Problem Set # 3

1. SUBSTRATE TE MODE OF A SLAB WAVEGUIDE

A general three-layer step-index slab waveguide is characterized by n_s, n_f, n_c and h. Assume a TE substrate mode with the following representation of its electric field:

$$\begin{aligned}
\vec{E}_{cover} &= \hat{y}E_c \exp[-\gamma_c(x-h)] \exp(-j\beta z), \\
\vec{E}_{film} &= \hat{y}[E_{f1} \exp(jk_{fx}x) + E_{f2} \exp(-jk_{fx}x)] \exp(-j\beta z), \\
\vec{E}_{substrate} &= \hat{y}[E_{s1} \exp(jk_{sx}x) + E_{s2} \exp(-jk_{sx}x)] \exp(-j\beta z).
\end{aligned}$$

Show that the substrate modes form a continuum for values of β between $k_0 n_c$ and $k_0 n_s$.

2. FOUR LAYER ASYMMETRIC WAVEGUIDE

A four-layer waveguide has two film layers of refractive indices n_1 and n_2 with corresponding thicknesses h_1 and h_2, respectively. The cover and substrate layers have refractive indices n_c and n_s, respectively. Assume that the refractive indices satisfy the inequality $n_1 > n_2 > n_s > n_c$. For given waveguide parameters show that the dispersion equation for TE modes with β between $k_0 n_2$ and $k_0 n_s$ is given by

$$\kappa_1 h_1 - \tan^{-1}\left(\frac{\gamma_s}{\kappa_1}\right) - \tan^{-1}\left[\frac{\kappa_2}{\kappa_1} \tan\left(\tan^{-1}(\gamma_c/\kappa_2) - \kappa_2 h_2\right)\right] = \nu\pi,$$

where ν is a positive integer and κ_1, κ_2, γ_c, γ_s are as defined in class notes. What is the corresponding equation for TM modes? How are these dispersion equations modified when β is between $k_0 n_2$ and $k_0 n_1$?

3. MODE ORTHOGONALITY

Assume a three-layer step-index slab waveguide. The orthogonality condition for TE modes can be written in the form

$$\begin{aligned}
\langle E_{my}, E_{ny} \rangle = \frac{\beta_n}{2\omega\mu_0} \int_{-\infty}^{+\infty} E_{my} E_{ny}^* dx &= P\delta_{mn}, \\
\langle E_y(\beta), E_y(\beta') \rangle = \frac{\beta'}{2\omega\mu_0} \int_{-\infty}^{+\infty} E_y(\beta) E_y^*(\beta') dx &= P\delta(\beta - \beta'),
\end{aligned}$$

for the guided and for the radiation (including substrate) modes, respectively. In addition the orthogonality between any guided and any radiation (or substrate mode) can be described by $\langle E_{my}, E_y(\beta) \rangle = \langle E_y(\beta), E_{my} \rangle = 0$. A TE electric field distribution E_y can be expressed in the form

$$E_y = \sum_n a_n E_{ny} + \int q(\beta) E_y(\beta) d\beta$$

Find the expansion coefficients a_n and $q(\beta)$. Express the time average power per unit length in the y direction that propagates along the waveguide as a function of the expansion coefficients and the constant P.