CARTESIAN OVOIDS†

The Cartesian ovoids are plane curves that have been first described by Rene Descartes (in 1637 AD) and represent the geometric locus of points that have the same linear combination of distances from two fixed points. Let O (object in optical problems) and I (image in optical problems) be the two fixed points in the xy-plane, and let $d(OS)$ and $d(IS)$ denote the Euclidean distances from these points to a third variable point S. Let n_o, n_i (refractive indices) and $A = n_o s_o + n_i s_i$ be arbitrary real numbers. Then the Cartesian ovoid is the locus of points S satisfying $n_o d(OS) + n_i d(IS) = A$. The two ovoids formed by the four equations $n_o d(OS) + n_i d(IS) = \pm A$ and $n_o d(OS) - n_i d(IS) = \pm A$ are closely related; together they form a quartic plane curve called the *ovals of Descartes* (from Wikipedia).

A. Cartesian Ovoids in the case $n_o < n_i$

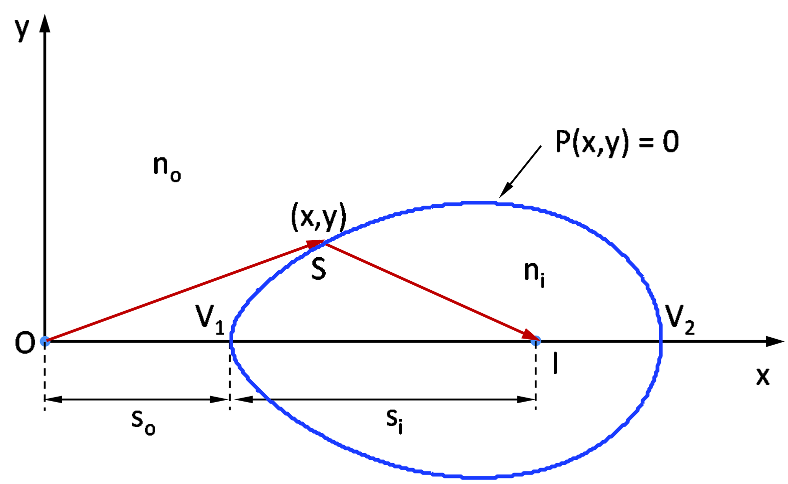

Figure 1: A Cartesian ovoid of refractive index n_i is shown in a surrounding medium of refractive index n_o. The shape shown is valid for $n_o < n_i$. Every ray from the point object O ends up in the image point I. The function of the ovoid is $P(x, y) = 0$. The distances s_o and s_i are the object and image distances from the ovoid's left vertex.

The basic problem of determining the Cartesian ovoid as a refractive surface for spherical-

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aberration-free point-to-point imaging (shown in Fig. 1) is summarized next. In the following analysis it is assumed that $n_i > n_o$ except if otherwise is stated. The basic equation of the Cartesian ovoid can be written in the following form [replacing the distances $d(OS)$ and $d(IS)$ with the xy-plane expressions and assuming that point O coincides with the origin of the xy coordinate system]

$$n_o\sqrt{x^2+y^2} + n_i\sqrt{(s_o+s_i-x)^2+y^2} = n_o s_o + n_i s_i = A, \tag{1}$$

where s_o and s_i the object and image distances as shown in Fig. 1. The vertex points V_1 and V_2 are the points of the ovoid for $y=0$. These points can be determined by setting $y=0$ in the previous equation which results in the following

$$n_o|x| + n_i|s_o+s_i-x| = A, \tag{2}$$

which has the following solutions

$$x = s_0, \qquad \text{for} \quad 0 < x < s_o + s_i \tag{3}$$
$$x = \frac{A + n_i(s_o+s_i)}{n_o+n_i} = s_o + \frac{2n_i}{n_o+n_i}s_i, \qquad \text{for} \quad s_o+s_i < x < \infty. \tag{4}$$

The first of the previous equations [Eq. (3)] shows the position of V_1 and the second [Eq. (4)] shows the position of V_2. For example, if the values of the parameters are $s_0 = 5\,\text{cm}$, and $s_i = 10, 15, 20\,\text{cm}$, $n_o = 1.0$, and $n_i = 1.50$ the values of V_2, as measured from $x=0$, (where the object O is) are 17, 23, and 29 cm, respectively.

In order to find the equation of the ovoid, $P(x,y)=0$, Eq. (1) must be solved. This equation can be transformed by moving one square root to the right hand side and raising to the second power both sides twice. Then the square roots are eliminated and a polynomial form is obtained

$$\begin{aligned} a_1 y^4 &+ b_1(x)y^2 + c_1(x) = 0, \quad \text{with} \qquad (5)\\ a_1 &= (n_o^2 - n_i^2)^2,\\ b_1(x) &= 2\left[A^2 + B(x)\right](n_o^2-n_i^2) - 4A^2n_o^2 = B_2x^2 + B_1x + B_0 =\\ &= \underbrace{2(n_o^2-n_i^2)^2}_{B_2}x^2 + \underbrace{4n_i^2(n_o^2-n_i^2)(s_o+s_i)}_{B_1}x \underbrace{-2(n_o^2+n_i^2)(n_os_o+n_is_i)^2 - 2n_i^2(n_o^2-n_i^2)(s_o+s_i)^2}_{B_0}, \end{aligned}$$

$$
\begin{aligned}
c_1(x) &= \Big[A^2 + B(x)\Big]^2 - 4A^2 n_o^2 x^2 = C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0 = \\
&= \underbrace{(n_o^2 - n_i^2)^2}_{C_4} x^4 + \underbrace{4n_i^2(n_o^2 - n_i^2)(s_o + s_i)}_{C_3} x^3 + \\
&\quad \underbrace{-[2(n_o^4 + 2n_o^2 n_i^2 - 3n_i^4)s_o^2 + 4(n_o^3 n_i + n_o^2 n_i^2 + n_o n_i^3 - 3n_i^4)s_o s_i + 4(n_o^2 n_i^2 - n_i^4)s_i^2]}_{C_2} x^2 + \\
&\quad \underbrace{4n_i^2 s_o (s_o + s_i)(n_o - n_i)(n_o s_o + n_i s_o + 2n_i s_i)}_{C_1} x + \underbrace{s_o^2 (n_o - n_i)^2 (n_o s_o + n_i s_o + 2n_i s_i)^2}_{C_0}, \\
A &= n_o s_o + n_i s_i, \\
B(x) &= n_o^2 x^2 - n_i^2 (s_o + s_i - x)^2.
\end{aligned}
$$

The solutions to Eq. (5) are

$$
y_1(x) = \pm \left\{ \frac{-b_1(x) + \sqrt{b_1(x)^2 - 4a_1 c_1(x)}}{2a_1} \right\}^{1/2}, \tag{6}
$$

$$
y_2(x) = \pm \left\{ \frac{-b_1(x) - \sqrt{b_1(x)^2 - 4a_1 c_1(x)}}{2a_1} \right\}^{1/2}. \tag{7}
$$

The negative solutions show that the ovoid is symmetric with respect to the x-axis. From the two solutions the smaller in absolute value is the one that specifies the ovoid that is seeked $[n_o d(OS) + n_i d(IS) = A]$. I.e. the solution for $y(x)$ that satisfies $P(x, y(x)) = 0$ shown in Fig. 1 is the $|y(x)| = \min\{|y_1(x)|, |y_2(x)|\}$. The resulting ovoids for the parameters $s_0 = 5\,\text{cm}$, and $s_i = 10, 15, 20\,\text{cm}$ are shown in Fig. 2a. The larger ovoids are also shown in Fig. 2b along with their inner counterparts for comparison. These correspond to the solution $n_o d(OS) - n_i d(IS) = -A$. It is observed that the left vertices remain constant as s_i varies while the right vertices increase with increasing s_i.

In order to find the vertices of the larger ovoid the following equation can be derived if $y = 0$ is set into Eq. (5). The resulting equation with respect to x is given by

$$
D_4 x^4 + D_3 x^3 + D_2 x^2 + D_1 x + D_0 = 0, \tag{8}
$$

where the coefficients D_i $(i = 1, 2, \cdots, 5)$ are given by

$$
\begin{aligned}
D_4 &= (n_o^2 - n_i^2)^2, \\
D_3 &= 4n_i^2 (n_o^2 - n_i^2)(s_o + s_i),
\end{aligned}
$$

$$
\begin{aligned}
D_2 &= 2A^2(n_o^2 - n_i^2) + 4n_i^4(s_o+s_i)^2 - 4A^2 * n_o^2 - 2(n_o^2-n_i^2)n_i^2(s_o+s_i)^2,\\
&= 4n_i^4(s_o+s_i)^2 - 4n_o^2(n_os_o+n_is_i)^2 + 2(n_o^2-n_i^2)(n_os_o+n_is_i)^2 - n_i^2(s_o+s_i)^2(2n_o^2-2n_i^2),\\
D_1 &= 4A^2n_i^2(s_o+s_i) - 4n_i^4(s_o+s_i)^3,\\
&= 4n_i^2(s_o+s_i)(n_os_o+n_is_i)^2 - 4n_i^4(s_o+s_i)^3, \qquad \text{and}\\
D_0 &= A^4 - 2A^2n_i^2(s_o+s_i)^2 + n_i^4(s_o+s_i)^4 = s_o^2(n_o-n_i)^2(n_os_o+n_is_o+2n_is_i)^2.
\end{aligned}
$$

Equation (9) can be solved either analytically (since there are analytical solutions up to the 4-th order polynomials) or it can be solved numerically using for example MatLab. Let the solutions be $\rho_1 > \rho_2 > \rho_3 > \rho_4$. The largest and the smallest solutions are the ones that correspond to V_2' and V_1' (ρ_1 and ρ_4) of the larger ovoid, while the intermediate ones ρ_2 and ρ_3 correspond to the vertices V_2 and V_1 of the inner (useful) ovoid. For the outer ovoid the points for which there is only one solution for $y_1(x)$ or $y_2(x)$ are the ones for which the discriminant $b_1^2 - 4a_1c_1$ of Eq. (5) becomes zero. In this case a line parallel to the y axis is tangential to the outer ovoid from the left. These points can be found by solving the equation:

$$
b_1^2(x) - 4a_1c_1(x) = 0, \tag{9}
$$

Using the previous definitions Eq. (9) can be written in the form

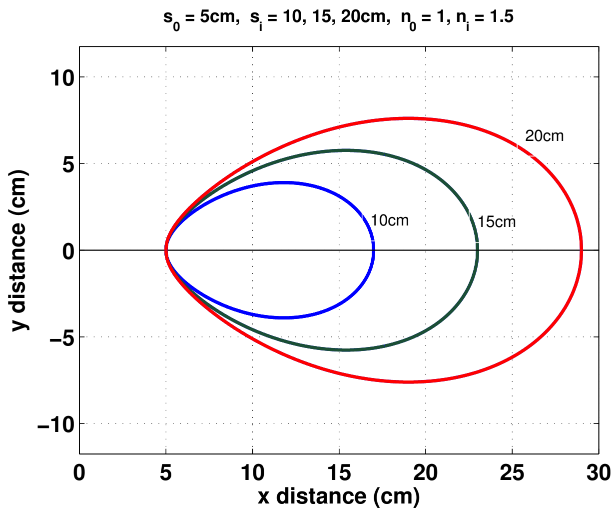
$$
\begin{aligned}
(B_2^2 &- 4a_1C_4)x^4 + (2B_2B_1 - 4a_1C_3)x^3 + (B_1^2 + 2B_0B_2 - 4a_1C_2)x^2\\
&+ (2B_0B_1 - 4a_1C_1)x + (B_0^2 - 4a_1C_0) = 0 \Longrightarrow\\
&\Longrightarrow W_1x + W_0 = 0 \Longrightarrow \rho' = -\frac{W_0}{W_1}, \qquad \text{where} \qquad (10)\\
W_1 &= -32n_o^2n_i^2(s_o+s_i)(n_o^2-n_i^2)(n_os_o+n_is_i)^2,\\
W_0 &= 16n_o^2n_i^2(n_os_o+n_is_i)^2(2n_o^2s_o^2 + 2n_o^2s_os_i + n_o^2s_i^2 + 2n_on_is_os_i - n_i^2s_o^2 - 2n_i^2s_os_i).
\end{aligned}
$$

The above polynomial has only one solution ρ', since its coefficients of forth, third, and second order with respect to x become zero. This real root should be less than V_1' (I.e. $\rho' < \rho_4$). For the example cases used before the roots are summarized in Table 1.

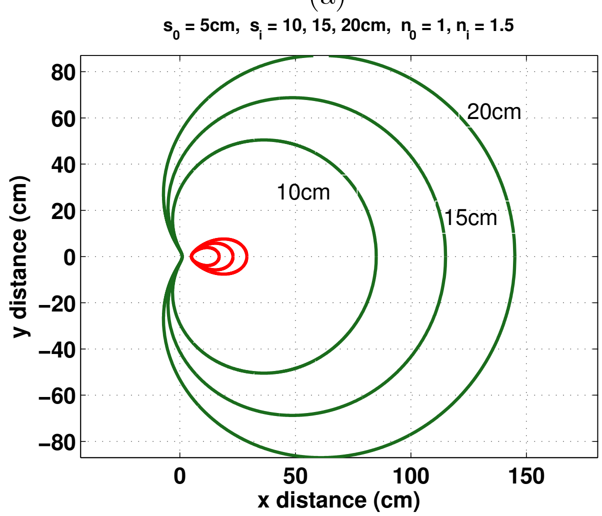
In order to find the maximum angle of rays emanating from O incident on the refractive surface of the ovoid $P(x,y) = 0$ and getting refracted passing through the image point I, it is necessary to find the tangent from point O, OS to the ovoid surface. This is schematically

Table 1: *Case of $n_o < n_i$. Vertices V_1' and V_2' of the outer ovoid and V_1 and V_2 of the inner ovoid (boldface). The $x = \rho'$ for outer ovoid.*

s_0 (cm)	s_i (cm)	ρ_1 [V_1'] (cm)	ρ_2 [V_2] (cm)	ρ_3 [V_1] (cm)	ρ_4 [V_2'] (cm)	ρ' (cm)
5	10	85	**17**	**5**	1	-3.1667
5	15	115	**23**	**5**	1	-5.1250
5	20	145	**29**	**5**	1	-7.1000



(a)



(b)

Figure 2: The three generated Cartesian ovoids are shown for the parameters $n_o = 1.0$, $n_i = 1.5$, $s_o = 5$ cm and $s_i = 10, 15,$ and 20 cm. Just for information the other solution that corresponds to the larger root of Eq. (5) is also shown in Fig. 2b along with the smaller solutions in the same scale. The second (larger) solution is not accepted from a physical point of view. These ovoids are known in the literature as Cartesian ovoids.

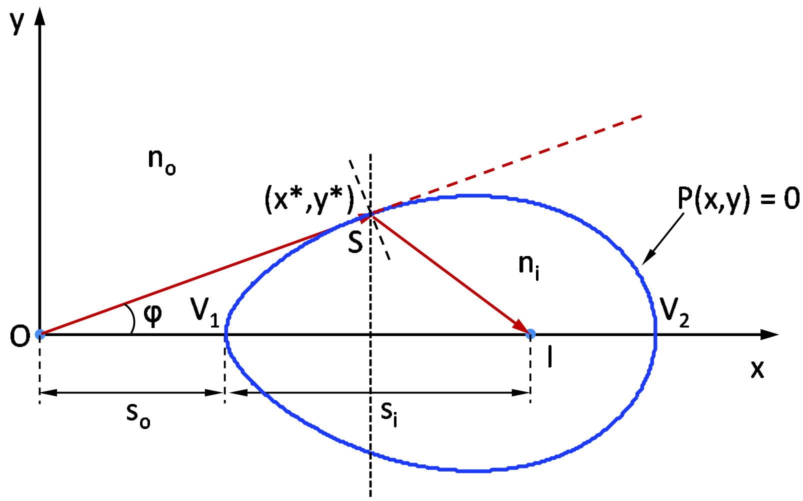


Figure 3: The same Cartesian ovoid with the incident from O beam being tangential to the ovoid. This specifies the maximum angle ϕ that by refraction should pass from point I. The (x^*, y^*) denotes the point on the ovoid that the beam is tangential.

shown in Fig. 3. The angle ϕ of the tangent can be found if the following equation is solved ($\dot{y} = dy/dx$)

$$\tan\phi = \frac{y(x^*)}{x^*} = \frac{dy}{dx}\Big|_{x^*} = \dot{y}(x^*). \tag{11}$$

In order to solve the Eq. (11) it is necessary to find the derivative $\dot{y}(x)$ for the ovoid. This can be found by differentiating Eq. (1) with respect to the variable x while retaining $y = y(x)$. After some algebra the equation for determining the derivative $\dot{y}(x)$ at any point in the ovoid is given by

$$\underbrace{\Big[[(A_1(x,y) - A_2(x,y)]y^2\Big]}_{F_2}\dot{y}^2 \quad + \quad \underbrace{2[A_1(x,y)xy + A_2(x,y)(s_o + s_i - x)y]}_{F_1}\dot{y} + \underbrace{A_1(x,y)x^2 - A_2(x,y)(s_o + s_i - x)^2}_{F_0} = 0, \tag{12}$$

where

$$\begin{aligned} A_1(x,y) &= n_o^2[(s_o + s_i - x)^2 + y^2], \\ A_2(x,y) &= n_i^2(x^2 + y^2), \\ F_2 &= n_o^2 y^2[(s_o + s_i - x)^2 + y^2] - n_i^2 y^2(x^2 + y^2), \end{aligned}$$

$$F_1 = 2xy[(s_o+s_i-x)^2+y^2]n_o^2 + 2y(x^2+y^2)(s_o+s_i-x)n_i^2,$$
$$F_0 = n_o^2x^2[(s_o+s_i-x)^2+y^2] - n_i^2(x^2+y^2)(s_o+s_i-x)^2.$$

Then Eq. (11) can be solved graphically if $f_1(x) = y(x)/x$ and $f_2(x) = \dot{y}(x)$ are plotted as functions of x. The resulting curves and intersections points are shown in Fig. 4. When the numerical solution point (x^*, y^*) is specified the angle ϕ can also be calculated from Eq. (11). The results are tabulated in Table 2.

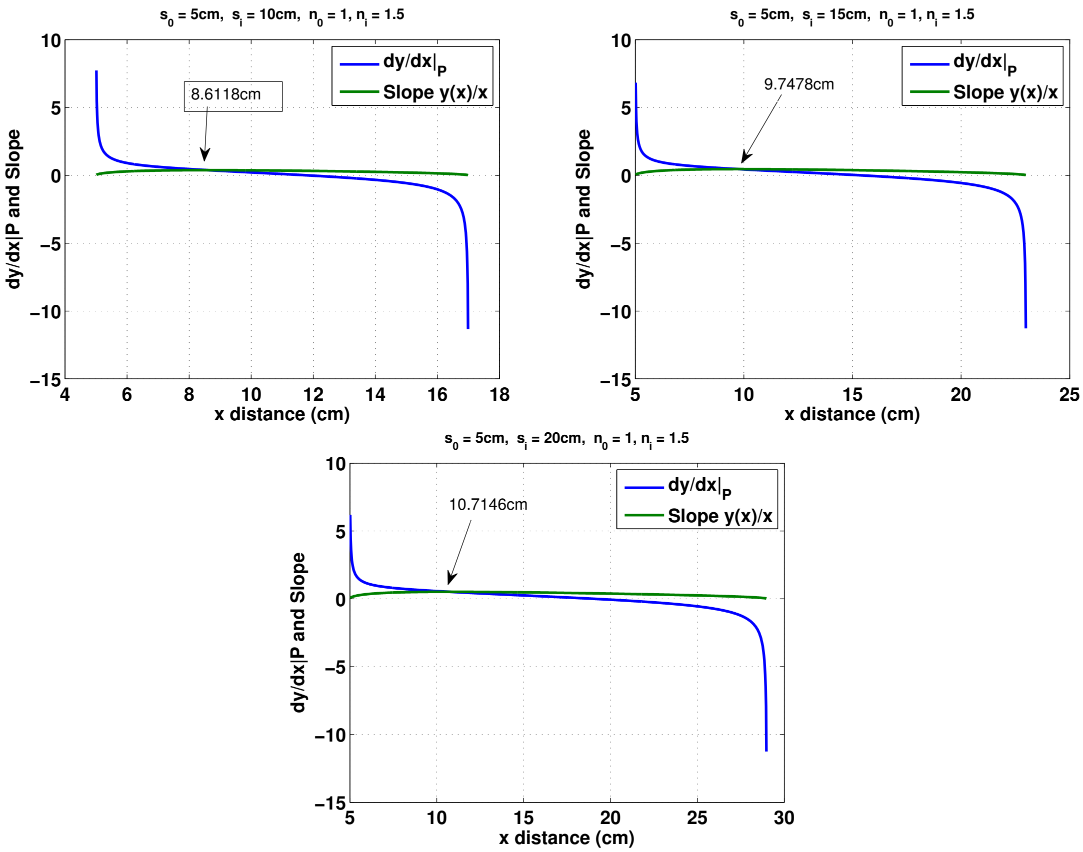


Figure 4: Graphical solution for determining the point for which the incident beam from O is tangential to the ovoid. The intersection of the two curves gives the point x^* and then y^* can be determined from the equation of the ovoid $P(x^*, y^*) = 0$.

<u>Analytical solution for tangent to the Ovoid:</u>

Proposed by **Mr. Orfeas Voutiras** (Optical Science & Engineering Class - Spring 2015)

From Eq. (11) $tan\phi = \alpha = y^*/x^*$. Therefore $y = \alpha x$ for the point (x^*, y^*) that is tangential to ovoid. Inserting this relation into Eq. (1) the following equation is derived

Table 2: *Tangent points (x^*, y^*) from O to the inner ovoid*

s_0 (cm)	s_i (cm)	x^* (cm)	y^* (cm)	ϕ (deg)
5	10	8.6116	3.2925	20.9235
5	15	9.7478	4.4697	24.6331
5	20	10.7146	5.4948	27.1502

(setting $s_0 + s_i = D$):

$$\sqrt{x^2 + \alpha^2 x^2} + \frac{n_i}{n_0}\sqrt{(D-x)^2 + \alpha^2 x^2} = \frac{n_0 s_0 + n_i s_i}{n_0} = A'. \tag{13}$$

Re-arranging the above equation and squaring both sides it is straightforward to derive the following second order polynomial equation with respect to x (setting $\kappa = n_i/n_0$):

$$(1+\alpha^2)(\kappa^2 - 1)x^2 + \left[2A'(1+\alpha^2)^{1/2} - 2D\kappa^2\right] x + D^2\kappa^2 - A'^2 = 0. \tag{14}$$

The last equation should have a single solution with respect to x in order to correspond to the tangent to the ovoid. Therefore, the discriminant of the polynomial of Eq. (13) should be zero. This can be expressed from the following equation:

$$\mathscr{D} = 4\left[A'(1+\alpha^2)^{1/2} - D\kappa^2\right]^2 - 4(D^2\kappa^2 - A'^2)(1+\alpha^2)(\kappa^2-1) = 0. \tag{15}$$

Setting $p = (1+\alpha^2)^{1/2}$ the previous equation is written as

$$\begin{aligned} \left[A'^2 - (D^2\kappa^2 - A'^2)(\kappa^2-1)\right] p^2 - \left[2A'D\kappa^2\right] p + D^2\kappa^4 &= 0 \Longrightarrow \\ a_1 p^2 + a_2 p + a_3 &= 0 \Longrightarrow \end{aligned} \tag{16}$$

$$p = (1+\alpha^2)^{1/2} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1} \Longrightarrow \alpha = \pm\sqrt{p^2 - 1}, \tag{17}$$

where the root for $|\alpha| < 1$ is the one that corresponds to the solution of interest. Using the above analytical solutions the following results can be derived (see Table 3). It is reminded that $\phi = \tan^{-1}(\alpha)$.

Of course the slope does not immediately specify the point (x^*, y^*). However, this can be found solving Eq. (14) for x^* if $y^* = \alpha x^*$ and $\dot{y} = \alpha$. It is given by

$$x^* = \frac{A'(1+\alpha^2)^{1/2} - D\kappa^2}{(1+\alpha^2)(\kappa^2-1)}, \qquad y^* = \alpha x^* \tag{18}$$

Table 3: *Results for question (c) using the analytical approach:*

s_0 (cm)	s_i (cm)	α	ϕ (deg)
5	10	0.3823357	20.92364
5	15	0.4585373	24.63322
5	20	0.5128322	27.15022

Table 4: *Case of $n_o > n_i$. Vertices V_1' and V_2' of the outer ovoid and V_1 and V_2 of the inner ovoid (boldface). The $x = \rho'$ for outer ovoid.*

s_0 (cm)	s_i (cm)	ρ_1 [V_1'] (cm)	ρ_2 [V_1] (cm)	ρ_3 [V_2] (cm)	ρ_4 [V_2'] (cm)	ρ' (cm)
5	10	13	**5**	**-1**	-65	15.6667
5	15	17	**5**	**-1**	-85	20.1250
5	20	21	**5**	**-1**	-105	24.6000

B. Cartesian Ovoids in the case $n_o > n_i$

In this section the case of the higher to lower refractive index ($n_o > n_i$) Cartesian ovoid will be summarized. The case is shown in Fig. 5. Equation (1) is the same for this case too. However, the vertex points V_1 and V_2 (which are the points of the ovoid for $y = 0$) are given by slightly different equations due to the absolute values of Eq. (2) . In this case Eq. (2) has the the following solutions

$$x = s_0, \qquad \text{for} \quad x > 0 \tag{19}$$

$$x = -\frac{n_o - n_i}{n_o + n_i} s_o, \qquad \text{for} \quad -\infty < x < 0. \tag{20}$$

It is interesting to observe that in this case the positions of the two vertices V_1 and V_2 are independent of s_i and depend only on s_o and on the refractive indices. Example positions are shown in Table 4 for $s_o = 5\,\text{cm}$, $s_i = 10, 15, 20\,\text{cm}$, and $n_o = 1.5$, $n_i = 1.0$.

<u>Note:</u> The above Cartesian ovoid curves are special cases of the general equation describing the Cartesian ovoids. This is the equation $d(OS)+md(IS) = a$ where O and I are the object and image points that were used before and m and a are constants. In our case $m = n_i/n_o$

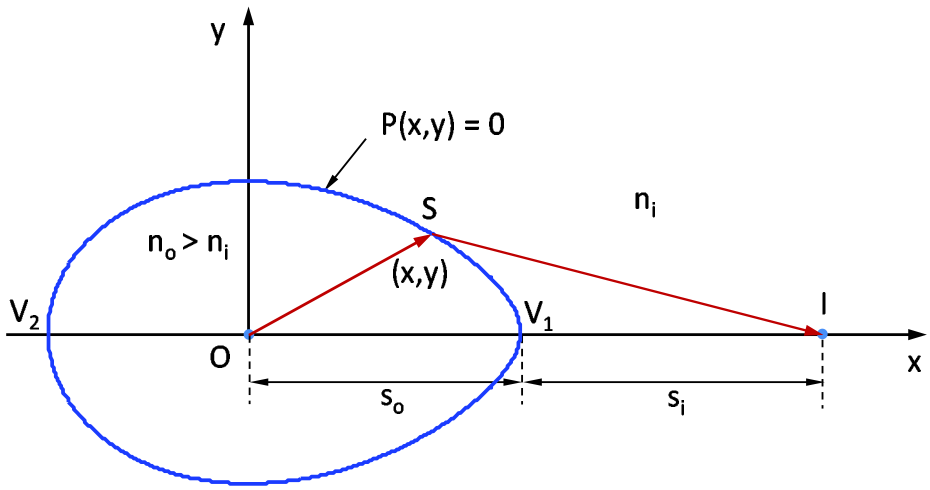


Figure 5: A Cartesian ovoid of refractive index n_o is shown in a surrounding medium of refractive index n_i. The shape shown is valid for $n_o > n_i$. Every ray from the point object O ends up in the image point I. The function of the ovoid is $P(x, y) = 0$. The distances s_o and s_i are the object and image distances from the ovoid's left vertex.

and $a = s_o + s_i(n_i/n_o)$. Then the resulting general equation in (x, y) is the following

$$\left[(1 - m^2)(x^2 + y^2) + 2m^2cx + a^2 - m^2c^2\right]^2 = 4a^2(x^2 + y^2), \tag{21}$$

which has solutions as the ovoids shown in Fig. 2.

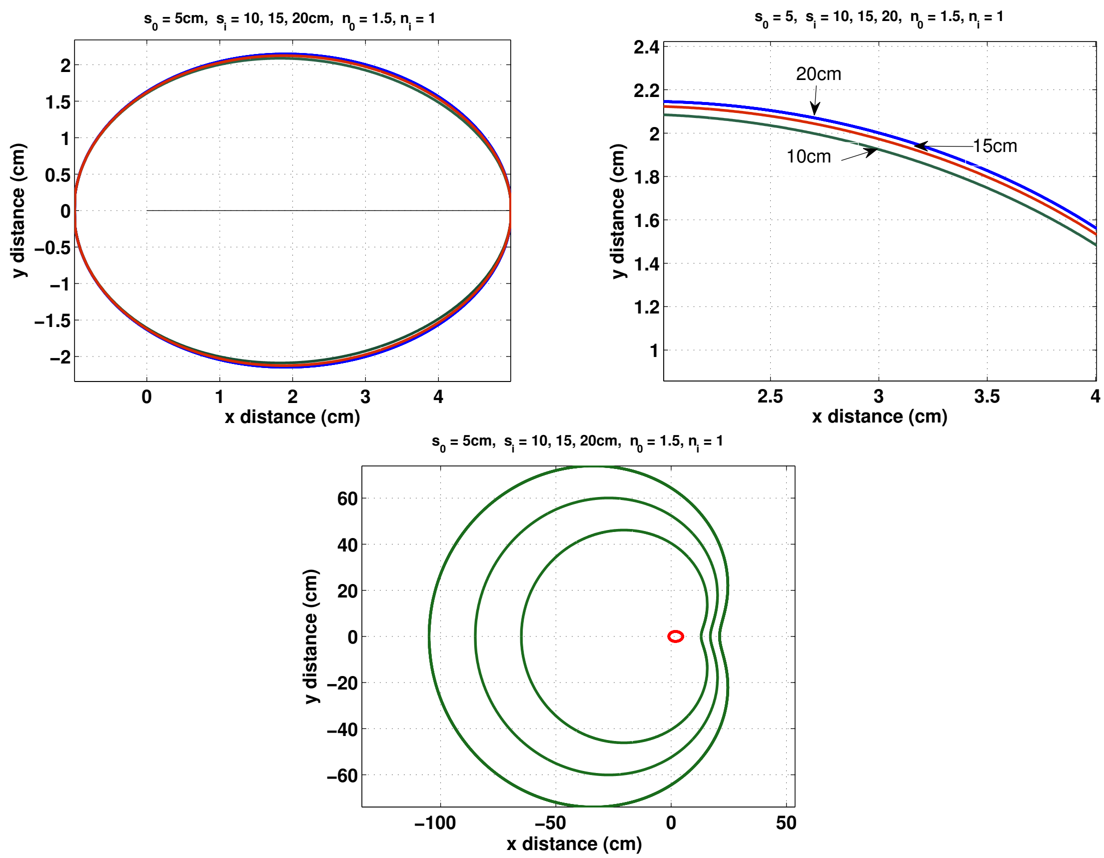


Figure 6: Cartesian ovoids for the case of $n_o > n_i$. These are the ovoids for $n_o = 1.5$, $n_i = 1.0$, $s_o = 5\,\text{cm}$, and $s_i = 10, 15, 20\,\text{cm}$. The useful ovoids are the egg-like ones that are very similar since the points V_1 and V_2 do not depend on s_i.