CARTESIAN OVOIDS†

The Cartesian ovoids are plane curves that have been first described by Rene Descartes (in 1637 AD) and represent the geometric locus of points that have the same linear combination of distances from two fixed points. Let \( O \) (object in optical problems) and \( I \) (image in optical problems) be the two fixed points in the \( xy \)-plane, and let \( d(OS) \) and \( d(IS) \) denote the Euclidean distances from these points to a third variable point \( S \). Let \( n_o, n_i \) (refractive indices) and \( A = n_o s_o + n_i s_i \) be arbitrary real numbers. Then the Cartesian ovoid is the locus of points \( S \) satisfying \( n_o d(OS) + n_i d(IS) = A \). The two ovoids formed by the four equations \( n_o d(OS) + n_i d(IS) = \pm A \) and \( n_o d(OS) - n_i d(IS) = \pm A \) are closely related; together they form a quartic plane curve called the ovals of Descartes (from Wikipedia).

A. Cartesian Ovoids in the case \( n_o < n_i \)

![Figure 1: A Cartesian ovoid of refractive index \( n_i \) is shown in a surrounding medium of refractive index \( n_o \). The shape shown is valid for \( n_o < n_i \). Every ray from the point object \( O \) ends up in the image point \( I \). The function of the ovoid is \( P(x, y) = 0 \). The distances \( s_o \) and \( s_i \) are the object and image distances from the ovoid's left vertex.](image)

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aberration-free point-to-point imaging (shown in Fig. 1) is summarized next. In the following analysis it is assumed that $n_i > n_o$ except if otherwise is stated. The basic equation of the Cartesian ovoid can be written in the following form [replacing the distances $d(OS)$ and $d(IS)$ with the $xy$-plane expressions and assuming that point $O$ coincides with the origin of the $xy$ coordinate system]

$$n_o \sqrt{x^2 + y^2} + n_i \sqrt{(s_o + s_i - x)^2 + y^2} = n_os_o + n_is_i = A, \quad (1)$$

where $s_o$ and $s_i$ the object and image distances as shown in Fig. 1. The vertex points $V_1$ and $V_2$ are the points of the ovoid for $y = 0$. These points can be determined by setting $y = 0$ in the previous equation which results in the following

$$n_o|x| + n_is_o + s_i - x| = A, \quad (2)$$

which has the following solutions

$$x = s_o, \quad \text{for} \quad 0 < x < s_o + s_i \quad (3)$$

$$x = \frac{A + n_i(s_o + s_i)}{n_o + n_i} = s_o + \frac{2n_i}{n_o + n_i}s_i, \quad \text{for} \quad s_o + s_i < x < \infty. \quad (4)$$

The first of the previous equations [Eq. (3)] shows the position of $V_1$ and the second [Eq. (4)] shows the position of $V_2$. For example, if the values of the parameters are $s_0 = 5$ cm, and $s_i = 10, 15, 20$ cm, $n_o = 1.0$, and $n_i = 1.50$ the values of $V_2$, as measured from $x = 0$, (where the object $O$ is) are 17, 23, and 29 cm, respectively.

In order to find the equation of the ovoid, $P(x, y) = 0$, Eq. (1) must be solved. This equation can be transformed by moving one square root to the right hand side and raising to the second power both sides twice. Then the square roots are eliminated and a polynomial form is obtained

$$a_1y^4 + b_1(x)y^2 + c_1(x) = 0, \quad \text{with}$$

$$a_1 = (n_o^2 - n_i^2)^2,$$

$$b_1(x) = 2\left[A^2 + B(x)\right](n_o^2 - n_i^2) - 4A^2n_o^2 = B_2x^2 + B_1x + B_0 =$$

$$= \frac{2(n_o^2 - n_i^2)^2x^2}{B_2} + \frac{4n_i(n_o^2 - n_i^2)(s_o + s_i)x}{B_1} - \frac{2(n_o^2 + n_i^2)(n_os_o + n_is_i)^2 - 2n_i^2(n_o^2 - n_i^2)(s_o + s_i)^2}{B_0},$$

\[2\]
\[ c_1(x) = \left[A^2 + B(x)\right]^2 - 4A^2n_o^2x^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 = \]
\[
\begin{align*}
&= \frac{(n_o^2 - n_i^2)^2x^4 + 4n_i^2(n_o^2 - n_i^2)(s_o + s_i)x^3 +}{C_4}
&\quad - \left[2(n_o^4 + 2n_o^2n_i^2 - 3n_i^4)s_o^2 + 4(n_o^3n_i + n_o^2n_i^2 + n_o n_i^3 - 3n_i^4)s_o s_i + 4(n_o n_i^2 - n_i^4)s_i^2\right]x^2 +
&\quad \left[4n_o^2 s_o (s_o + s_i)(n_o - n_i)(n_o s_o + n_i s_o + 2n_i s_i)x + s_o^2 (n_o - n_i)^2 (n_o s_o + n_i s_o + 2n_i s_i)^2\right],
\end{align*}
\]

\[ \begin{align*}
A &= n_o s_o + n_i s_i, \\
B(x) &= n_o^2 x^2 - n_i^2 (s_o + s_i - x)^2.
\end{align*} \]

The solutions to Eq. (5) are
\[
\begin{align*}
y_1(x) &= \pm \left\{ \frac{-b_1(x) + \sqrt{b_1(x)^2 - 4a_1 c_1(x)}}{2a_1} \right\}^{1/2}, \\
y_2(x) &= \pm \left\{ \frac{-b_1(x) - \sqrt{b_1(x)^2 - 4a_1 c_1(x)}}{2a_1} \right\}^{1/2}.
\end{align*}
\]

The negative solutions show that the ovoid is symmetric with respect to the \(x\)-axis. From the two solutions the smaller in absolute value is the one that specifies the ovoid that is sought \([n_o d(OS) + n_i d(IS) = A]\). I.e. the solution for \(y(x)\) that satisfies \(P(x, y(x)) = 0\) shown in Fig. 1 is the \(|y(x)| = \min\{|y_1(x)|, |y_2(x)|\}\). The resulting ovoids for the parameters \(s_0 = 5\text{ cm}, \text{ and } s_i = 10, 15, 20\text{ cm}\) are shown in Fig. 2a. The larger ovoids are also shown in Fig. 2b along with their inner counterparts for comparison. These correspond to the solution \(n_o d(OS) - n_i d(IS) = -A\). It is observed that the left vertices remain constant as \(s_i\) varies while the right vertices increase with increasing \(s_i\).

In order to find the vertices of the larger ovoid the following equation can be derived if \(y = 0\) is set into Eq. (5). The resulting equation with respect to \(x\) is given by
\[
D_4 x^4 + D_3 x^3 + D_2 x^2 + D_1 x + D_0 = 0,
\]

where the coefficients \(D_i (i = 1, 2, \cdots, 5)\) are given by
\[
\begin{align*}
D_4 &= (n_o^2 - n_i^2)^2, \\
D_3 &= 4n_i^2(n_o^2 - n_i^2)(s_o + s_i),
\end{align*}
\]
Using the previous definitions Eq. (9) can be written in the form:

\[ D_2 = 2A^2(n_o^2 - n_i^2) + 4n_i^4(s_o + s_i)^2 - 4A^2 * n_o^2 - 2(n_o^2 - n_i^2)n_i^2(s_o + s_i)^2, \]
\[ = 4n_i^4(s_o + s_i)^2 - 4n_o^2(n_o s_o + n_i s_i)^2 + 2(n_o^2 - n_i^2)(n_o s_o + n_i s_i)^2 - n_i^2(s_o + s_i)^2(2n_o^2 - 2n_i^2), \]
\[ D_1 = 4A^2n_i^2(s_o + s_i - 4n_i^4(s_o + s_i)^3, \]
\[ = 4n_i^2(s_o + s_i)(n_o s_o + n_i s_i)^2 - 4n_i^4(s_o + s_i)^3, \]
\[ D_0 = A^4 - 2A^2n_i^2(s_o + s_i)^2 + n_i^4(s_o + s_i)^4 = s_o^2(n_o - n_i)^2(n_o s_o + n_i s_o + 2n_i s_i)^2. \]

Equation (9) can be solved either analytically (since there are analytical solutions up to the 4-th order polynomials) or it can be solved numerically using for example MatLab. Let the solutions be \( \rho_1 > \rho_2 > \rho_3 > \rho_4 \). The largest and the smallest solutions are the ones that correspond to \( V'_2 \) and \( V'_1 \) (\( \rho_1 \) and \( \rho_4 \)) of the larger ovoid, while the intermediate ones \( \rho_2 \) and \( \rho_3 \) correspond to the vertices \( V_2 \) and \( V_1 \) of the inner (useful) ovoid. For the outer ovoid the points for which there is only one solution for \( y_1(x) \) or \( y_2(x) \) are the ones for which the discriminant \( b_i^2 - 4a_i c_1 \) of Eq. (5) becomes zero. In this case a line parallel to the \( y \) axis is tangential to the outer ovoid from the left. These points can be found by solving the equation:

\[ b_i^2(x) - 4a_i c_1(x) = 0, \tag{9} \]

Using the previous definitions Eq. (9) can be written in the form:

\[ (B_2^2 - 4a_1 C_4)x^4 + (2B_2 B_1 - 4a_1 C_3)x^3 + (B_1^2 + 2B_0 B_2 - 4a_1 C_2)x^2 \]
\[ + (2B_0 B_1 - 4a_1 C_1)x + (B_0^2 - 4a_1 C_0) = 0 \implies \]
\[ W_1 x + W_0 = 0 \implies \rho' = -\frac{W_0}{W_1}, \quad \text{where} \]
\[ W_1 = -32n_o^2 n_i^2 (s_o + s_i)(n_o^2 - n_i^2)(n_o s_o + n_i s_i)^2, \]
\[ W_0 = 16n_o^2 n_i^2 (n_o s_o + n_i s_i)^2(2n_o^2 s_o^2 + 2n_o^2 s_o s_i + n_o^2 s_i^2 + 2n_o n_i s_o s_i - n_i^2 s_o^2 - 2n_i^2 s_o s_i). \]

The above polynomial has only one solution \( \rho' \), since its coefficients of forth, third, and second order with respect to \( x \) become zero. This real root should be less than \( V'_1 \) (i.e. \( \rho' < \rho_4 \)). For the example cases used before the roots are summarized in Table 1.

In order to find the maximum angle of rays emanating from \( O \) incident on the refractive surface of the ovoid \( P(x, y) = 0 \) and getting refracted passing through the image point \( I \), it is necessary to find the tangent from point \( O, OS \) to the ovoid surface. This is schematically
Table 1: Case of $n_o < n_i$. Vertices $V'_1$ and $V'_2$ of the outer ovoid and $V_1$ and $V_2$ of the inner ovoid (boldface). The $x = \rho'$ for outer ovoid.

<table>
<thead>
<tr>
<th>$s_0$ (cm)</th>
<th>$s_i$ (cm)</th>
<th>$\rho_1 [V'_1]$ (cm)</th>
<th>$\rho_2 [V_2]$ (cm)</th>
<th>$\rho_3 [V_1]$ (cm)</th>
<th>$\rho_4 [V'_2]$ (cm)</th>
<th>$\rho'$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>85</td>
<td>17</td>
<td>5</td>
<td>1</td>
<td>-3.1667</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>115</td>
<td>23</td>
<td>5</td>
<td>1</td>
<td>-5.1250</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>145</td>
<td>29</td>
<td>5</td>
<td>1</td>
<td>-7.1000</td>
</tr>
</tbody>
</table>

Figure 2: The three generated Cartesian ovoids are shown for the parameters $n_o = 1.0, n_i = 1.5, s_o = 5$ cm and $s_i = 10, 15$, and 20 cm. Just for information the other solution that corresponds to the larger root of Eq. (5) is also shown in Fig. 2b along with the smaller solutions in the same scale. The second (larger) solution is not accepted from a physical point of view. These ovoids are known in the literature as Cartesian ovoids.
Figure 3: The same Cartesian ovoid with the incident from $O$ beam being tangential to the ovoid. This specifies the maximum angle $\phi$ that by refraction should pass from point $I$. The $(x^*, y^*)$ denotes the point on the ovoid that the beam is tangential.

shown in Fig. 3. The angle $\phi$ of the tangent can be found if the following equation is solved

$$(\dot{y} = dy/dx)$$

$$\tan \phi = \frac{y(x^*)}{x^*} = \frac{dy}{dx} \bigg|_{x^*} = \dot{y}(x^*).$$

In order to solve the Eq. (11) it is necessary to find the derivative $\dot{y}(x)$ for the ovoid. This can be found by differentiating Eq. (1) with respect to the variable $x$ while retaining $y = y(x)$. After some algebra the equation for determining the derivative $\dot{y}(x)$ at any point in the ovoid is given by

$$\left\{ \frac{[(A_1(x, y) - A_2(x, y))y^2]}{F_2} \right\} \dot{y}^2 + \frac{2[A_1(x, y)xy + A_2(x, y)(s_o + s_i - x)y]}{F_1} \dot{y} + \frac{A_1(x, y)x^2 - A_2(x, y)(s_o + s_i - x)^2}{F_0} = 0,$$

(12)

where

$$A_1(x, y) = n_o^2[(s_o + s_i - x)^2 + y^2],$$

$$A_2(x, y) = n_i^2(x^2 + y^2),$$

$$F_2 = n_o^2 y^2[(s_o + s_i - x)^2 + y^2] - n_i^2 y^2(x^2 + y^2),$$

$$(A_1(x, y) - A_2(x, y)) = n_o^2[(s_o + s_i - x)^2 + y^2] - n_i^2(x^2 + y^2).$$
\[ F_1 = 2xy[(s_o + s_i - x)^2 + y^2]n_o^2 + 2y(x^2 + y^2)(s_o + s_i - x)n_i^2, \]
\[ F_0 = n_o^2x^2[(s_o + s_i - x)^2 + y^2] - n_i^2(x^2 + y^2)(s_o + s_i - x)^2. \]

Then Eq. (11) can be solved graphically if \( f_1(x) = y(x)/x \) and \( f_2(x) = \dot{y}(x) \) are plotted as functions of \( x \). The resulting curves and intersections points are shown in Fig. 4. When the numerical solution point \( (x^*, y^*) \) is specified the angle \( \phi \) can also be calculated from Eq. (11). The results are tabulated in Table 2.

**Figure 4:** Graphical solution for determining the point for which the incident beam from \( O \) is tangential to the ovoid. The intersection of the two curves gives the point \( x^* \) and then \( y^* \) can be determined from the equation of the ovoid \( P(x^*, y^*) = 0 \).

**Analytical solution for tangent to the Ovoid:**

Proposed by Mr. Orfeas Voutiras (Optical Science & Engineering Class - Spring 2015)

From Eq. (11) \( \tan \phi = \alpha = y^*/x^* \). Therefore \( y = \alpha x \) for the point \( (x^*, y^*) \) that is tangential to ovoid. Inserting this relation into Eq. (1) the following equation is derived
Table 2: Tangent points \((x^*, y^*)\) from \(O\) to the inner ovoid

<table>
<thead>
<tr>
<th>(s_0) (cm)</th>
<th>(s_i) (cm)</th>
<th>(x^*) (cm)</th>
<th>(y^*) (cm)</th>
<th>(\phi) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>8.6116</td>
<td>3.2925</td>
<td>20.9235</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>9.7478</td>
<td>4.4697</td>
<td>24.6331</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>10.7146</td>
<td>5.4948</td>
<td>27.1502</td>
</tr>
</tbody>
</table>

(setting \(s_0 + s_i = D\)):

\[
\sqrt{x^2 + \alpha^2x^2} + \frac{n_i}{n_0}(D - x)^2 + \alpha^2x^2 = \frac{n_0s_0 + n_is_i}{n_0} = A'.
\]  

(13)

Re-arranging the above equation and squaring both sides it is straightforward to derive the following second order polynomial equation with respect to \(x\) (setting \(\kappa = n_i/n_0\)):

\[
(1 + \alpha^2)(\kappa^2 - 1)x^2 + \left[2A'(1 + \alpha^2)^{1/2} - 2D\kappa^2\right]x + D^2\kappa^2 - A'^2 = 0.
\]

(14)

The last equation should have a single solution with respect to \(x\) in order to correspond to the tangent to the ovoid. Therefore, the discriminant of the polynomial of Eq. (13) should be zero. This can be expressed from the following equation:

\[
\mathcal{D} = 4 \left[A'(1 + \alpha^2)^{1/2} - D\kappa^2\right]^2 - 4(D^2\kappa^2 - A'^2)(1 + \alpha^2)(\kappa^2 - 1) = 0.
\]

(15)

Setting \(p = (1 + \alpha^2)^{1/2}\) the previous equation is written as

\[
\left[A'^2 - (D^2\kappa^2 - A'^2)(\kappa^2 - 1)\right]p^2 - \left[2A'D\kappa^2\right]p + D^2\kappa^4 = 0 \implies \quad a_1p^2 + a_2p + a_3 = 0 \implies
\]

(16)

\[
p = (1 + \alpha^2)^{1/2} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1a_3}}{2a_1} \implies \alpha = \pm \sqrt{p^2 - 1},
\]

(17)

where the root for \(|\alpha| < 1\) is the one that corresponds to the solution of interest. Using the above analytical solutions the following results can be derived (see Table 3). It is reminded that \(\phi = \tan^{-1}(\alpha)\).

Of course the slope does not immediately specify the point \((x^*, y^*)\). However, this can be found solving Eq. (14) for \(x^*\) if \(y^* = \alpha x^*\) and \(\dot{y} = \alpha\). It is given by

\[
x^* = \frac{A'(1 + \alpha^2)^{1/2} - D\kappa^2}{(1 + \alpha^2)(\kappa^2 - 1)}, \quad y^* = \alpha x^*
\]

(18)
Table 3: Results for question (c) using the analytical approach:

<table>
<thead>
<tr>
<th>$s_0$ (cm)</th>
<th>$s_i$ (cm)</th>
<th>$\alpha$</th>
<th>$\phi$ (deg)</th>
</tr>
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<tbody>
<tr>
<td>5</td>
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<td>0.3823</td>
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</tr>
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<td>5</td>
<td>20</td>
<td>0.5128</td>
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</table>

Table 4: Case of $n_o > n_i$. Vertices $V'_1$ and $V'_2$ of the outer ovoid and $V_1$ and $V_2$ of the inner ovoid (boldface). The $x = \rho'$ for outer ovoid.

<table>
<thead>
<tr>
<th>$s_0$ (cm)</th>
<th>$s_i$ (cm)</th>
<th>$\rho_1$ [$V'_1$] (cm)</th>
<th>$\rho_2$ [$V_1$] (cm)</th>
<th>$\rho_3$ [$V_2$] (cm)</th>
<th>$\rho_4$ [$V'_2$] (cm)</th>
<th>$\rho'$ (cm)</th>
</tr>
</thead>
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</tr>
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<td>-1</td>
<td>-85</td>
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</tr>
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<td>21</td>
<td>5</td>
<td>-1</td>
<td>-105</td>
<td>24.6000</td>
</tr>
</tbody>
</table>

B. Cartesian Ovoids in the case $n_o > n_i$

In this section the case of the higher to lower refractive index ($n_o > n_i$) Cartesian ovoid will be summarized. The case is shown in Fig. 5. Equation (1) is the same for this case too. However, the vertex points $V_1$ and $V_2$ (which are the points of the ovoid for $y = 0$) are given by slightly different equations due to the absolute values of Eq. (2). In this case Eq. (2) has the following solutions

$$x = s_0, \quad \text{for } x > 0$$  \hspace{1cm} (19)

$$x = -\frac{n_o - n_i}{n_o + n_i}s_0, \quad \text{for } -\infty < x < 0.$$  \hspace{1cm} (20)

It is interesting to observe that in this case the positions of the two vertices $V_1$ and $V_2$ are independent of $s_i$ and depend only on $s_o$ and on the refractive indices. Example positions are shown in Table 4 for $s_o = 5$ cm, $s_i = 10, 15, 20$ cm, and $n_o = 1.5, n_i = 1.0$.

Note: The above Cartesian ovoid curves are special cases of the general equation describing the Cartesian ovoids. This is the equation $d(OS) + md(IS) = a$ where $O$ and $I$ are the object and image points that were used before and $m$ and $a$ are constants. In our case $m = n_i/n_o$. 

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Figure 5: A Cartesian ovoid of refractive index $n_o$ is shown in a surrounding medium of refractive index $n_i$. The shape shown is valid for $n_o > n_i$. Every ray from the point object $O$ ends up in the image point $I$. The function of the ovoid is $P(x, y) = 0$. The distances $s_o$ and $s_i$ are the object and image distances from the ovoid’s left vertex.

and $a = s_o + s_i(n_i/n_o)$. Then the resulting general equation in $(x, y)$ is the following

$$\left[(1 - m^2)(x^2 + y^2) + 2m^2cx + a^2 - m^2c^2\right]^2 = 4a^2(x^2 + y^2),$$

(21) which has solutions as the ovoids shown in Fig. 2.
Figure 6: Cartesian ovoids for the case of $n_o > n_i$. These are the ovoids for $n_o = 1.5$, $n_i = 1.0$, $s_o = 5 \text{ cm}$, and $s_i = 10, 15, 20 \text{ cm}$. The useful ovoids are the egg-like ones that are very similar since the points $V_1$ and $V_2$ do not depend on $s_i$. 