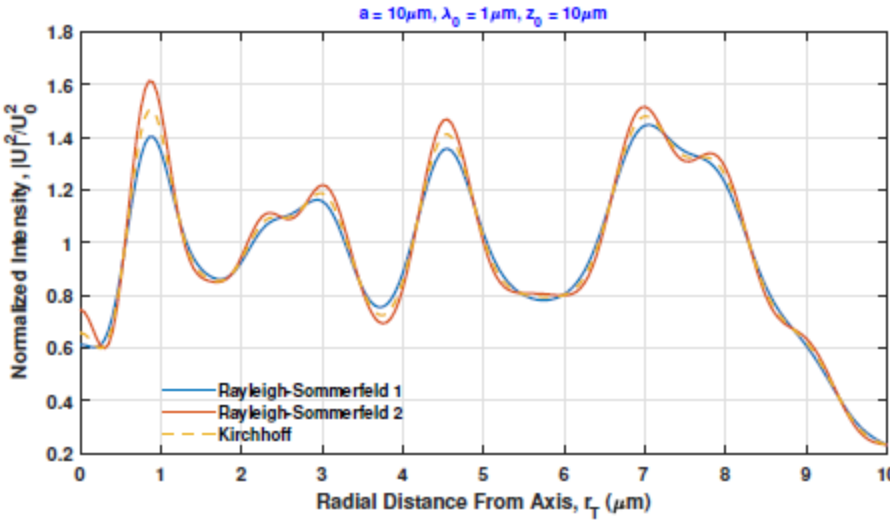
$$\begin{aligned} U(x_0,y_0,z_0) &= jk\frac{\exp(-jkz_0)}{2\pi z_0} \int_0^a \int_0^{2\pi} \exp\left\{-jk\frac{r_T^2 + r_T'^2 - 2r_T r'_T \cos(\phi-\phi')}{2z_0}\right\} U_0 r'_T dr'_T d\phi' = \\ &= jk\frac{\exp(-jkz_0)}{z_0} \int_0^a \exp\left(-jk\frac{r_T^2 + r_T'^2}{2z_0}\right) U_0 J_0\left(\frac{kr_T r'_T}{z_0}\right) r'_T dr'_T, \end{aligned}$$

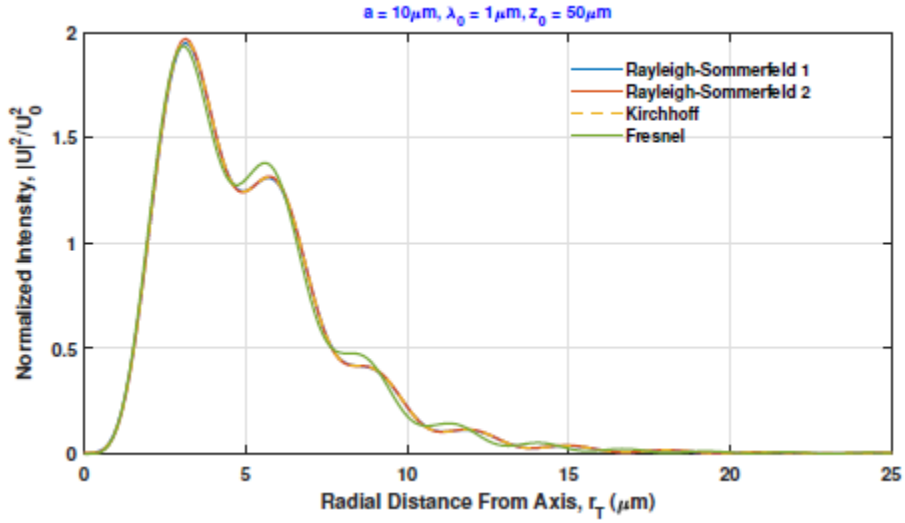
where

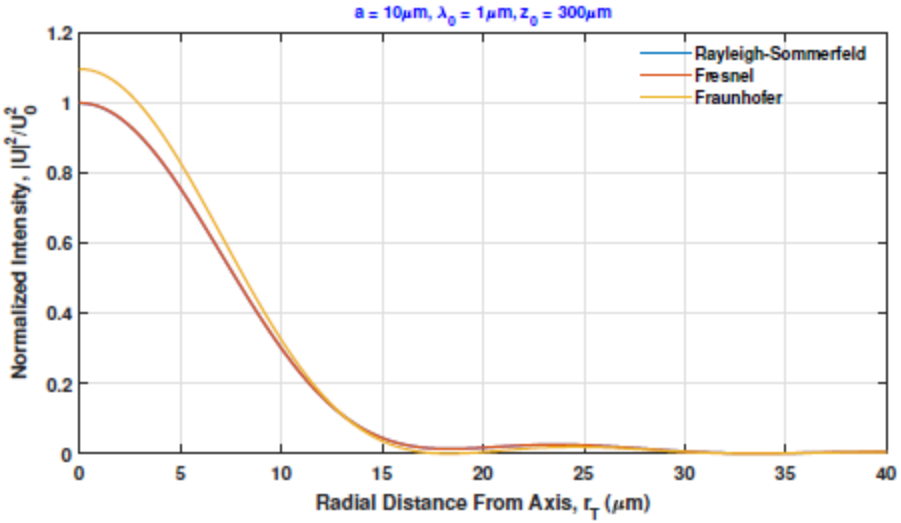
$$R = \left[r_T^2 + r_T'^2 - 2r_T r'_T \cos(\phi-\phi') + z_0^2\right]^{1/2}, \qquad \text{and,}$$

$$r_T = (x_0^2 + y_0^2)^{1/2}, \qquad \phi = \tan^{-1}\left(\frac{y_0}{x_0}\right).$$

Diffraction from a Circular Aperture







Diffraction from a Circular Aperture

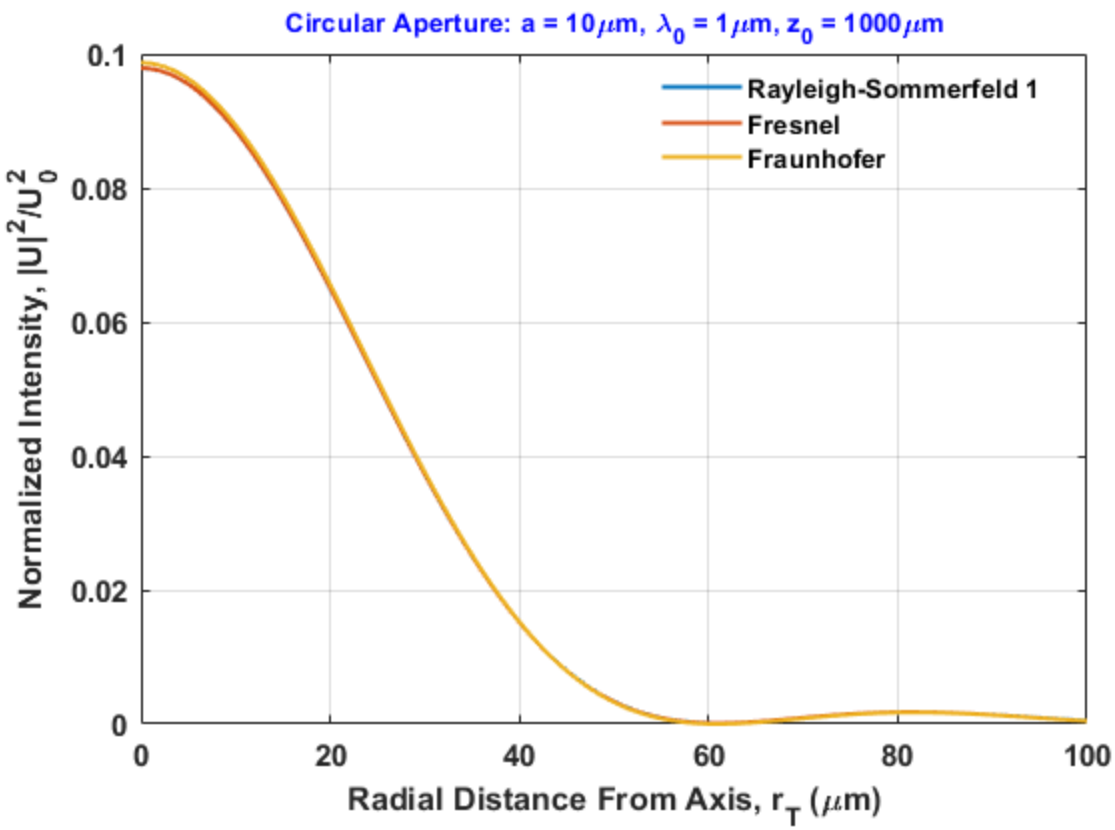
Fraunhofer Approximation

$$
\begin{aligned}
U_{ff} &= jk\frac{\exp(-jkz_0)}{2\pi z_0}\exp\left(-jk\frac{r_T^2}{2z_0}\right)\int_0^a\int_0^{2\pi}\left[\exp\left\{+jk\left(\frac{r_T r_T'\cos(\phi-\phi')}{z_0}\right)\right\}U_0\right]r_T' dr_T' d\phi' = \\
&= jk\frac{\exp(-jkz_0)}{2\pi z_0}\exp\left(-jk\frac{r_T^2}{2z_0}\right)U_0\int_0^a 2\pi J_0\left(\frac{kr_T r_T'}{z_0}\right)r_T' dr_T' = \\
&= jk\frac{\exp(-jkz_0)}{2\pi z_0}\exp\left(-jk\frac{r_T^2}{2z_0}\right)U_0 2\pi\frac{aJ_1\left(kar_T/z_0\right)}{kr_T/z_0} \Longrightarrow
\end{aligned}
$$

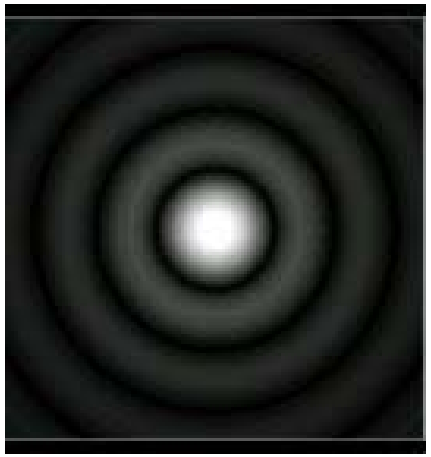
$$
\begin{aligned}
U_{ff} &= jU_0\exp(-jkz_0)\exp\left[-jk\frac{r_T^2}{2z_0}\right]\frac{aJ_1\left(kar_T/z_0\right)}{r_T}, \qquad \text{and the intensity is} \\
I_{ff} &= \frac{|U_0|^2}{4Z}\frac{k^2a^4}{z_0^2}\left(\frac{J_1\left(kar_T/z_0\right)}{kar_T/z_0}\right)^2 = I_0\left(\frac{2J_1\left(kar_T/z_0\right)}{kar_T/z_0}\right)^2,
\end{aligned}
$$

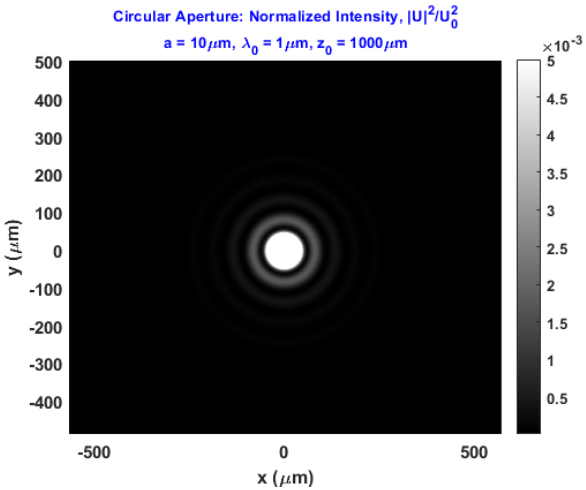
Diffraction from a Circular Aperture

Fraunhofer Regime

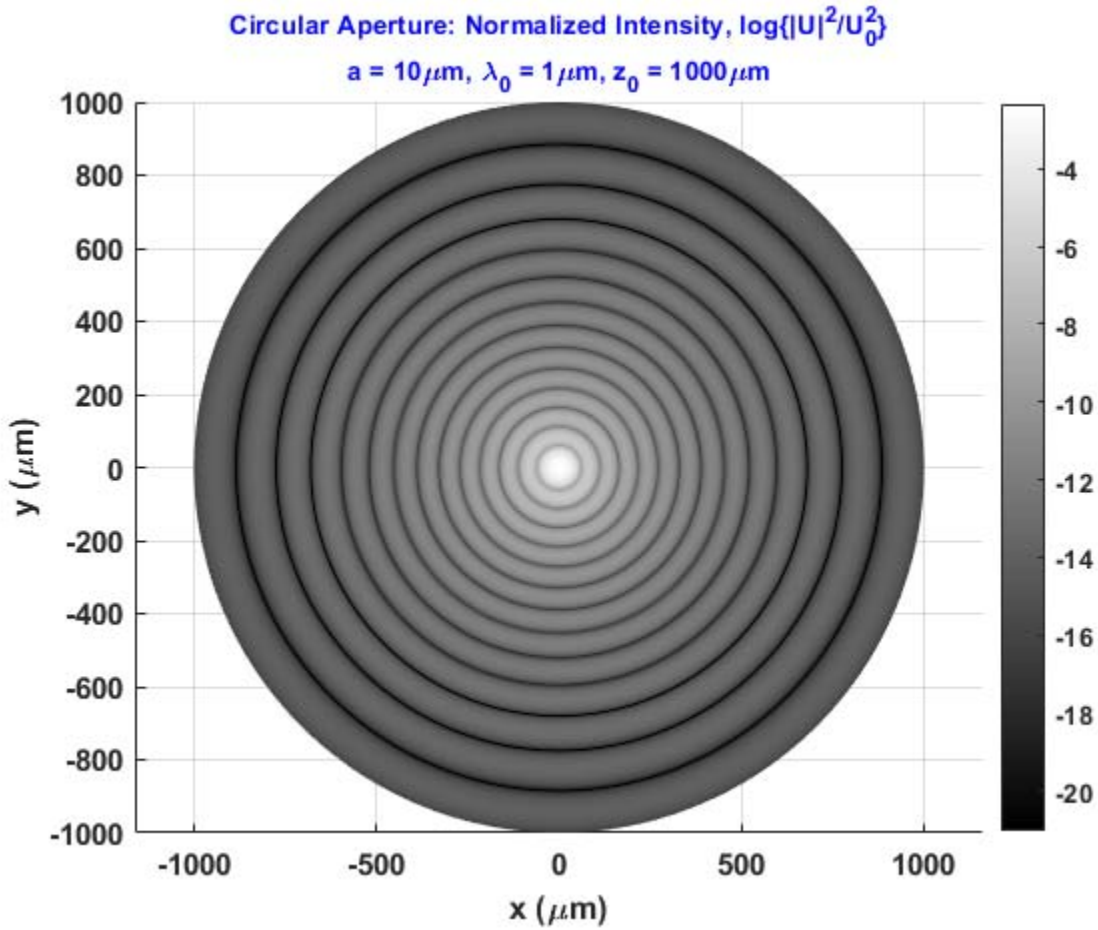


Airy Pattern

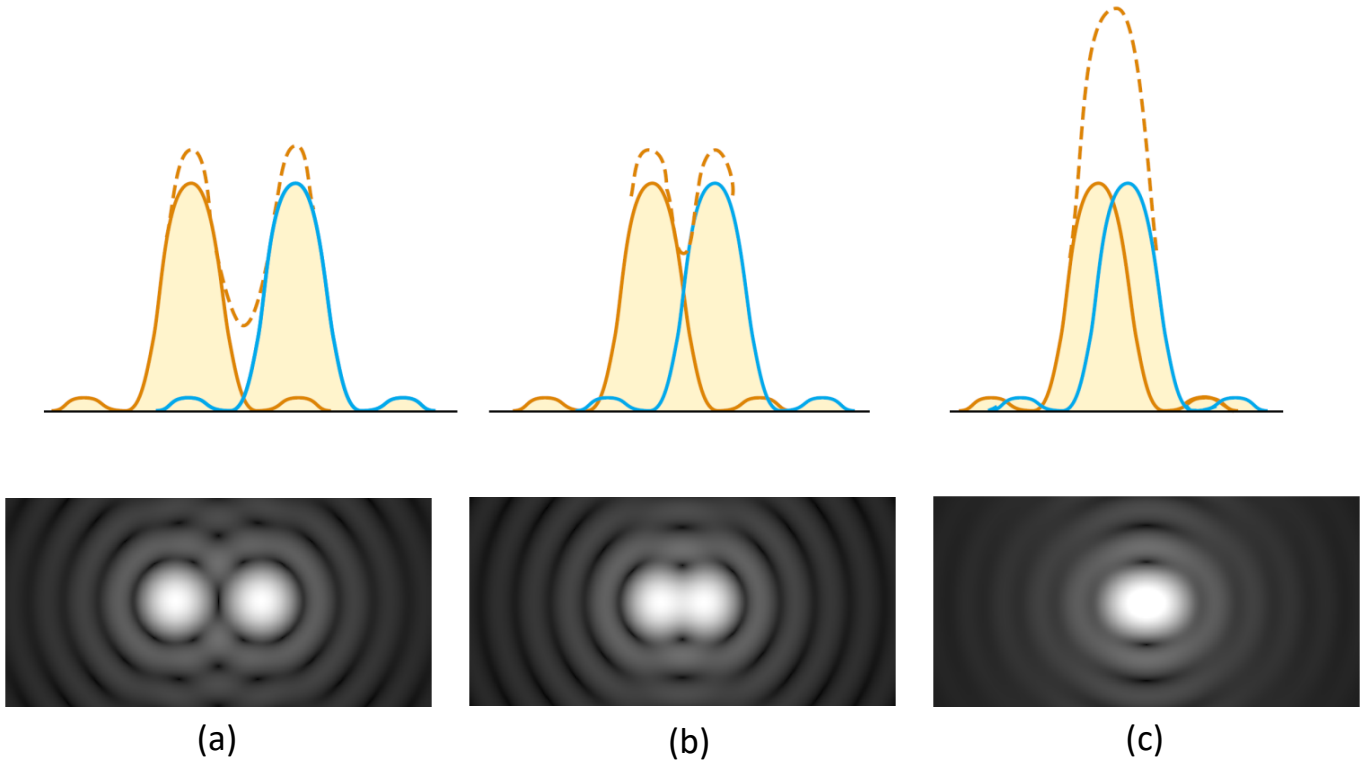




Diffraction from a Circular Aperture



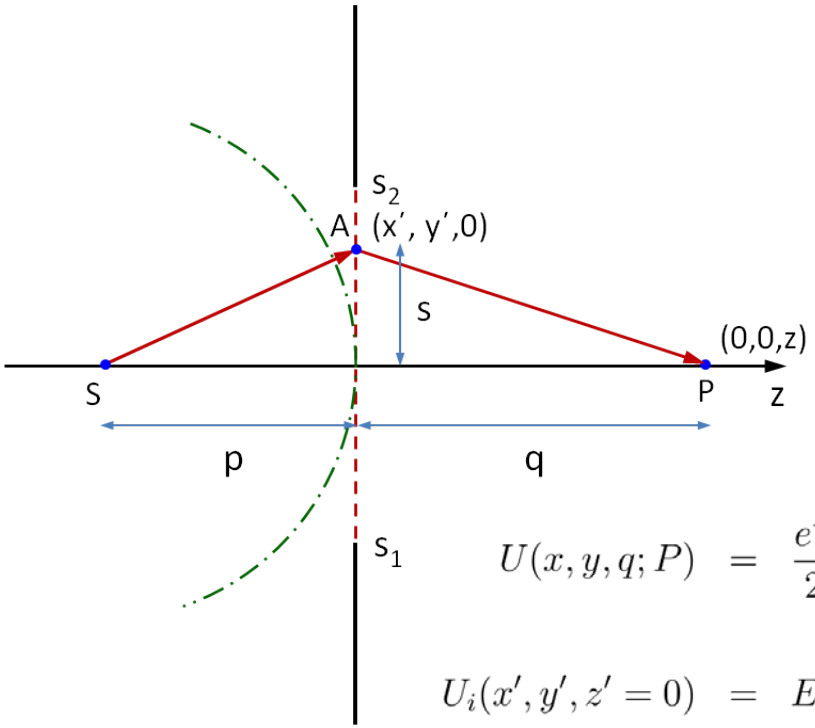
Rayleigh's Criterion



Individual diffraction patterns of two point sources (solid curves) and the resultant patterns (dashed curves) for various angular separations of the sources. In each case, the dashed curve is the sum of the two solid curves. (a) The sources are far apart, and the patterns are well resolved. (b) The sources are closer together such that the angular separation just satisfies Rayleigh's criterion, and the patterns are just resolved. (c) The sources are so close together that the patterns are not resolved.

https://en.wikipedia.org/wiki/Angular_resolution

Fresnel Diffraction – Cornu Spiral



Fresnel Approximation

$$U(x,y,q;P) = \frac{e^{-jkq}}{2\pi q} jk \iint \exp\left[-jk\frac{(x-x')^2+(y-y')^2}{2q}\right] U_i(x',y',z'=0)dx'dy'$$

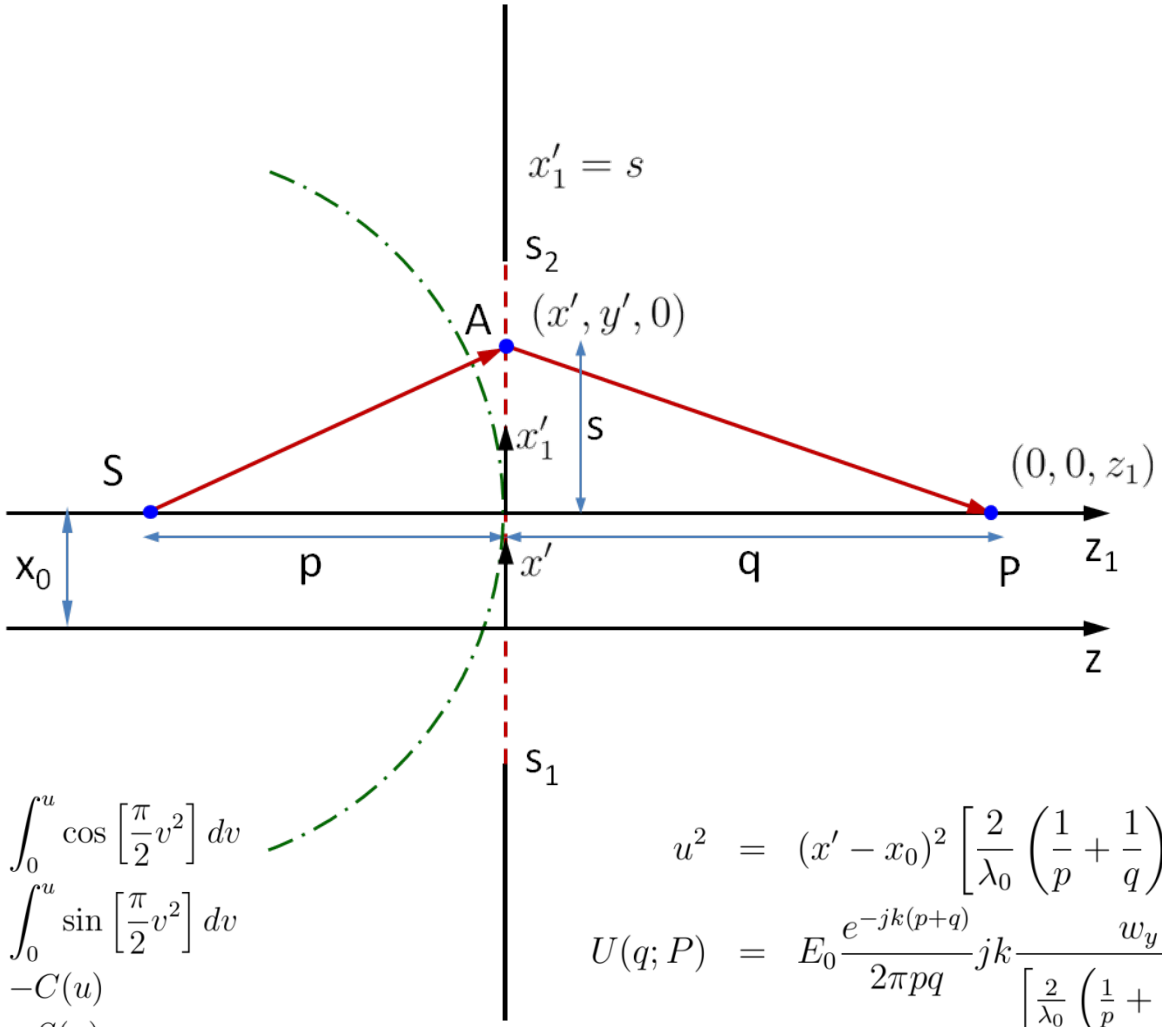
$$U_i(x',y',z'=0) = E_0 \frac{e^{-jk(x'^2+y'^2+p^2)^{1/2}}}{(x'^2+y'^2+p^2)^{1/2}} \simeq E_0 \frac{e^{-jkp}}{p} e^{-jk\left(\frac{x'^2+y'^2}{2p}\right)}$$

$$U(0,0,q;P) \simeq E_0 \frac{e^{-jk(p+q)}}{2\pi pq} jk \iint \exp\left[-jk\frac{x'^2+y'^2}{2}\left(\frac{1}{p}+\frac{1}{q}\right)\right] dx'dy'$$

$$u^2 = s^2\left[\frac{2}{\lambda_0}\left(\frac{1}{p}+\frac{1}{q}\right)\right]$$

$$U(q;P) = E_0 \frac{e^{-jk(p+q)}}{2\pi pq} jk \frac{w_y}{\left[\frac{2}{\lambda_0}\left(\frac{1}{p}+\frac{1}{q}\right)\right]^{1/2}} \int_{u_1}^{u_2} \exp\left[-j\frac{\pi}{2}u^2\right] du$$

Fresnel Diffraction – Cornu Spiral



$$C(u) = \int_0^u \cos\left[\frac{\pi}{2}v^2\right] dv$$

$$S(u) = \int_0^u \sin\left[\frac{\pi}{2}v^2\right] dv$$

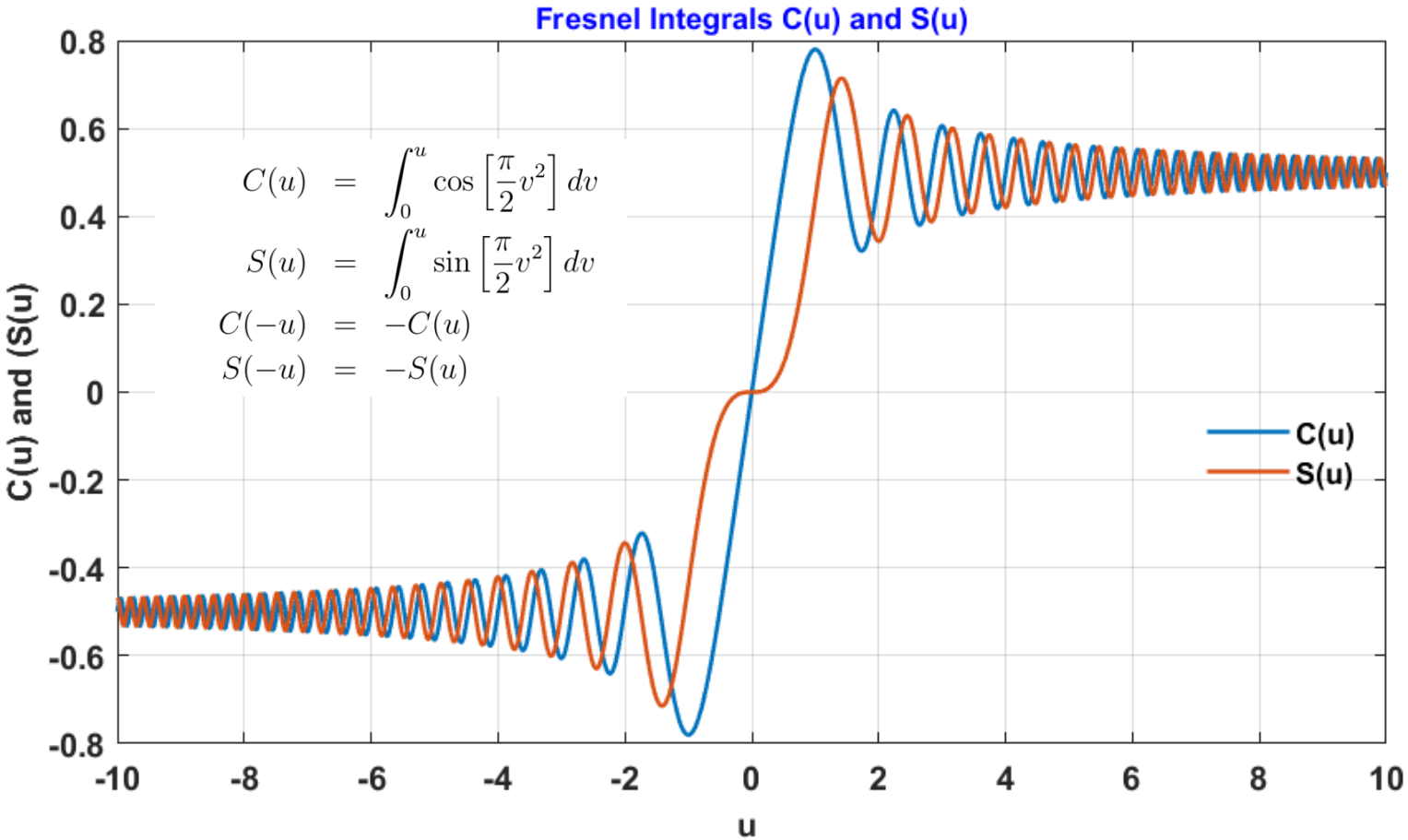
$$C(-u) = -C(u)$$

$$S(-u) = -S(u)$$

$$u^2 = (x' - x_0)^2 \left[\frac{2}{\lambda_0}\left(\frac{1}{p} + \frac{1}{q}\right)\right]$$

$$U(q; P) = E_0 \frac{e^{-jk(p+q)}}{2\pi pq} jk \frac{w_y}{\left[\frac{2}{\lambda_0}\left(\frac{1}{p} + \frac{1}{q}\right)\right]^{1/2}} \int_{u_1}^{u_2} \exp\left[-j\frac{\pi}{2}u^2\right] du$$

Fresnel Integrals



$$C(u) = \int_0^u \cos\left[\frac{\pi}{2}v^2\right] dv$$

$$S(u) = \int_0^u \sin\left[\frac{\pi}{2}v^2\right] dv$$

$$C(-u) = -C(u)$$

$$S(-u) = -S(u)$$

Cornu Spiral

$$C(u) = \int_0^u \cos\left[\frac{\pi}{2}v^2\right] dv$$

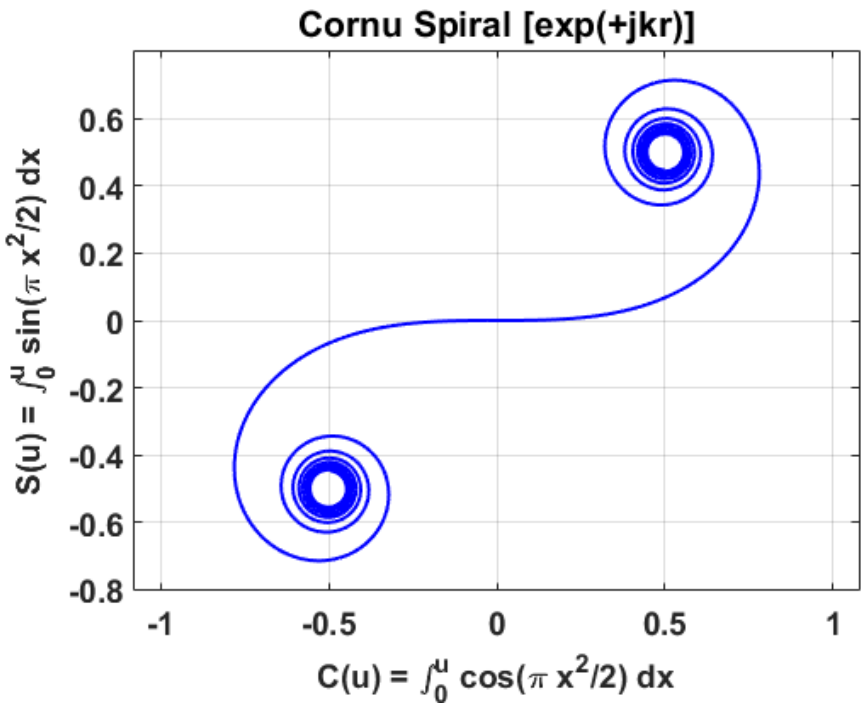
$$S(u) = \int_0^u \sin\left[\frac{\pi}{2}v^2\right] dv$$

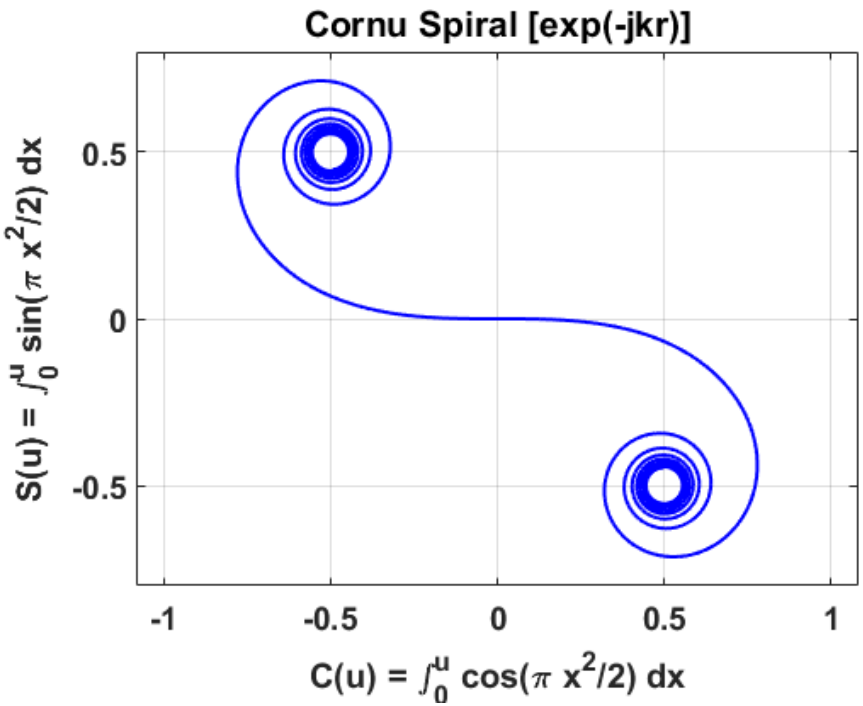
$$E(P) = E_0\Big[[C(u_2) - C(u_1)] \pm j[(S(u_2) - S(u_1)]\Big]$$

$$I(P) = |E_0|^2\Big[[C(u_2) - C(u_1)]^2 + [(S(u_2) - S(u_1)]^2\Big]$$

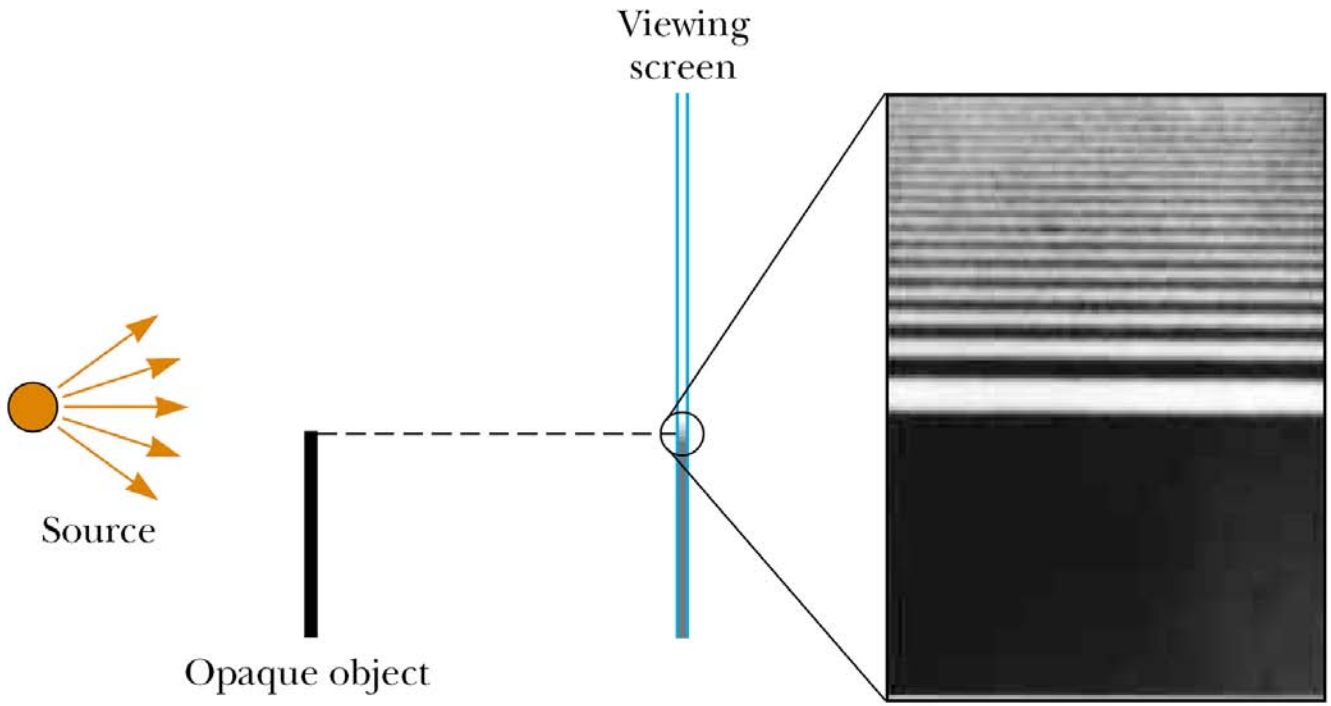
$$E(P) = E_{unobstructed} = E_u = E_0[1 \pm j]$$

$$I(P) = |E_{unobstructed}|^2 = |E_u|^2 = 2|E_0|^2$$





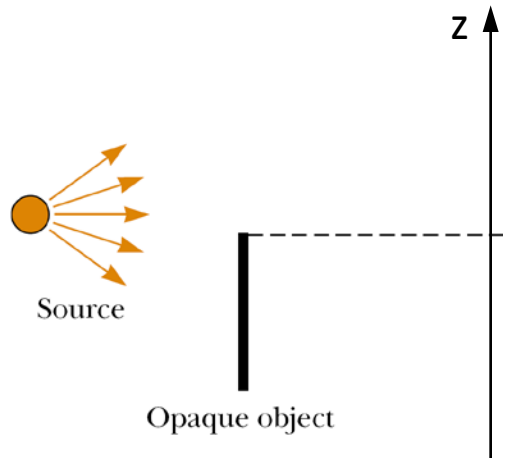
Diffraction by an Edge

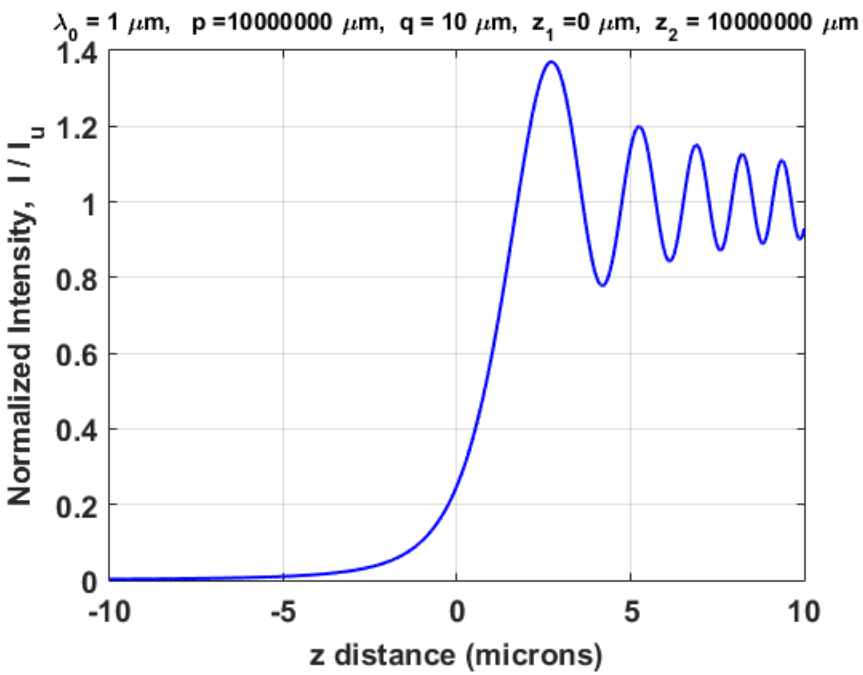


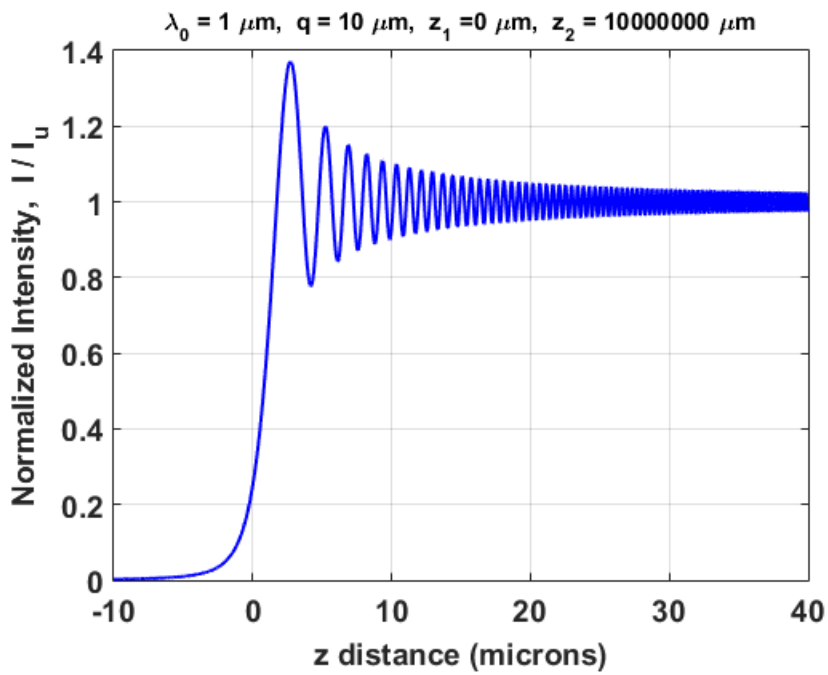
Light from a small source passes by the edge of an opaque object and continues on to a screen. A diffraction pattern consisting of bright and dark fringes appears on the screen in the region above the edge of the object.

R.A. Serway and J. W. Jewett, "Physics for Scientists and Engineers", 6th Ed., Thomson Brooks/Cole, 2004

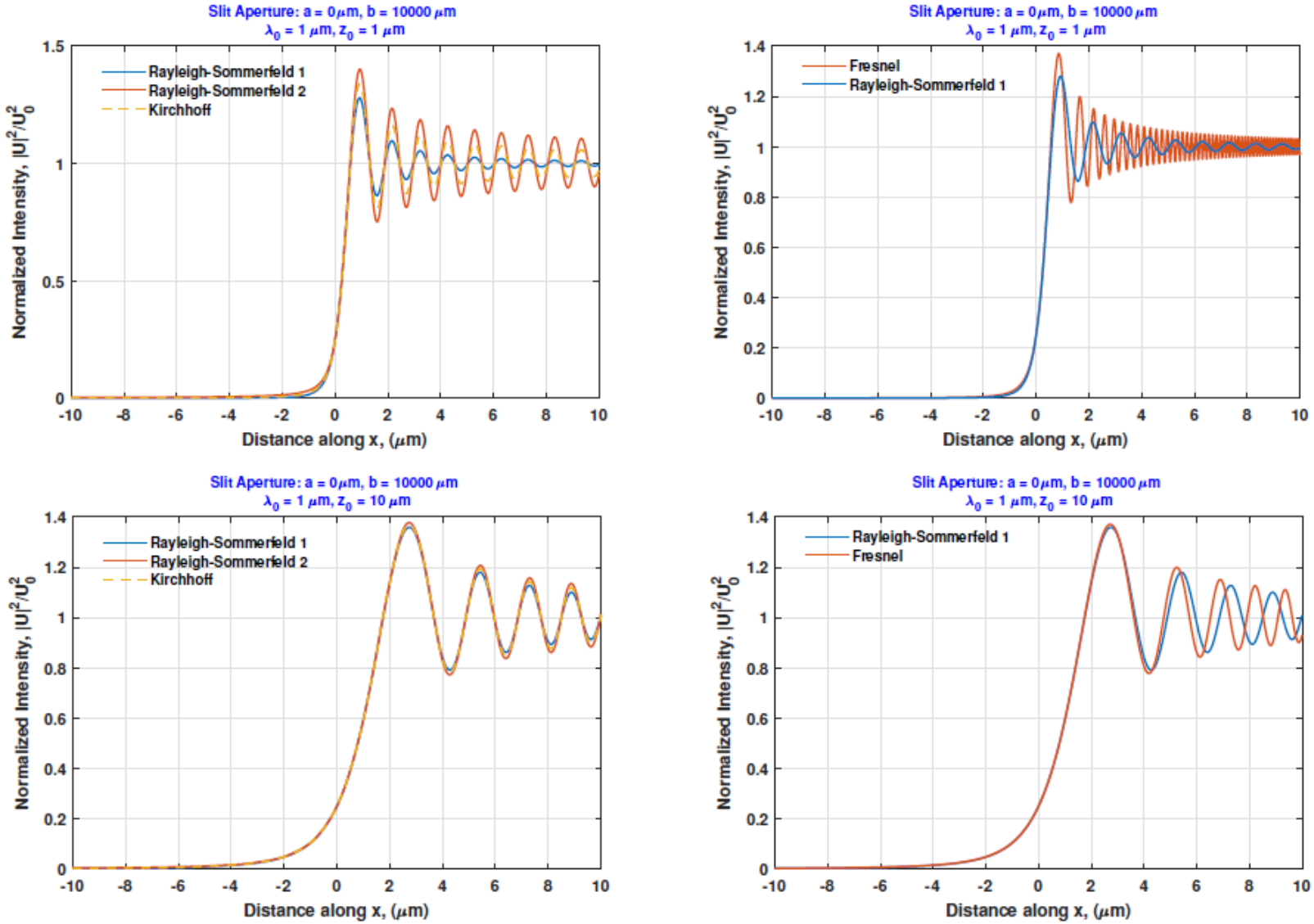
Diffraction by an Edge



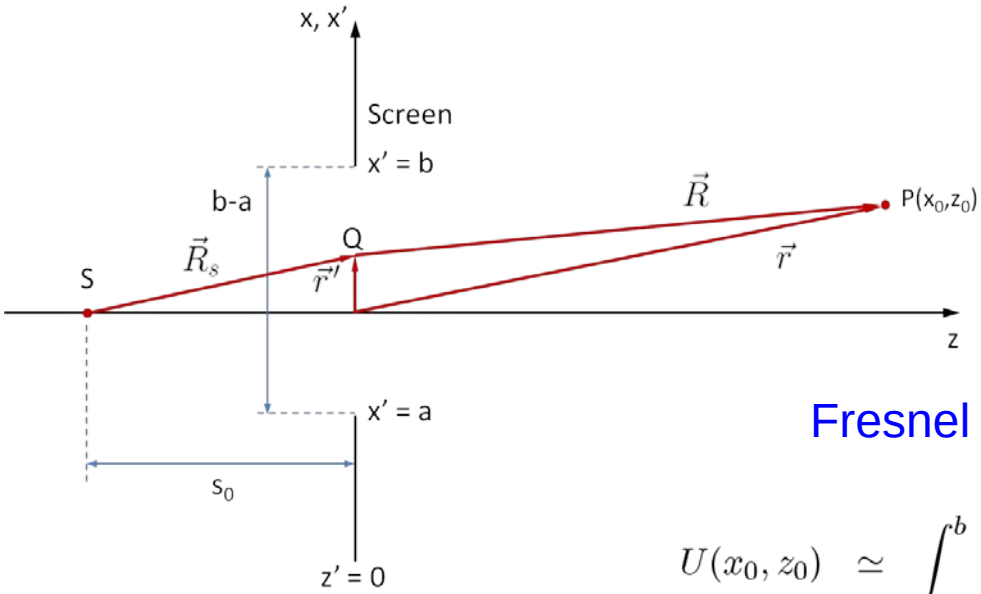




Diffraction by an Edge



Cornu Spiral: 1D Slit



Fresnel Approximation - Incident Plane Wave

$$U(x_0, z_0) \simeq \int_a^b \left[U_{inc} \frac{k}{2j} \sqrt{\frac{2}{\pi k R_T}} \exp\left[-j \left(k R_T - \frac{3\pi}{4} \right) \right] \frac{z_0}{R_T} \right]_{z'=0} dx'$$

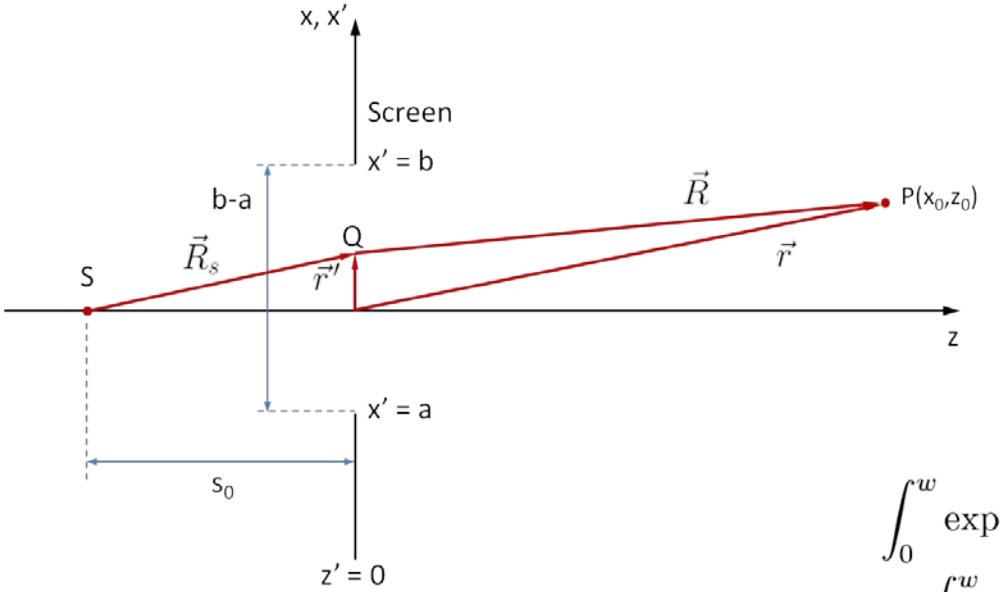
$$\simeq U_0 \frac{k}{2j} \sqrt{\frac{2}{\pi k z_0}} e^{j3\pi/4} e^{-jkz_0} \int_a^b \exp\left[-j \frac{k}{2z_0} (x_0 - x')^2 \right] dx'.$$

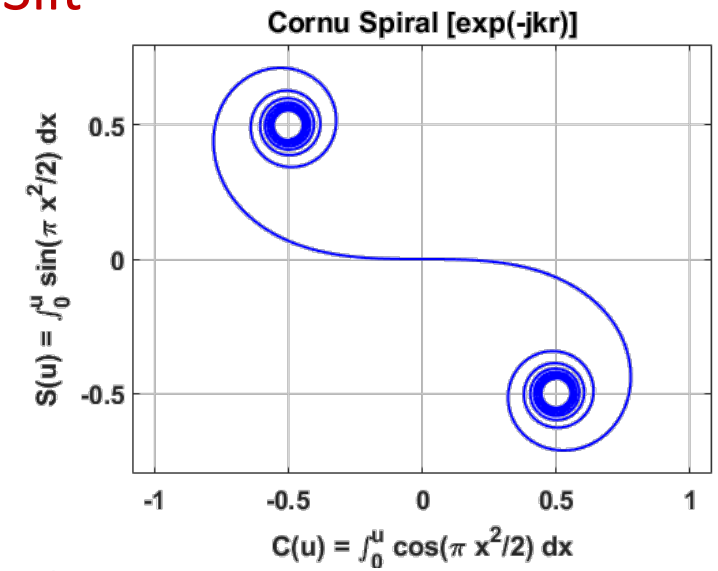
$$\frac{\pi}{2} u^2 = \frac{k}{2z_0} (x_0 - x')^2, \qquad \text{resulting in}$$

$$\int_a^b \exp\left[-j \frac{k}{2z_0} (x_0 - x')^2 \right] dx' = \sqrt{\frac{\lambda z_0}{2}} \int_{u_1}^{u_2} \exp\left(-j \frac{\pi}{2} u^2 \right) du, \qquad \text{where,}$$

$$u_1 = \sqrt{\frac{2}{\lambda z_0}} (x_0 - b), \qquad u_2 = \sqrt{\frac{2}{\lambda z_0}} (x_0 - a).$$

Cornu Spiral: 1D Slit





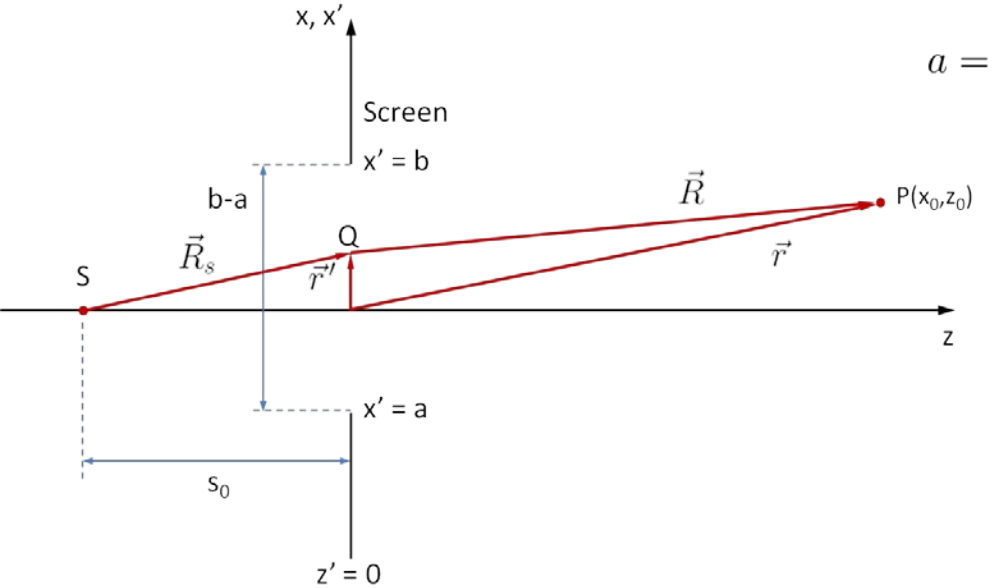
$$\int_0^w \exp\left(-j\frac{\pi}{2}u^2\right) du = C(w) - jS(w), \qquad \text{where}$$

$$C(w) = \int_0^w \cos\left(\frac{\pi}{2}u^2\right) du, \qquad S(w) = \int_0^w \sin\left(\frac{\pi}{2}u^2\right) du.$$

$$\begin{aligned} U(x_0, z_0) &= U_0 e^{j\pi/4}\sqrt{\frac{k}{2\pi z_0}}\sqrt{\frac{\lambda z_0}{2}}e^{-jkz_0}\Big\{ [C(u_2) - C(u_1)] - j\,[S(u_2) - S(u_1)] \Big\} = \\ &= \frac{U_0}{\sqrt{2}}e^{j\pi/4}e^{-jkz_0}\Big\{ [C(u_2) - C(u_1)] - j\,[S(u_2) - S(u_1)] \Big\}. \end{aligned}$$

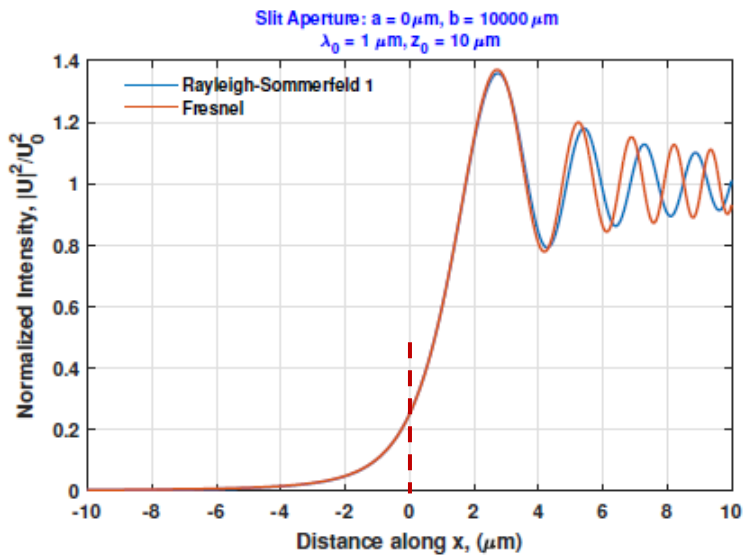
$$\begin{aligned} I(x_0, z_0) &= \frac{|U_0|^2}{4Z}\frac{1}{2}\Big\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\Big\} = \\ &= \frac{I_u}{2}\Big\{[C(u_2) - C(u_1)]^2 + [S(u_2) - S(u_1)]^2\Big\}, \end{aligned}$$

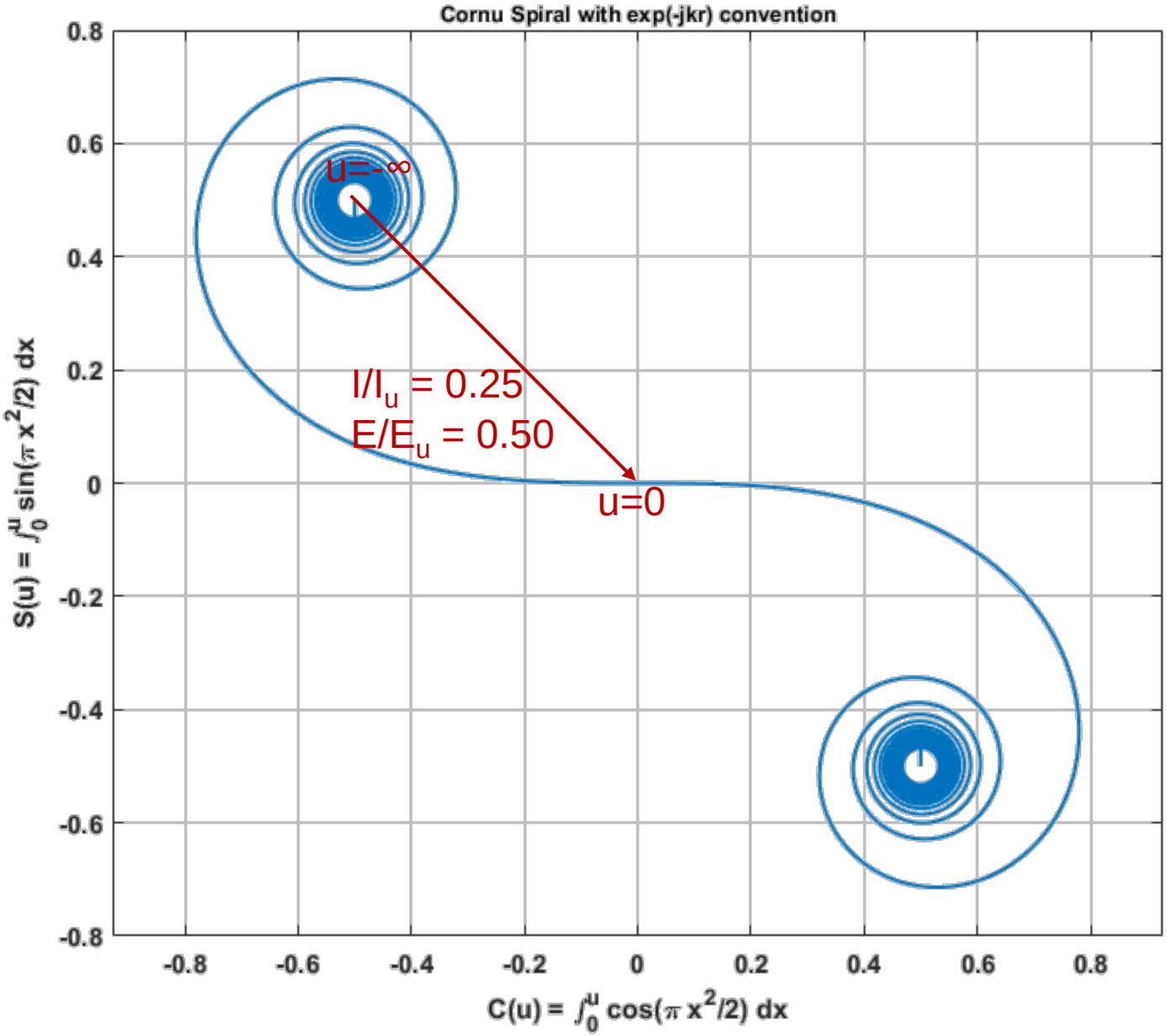
Cornu Spiral: Example of Diffraction by an Edge



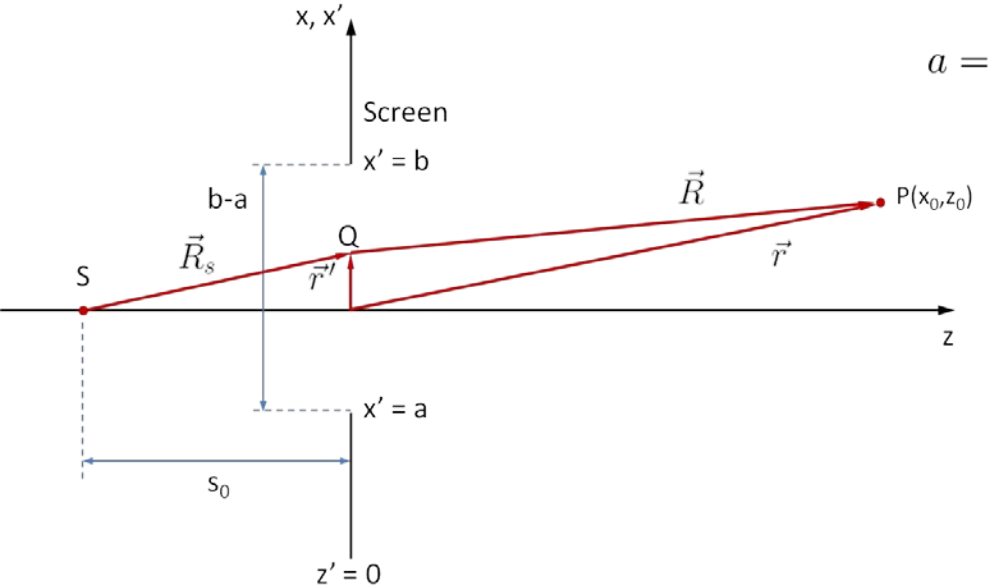
$$a = 0, \qquad b = +\infty, \qquad \lambda_0 = 1\,\mu m, \qquad z_0 = 10\,\mu m$$

$$\begin{aligned}
x_0 &= 0 \\
u_1 &= \sqrt{\frac{2}{\lambda_0 z_0}}(x_0 - b) = -\infty \\
u_2 &= \sqrt{\frac{2}{\lambda_0 z_0}}(x_0 - a) = 0 \\
C(u_1) &= -0.50 \\
S(u_1) &= -0.50 \\
C(u_2) &= 0 \\
S(u_2) &= 0 \\
U(x_0, z_0) &= \frac{U_0}{\sqrt{2}} e^{j\pi/4} e^{-jkz_0}\left[0.5 - j0.5\right] \\
|U(x_0, z_0)| &= 0.5U_0 = 0.5E_u \\
I(x_0, z_0) &= \frac{I_u}{2}\left[0.5^2 + 0.5^2\right] = \frac{I_u}{4}
\end{aligned}$$

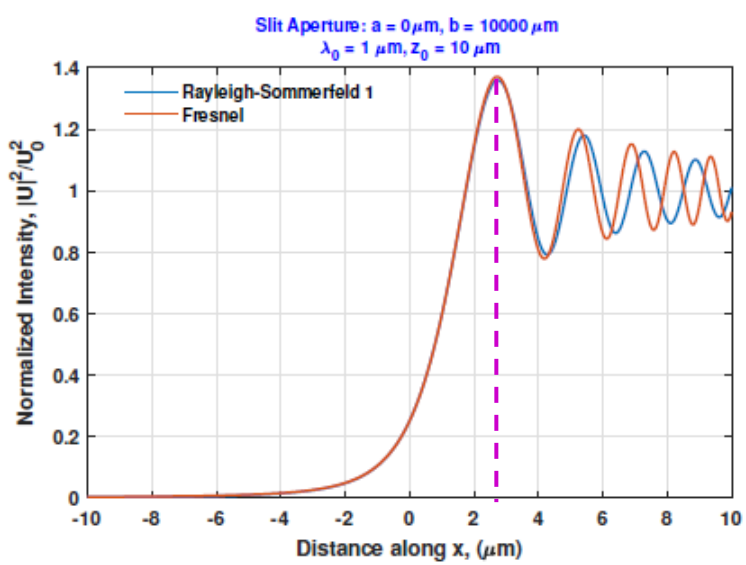



Cornu Spiral with exp(-jkr) convention
u=-∞
I/I_u = 0.25
E/E_u = 0.50
u=0
S(u) = ∫_0^u sin(π x^2/2) dx
C(u) = ∫_0^u cos(π x^2/2) dx
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

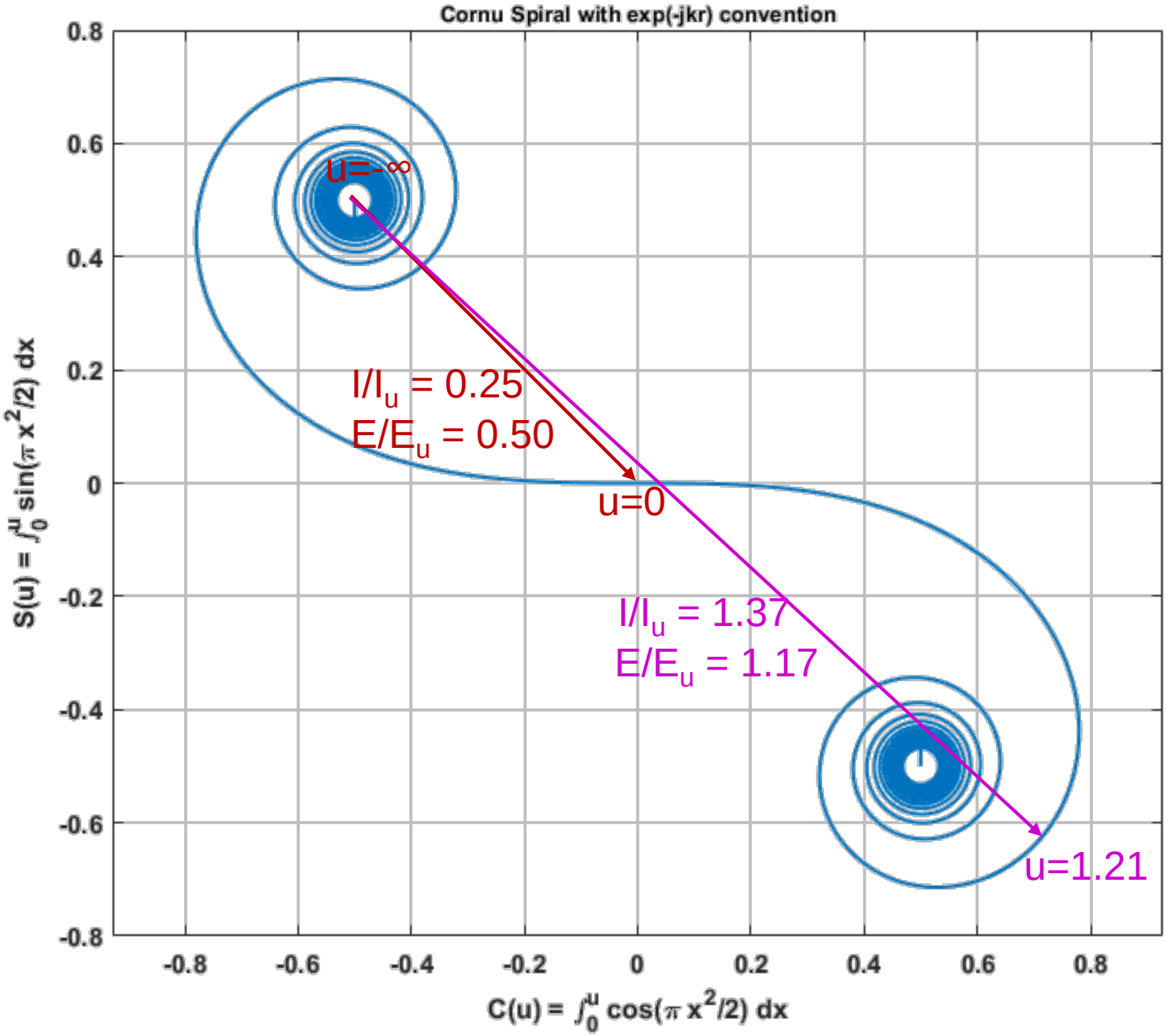
Cornu Spiral: Example of Diffraction by an Edge



$$a = 0, \qquad b = +\infty, \qquad \lambda_0 = 1\,\mu m, \qquad z_0 = 10\,\mu m$$

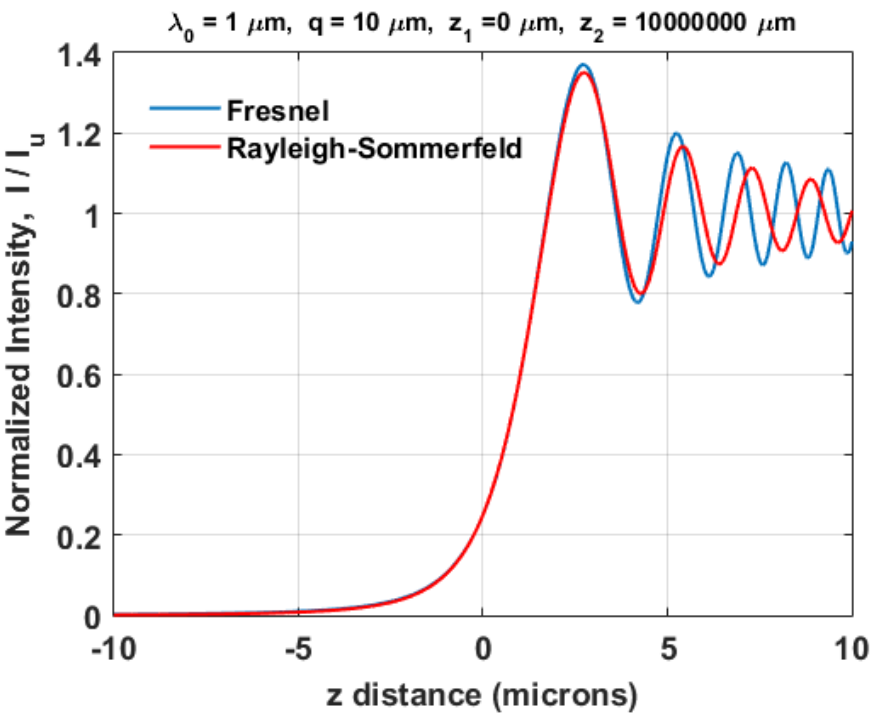


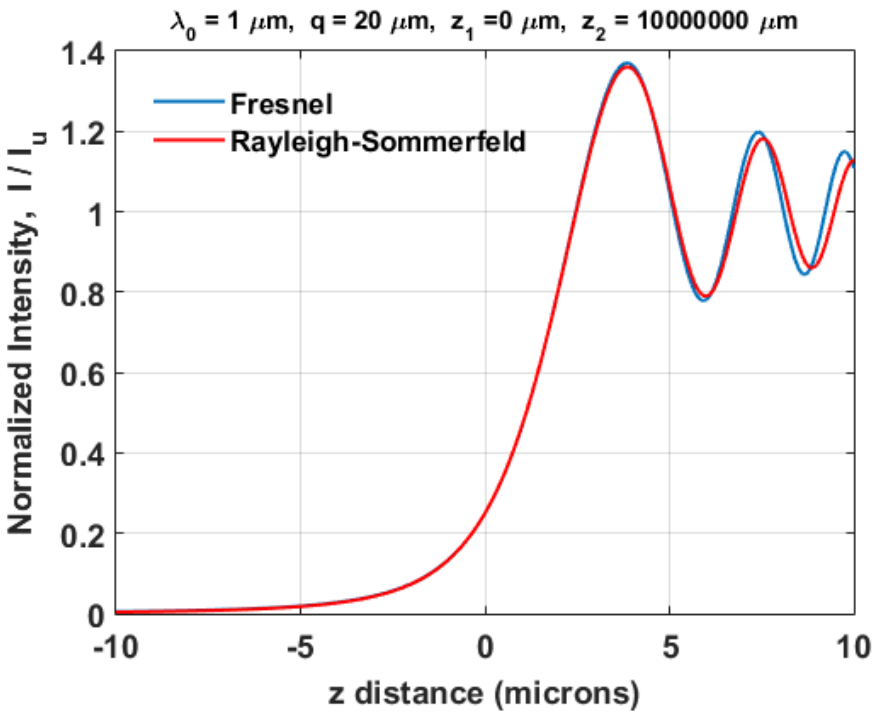
$$\begin{aligned}
x_0 &= 2.7056 \\
u_1 &= \sqrt{\frac{2}{\lambda_0 z_0}}(x_0 - b) = -\infty \\
u_2 &= \sqrt{\frac{2}{\lambda_0 z_0}}(x_0 - a) = 1.21 \\
C(u_1) &= -0.50 \\
S(u_1) &= -0.50 \\
C(u_2) &= 0.7089 \\
S(u_2) &= 0.6310 \\
U(x_0, z_0) &= \frac{U_0}{\sqrt{2}} e^{j\pi/4} e^{-jkz_0} \left[(0.7089 + 0.5) - j(0.6310 + 0.5)\right] \\
|U(x_0, z_0)| &= 1.1706 U_0 = 1.1706 E_u \\
I(x_0, z_0) &= \frac{I_u}{2}\left[1.2089^2 + 1.1310^2\right] = 1.3703 I_u
\end{aligned}$$


Cornu Spiral with exp(-jkr) convention
u=-∞
I/I_u = 0.25
E/E_u = 0.50
u=0
I/I_u = 1.37
E/E_u = 1.17
u=1.21
S(u) = ∫_0^u sin(π x^2/2) dx
C(u) = ∫_0^u cos(π x^2/2) dx
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Diffraction by an Edge

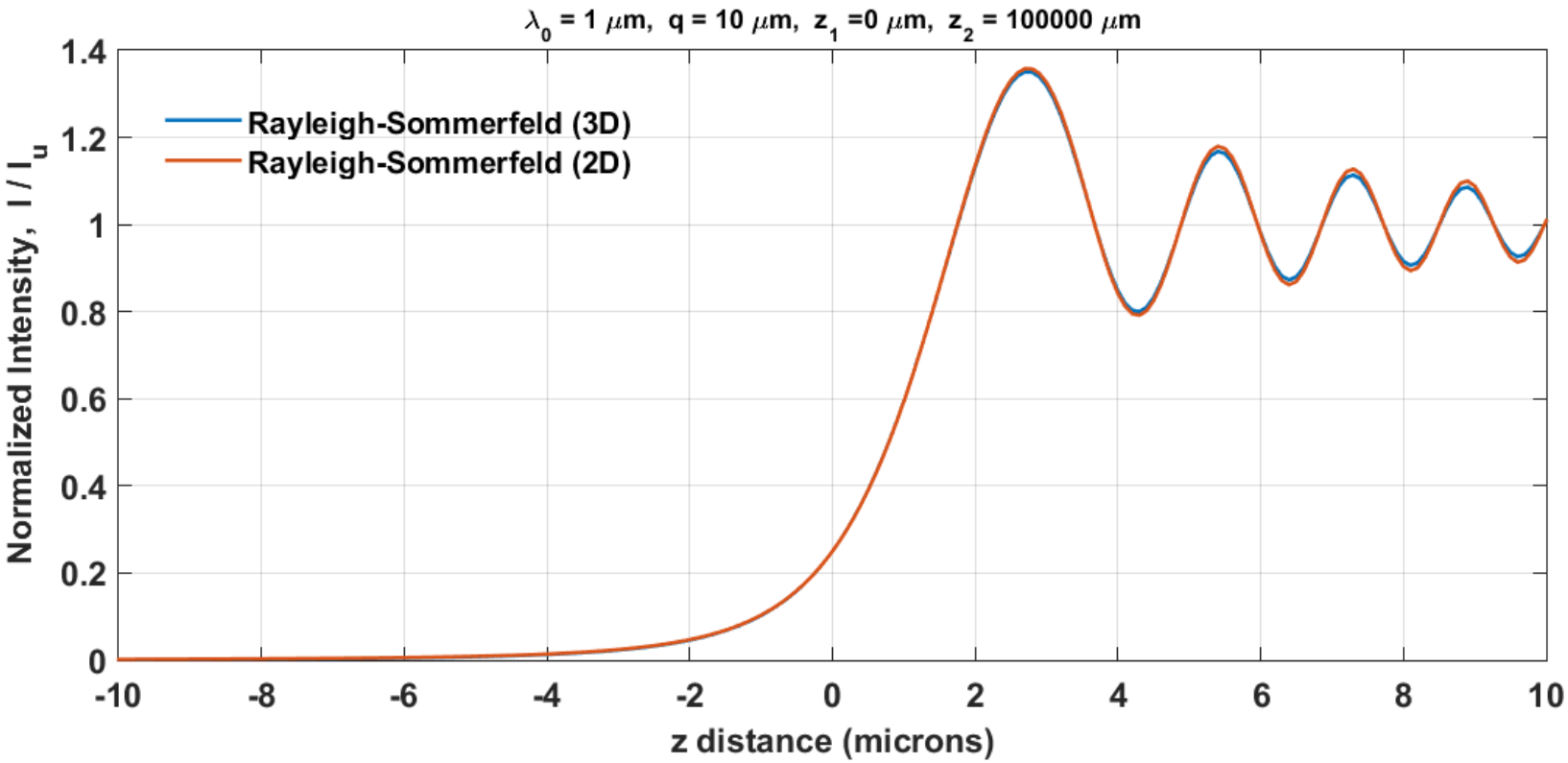
Comparison of Rayleigh-Sommerfeld-1 and Fresnel Approximation



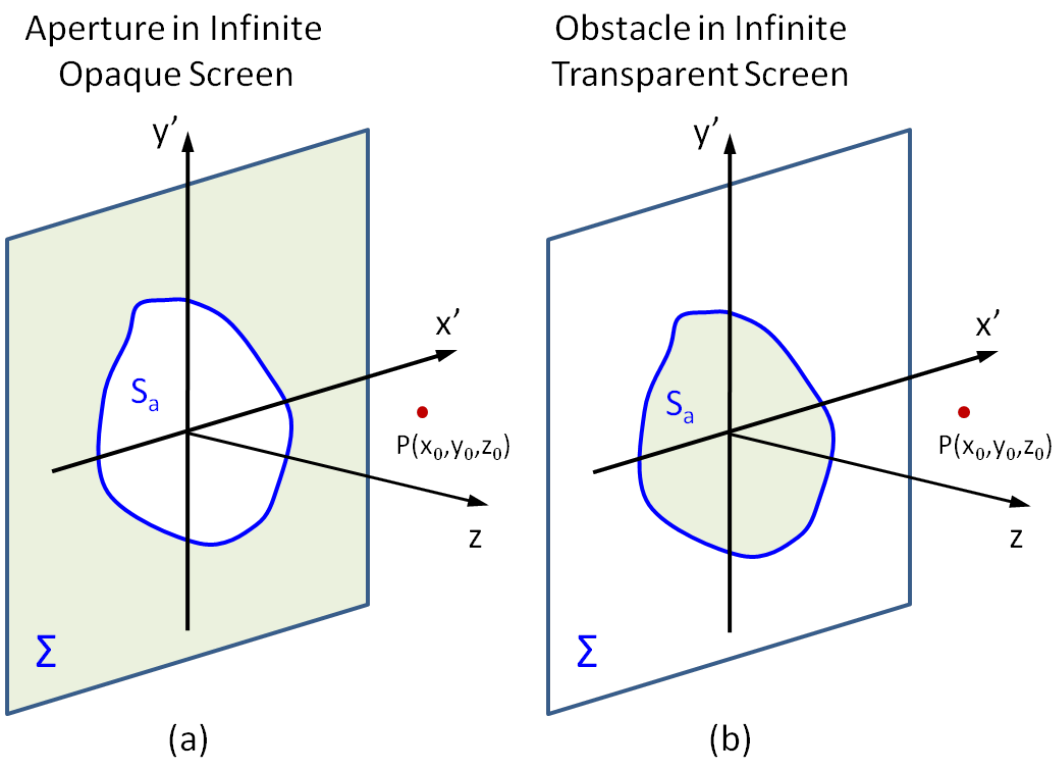


Diffraction by an Edge

Comparison of Rayleigh-Sommerfeld-1 with 2D and 3D Green's Function

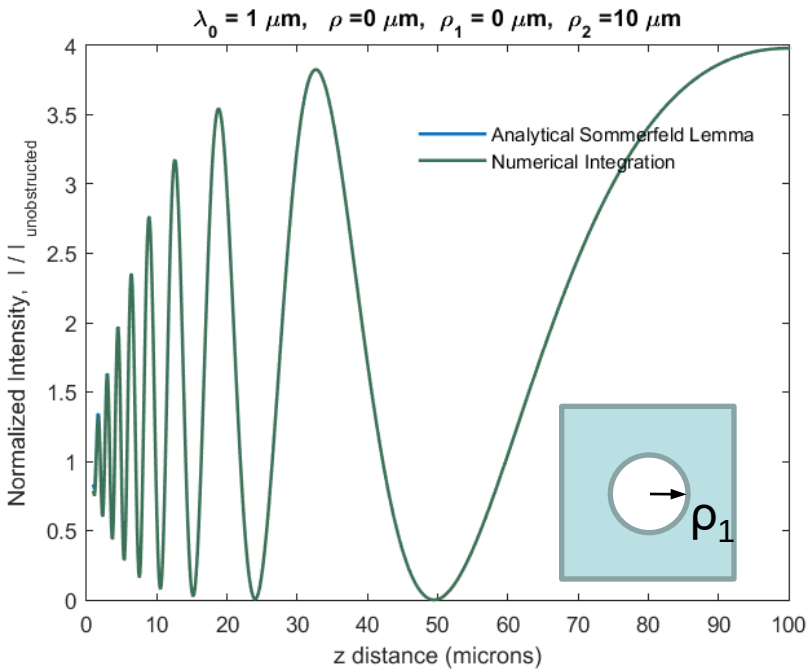


Babinet's Principle

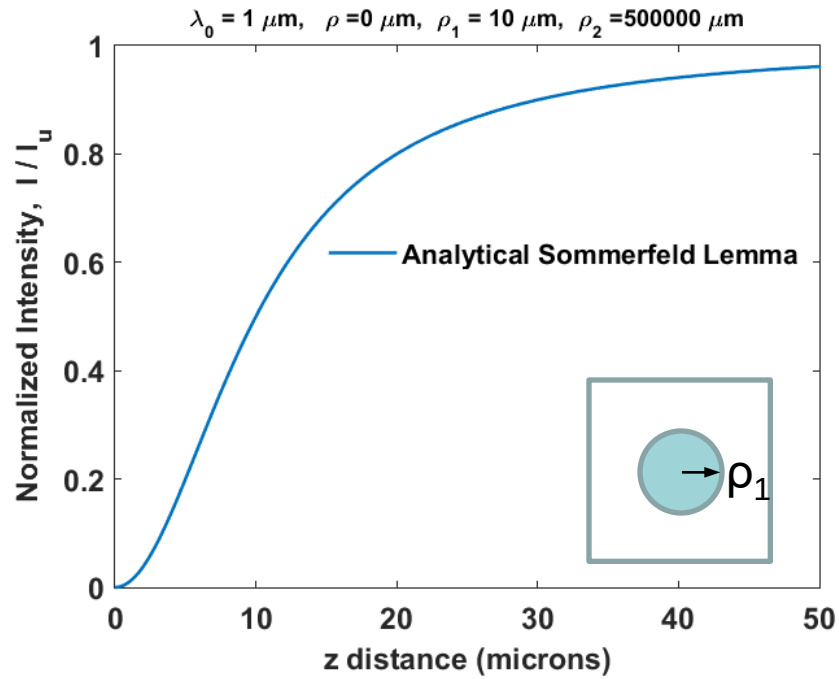


$$U_1(x_0, y_0, z_0) + U_2(x_0, y_0, z_0) = U_{inc}(x_0, y_0, z_0).$$

Diffraction from a Circular Aperture



Diffraction from an Opaque Disk



$$U(\vec{r}) = j\frac{2\pi z}{\lambda}\int_{\rho_1}^{\rho_2}\frac{e^{-jkR_0}}{R_0^2}J_0\left(\frac{k\rho\rho'}{R_0}\right)\rho' d\rho', \qquad I = |U|^2$$

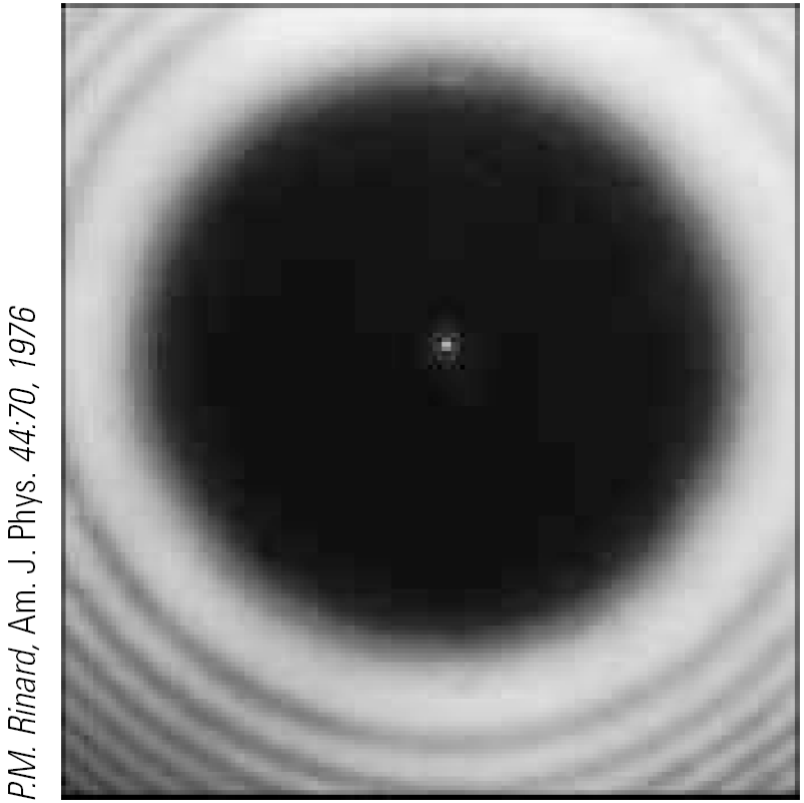
$$R_0 = \sqrt{\rho^2 + \rho'^2 + z^2}$$

Sommerfeld's Lemma*

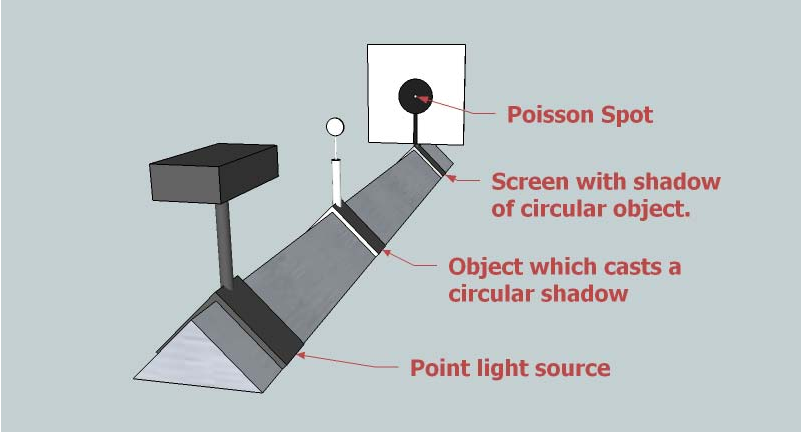
$$E_1 = E_u\left[e^{-jkz} - \frac{e^{-jkz\sqrt{1+(\rho_1/z)^2}}}{\sqrt{1+(\rho_1/z)^2}}\right] \qquad E_2 = E_u\frac{e^{-jkz\sqrt{1+(\rho_1/z)^2}}}{\sqrt{1+(\rho_1/z)^2}}$$

* R. E. Lucke, "Rayleigh–Sommerfeld diffraction and Poisson's spot", ***Eur. J. Phys.*** vol. 27, pp. 193-204, 2006.

Diffraction from an Opaque Disk



P.M. Rinard, Am. J. Phys. *44:70, 1976*



http://en.wikipedia.org/wiki/Arago_spot

Diffraction pattern created by the illumination of an opaque disk (U.S. penny), with the disk positioned midway between screen and light source. Note the bright spot at the center (Poisson Spot)

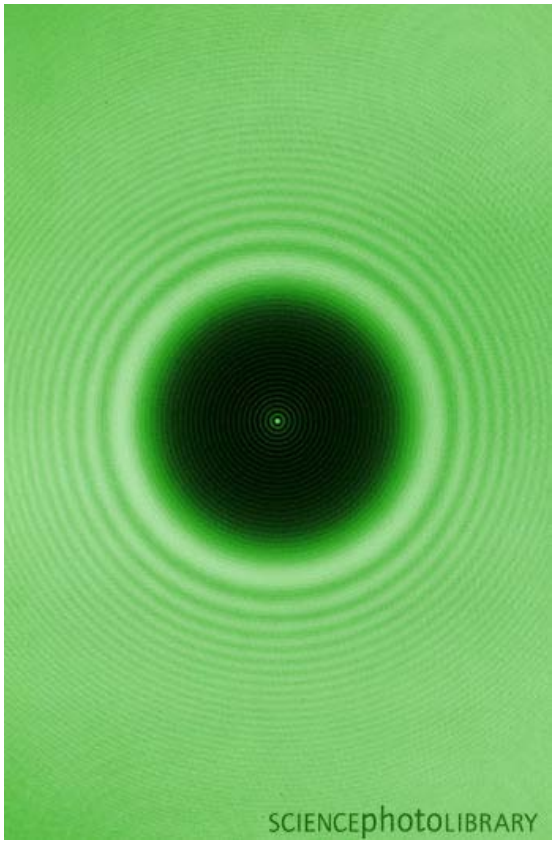
R.A. Serway and J. W. Jewett, "Physics for Scientists and Engineers", 6th Ed., Thomson Brooks/Cole, 2004

Diffraction from an Opaque Disk

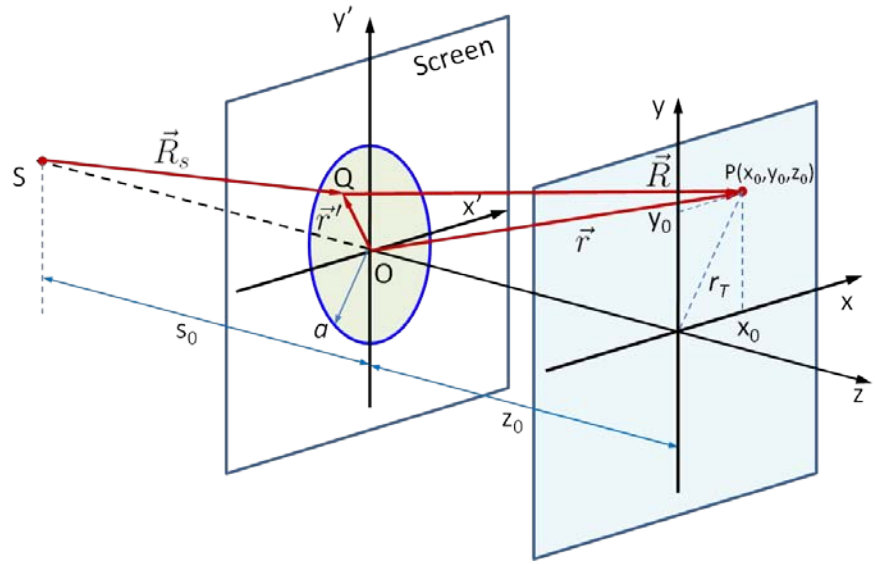
In 1818, Augustin Fresnel submitted a paper on the theory of diffraction for a competition sponsored by the French Academy. Simeon D. Poisson, a member of the judging committee for the competition, was very critical of the wave theory of light. Using Fresnel's theory, Poisson deduced the seemingly absurd prediction that a bright spot should appear behind a circular obstruction, a prediction he felt was the last nail in the coffin for Fresnel's theory.

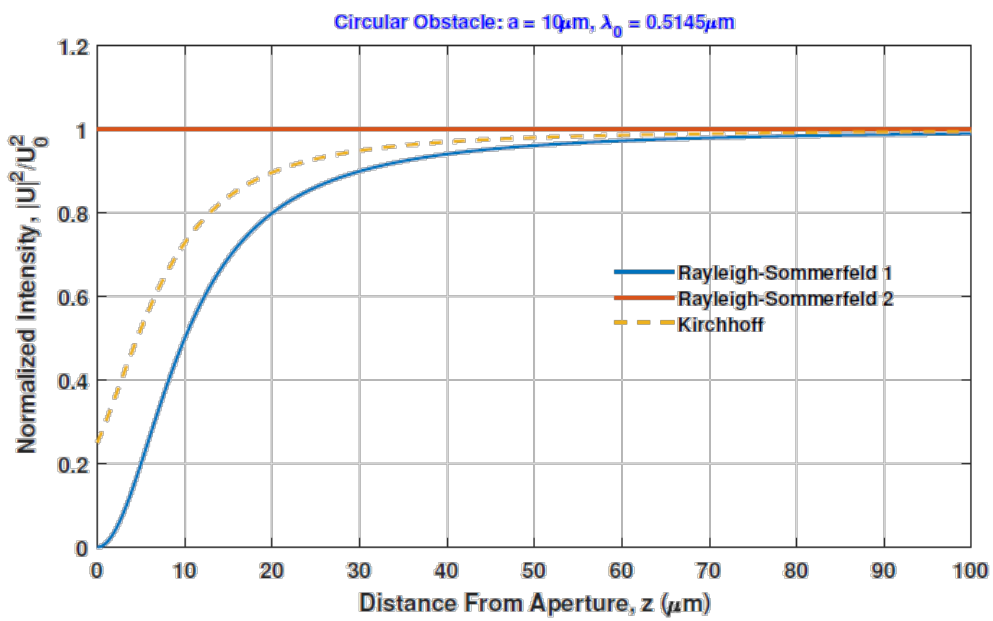
However, Dominique Arago, another member of the judging committee, almost immediately verified the spot experimentally. Fresnel won the competition, and, although it may be more appropriate to call it "the Spot of Arago," the spot goes down in history with the name "Poisson's bright spot".

Poisson's bright spot

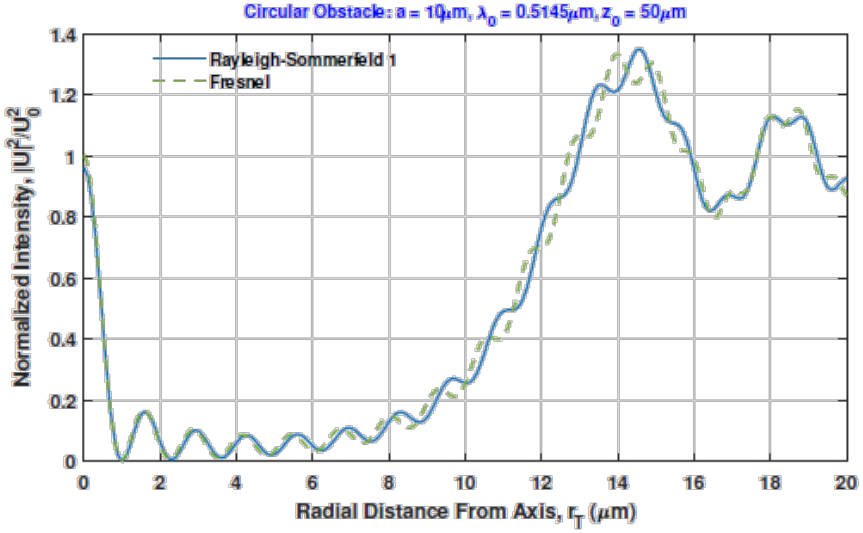


Diffraction from an Opaque Disk

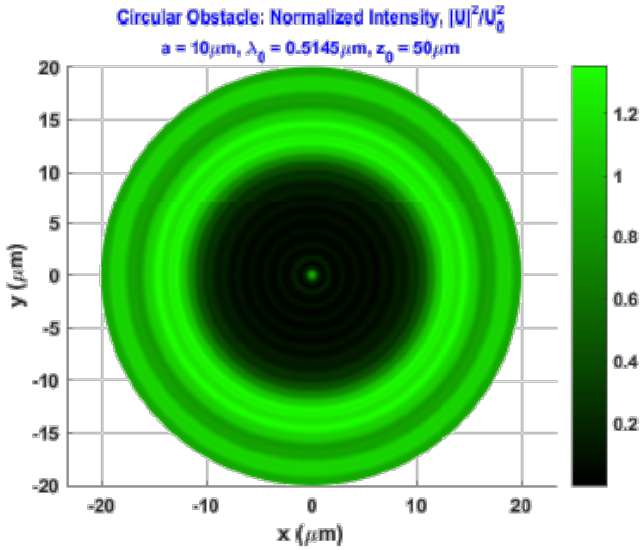




Diffraction from an Opaque Disk



(a)

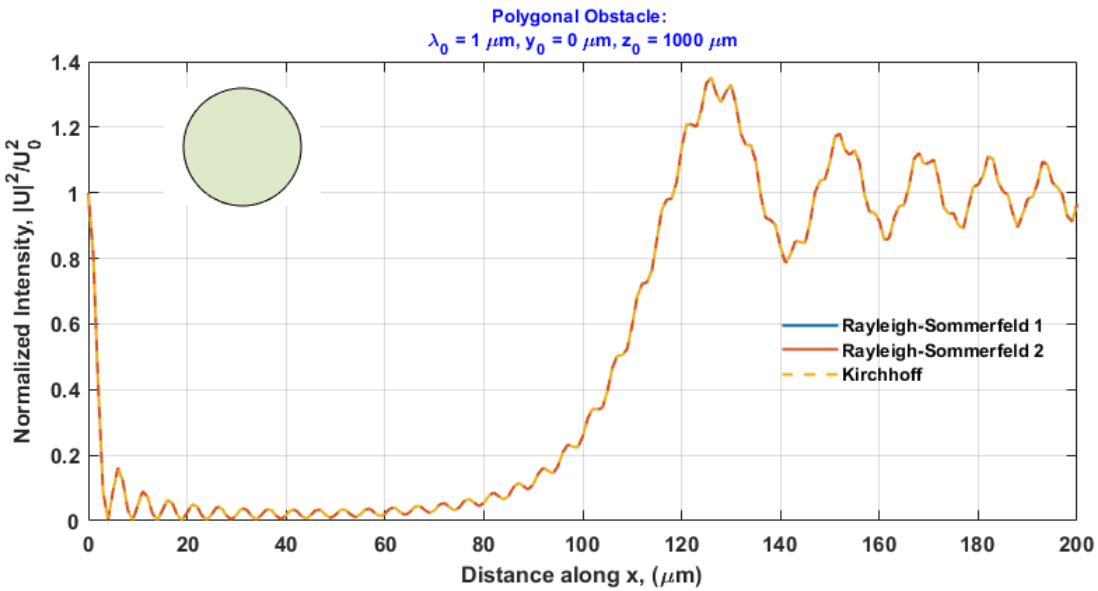


(b)

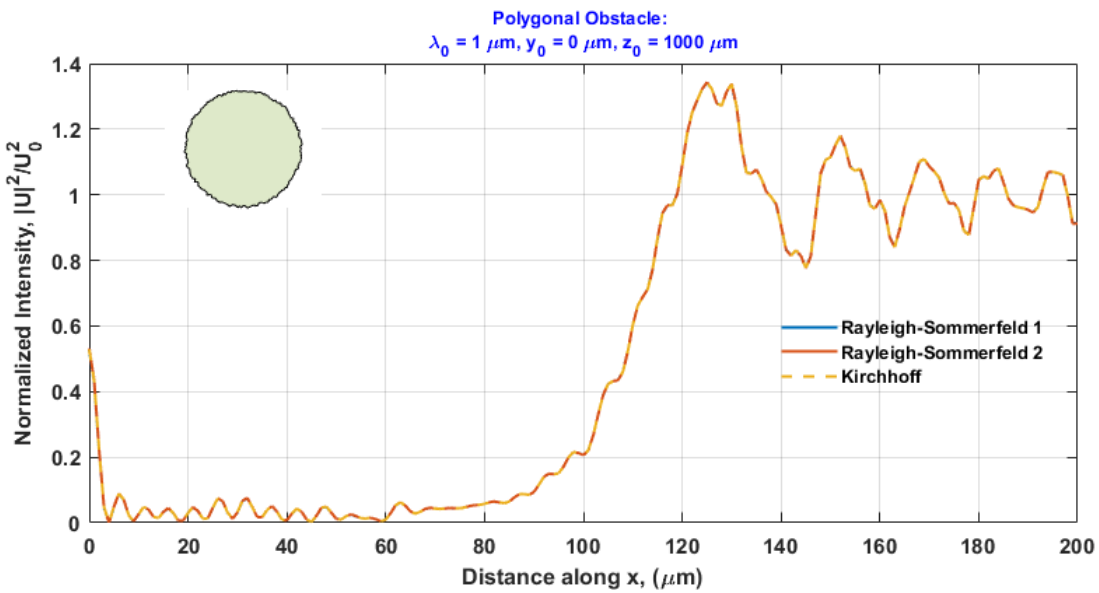
Diffraction from an Opaque Disk with Randomness at its Edge

0.0% radius variation

a=100μm



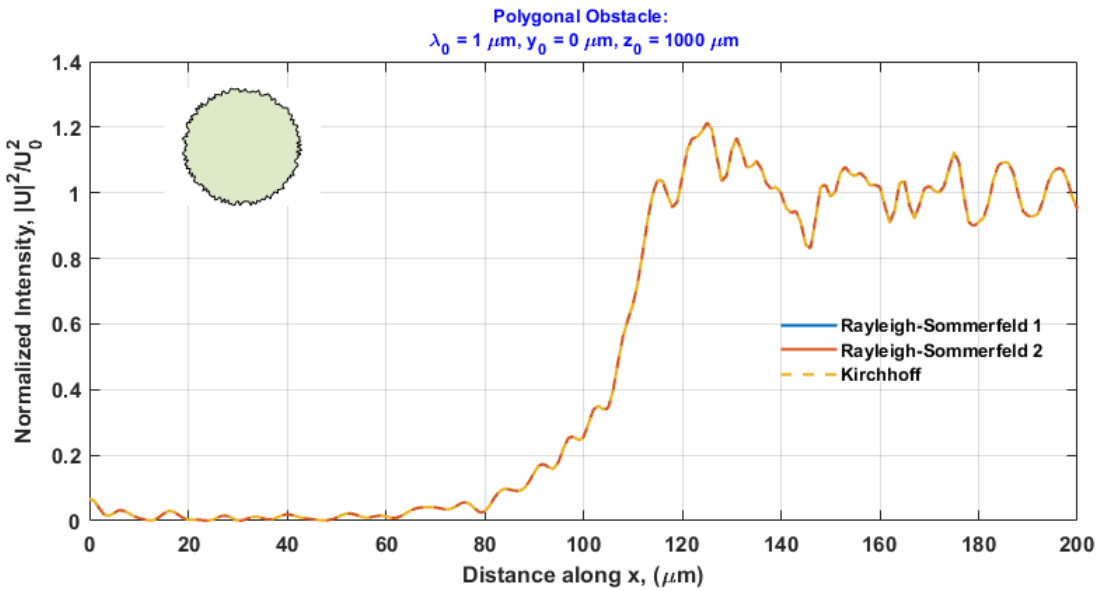
2.5% radius variation



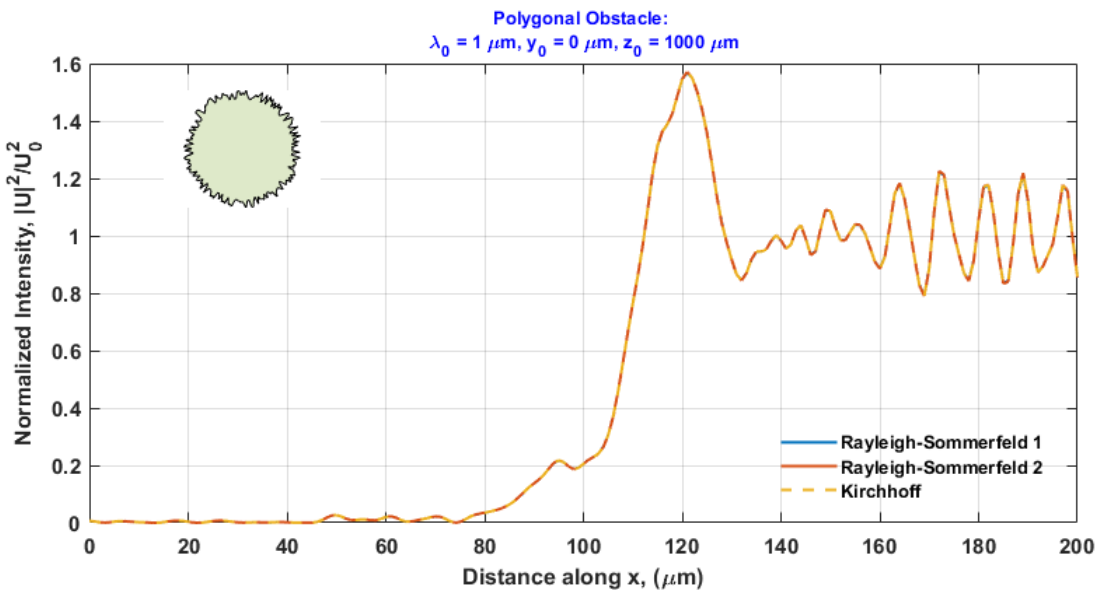
Diffraction from an Opaque Disk with Randomness at its Edge

5.0% radius variation

a=100μm

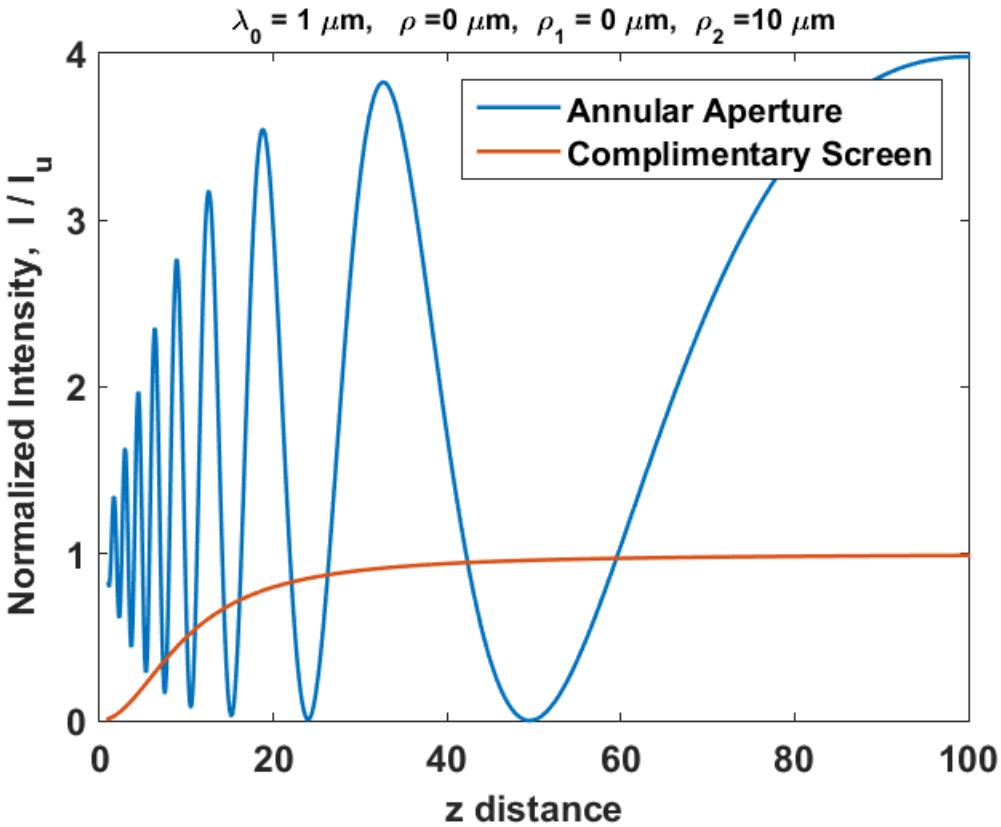


10.0% radius variation

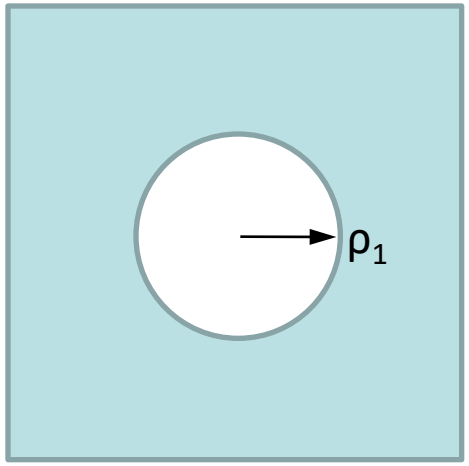


Babinet's Principle

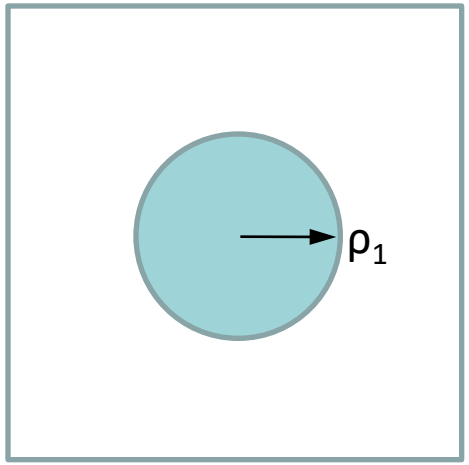
$$E_1 + E_2 = E_u$$



Circular Aperture



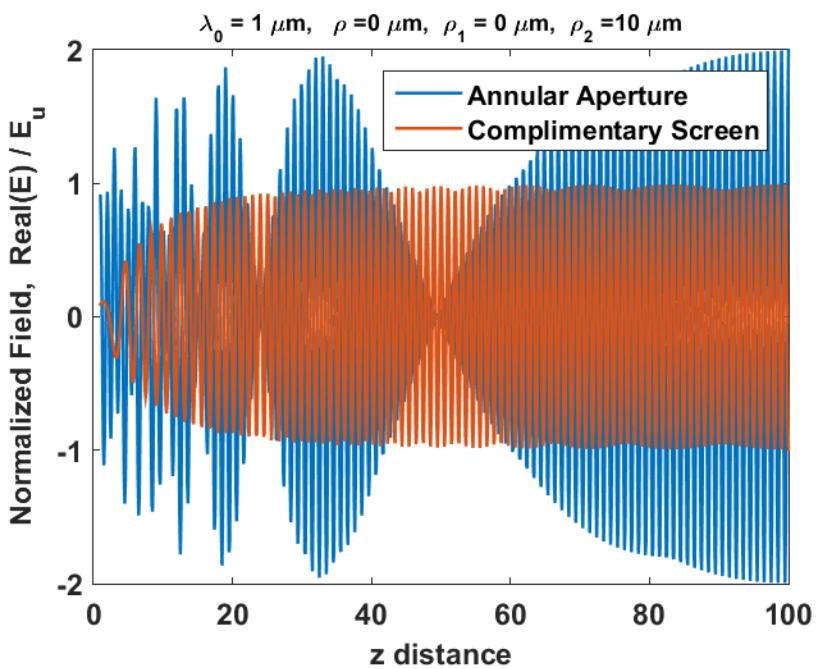
Opaque Disk

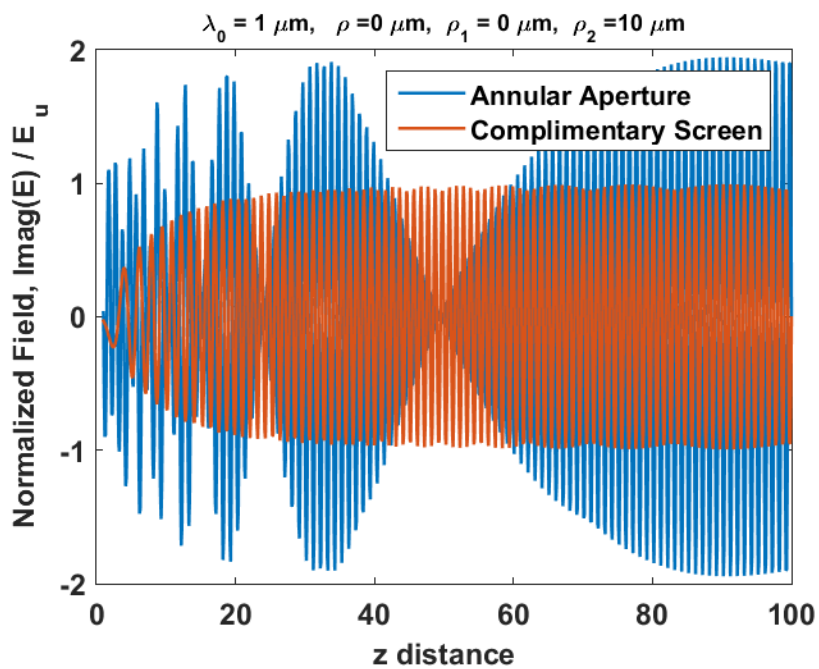


Babinet's Principle

$$E_1 = E_u\left[e^{-jkz} - \frac{e^{-jkz\sqrt{1+(\rho_1/z)^2}}}{\sqrt{1+(\rho_1/z)^2}}\right]$$

$$E_2 = E_u\frac{e^{-jkz\sqrt{1+(\rho_1/z)^2}}}{\sqrt{1+(\rho_1/z)^2}}$$

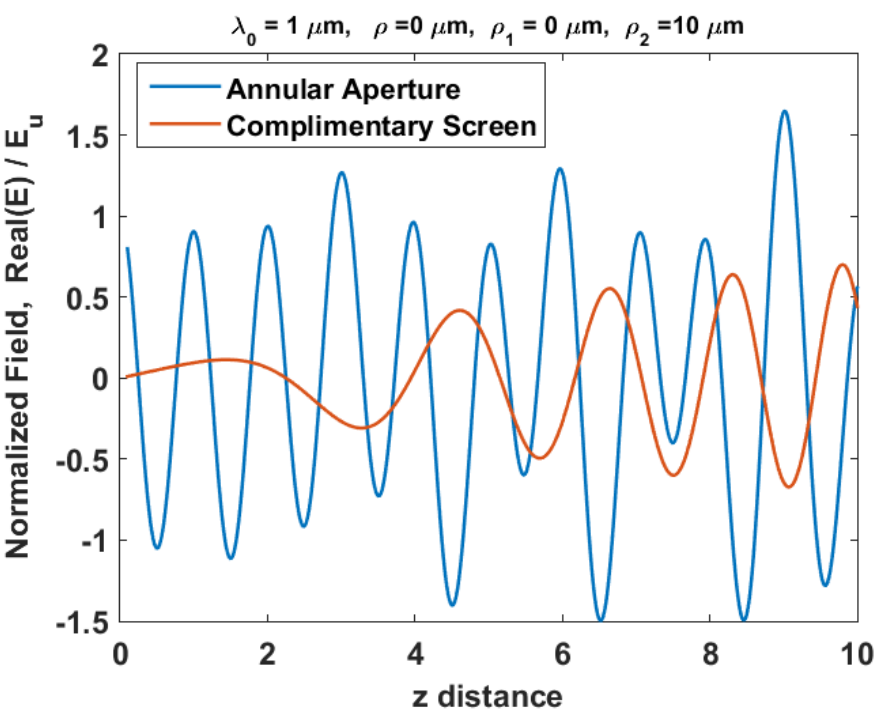


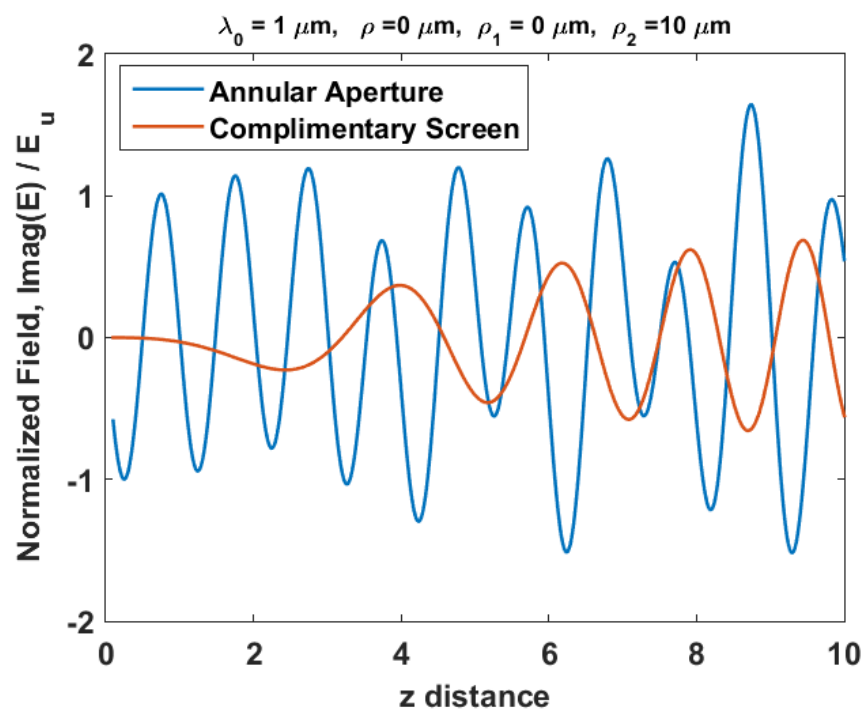


Babinet's Principle

$$E_1 = E_u\left[e^{-jkz} - \frac{e^{-jkz\sqrt{1+(\rho_1/z)^2}}}{\sqrt{1+(\rho_1/z)^2}}\right]$$

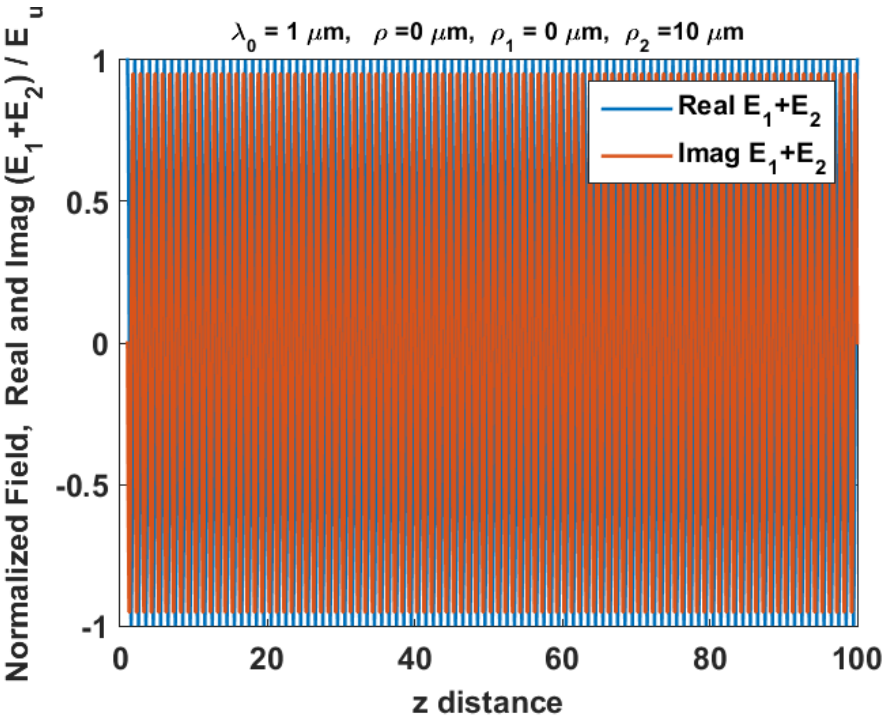
$$E_2 = E_u\frac{e^{-jkz\sqrt{1+(\rho_1/z)^2}}}{\sqrt{1+(\rho_1/z)^2}}$$

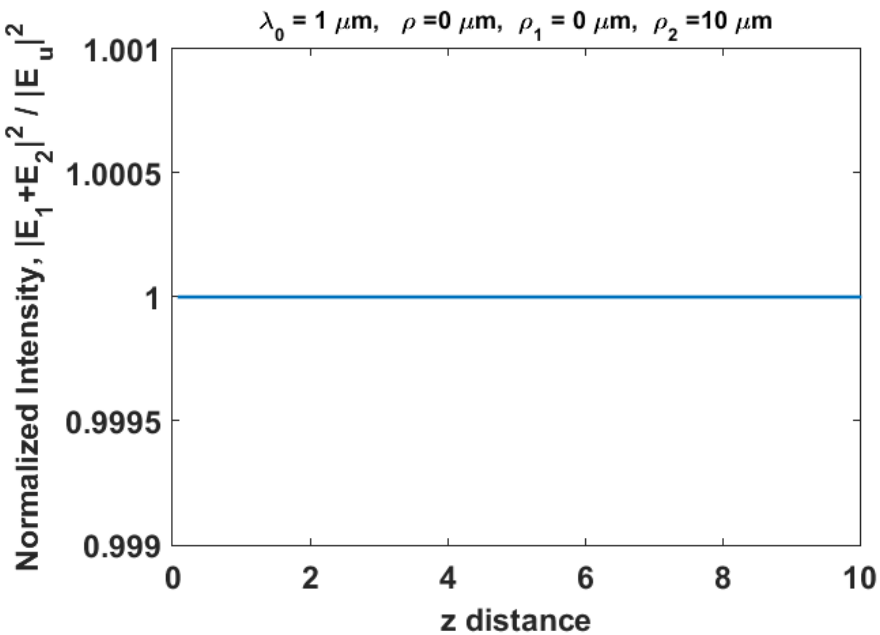




Babinet's Principle

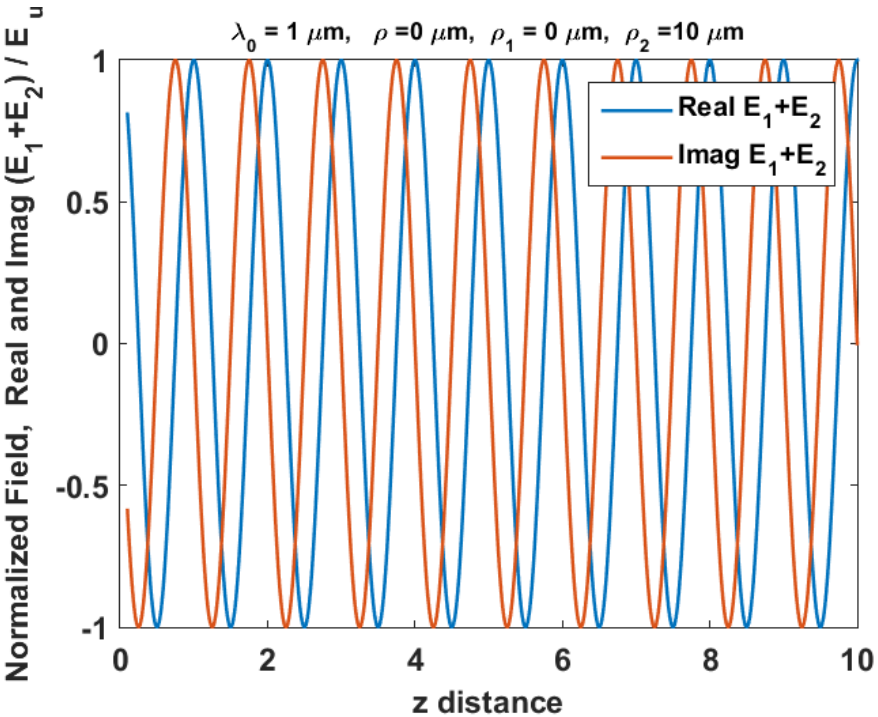
$$E_1 + E_2 = E_u$$

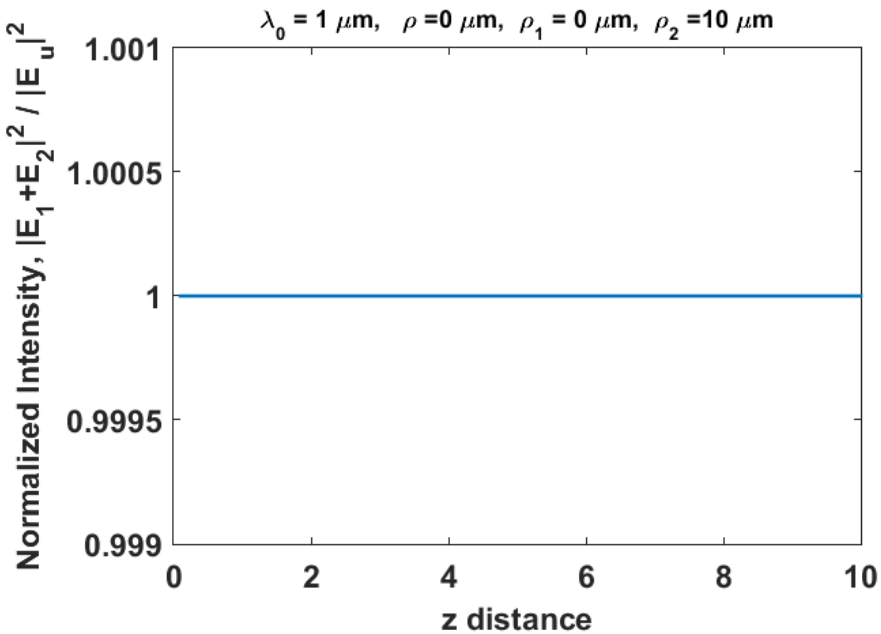




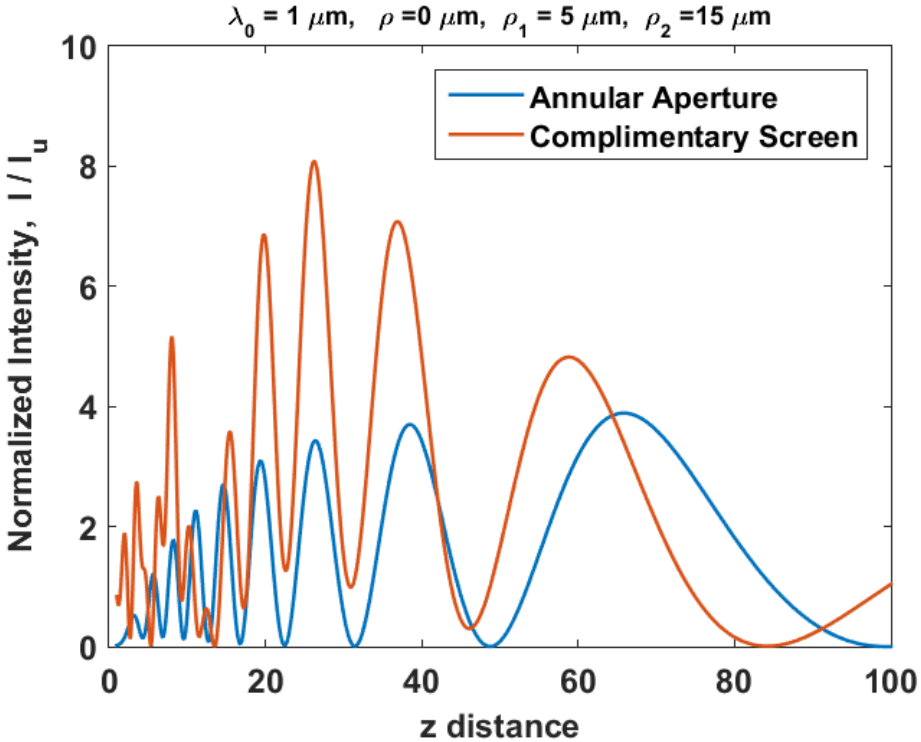
Babinet's Principle

$$E_1 + E_2 = E_u$$

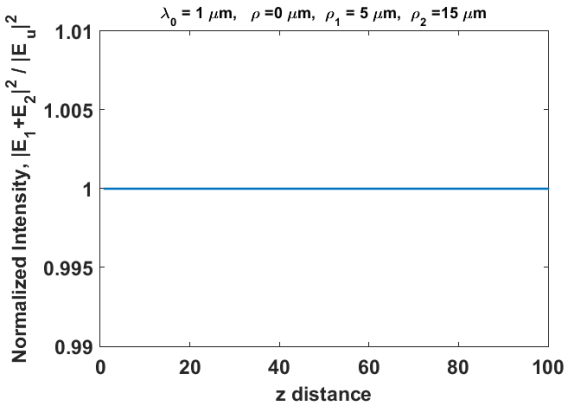




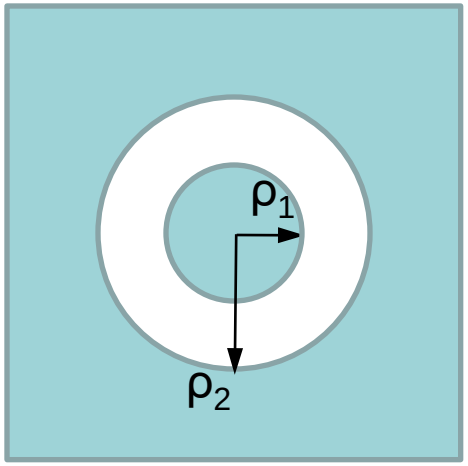
Babinet's Principle



$E_1 + E_2 = E_u$

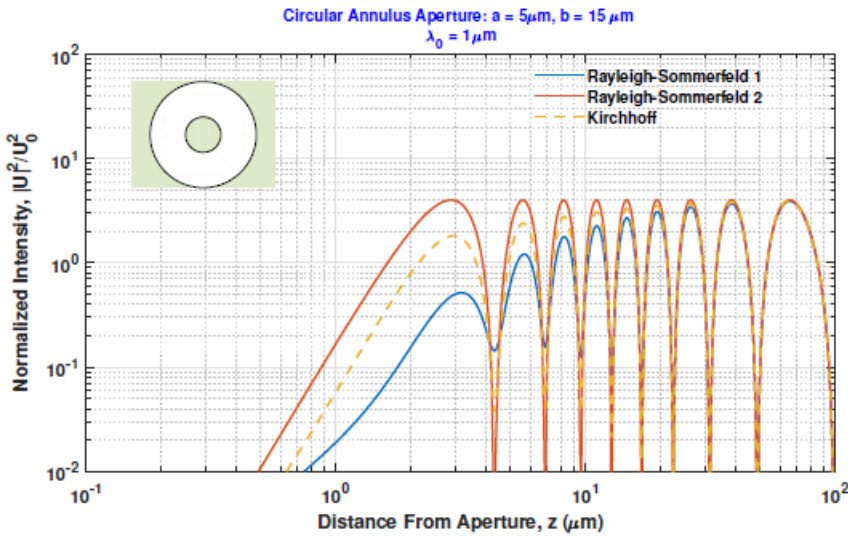


Annular Aperture

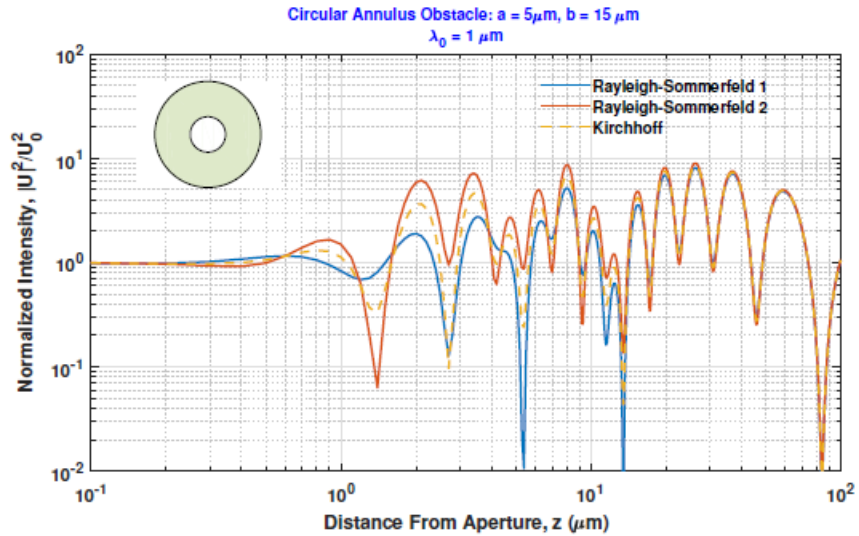


Opaque Annular Ring

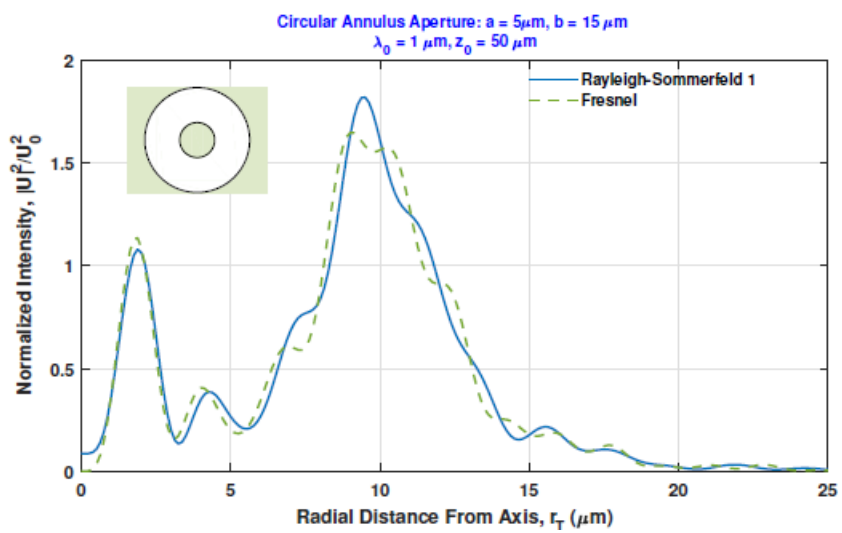
Babinet's Principle

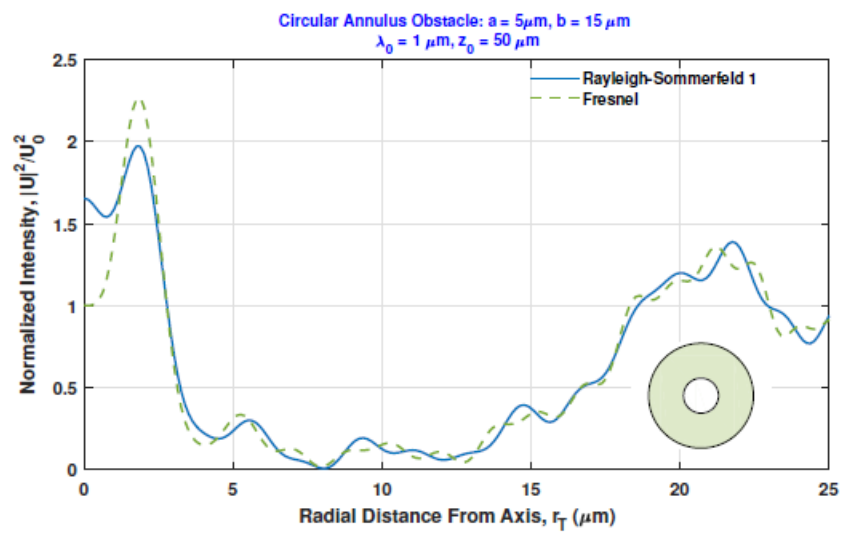


(a)



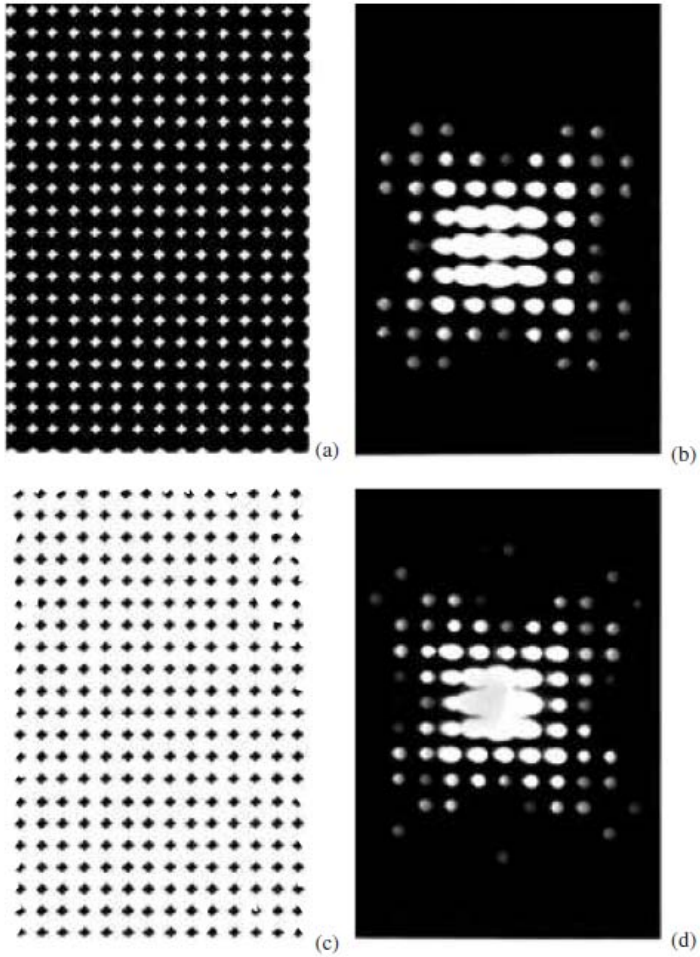
(a)





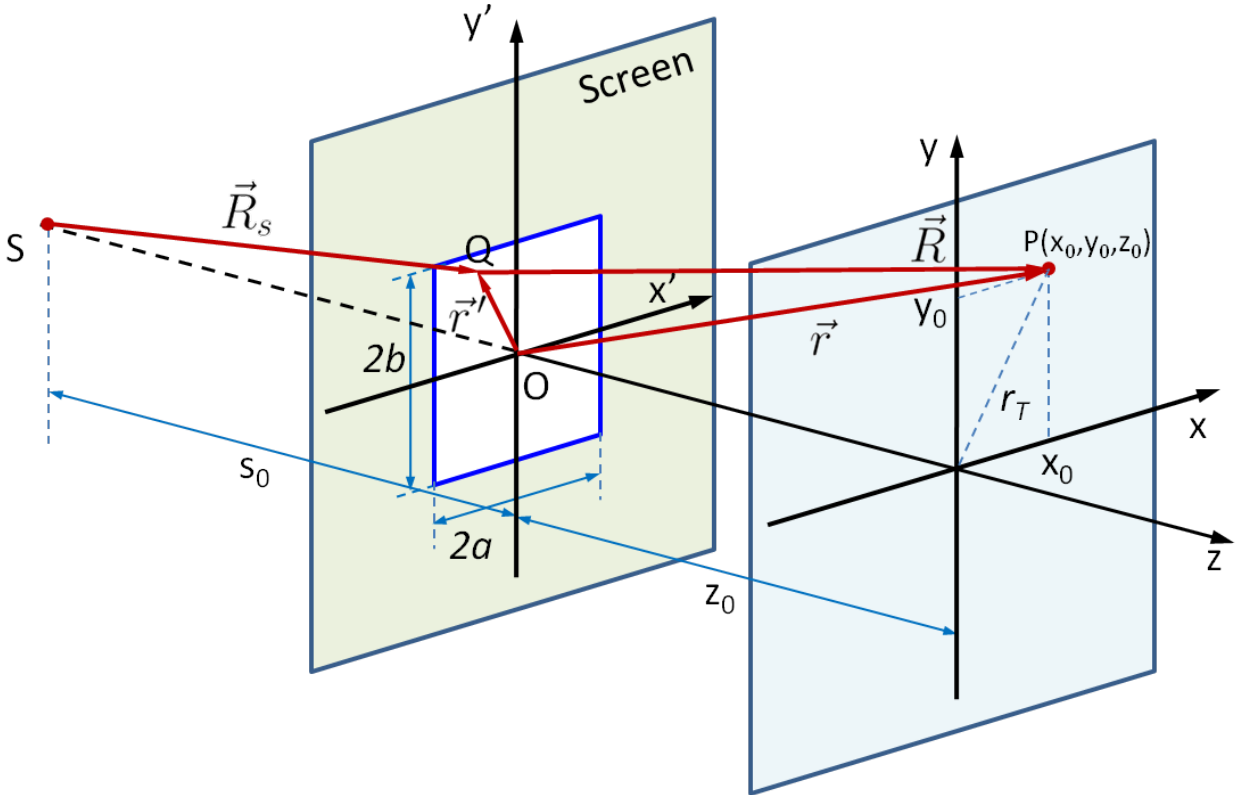
Babinet's Principle

Complementary Screen Diffraction Patterns in *Fraunhofer* Regime



E. Hecht, "Optics", 4th Ed., Addison Wesley, 2002

Diffraction by a Rectangular Aperture



$$U_{RS1}(P_0) = \int_{-a}^{+a}\int_{-b}^{+b}\left[U_{inc}\left(jk+\frac{1}{R}\right)\frac{\exp(-jkR)}{2\pi R}\frac{z_0}{R}\right]_{z'=0} dx'dy',$$

$$U_{RS2}(P_0) = \int_{-a}^{+a}\int_{-b}^{+b}\left[\frac{\exp(-jkR)}{2\pi R}\frac{\partial U_{inc}}{\partial n}\right]_{z'=0} dx'dy',$$

$$\text{with} \qquad R = \left[(x_0-x')^2+(y_0-y')^2+z_0^2\right]^{1/2}.$$

Diffraction by a Rectangular Aperture

Fresnel Approximation

$$
\begin{aligned}
U(x_0, y_0, z_0) &= jk\frac{\exp(-jkz_0)}{2\pi z_0}\int_{-a}^{+a}\int_{-b}^{+b}\left[\exp\left\{-jk\left[\frac{(x_0-x')^2}{2z_0}+\frac{(y_0-y')^2}{2z_0}\right]\right\}U_0\right]dx'dy' = \\
&= jk\frac{\exp(-jkz_0)}{2\pi z_0}U_0\underbrace{\int_{-a}^{+a}\exp\left[-jk\frac{(x_0-x')^2}{2z_0}\right]dx'}_{U_x}\underbrace{\int_{-b}^{+b}\exp\left[-jk\frac{(y_0-y')^2}{2z_0}\right]dy'}_{V_y}.
\end{aligned}
$$

$$
\begin{aligned}
U_x &= U_x(x_0, z_0) = \sqrt{\frac{\lambda z_0}{2}}\Big\{C(u_2)-C(u_1)-j\left[S(u_2)-S(u_2)\right]\Big\}, \\
\text{with} \quad & u_2 = \sqrt{\frac{2}{\lambda z_0}}(x_0+a), \qquad u_1 = \sqrt{\frac{2}{\lambda z_0}}(x_0-a), \\
V_y &= V_y(y_0, z_0) = \sqrt{\frac{\lambda z_0}{2}}\Big\{C(v_2)-C(v_1)-j\left[S(v_2)-S(v_2)\right]\Big\}, \\
\text{with} \quad & v_2 = \sqrt{\frac{2}{\lambda z_0}}(y_0+b), \qquad v_1 = \sqrt{\frac{2}{\lambda z_0}}(y_0-b).
\end{aligned}
$$

$$
\begin{aligned}
U(x_0, y_0, z_0) &= j\frac{U_0}{2}\exp\left(-jkz_0\right)F(u_2, u_1)F(v_2, v_1), \\
\text{where} \quad F(u_2, u_1) &= \Big\{C(u_2)-C(u_1)-j\left[S(u_2)-S(u_2)\right]\Big\}, \\
\text{and} \quad F(v_2, v_1) &= \Big\{C(v_2)-C(v_1)-j\left[S(v_2)-S(v_2)\right]\Big\}.
\end{aligned}
$$

Diffraction by a Rectangular Aperture

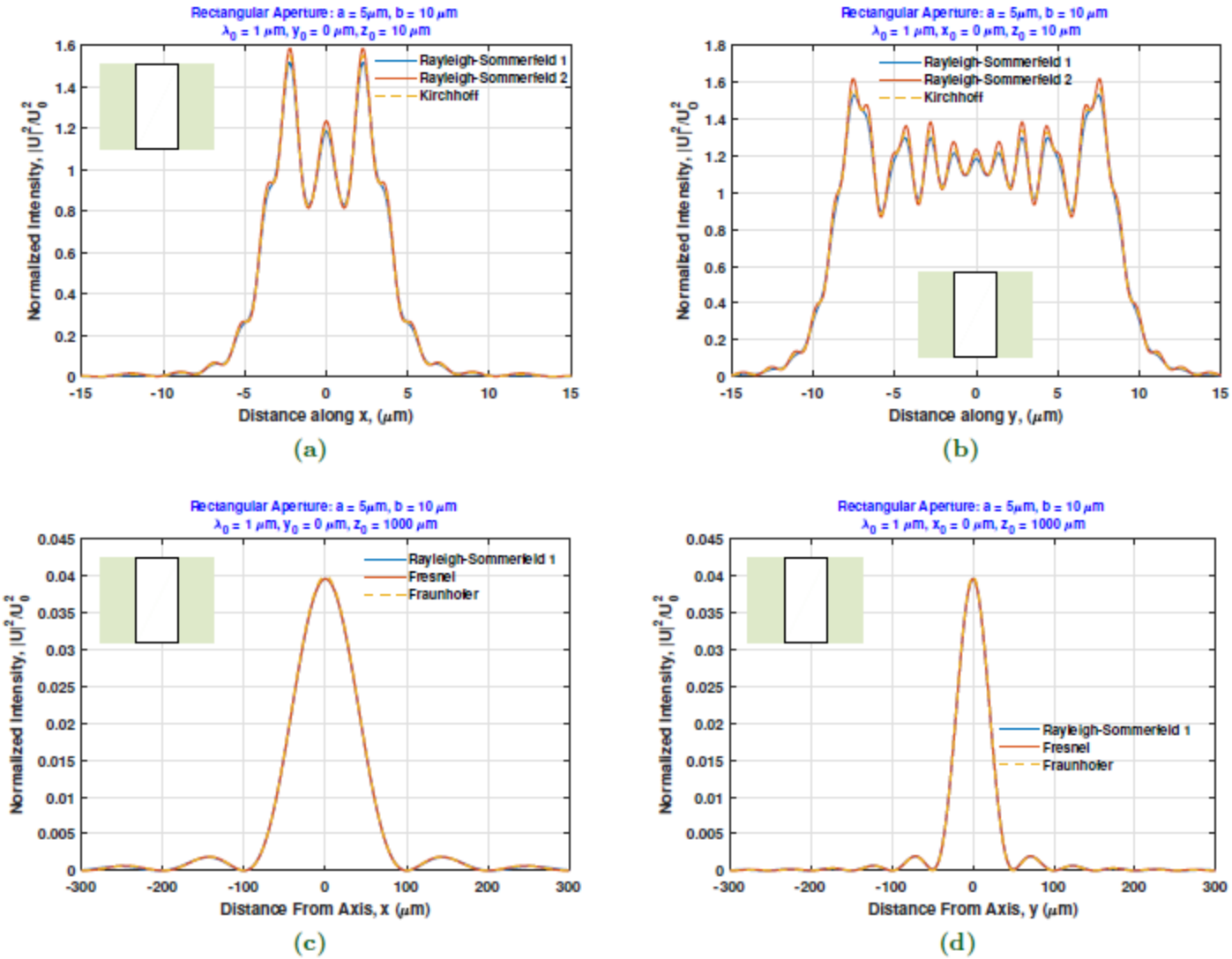
Fraunhofer Approximation

$$U_{ff}(x_0, y_0, z_0) = jkU_0 \frac{e^{-jkz_0}}{2\pi z_0} \exp\left[-j\frac{k}{2z_0}\left(x_0^2 + y_0^2\right)\right] \left\{ 4ab \frac{\sin\left(a\frac{kx_0}{z_0}\right)}{a\frac{kx_0}{z_0}} \frac{\sin\left(b\frac{ky_0}{z_0}\right)}{b\frac{ky_0}{z_0}} \right\}$$

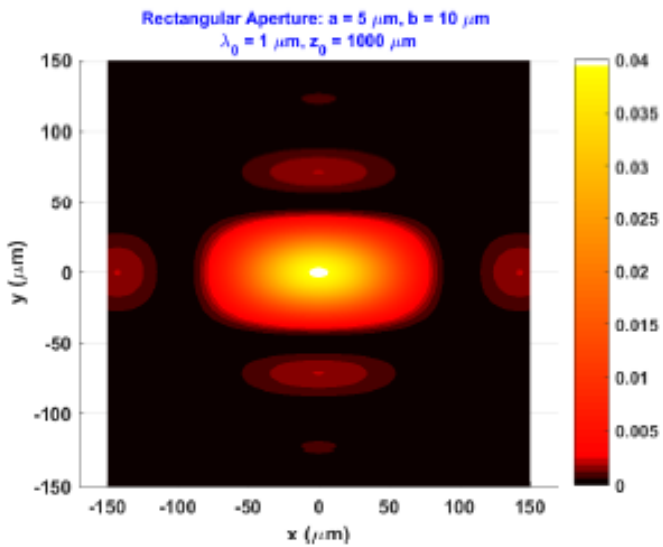
$$\boxed{\begin{aligned} I_{ff}(x_0, y_0, z_0) &= \underbrace{\frac{|U_0|^2}{4Z}\left(\frac{4ab}{\lambda z_0}\right)^2}_{I_0(z_0)} \left\{ \frac{\sin\left(a\frac{kx_0}{z_0}\right)}{a\frac{kx_0}{z_0}} \right\}^2 \left\{ \frac{\sin\left(b\frac{ky_0}{z_0}\right)}{b\frac{ky_0}{z_0}} \right\}^2 = \\ &= I_0(z_0)\ \mathrm{sinc}^2\left(2a\frac{x_0}{\lambda z_0}\right) \mathrm{sinc}^2\left(2b\frac{y_0}{\lambda z_0}\right) \end{aligned}}$$

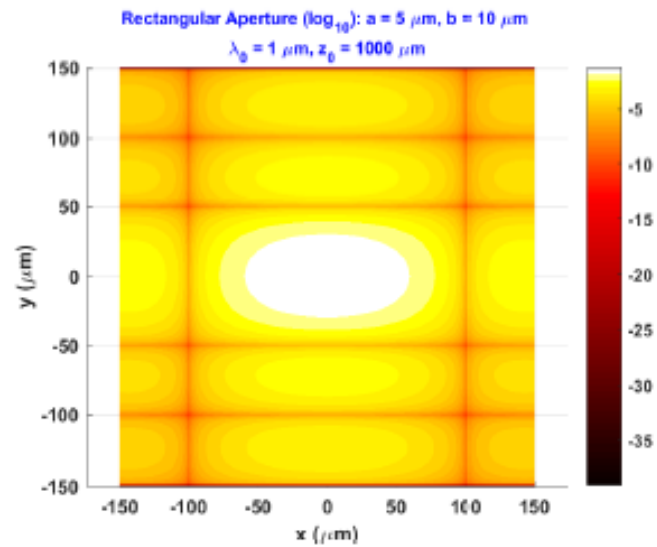
$$\mathrm{sinc}(u) = \sin(\pi u)/(\pi u)$$

Diffraction by a Rectangular Aperture

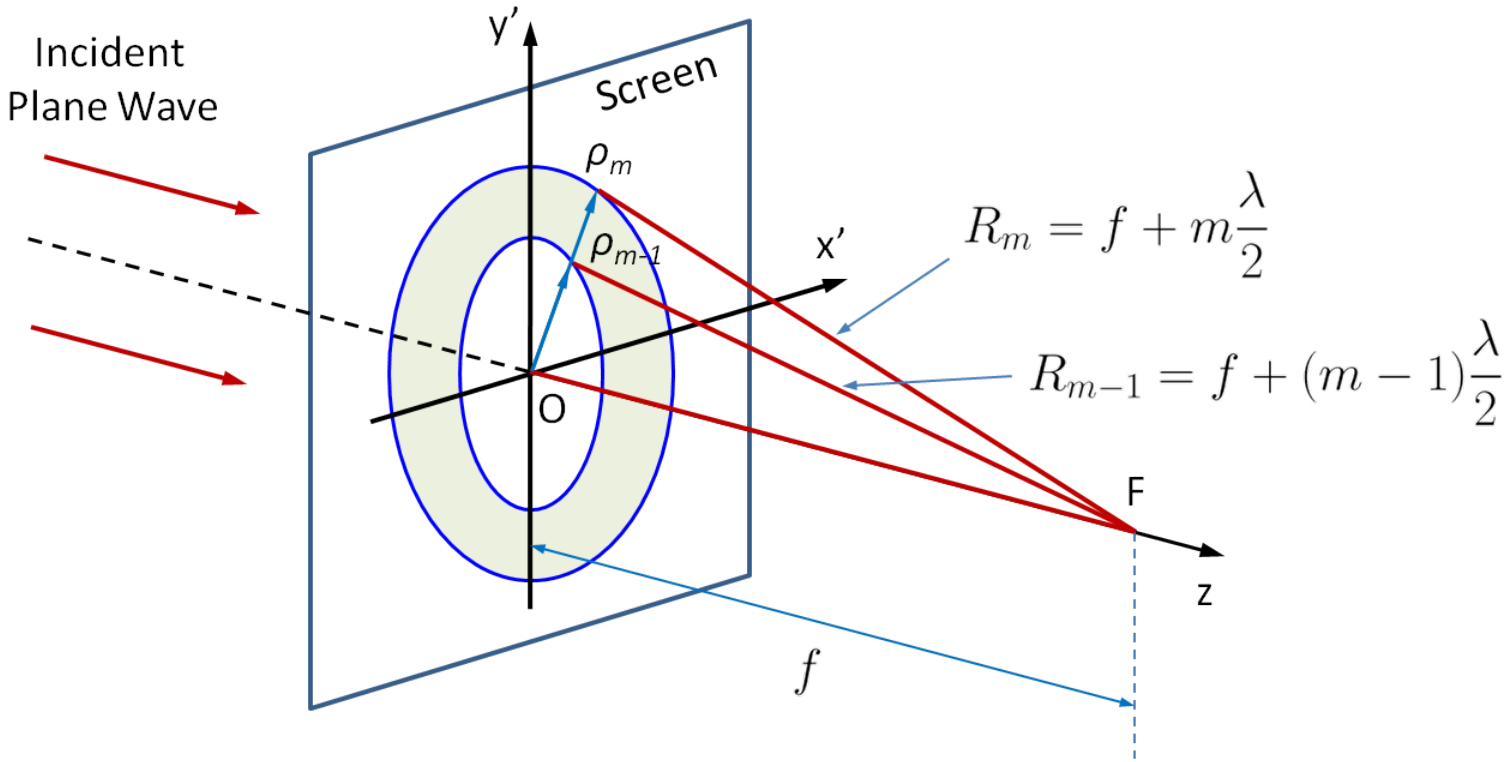


Diffraction by a Rectangular Aperture





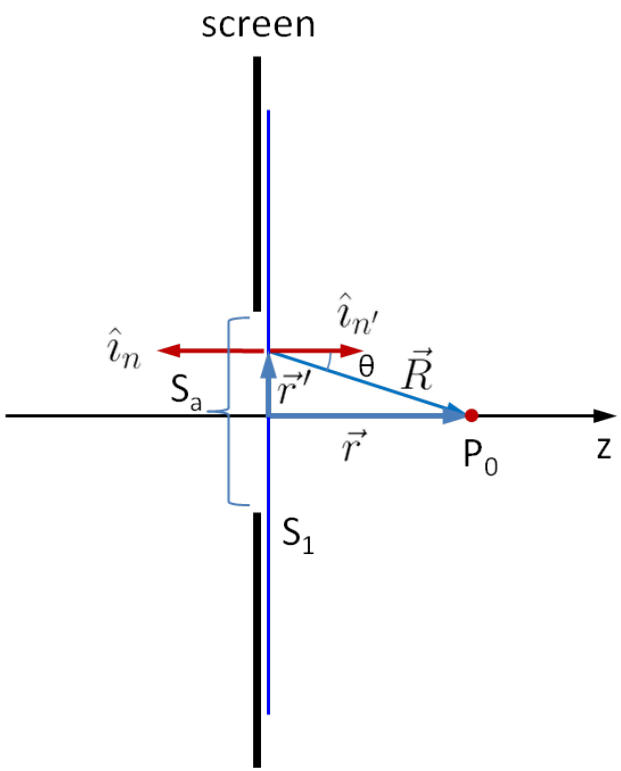
Fresnel Zone Plate



$$\rho_m^2 + f^2 = R_m^2 = \left(f + m\frac{\lambda}{2}\right)^2 \Longrightarrow$$

$$\rho_m = \left(m\lambda f + m^2\frac{\lambda^2}{4}\right)^{1/2}, \qquad m = 0, 1, 2, \cdots$$

Fresnel Zone Plate

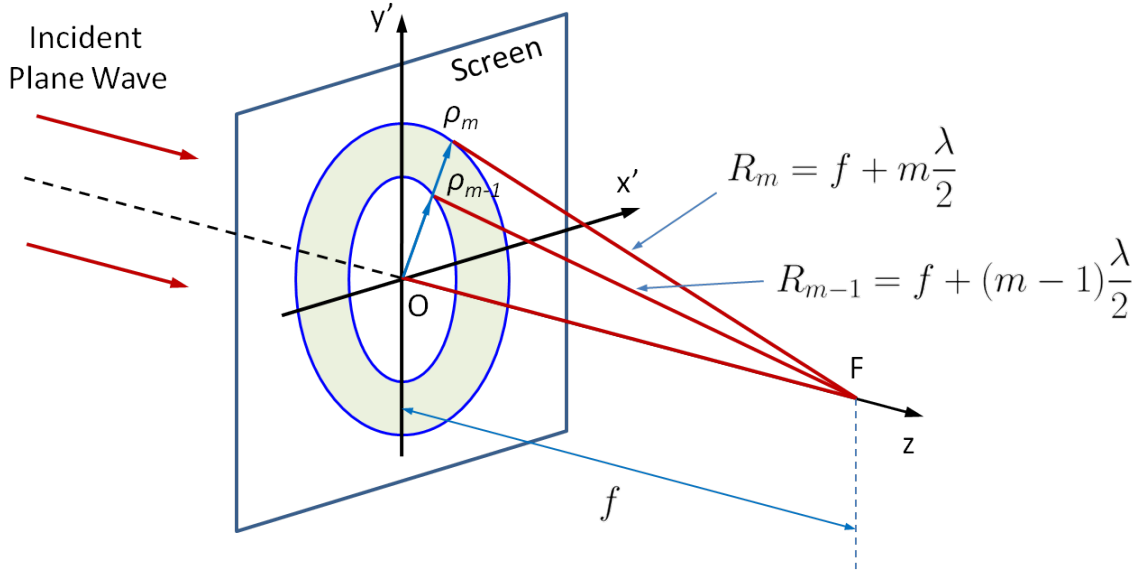


Kirchhoff's Formulation for a Planar Screen

$$\boxed{\begin{aligned} U(P_0) &= \iint_{S_1} \left(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n} \right) ds = \\ &= \iint_{S_1} \left[\frac{e^{-jkR}}{4\pi R}\frac{\partial U}{\partial n} - U\left(jk + \frac{1}{R}\right)\frac{e^{-jkR}}{4\pi R}\left(\hat{\imath}_R \cdot \hat{\imath}_n\right) \right] ds. \end{aligned}}$$

$$\begin{aligned} U_K(P_0) = U(P_0) &\simeq \iint_{S_a} \left[\frac{e^{-jkR}}{4\pi R}\frac{\partial U_{inc}}{\partial n} - U_{inc}\left(jk + \frac{1}{R}\right)\frac{e^{-jkR}}{4\pi R}\left(\hat{\imath}_R \cdot \hat{\imath}_n\right) \right] ds = \\ &= \iint_{S_a} \frac{e^{-jkR}}{4\pi R} \left[\frac{\partial U_{inc}(x', y', 0)}{\partial n} + U_{inc}(x', y', 0)\left(jk + \frac{1}{R}\right)\cos\theta \right] dx'dy', \\ \text{where} \qquad & \cos\theta = \frac{z_0}{R}. \end{aligned}$$

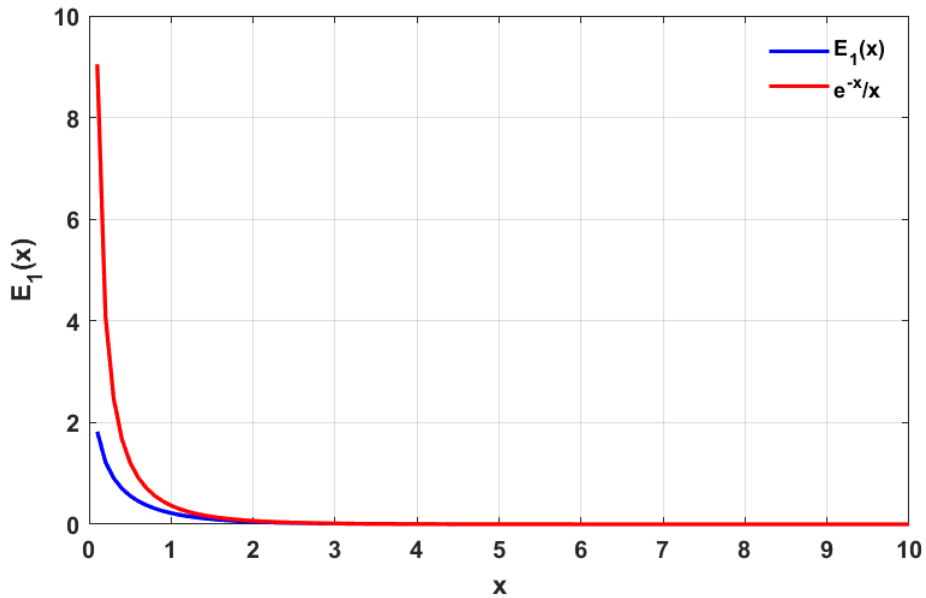
Fresnel Zone Plate

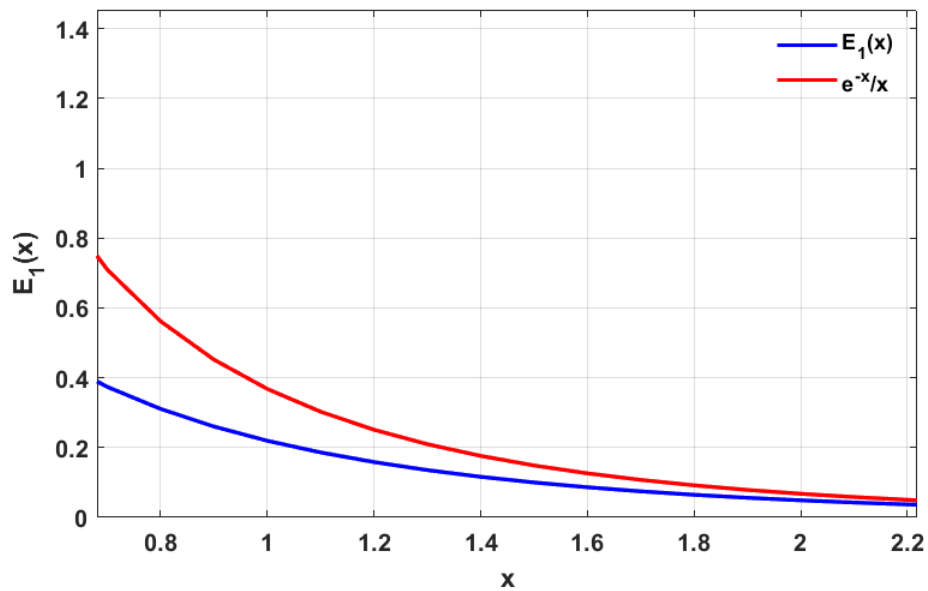


$$
\begin{aligned}
U_K(f) &= U_0 jk \iint_{S_a} \frac{e^{-jkR}}{4\pi R}\left(1+\frac{f}{R}\right) dS \\
&= \frac{U_0 jk}{2}\sum_{m=1}^{N}\int_{\rho_{m-1}}^{\rho_m} \frac{e^{-jkR}}{R}\left(1+\frac{f}{R}\right) r_T dr_T = \frac{U_0 jk}{2}\sum_{m=1}^{N}\int_{\rho_{m-1}}^{\rho_m} \frac{e^{-jkR}}{R}\left(1+\frac{f}{R}\right) R dR \\
&= \frac{U_0 jk}{2}\sum_{m=1}^{N}\left\{\frac{\exp(-jkR_{m-1})-\exp(-jkR_m)}{jk} + f\left[E_1(kR_{m-1})-E_1(kR_m)\right]\right\},
\end{aligned}
$$

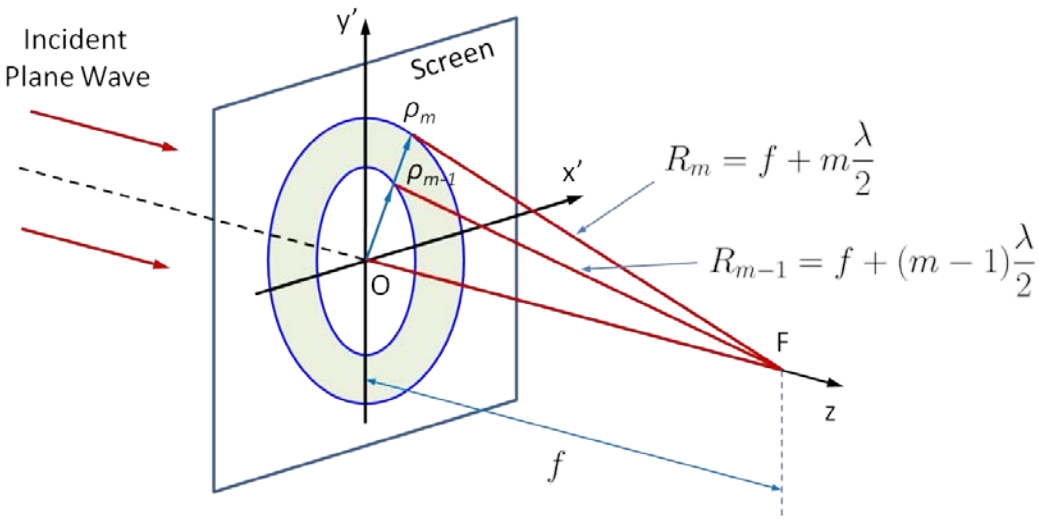
where $\quad E_1(w) = \int_w^{+\infty} \frac{e^{-t}}{t} dt.$

Exponential Integral Function





Fresnel Zone Plate



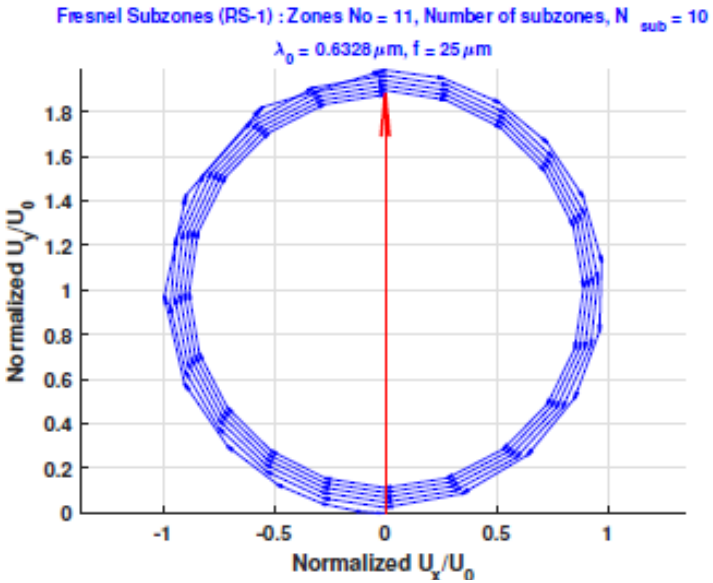
$$E_1(w) \simeq e^{-w}/w$$

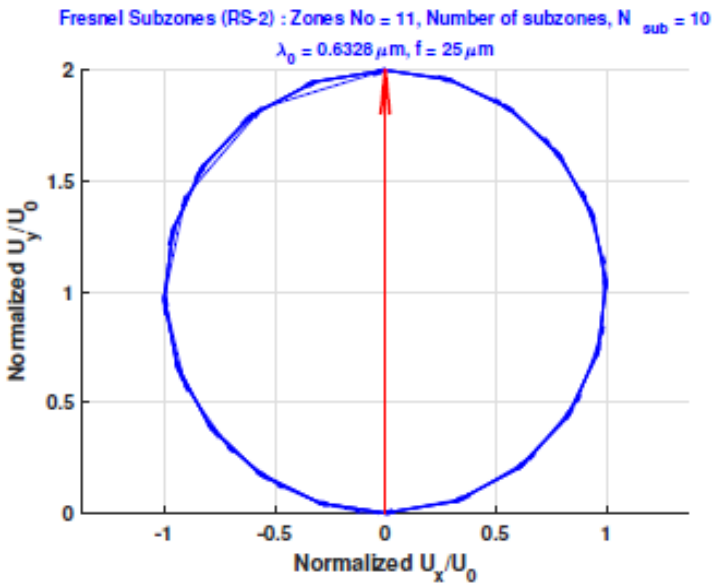
$$\begin{aligned} kR_{m-1} &= kf + (m-1)\pi \\ kR_m &= kf + m\pi \end{aligned}$$

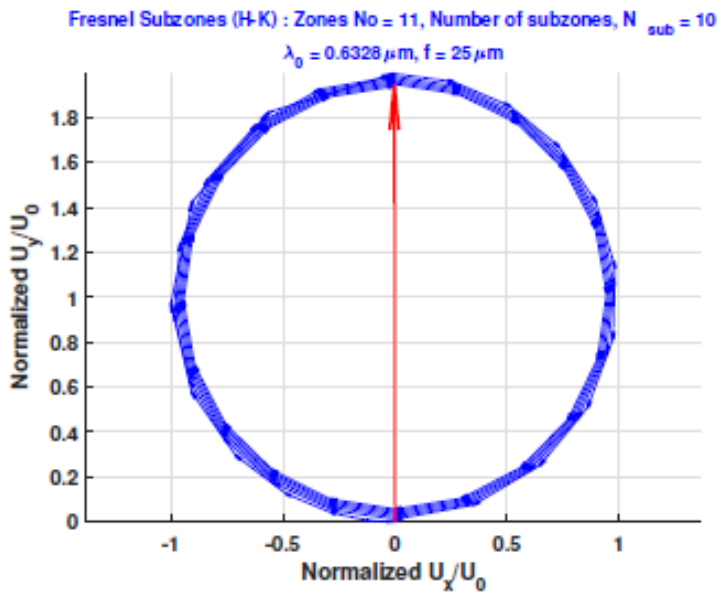
$$\begin{aligned} U_K(f) &\simeq \frac{U_0 e^{-jkf}}{2} \sum_{m=1}^{N} \{[\exp(-j(m-1)\pi) - \exp(-jm\pi)] + [\exp(-j(m-1)\pi) - \exp(-jm\pi)]\} \\ &= U_0 e^{-jkf} \sum_{m=1}^{N} \underbrace{[\exp(-j(m-1)\pi) - \exp(-jm\pi)]}_{V_{m-1}^{m}} \\ &= U_0 e^{-jkf} \left\{ \underbrace{[1-(-1)]}_{V_0^1} + \underbrace{[(-1)-1]}_{V_1^2} + \underbrace{[1-(-1)]}_{V_2^3} + \cdots \right\} = \sum_{m=1}^{N} [U_K]_{m-1}^{m} \Longrightarrow \end{aligned}$$

$$U_K(f) \simeq \begin{cases} 2U_0 e^{-jkf}, & \text{for odd } N, \\ 0, & \text{for even } N, \end{cases}$$

Fresnel Zone Plate

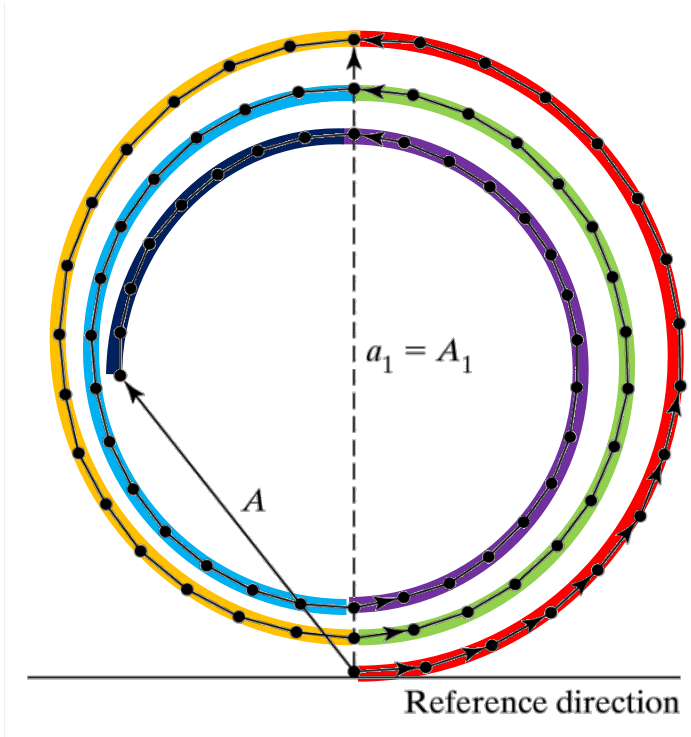






Fresnel Zone Plate

Each zone is subdivided into 15 subzones



1st Zone

2nd Zone

3rd Zone

4th Zone

5th Zone

5½ Zone

$$A_n = a_1 + a_2 e^{i\pi} + a_3 e^{i2\pi} + a_4 e^{i3\pi} + \ldots + a_n e^{i(n-1)\pi}$$

$$A_n = a_1 - a_2 + a_3 - a_4 + \ldots + e^{i(n-1)\pi} a_n$$

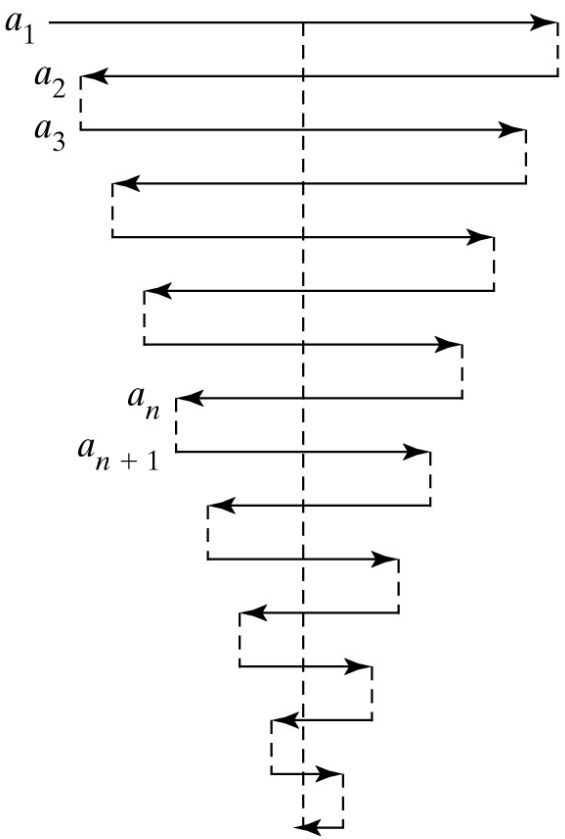
Prof. F.A. van Goor, Twente University

http://edu.tnw.utwente.nl/inlopt/overhead_sheets/

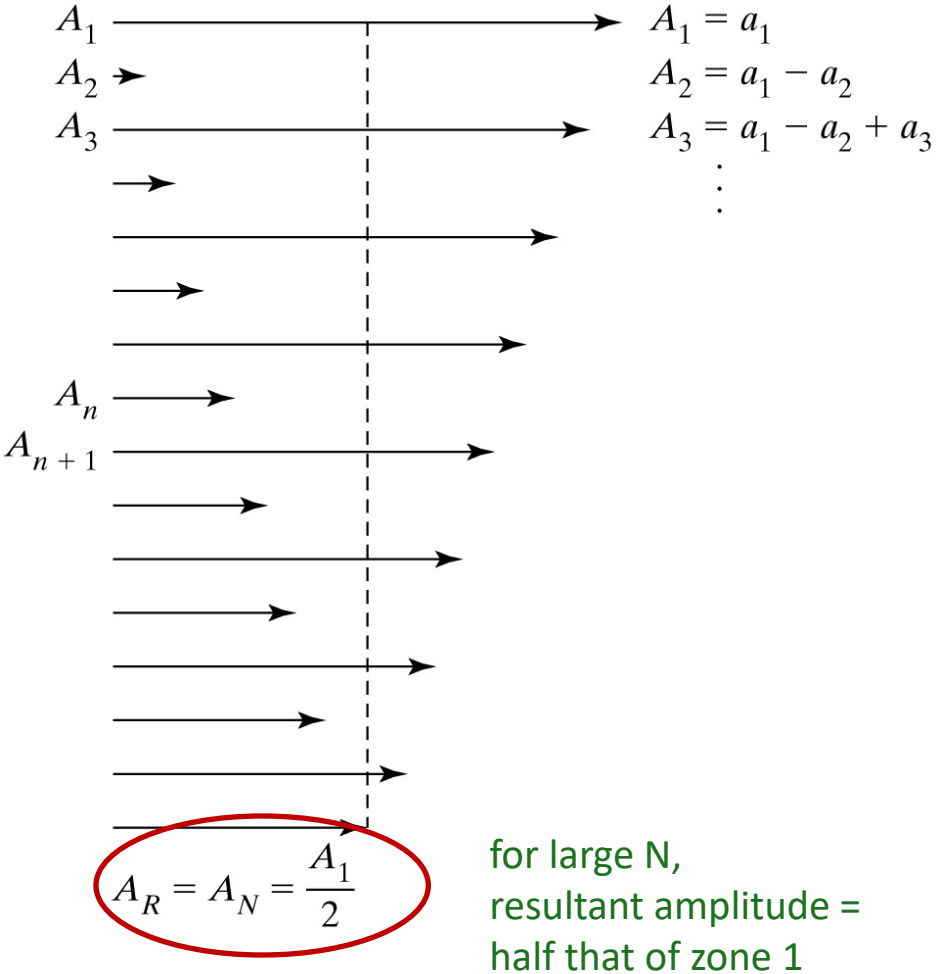
F. L. Pedrotti et al., "Introduction to Optics", 3nd Ed., Prentice Hall, New Jersey,2006

Adding up the zones

individual phasors



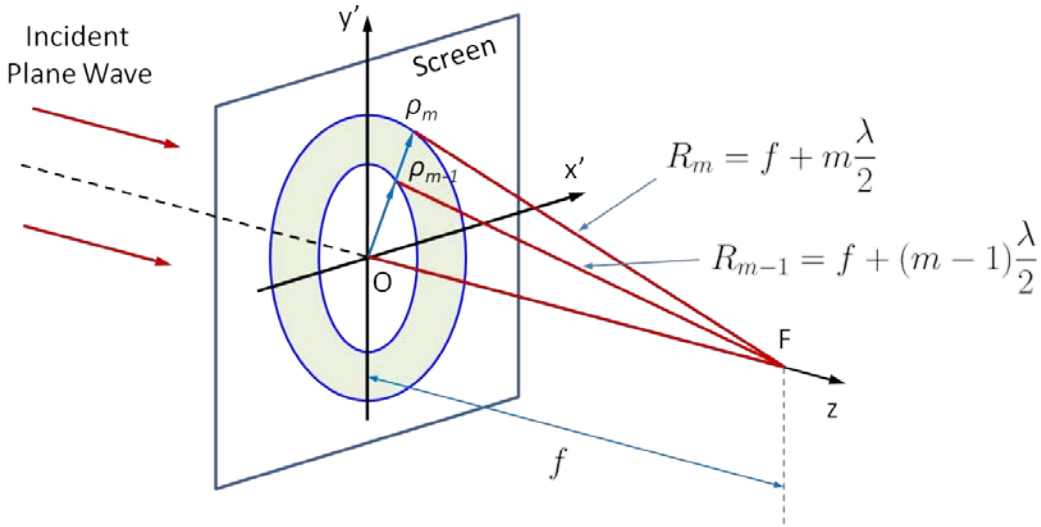
composite phasors



for large N,
resultant amplitude =
half that of zone 1

F. L. Pedrotti et al., "Introduction to Optics", 3nd Ed., Prentice Hall, New Jersey,2006

Fresnel Zone Plate



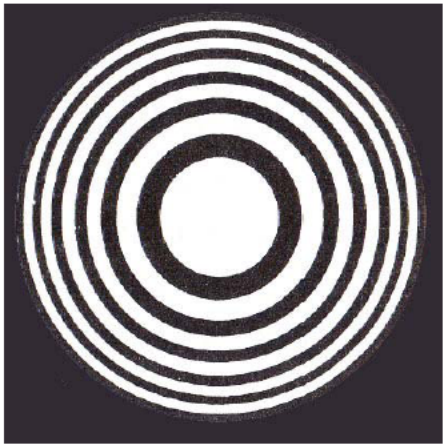
$$U_{FK}^{odd}(f) \simeq \begin{cases} 2U_0 e^{-jkf}\dfrac{N}{2} = NU_0 e^{-jkf}, & \text{for even } N, \\ 2U_0 e^{-jkf}\dfrac{N+1}{2} = (N+1)U_0 e^{-jkf}, & \text{for odd } N. \end{cases}$$

$$U_{FK}^{even}(f) \simeq \begin{cases} 2U_0 e^{-jkf}\dfrac{N}{2} = NU_0 e^{-jkf}, & \text{for even } N, \\ 2U_0 e^{-jkf}\dfrac{N-1}{2} = (N-1)U_0 e^{-jkf}, & \text{for odd } N. \end{cases}$$

$$\boxed{\begin{aligned} I_{FK}^{odd}(f) &\simeq \begin{cases} \dfrac{|U_0|^2}{4Z}N^2 = N^2 I_0, & \text{for even } N, \\ \dfrac{|U_0|^2}{4Z}(N+1)^2 = (N+1)^2 I_0, & \text{for odd } N, \end{cases} \\ I_{FK}^{even}(f) &\simeq \begin{cases} \dfrac{|U_0|^2}{4Z}N^2 = N^2 I_0, & \text{for even } N, \\ \dfrac{|U_0|^2}{4Z}(N-1)^2 = (N-1)^2 I_0, & \text{for odd } N. \end{cases} \end{aligned}}$$

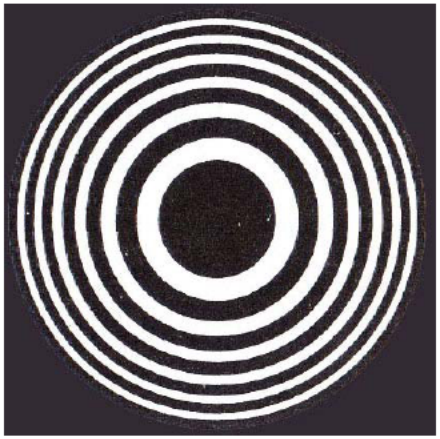
Fresnel Zone Plate

$$E_{tot} = |E_1| + |E_3| + \ldots + |E_{2m-1}| \approx m|E_i|$$



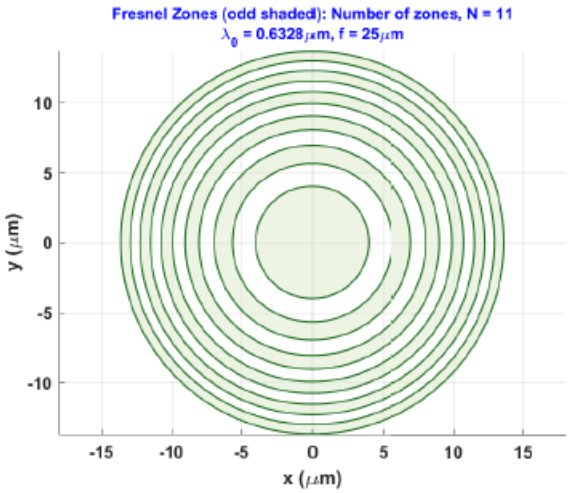
Odd (positive) Fresnel Plate

$$E_{tot} = -|E_2| - |E_4| - \ldots - |E_{2m}| = -m|E_i|$$

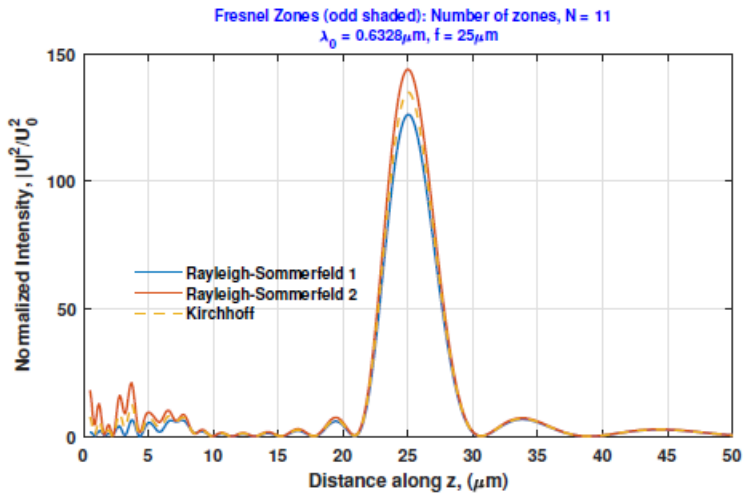


Even (negative) Fresnel Plate

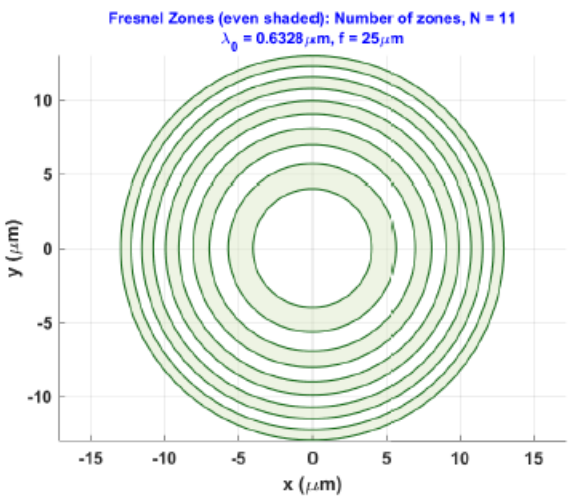
Fresnel Zone Plate



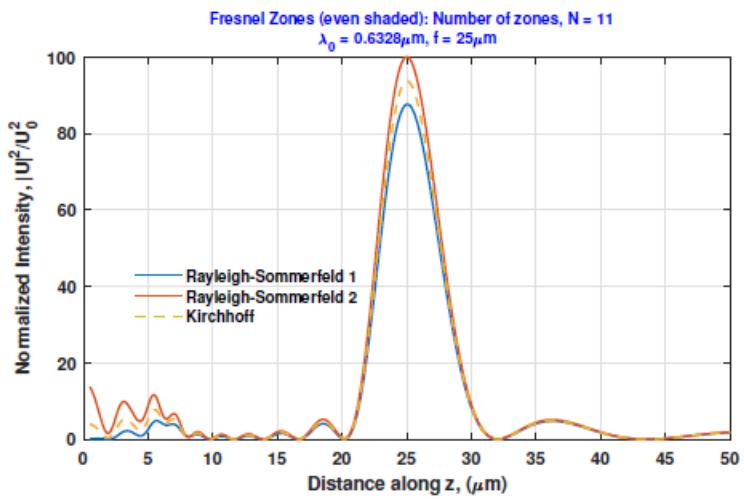
(a)



(b)

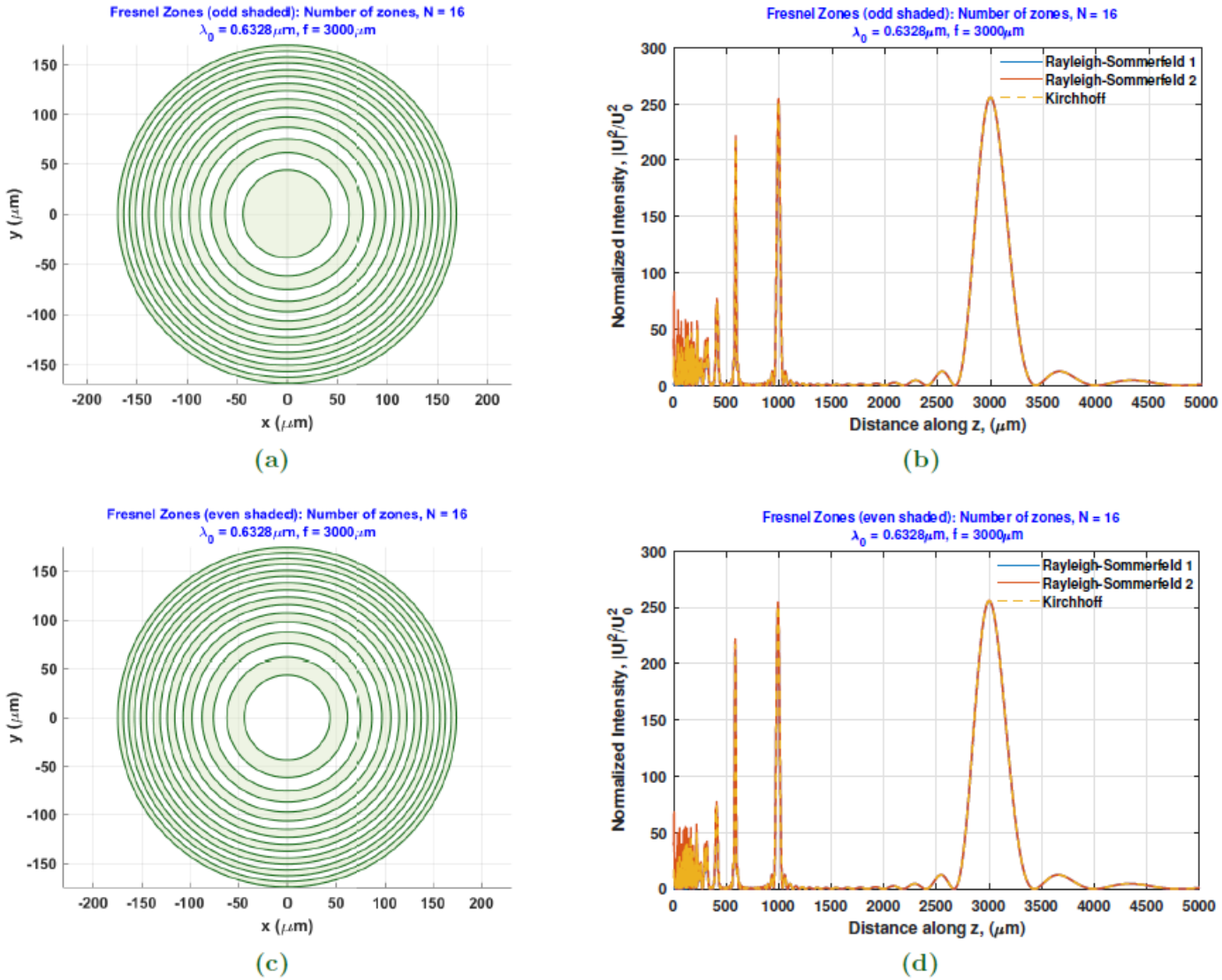


(c)

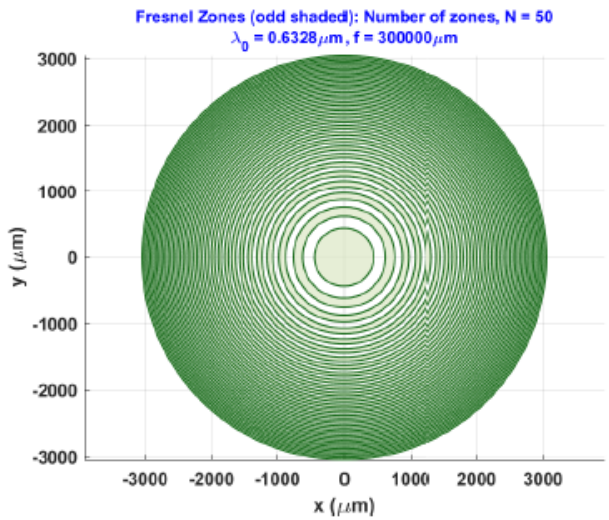


(d)

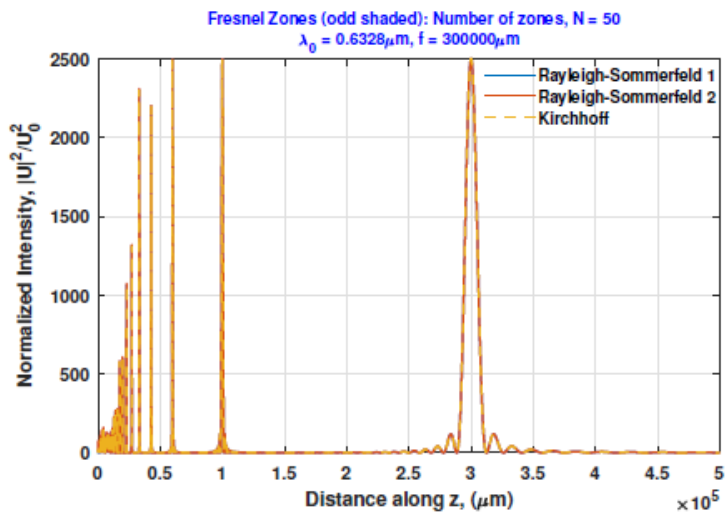
Fresnel Zone Plate



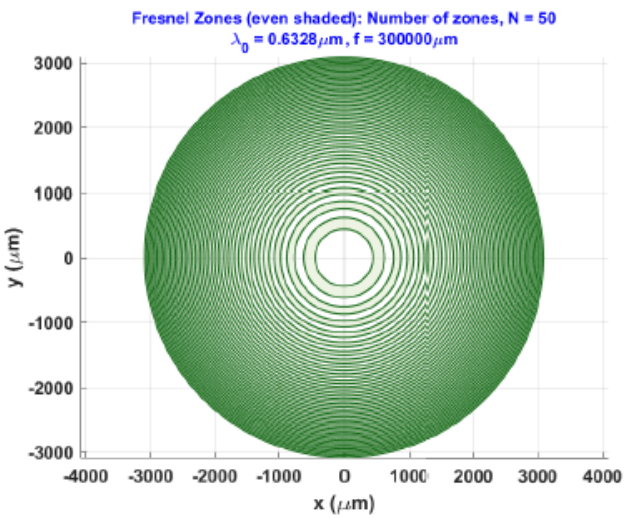
Fresnel Zone Plate



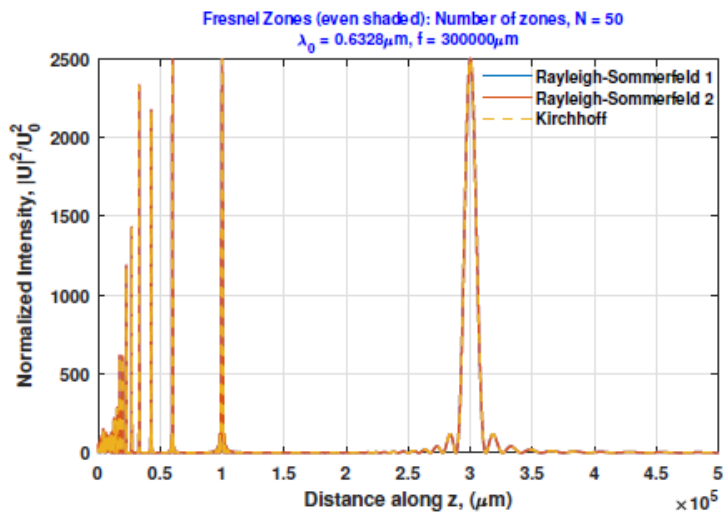
(a)



(b)



(c)



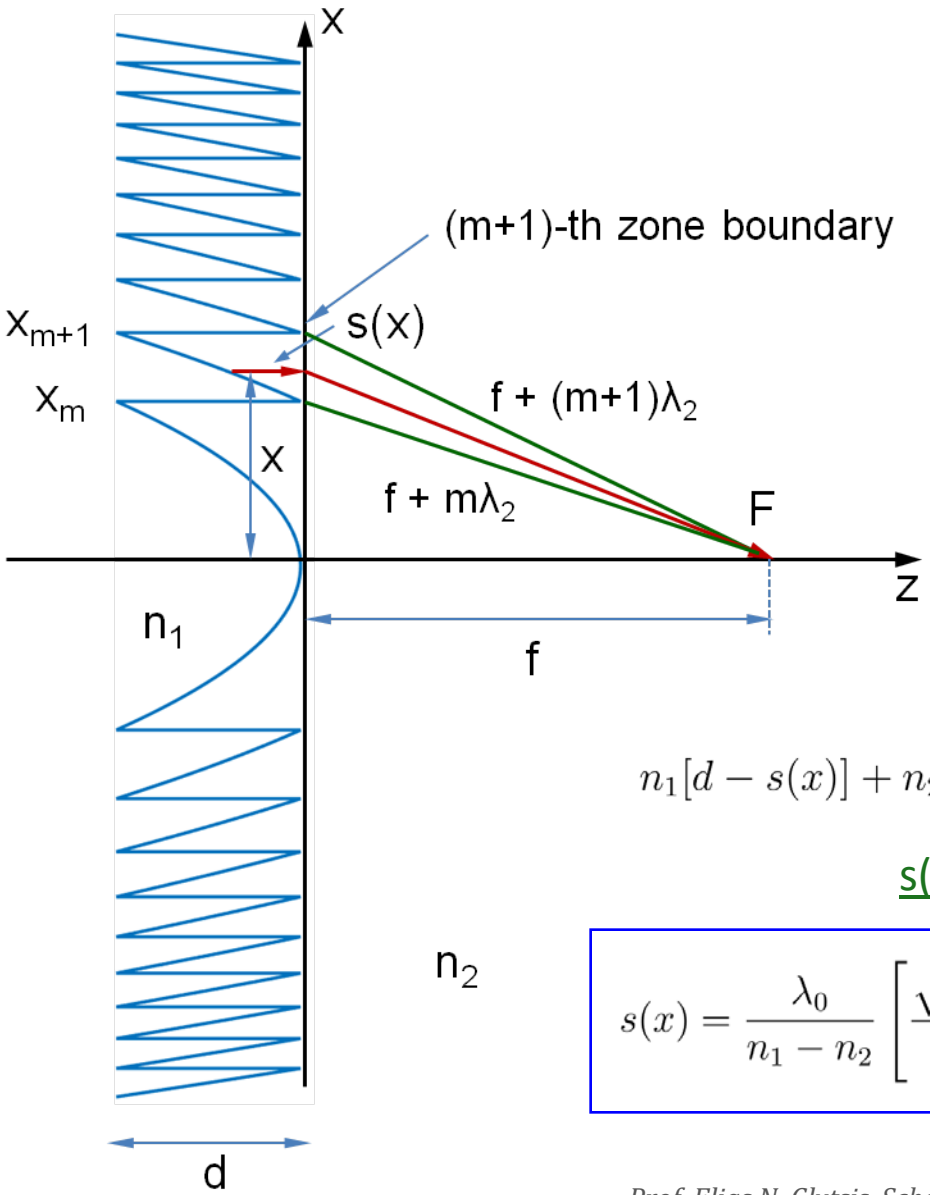
(d)

Fresnel Lenses

http://www.usglassmanufacturer.com/images/products/fresnel/625mm-RD-Lens-364.jpg

http://www.fresneloptic.com/images/solar_fresnel_lens.jpg

Diffractive Lens Design (Non-parabolic)



Zone Boundaries

$$x_m = \sqrt{2fm\lambda_2 + m^2\lambda_2^2} \qquad \lambda_2 = \frac{\lambda_0}{n_2}$$

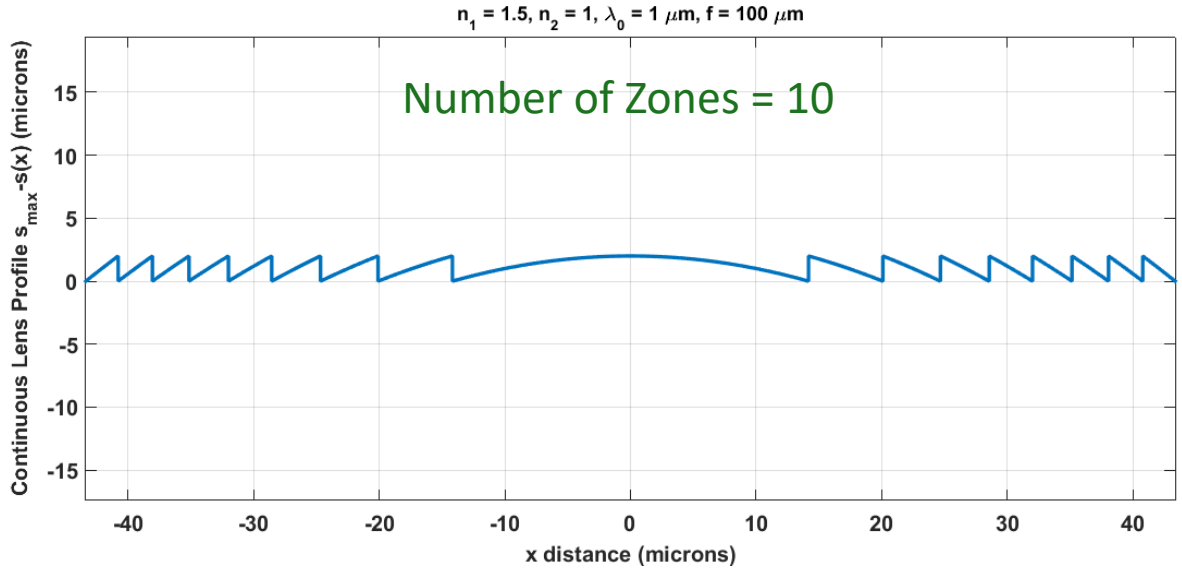
$$n_1[d - s(x)] + n_2 s(x) + n_2\sqrt{f^2 + x^2} = n_1 d + f + m\lambda_2$$

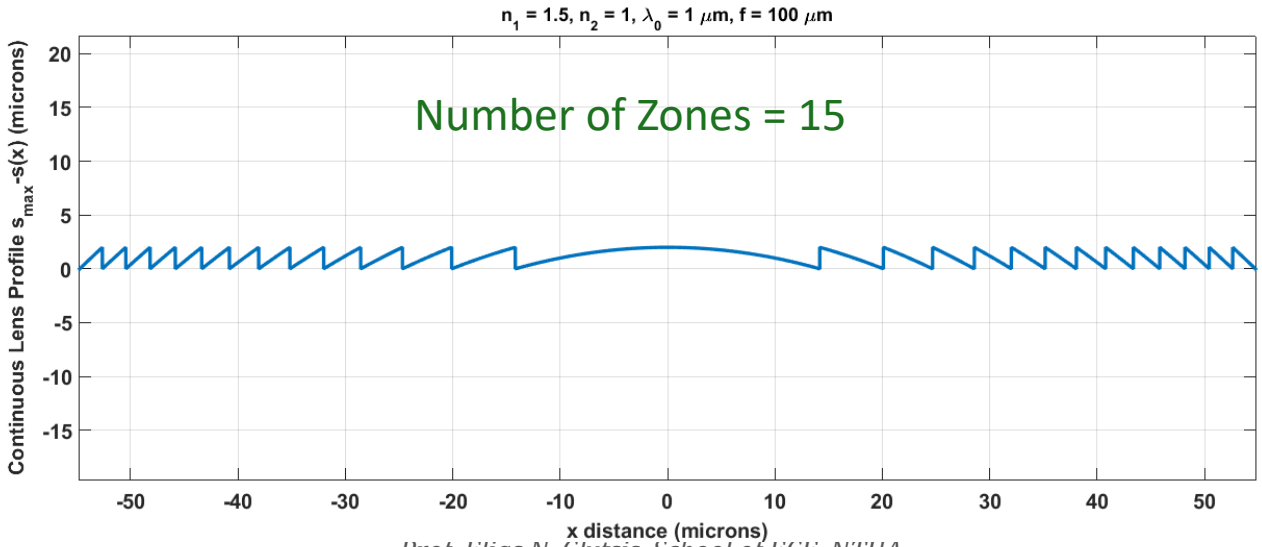
s(x) Determination

$$s(x) = \frac{\lambda_0}{n_1 - n_2}\left[\frac{\sqrt{f^2 + x^2} - f}{\lambda_2} - m\right] \qquad x_m \le x < x_{m+1}$$

Diffractive Lens Design Example

$\lambda_0 = 1\mu m$, $n_1 = 1.5$, $n_2 = 1.0$, $f = 100\mu m$,



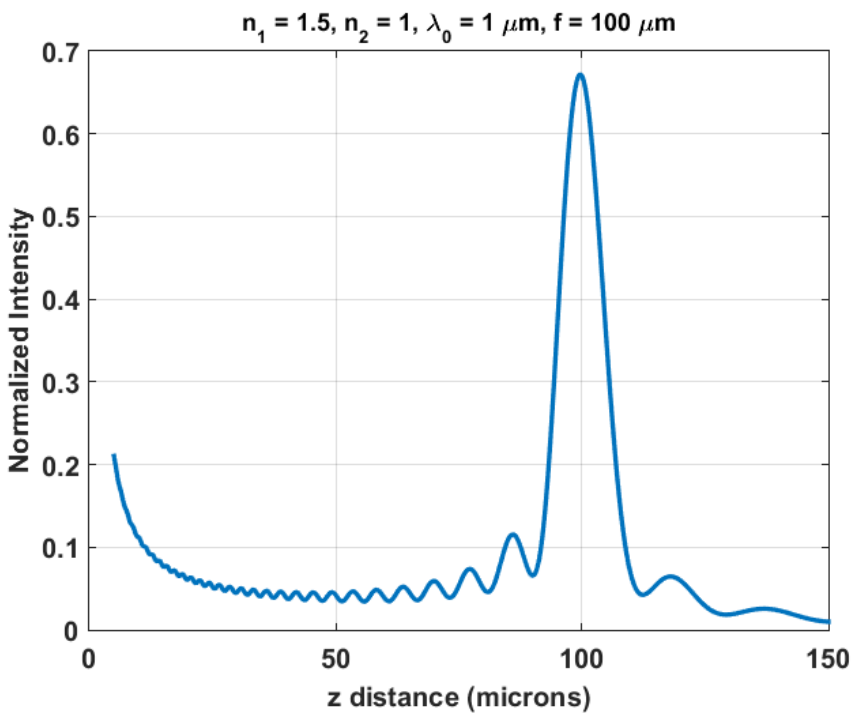


Diffractive Lens Performance

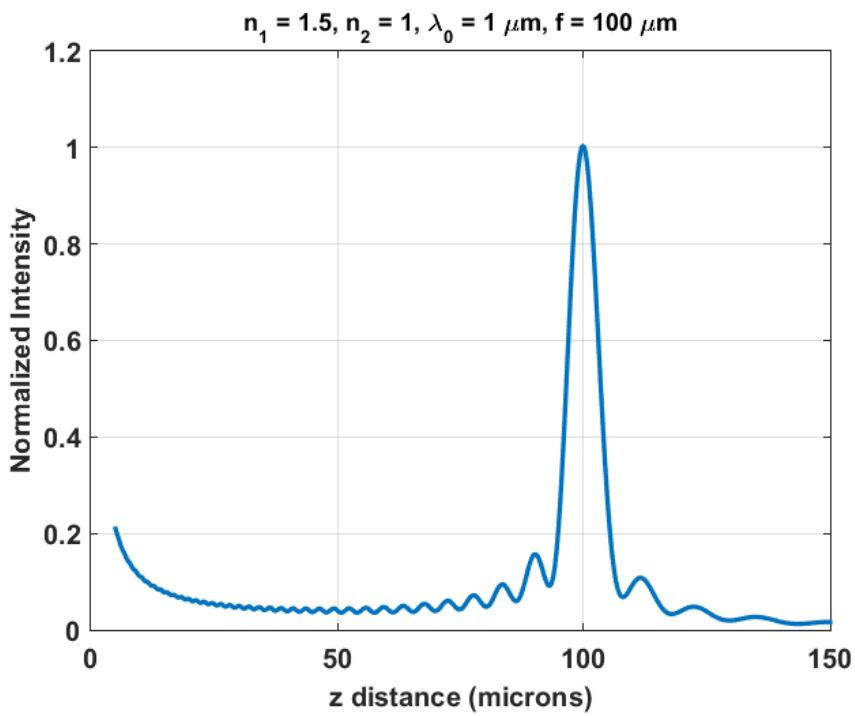
$$
\begin{aligned}
U(x,z) &= \int_{-D/2}^{D/2} \frac{\exp(-jkR)}{2\pi R}\left(jk+\frac{1}{R}\right)\frac{z}{R}U_i(x',z'=0)dx' \\
R &= [(x-x')^2+z^2]^{1/2} \\
U_i(x',z'=0) &= E_0 e^{-j\Phi(x')} \\
\Phi(x') &= -k_0(n_1-n_2)s(x')
\end{aligned}
$$

Diffractive Lens Performance

Number of Zones = 10

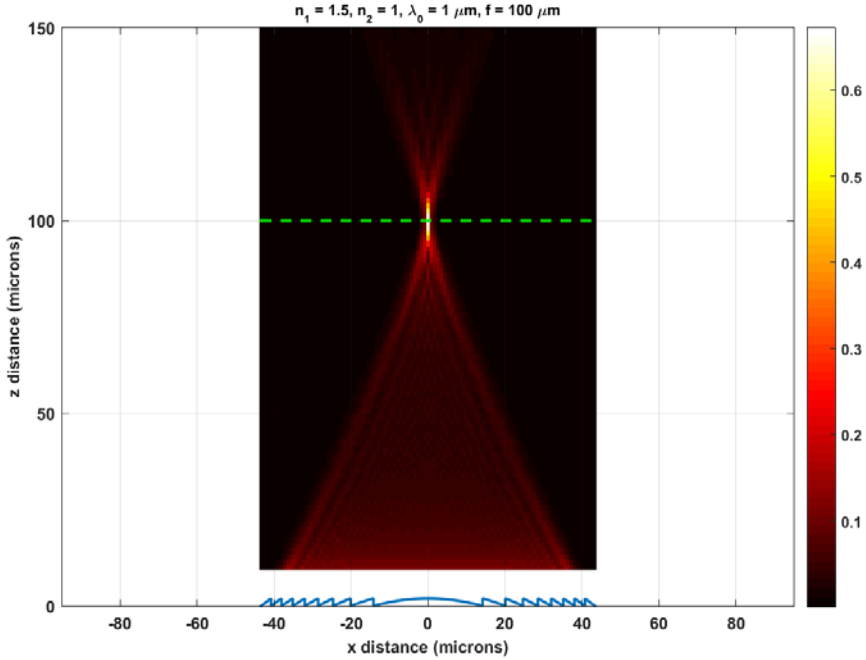


Number of Zones = 15

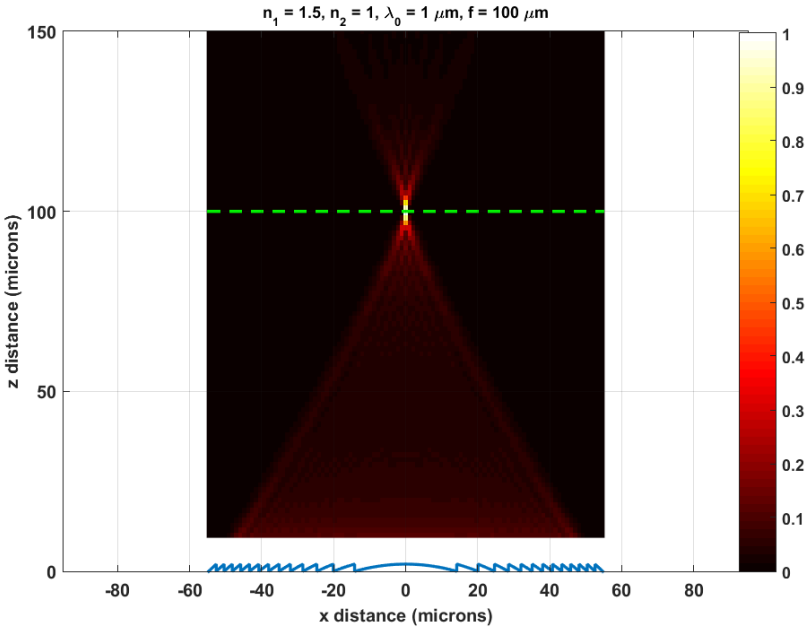


Diffractive Lens Performance

Number of Zones = 10

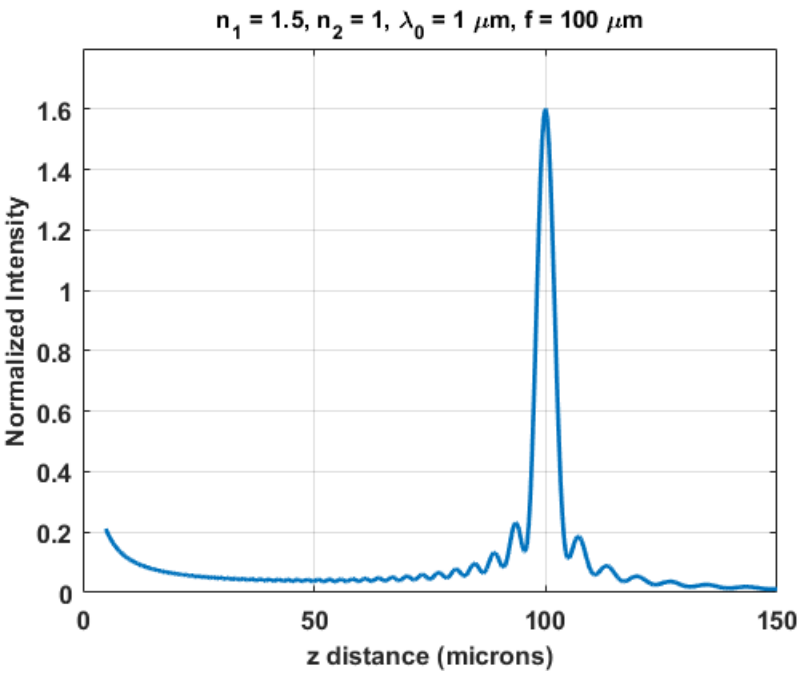


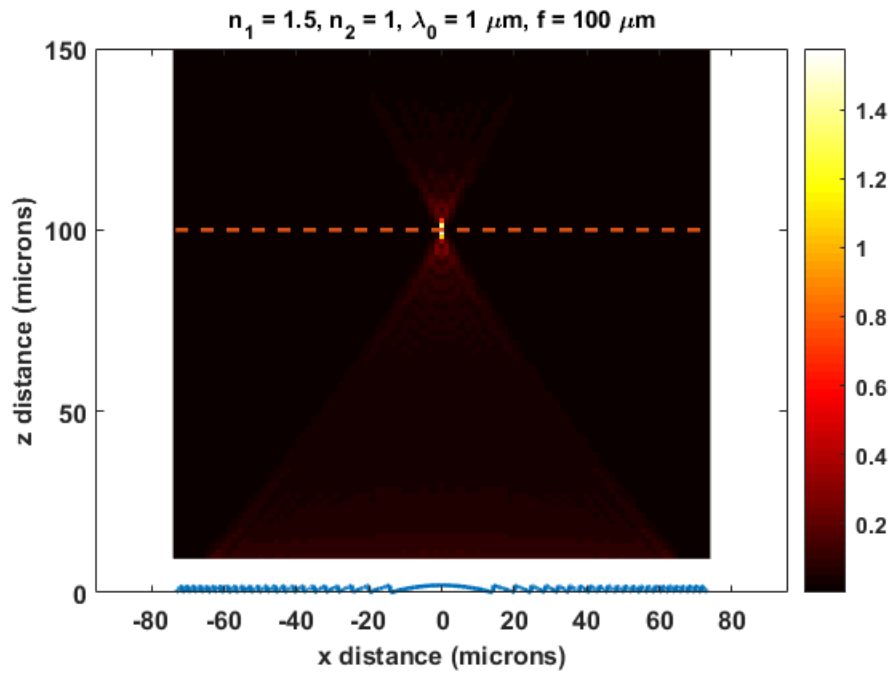
Number of Zones = 15



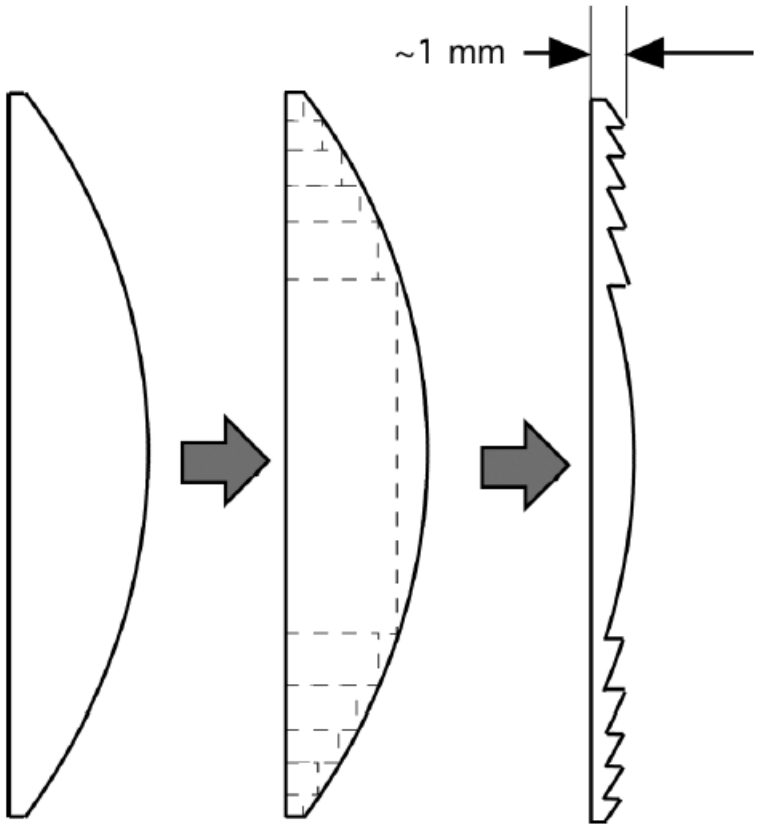
Diffractive Lens Performance

Number of Zones = 25

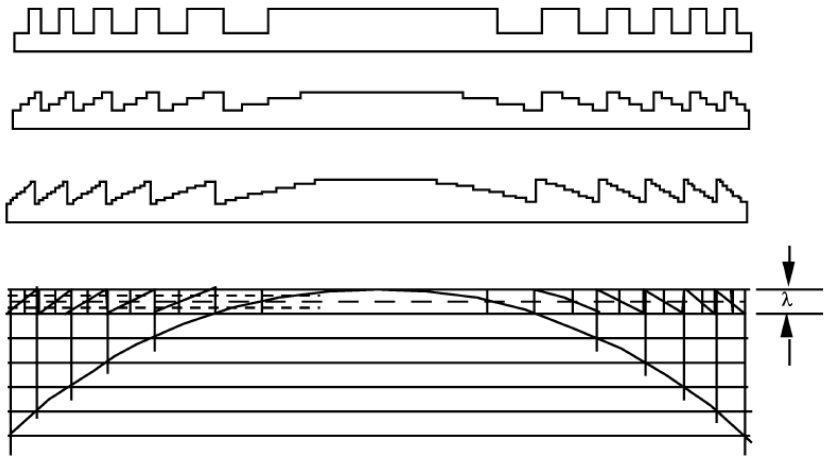


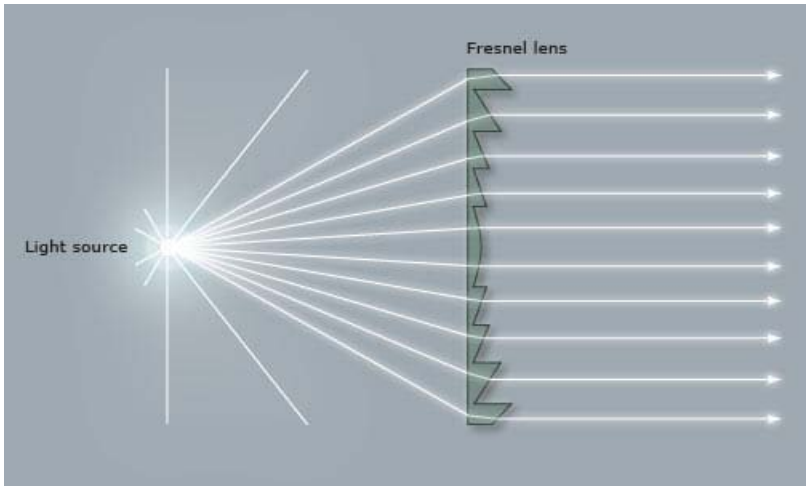


Diffractive Lenses



D. O'Shea et al., Diffractive Optics: Design, Fabrication & Testing, SPIE Press, 2004.

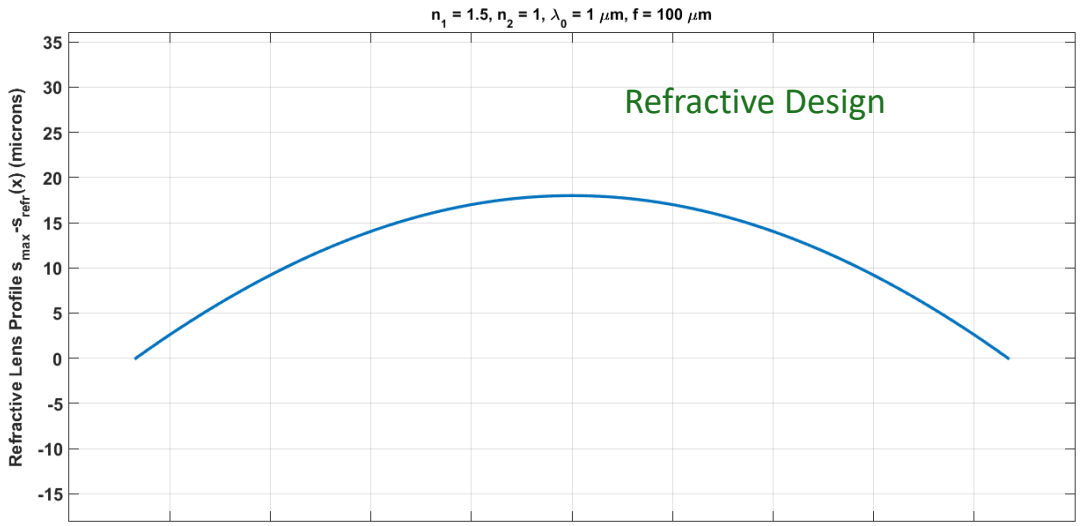


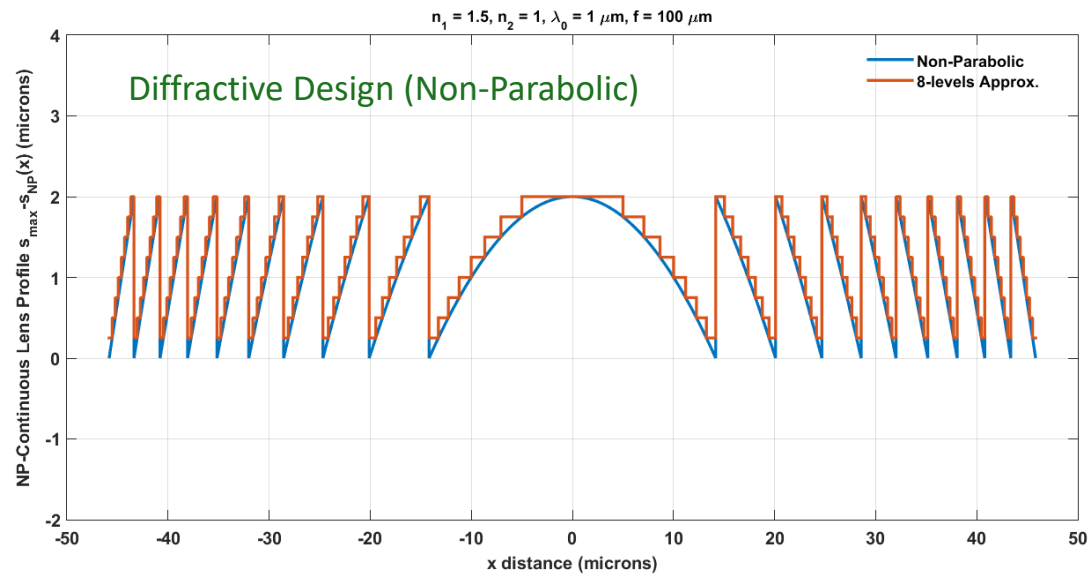


http://www.teara.govt.nz/files/di-6675-enz.jpg

N-Level Diffractive Lens Design Example

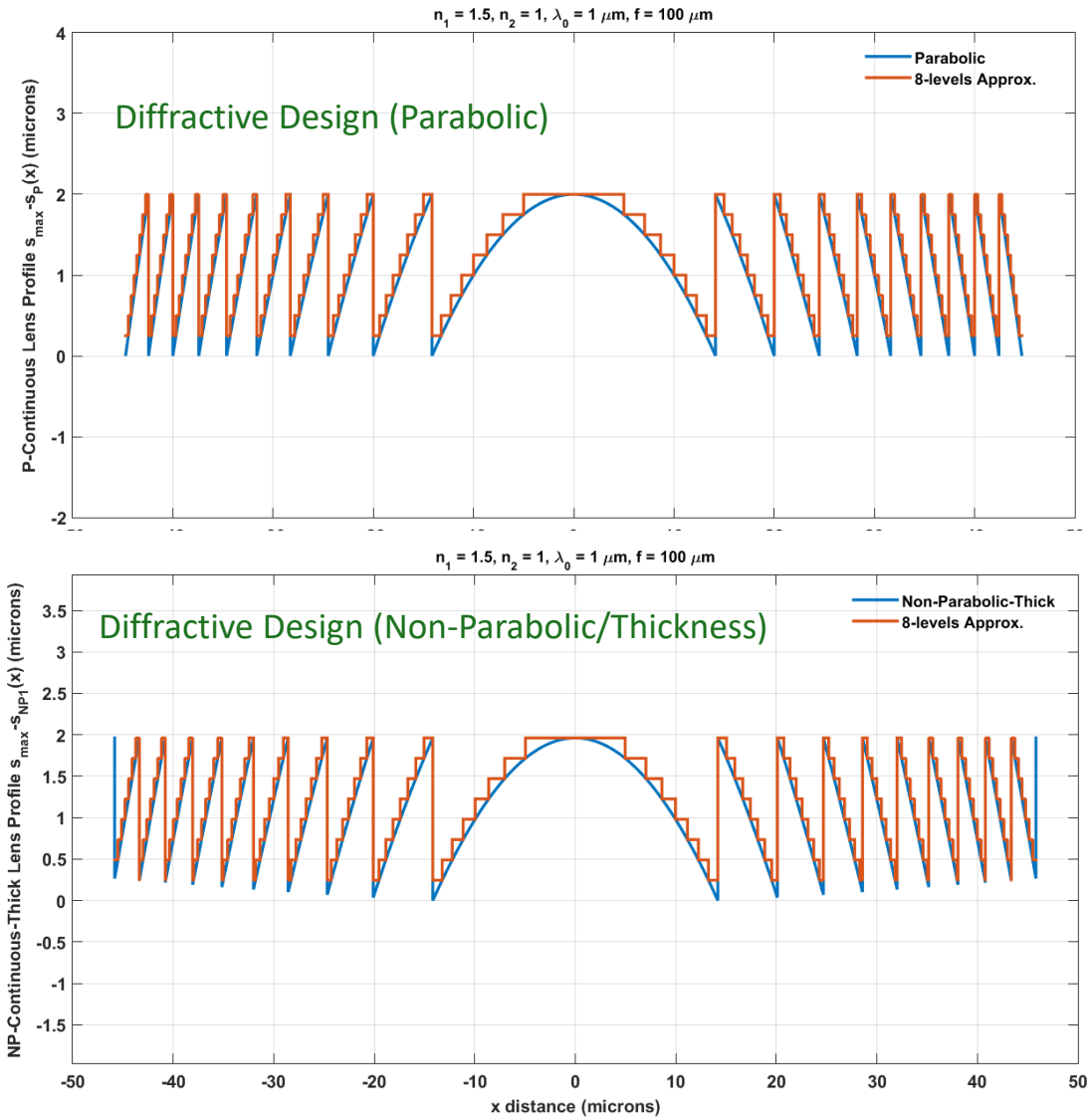
$\lambda_0 = 1\mu m$, $n_1 = 1.5$, $n_2 = 1.0$, $f = 100\mu m$,





N-Level Diffractive Lens Design Example

$\lambda_0 = 1\mu m$, $n_1 = 1.5$, $n_2 = 1.0$, $f = 100\mu m$,



Fresnel Lighthouse Lens

other applications:

overhead projectors
automobile headlights
solar collectors
traffic lights

Prof. F.A. van Goor, Twente University

http://edu.tnw.utwente.nl/inlopt/overhead_sheets/