Optical Waveguides Mode Orthogonality & Mode Matching

Integrated Optics

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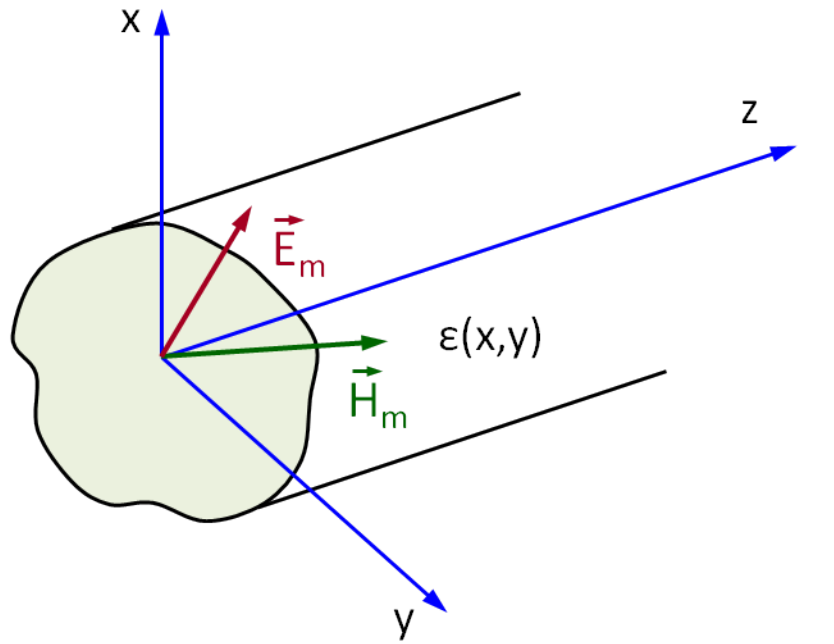
Mode Orthogonality & Completeness in Dielectric Waveguides



Waveguide Modes (*m*-th and *n*-th):

$$\begin{bmatrix} \vec{E}_m \\ \vec{H}_m \end{bmatrix} = \begin{bmatrix} \vec{\mathcal{E}}_m(x,y) \\ \vec{\mathcal{H}}_m(x,y) \end{bmatrix} e^{-j\beta_m z}, \qquad \begin{bmatrix} \vec{E}_n \\ \vec{H}_n \end{bmatrix} = \begin{bmatrix} \vec{\mathcal{E}}_n(x,y) \\ \vec{\mathcal{H}}_n(x,y) \end{bmatrix} e^{-j\beta_n z}$$

Mode Orthogonality & Completeness in Dielectric Waveguides

z-Reversal Symmetry:

$$
\begin{aligned}
\vec{\nabla} \times \vec{E}_\ell &= -j\omega\mu_0 \vec{H}_\ell, \\
\vec{\nabla} \times \vec{H}_\ell &= +j\omega\epsilon(x,y)\vec{E}_\ell.
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla} &= \vec{\nabla}_t + \frac{\partial}{\partial z}\hat{z} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right) + \frac{\partial}{\partial z}\hat{z}, \\
\vec{\mathcal{E}}_\ell &= \vec{\mathcal{E}}_{t,\ell} + \mathcal{E}_{z,\ell}\hat{z} = (\mathcal{E}_{x,\ell}\hat{x} + \mathcal{E}_{y,\ell}\hat{y}) + \mathcal{E}_{z,\ell}\hat{z}, \\
\vec{\mathcal{H}}_\ell &= \vec{\mathcal{H}}_{t,\ell} + \mathcal{H}_{z,\ell}\hat{z} = (\mathcal{H}_{x,\ell}\hat{x} + \mathcal{H}_{y,\ell}\hat{y}) + \mathcal{H}_{z,\ell}\hat{z}.
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}_t \times \vec{\mathcal{E}}_{t,\ell} &= -j\omega\mu_0 \mathcal{H}_{z,\ell}\hat{z}, \\
\vec{\nabla}_t \times (\mathcal{E}_{z,\ell}\hat{z}) - j\beta_\ell\left(\hat{z} \times \vec{\mathcal{E}}_{t,\ell}\right) &= -j\omega\mu_0 \vec{\mathcal{H}}_{t,\ell}, \\
\vec{\nabla}_t \times \vec{\mathcal{H}}_{t,\ell} &= +j\omega\epsilon(x,y)\mathcal{E}_{z,\ell}\hat{z}, \\
\vec{\nabla}_t \times (\mathcal{H}_{z,\ell}\hat{z}) - j\beta_\ell\left(\hat{z} \times \vec{\mathcal{H}}_{t,\ell}\right) &= +j\omega\epsilon(x,y)\vec{\mathcal{E}}_{t,\ell}.
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\vec{\mathcal{E}}_{t,-\ell} &= \vec{\mathcal{E}}_{t,\ell}, \\
\mathcal{E}_{z,-\ell} &= -\mathcal{E}_{z,\ell}, \\
\vec{\mathcal{H}}_{t,-\ell} &= -\vec{\mathcal{H}}_{t,\ell}, \\
\mathcal{H}_{z,-\ell} &= \mathcal{H}_{z,\ell}, \\
\beta_{-\ell} &= -\beta_\ell,
\end{aligned}
}
$$

Mode Orthogonality & Completeness in Dielectric Waveguides

Time-Reversal Symmetry:

$$
\begin{aligned}
\vec{\nabla} \times \left(\vec{\mathcal{E}}_\ell^* e^{+j\beta_\ell z} \right) &= -j\omega\mu_0 \left(-\vec{\mathcal{H}}_\ell^* e^{+j\beta_\ell z} \right), \\
\vec{\nabla} \times \left(-\vec{\mathcal{H}}_\ell^* e^{+j\beta_\ell z} \right) &= +j\omega\epsilon(x,y) \left(\vec{\mathcal{E}}_\ell^* e^{+j\beta_\ell z} \right).
\end{aligned}
$$

$$
\begin{aligned}
\vec{E}_{-\ell} &= \left(\vec{\mathcal{E}}_{t,-\ell} + \hat{z}\mathcal{E}_{z,-\ell} \right) e^{+j\beta_\ell z} = \left(\vec{\mathcal{E}}_{t,\ell} - \hat{z}\mathcal{E}_{z,\ell} \right) e^{+j\beta_\ell z} = \left(\vec{\mathcal{E}}_{t,\ell}^* + \hat{z}\mathcal{E}_{z,\ell}^* \right) e^{+j\beta_\ell z}, \\
\vec{H}_{-\ell} &= \left(\vec{\mathcal{H}}_{t,-\ell} + \hat{z}\mathcal{H}_{z,-\ell} \right) e^{+j\beta_\ell z} = \left(-\vec{\mathcal{H}}_{t,\ell} + \hat{z}\mathcal{H}_{z,\ell} \right) e^{+j\beta_\ell z} = -\left(\vec{\mathcal{H}}_{t,\ell}^* + \hat{z}\mathcal{H}_{z,\ell}^* \right) e^{+j\beta_\ell z}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\vec{\mathcal{E}}_{t,\ell} &= \vec{\mathcal{E}}_{t,\ell}^* \\
\mathcal{E}_{z,\ell} &= -\mathcal{E}_{z,\ell}^*, \\
\vec{\mathcal{H}}_{t,\ell} &= \vec{\mathcal{H}}_{t,\ell}^* \\
\mathcal{H}_{z,\ell} &= -\mathcal{H}_{z,\ell}^*,
\end{aligned}
}
$$

Mode Orthogonality & Completeness in Dielectric Waveguides

$$
\begin{aligned}
P_z &= \frac{1}{2}\iint_S \mathrm{Re}\left\{\vec{E}\times\vec{H}^*\right\}\cdot\hat{z}dxdy = \frac{1}{4}\iint_S\left[\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\right]\cdot\hat{z}dxdy \\
&= \frac{1}{4}\iint_S\left[\sum_m\sum_n a_m a_n^* \vec{\mathcal{E}}_m\times\vec{\mathcal{H}}_n^* e^{-j(\beta_m-\beta_n)z} + \sum_m\sum_n a_m^* a_n \vec{\mathcal{E}}_m^*\times\vec{\mathcal{H}}_n e^{+j(\beta_m-\beta_n)z}\right]\cdot\hat{z}dxdy \\
&= \frac{1}{4}\sum_m\sum_n a_m a_n^* e^{-j(\beta_m-\beta_n)z}\iint_S\left[\vec{\mathcal{E}}_m\times\vec{\mathcal{H}}_n^* + \vec{\mathcal{E}}_n^*\times\vec{\mathcal{H}}_m\right]\cdot\hat{z}dxdy,
\end{aligned}
$$

$$
\frac{1}{4}\sum_m\sum_n a_m a_n^*\left[-j(\beta_m-\beta_n)\right]e^{-j(\beta_m-\beta_n)z}\iint_S\left[\vec{\mathcal{E}}_m\times\vec{\mathcal{H}}_n^* + \vec{\mathcal{E}}_n^*\times\vec{\mathcal{H}}_m\right]\cdot\hat{z}dxdy = 0.
$$

$$
\boxed{\frac{1}{4}\iint_S\left[\vec{\mathcal{E}}_m\times\vec{\mathcal{H}}_n^* + \vec{\mathcal{E}}_n^*\times\vec{\mathcal{H}}_m\right]\cdot\hat{z}dxdy = 0, \;\longrightarrow\; \frac{1}{4}\iint_S\left[\vec{\mathcal{E}}_{t,m}\times\vec{\mathcal{H}}_{t,n}^* + \vec{\mathcal{E}}_{t,n}^*\times\vec{\mathcal{H}}_{t,m}\right]\cdot\hat{z}dxdy = 0.}
$$

Mode Orthogonality & Completeness in Dielectric Waveguides

$$\frac{1}{4}\iint_S \left[\vec{\mathcal{E}}_{t,-m}\times\vec{\mathcal{H}}^*_{t,n}+\vec{\mathcal{E}}^*_{t,n}\times\vec{\mathcal{H}}_{t,-m}\right]\cdot\hat{z}dxdy=0,\Longrightarrow$$
$$\frac{1}{4}\iint_S \left[\vec{\mathcal{E}}_{t,m}\times\vec{\mathcal{H}}^*_{t,n}+\vec{\mathcal{E}}^*_{t,n}\times(-\vec{\mathcal{H}}_{t,m})\right]\cdot\hat{z}dxdy=0,\Longrightarrow$$
$$\frac{1}{4}\iint_S \left[\vec{\mathcal{E}}_{t,m}\times\vec{\mathcal{H}}^*_{t,n}-\vec{\mathcal{E}}^*_{t,n}\times\vec{\mathcal{H}}_{t,m}\right]\cdot\hat{z}dxdy=0.$$

Guided Modes

$$\begin{aligned}\langle\vec{\mathcal{E}}_m,\vec{\mathcal{H}}_n\rangle=\frac{1}{2}\iint_S\left[\vec{\mathcal{E}}_m\times\vec{\mathcal{H}}^*_n\right]\cdot\hat{z}dxdy &= \frac{1}{2}\iint_S\left[\vec{\mathcal{E}}_{t,m}\times\vec{\mathcal{H}}^*_{t,n}\right]\cdot\hat{z}dxdy=\\ \frac{1}{2}\iint_S \mathrm{Re}\left\{\vec{\mathcal{E}}_{t,m}\times\vec{\mathcal{H}}^*_{t,n}\right\}\cdot\hat{z}dxdy &= \frac{\beta_m}{|\beta_m|}P_m\delta_{|m||n|},\end{aligned}$$

Radiation Modes

$$\begin{aligned}\langle\vec{\mathcal{E}}_\beta,\vec{\mathcal{H}}_{\beta'}\rangle=\frac{1}{2}\iint_S\left[\vec{\mathcal{E}}_\beta\times\vec{\mathcal{H}}^*_{\beta'}\right]\cdot\hat{z}dxdy &= \frac{1}{2}\iint_S\left[\vec{\mathcal{E}}_{t,\beta}\times\vec{\mathcal{H}}^*_{t,\beta'}\right]\cdot\hat{z}dxdy=\\ \frac{1}{2}\iint_S \mathrm{Re}\left\{\vec{\mathcal{E}}_{t,\beta}\times\vec{\mathcal{H}}^*_{t,\beta'}\right\}\cdot\hat{z}dxdy &= \frac{\beta}{|\beta|}P_\beta\delta(\beta-\beta').\end{aligned}$$

Mode Orthogonality & Completeness in Dielectric Waveguides

Power Considerations

$$\left[\begin{array}{c}\vec{E}\\ \vec{H}\end{array}\right] = \sum_m a_m \left[\begin{array}{c}\vec{\mathcal{E}}_m(x,y)\\ \vec{\mathcal{H}}_m(x,y)\end{array}\right] e^{-j\beta_m z} + \int_\beta q(\beta)\left[\begin{array}{c}\vec{\mathcal{E}}_\beta(x,y)\\ \vec{\mathcal{H}}_\beta(x,y)\end{array}\right] e^{-j\beta z} d\beta.$$

$$P_z = \iint_S \frac{1}{2}\mathrm{Re}\{\vec{E}\times\vec{H}^*\}\cdot\hat{z}\,dxdy,$$

$$P_z = \sum_m |a_m|^2 \frac{\beta_m}{|\beta_m|} P_m + \int_\beta |q(\beta)|^2 \frac{\beta}{|\beta|} P_\beta d\beta.$$

$$P_z = P_0\left[\sum_m |a_m|^2 \frac{\beta_m}{|\beta_m|} + \int_\beta |q(\beta)|^2\frac{\beta}{|\beta|}d\beta\right],$$

Mode Orthogonality in Slab Waveguides

TE Modes

$$\begin{aligned}
\vec{E}_\ell &= \hat{y}\mathcal{E}_{y,\ell}(x)e^{-j\beta_\ell z},\\
\vec{H}_\ell &= \left[\hat{x}\mathcal{H}_{x,\ell}(x) + \hat{z}\mathcal{H}_{z,\ell}\right]e^{-j\beta_\ell z} = \left[-\hat{x}\frac{\beta_\ell}{\omega\mu_0}\mathcal{E}_{y,\ell} - \hat{z}\frac{1}{j\omega\mu_0}\frac{d\mathcal{E}_{y,\ell}}{dx}\right]e^{-j\beta_\ell z}.
\end{aligned}$$

$$\boxed{\langle\vec{\mathcal{E}}_m, \vec{\mathcal{E}}_n\rangle = \frac{\beta_n}{2\omega\mu_0}\int_{-\infty}^{+\infty}\mathcal{E}_{y,m}(x)\mathcal{E}^*_{y,n}(x)dx = \frac{\beta_m}{|\beta_m|}P_m\delta_{|m||n|},}$$

$$\boxed{\langle\vec{\mathcal{E}}_\beta, \vec{\mathcal{E}}_{\beta'}\rangle = \frac{\beta'}{2\omega\mu_0}\int_{-\infty}^{+\infty}\mathcal{E}_{y,\beta}(x)\mathcal{E}^*_{y,\beta'}(x)dx = \frac{\beta}{|\beta|}P_\beta\delta(\beta-\beta').}$$

TE Field Expansion

$$\boxed{\vec{E} = \hat{y}E_y(x,z_0) = \hat{y}\left[\sum_m a_m\mathcal{E}_{y,m}e^{-j\beta_m z_0} + \int q(\beta)E_{y,\beta}e^{-j\beta z_0}d\beta\right]}$$

$$\begin{aligned}
a_m &= \frac{1}{P_m}\langle E_y(x,z_0), \mathcal{E}_{y,m}(x)\rangle = \frac{1}{P_m}\int_{-\infty}^{+\infty}E_y(x,z_0)\mathcal{E}^*_{y,m}(x)dx,\\
q(\beta) &= \frac{1}{P_\beta}\langle E_y(x,z_0), \mathcal{E}_{y,\beta}(x)\rangle = \frac{1}{P_\beta}\int_{-\infty}^{+\infty}E_y(x,z_0)\mathcal{E}^*_{y,\beta}(x)dx,
\end{aligned}$$

Mode Orthogonality in Slab Waveguides

TM Modes

$$
\begin{aligned}
\vec{H}_\ell &= \hat{y}\mathcal{H}_{y,\ell}(x)e^{-j\beta_\ell z}, \\
\vec{E}_\ell &= \left[\hat{x}\mathcal{E}_{x,\ell}(x) + \hat{z}\mathcal{E}_{z,\ell}\right] e^{-j\beta_\ell z} = \left[\hat{x}\frac{\beta_\ell}{\omega\epsilon}\mathcal{H}_{y,\ell} + \hat{z}\frac{1}{j\omega\epsilon}\frac{d\mathcal{H}_{y,\ell}}{dx}\right] e^{-j\beta_\ell z}.
\end{aligned}
$$

$$
\boxed{\langle\vec{\mathcal{H}}_m, \vec{\mathcal{H}}_n\rangle = \frac{\beta_n}{2\omega\epsilon_0}\int_{-\infty}^{+\infty}\frac{1}{n^2(x)}\mathcal{H}_{y,m}(x)\mathcal{H}^*_{y,n}(x)dx = \frac{\beta_m}{|\beta_m|}P_m\delta_{|m||n|},}
$$

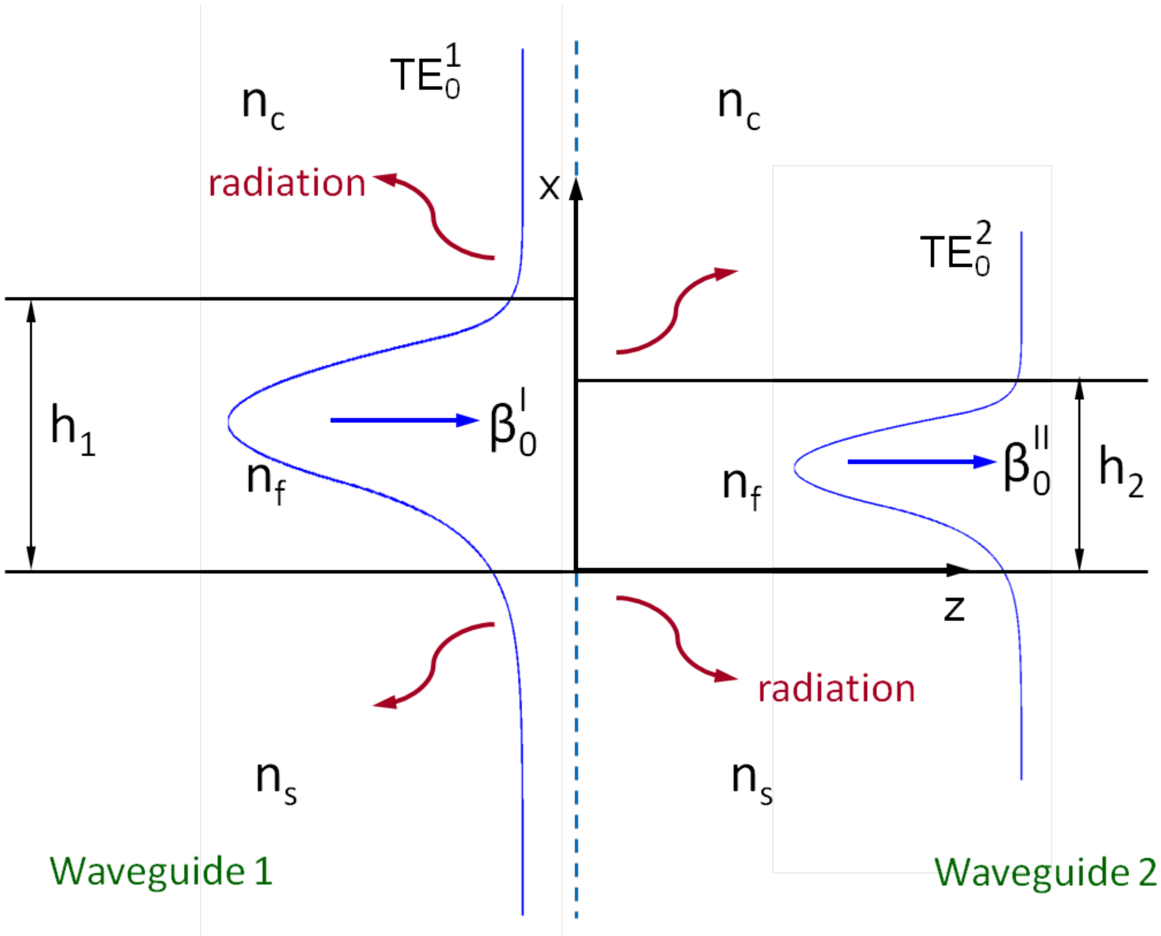
$$
\boxed{\langle\vec{\mathcal{H}}_\beta, \vec{\mathcal{H}}_{\beta'}\rangle = \langle\mathcal{H}_{y,\beta}, \mathcal{H}_{y,\beta'}\rangle = \frac{\beta'}{2\omega\epsilon_0}\int_{-\infty}^{+\infty}\frac{1}{n^2(x)}\mathcal{H}_{y,\beta}(x)\mathcal{H}^*_{y,\beta'}(x)dx = \frac{\beta}{|\beta|}P_\beta\delta(\beta-\beta')}
$$

TM Field Expansion

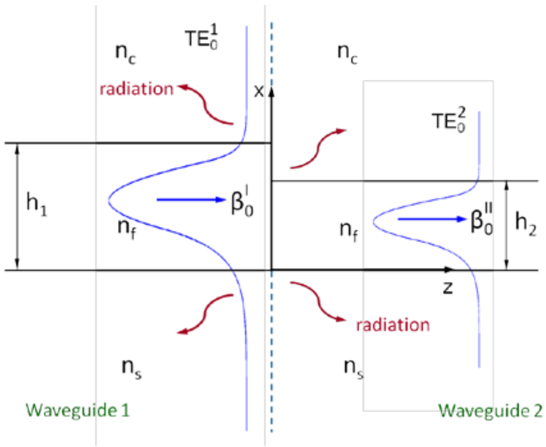
$$
\boxed{\vec{H} = \hat{y}H_y(x,z_0) = \hat{y}\left[\sum_m a_m\mathcal{H}_{y,m}e^{-j\beta_m z_0} + \int q(\beta)H_{y,\beta}e^{-j\beta z_0}d\beta\right],}
$$

$$
\begin{aligned}
a_m &= \frac{1}{P_m}\langle H_y(x,z_0), \mathcal{H}_{y,m}(x)\rangle = \frac{1}{P_m}\int_{-\infty}^{+\infty}\frac{1}{n^2(x)}H_y(x,z_0)\mathcal{H}^*_{y,m}(x)dx, \\
q(\beta) &= \frac{1}{P_\beta}\langle H_y(x,z_0), \mathcal{H}_{y,\beta}(x)\rangle = \frac{1}{P_\beta}\int_{-\infty}^{+\infty}\frac{1}{n^2(x)}H_y(x,z_0)\mathcal{H}^*_{y,\beta}(x)dx,
\end{aligned}
$$

Example Discontinuity in a Slab Waveguide



Example Discontinuity in a Slab Waveguide



$$E_y^{(1)} = E_0^I \mathcal{E}_{y,0}^I(x) e^{-j\beta_0^I z} + E_0^r \mathcal{E}_{y,-0}^I(x) e^{-j\beta_{-0}^I z} + \int_0^{k_0 n_s} q^I(\beta^I) \mathcal{E}_{y,-\beta}^I(x) e^{-j(-\beta^I)z} d\beta^I =$$

$$= E_0^I \mathcal{E}_{y,0}^I(x) e^{-j\beta_0^I z} + E_0^r \mathcal{E}_{y,0}^I(x) e^{+j\beta_0^I z} + \int_0^{k_0 n_s} q^I(\beta^I) \mathcal{E}_{y,\beta}^I(x) e^{+j\beta^I z} d\beta^I,$$

$$H_x^{(1)} = -\frac{\beta_0^I}{\omega\mu_0} E_0^I \mathcal{E}_{y,0}^I(x) e^{-j\beta_0^I z} + \frac{\beta_0^I}{\omega\mu_0} E_0^r \mathcal{E}_{y,0}^I(x) e^{+j\beta_0^I z} + \int_0^{k_0 n_s} \frac{\beta^I}{\omega\mu_0} q^I(\beta^I) \mathcal{E}_{y,\beta}^I(x) e^{+j\beta^I z} d\beta^I$$

$$E_y^{(2)} = E_0^t \mathcal{E}_{y,0}^{II}(x) e^{-j\beta_0^{II} z} + \int_0^{k_0 n_s} q^{II}(\beta^{II}) \mathcal{E}_{y,\beta}^{II}(x) e^{-j\beta^{II} z} d\beta^{II},$$

$$H_x^{(2)} = -\frac{\beta_0^{II}}{\omega\mu_0} E_0^t \mathcal{E}_{y,0}^{II}(x) e^{-j\beta_0^{II} z} - \int_0^{k_0 n_s} \frac{\beta^{II}}{\omega\mu_0} q^{II}(\beta^{II}) \mathcal{E}_{y,\beta}^{II}(x) e^{-j\beta^{II} z} d\beta^{II},$$

Boundary Conditions

$$\left(E_0^I + E_0^r\right) \mathcal{E}_{y,0}^I(x) + \int_0^{k_0 n_s} q^I(\beta^I) \mathcal{E}_{y,\beta}^I(x) d\beta^I = E_0^t \mathcal{E}_{y,0}^{II}(x) + \int_0^{k_0 n_s} q^{II}(\beta^{II}) \mathcal{E}_{y,\beta}^{II}(x) d\beta^{II},$$

$$\frac{\beta_0^I}{\omega\mu_0} \left(-E_0^I + E_0^r\right) \mathcal{E}_{y,0}^I(x) + \int_0^{k_0 n_s} \frac{q^I(\beta^I)\beta_I}{\omega\mu_0} \mathcal{E}_{y,\beta}^I(x) d\beta^I = -\frac{\beta_0^{II}}{\omega\mu_0} E_0^t \mathcal{E}_{y,0}^{II}(x) - \int_0^{k_0 n_s} \frac{q^{II}(\beta^{II})\beta^{II}}{\omega\mu_0} \mathcal{E}_{y,\beta}^{II}(x) d\beta^{II}.$$

Example Discontinuity in a Slab Waveguide

Boundary Conditions (neglecting radiation/substrate modes)

Formulation 1

$$
\begin{aligned}
\left(E_0^I + E_0^r\right)\mathcal{E}_{y,0}^I(x) &= E_0^t \mathcal{E}_{y,0}^{II}(x), \\
\beta_0^I\left(-E_0^I + E_0^r\right)\mathcal{E}_{y,0}^I(x) &= -\beta_0^{II} E_0^t \mathcal{E}_{y,0}^{II}(x).
\end{aligned}
$$

$$
\begin{aligned}
(1 + r_0)\,\mathcal{E}_{y,0}^I(x) &= t_0 \mathcal{E}_{y,0}^{II}(x), \\
\beta_0^I(-1 + r_0)\,\mathcal{E}_{y,0}^I(x) &= -\beta_0^{II} t_0 \mathcal{E}_{y,0}^{II}(x), \\
(1 + r_0)\,\langle \mathcal{E}_{y,0}^I(x), \mathcal{E}_{y,0}^{II}(x)\rangle_{II} &= t_0 P_0^{II}, \\
\beta_0^I(-1 + r_0)\,\langle \mathcal{E}_{y,0}^I(x), \mathcal{E}_{y,0}^{II}(x)\rangle_{II} &= -\beta_0^{II} t_0 P_0^{II},
\end{aligned}
$$

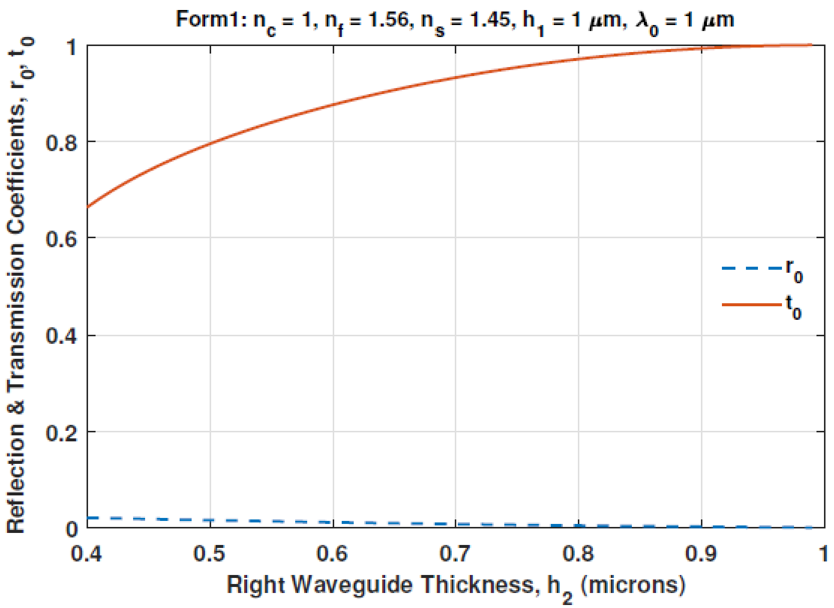
$$
\boxed{
\begin{aligned}
r_0 &= \frac{\beta_0^I - \beta_0^{II}}{\beta_0^I + \beta_0^{II}}, \\
t_0 &= \frac{1}{P_0^{II}}\,\frac{2\beta_0^I}{\beta_0^I + \beta_0^{II}}\,\langle \mathcal{E}_{y,0}^I(x), \mathcal{E}_{y,0}^{II}(x)\rangle_{II}, \\
\langle \mathcal{E}_{y,0}^I(x), \mathcal{E}_{y,0}^{II}(x)\rangle_{II} &= \frac{\beta_0^{II}}{2\omega\mu_0}\int_{-\infty}^{+\infty} \mathcal{E}_{y,0}^I(x)\left(\mathcal{E}_{y,0}^{II}(x)\right)^* dx,
\end{aligned}
}
$$

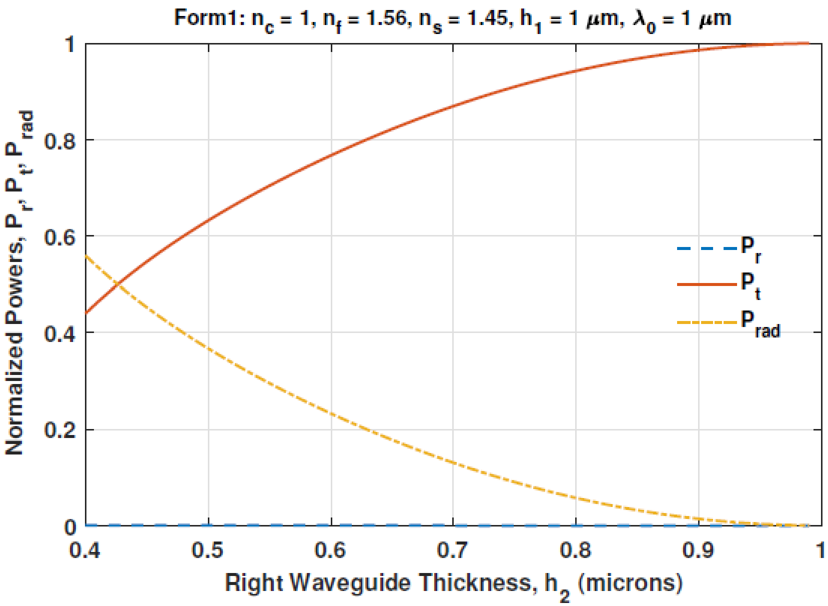
Example Discontinuity in a Slab Waveguide

Power Considerations (Formulation 1)

$$1-|r_0|^2-\int_0^{k_0 n_s}|q^I(\beta^I)|^2 d\beta^I = |t_0|^2+\int_0^{k_0 n_s}|q^{II}(\beta^{II})|^2 d\beta^{II},$$

$$1-|r_0|^2 \simeq |t_0|^2.$$





Example Discontinuity in a Slab Waveguide

Boundary Conditions (neglecting radiation/substrate modes)

Formulation 2

$$
\begin{aligned}
\left(E_0^I + E_0^r\right)\mathcal{E}_{y,0}^I(x) &= E_0^t \mathcal{E}_{y,0}^{II}(x),\\
\beta_0^I\left(-E_0^I + E_0^r\right)\mathcal{E}_{y,0}^I(x) &= -\beta_0^{II} E_0^t \mathcal{E}_{y,0}^{II}(x).
\end{aligned}
$$

$$
\begin{aligned}
\left(1 + r_0\right)\mathcal{E}_{y,0}^I(x) &= t_0 \mathcal{E}_{y,0}^{II}(x),\\
\beta_0^I\left(-1 + r_0\right)\mathcal{E}_{y,0}^I(x) &= -\beta_0^{II} t_0 \mathcal{E}_{y,0}^{II}(x),
\end{aligned}
$$

$$
\begin{aligned}
\left(1 + r_0\right)\langle \mathcal{E}_{y,0}^I(x), \mathcal{E}_{y,0}^{II}(x)\rangle_{II} &= t_0 P_0^{II},\\
\beta_0^I\left(-1 + r_0\right) P_0^I &= -\beta_0^{II} t_0 \langle \mathcal{E}_{y,0}^{II}(x), \mathcal{E}_{y,0}^I(x)\rangle_I,
\end{aligned}
$$

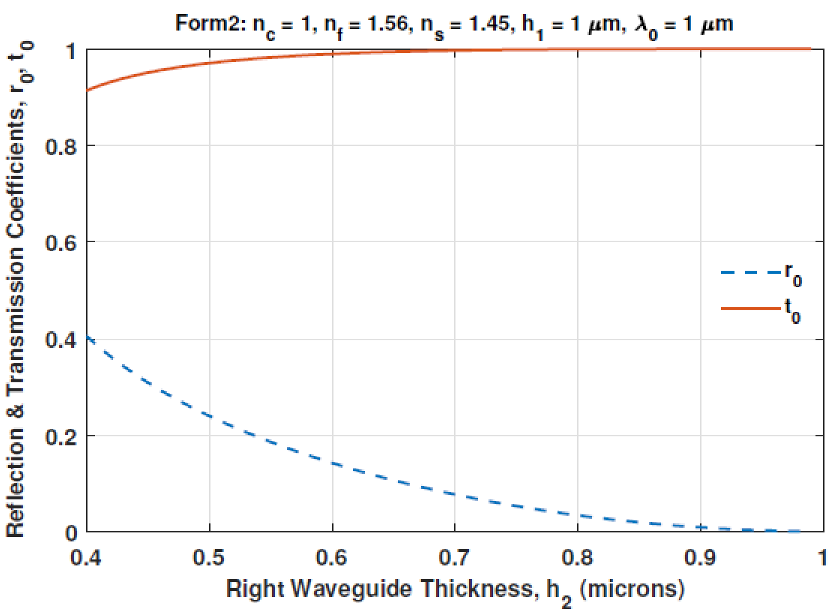
$$
\boxed{
\begin{aligned}
\begin{bmatrix} r_0 \\ t_0 \end{bmatrix} &= \begin{bmatrix} I_2 & -P_0^{II} \\ \beta_0^I P_0^I & \beta_0^{II} I_1 \end{bmatrix}^{-1} \begin{bmatrix} -I_2 \\ \beta_0^I P_0^I \end{bmatrix},\\
I_1 = \langle \mathcal{E}_{y,0}^{II}(x), \mathcal{E}_{y,0}^I(x)\rangle_I &= \frac{\beta_0^I}{2\omega\mu_0}\int_{-\infty}^{+\infty} \mathcal{E}_{y,0}^{II}(x)\left(\mathcal{E}_{y,0}^I(x)\right)^* dx,\\
I_2 = \langle \mathcal{E}_{y,0}^I(x), \mathcal{E}_{y,0}^{II}(x)\rangle_{II} &= \frac{\beta_0^{II}}{2\omega\mu_0}\int_{-\infty}^{+\infty} \mathcal{E}_{y,0}^I(x)\left(\mathcal{E}_{y,0}^{II}(x)\right)^* dx,
\end{aligned}
}
$$

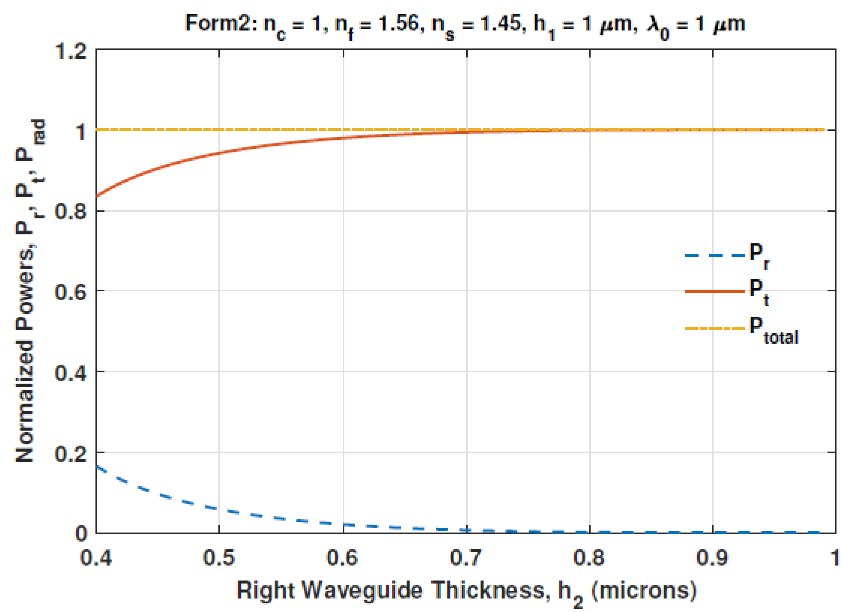
Example Discontinuity in a Slab Waveguide

Power Considerations (Formulation 2)

$$1 - |r_0|^2 - \int_0^{k_0 n_s} |q^I(\beta^I)|^2 d\beta^I = |t_0|^2 + \int_0^{k_0 n_s} |q^{II}(\beta^{II})|^2 d\beta^{II},$$

$$P_r + P_t = P_{inc} \implies \frac{P_r}{P_{inc}} + \frac{P_t}{P_{inc}} = 1 \qquad \frac{P_{rad}}{P_{inc}} = 1 - \frac{P_r}{P_{inc}} - \frac{P_t}{P_{inc}}$$





Example Discontinuity in a Slab Waveguide

Formulation 2 with Substrate & Radiation Modes

$$E_y^{(1)} = \mathcal{E}_{y,inc}^{I}(x)e^{-j\beta_0^I z} + \sum_{i=1}^{G_1} r_{gi}\mathcal{E}_{yg,i}^{I}(x)e^{+j\beta_{gi}^I z} + \sum_{i=1}^{S_1} r_{si}\mathcal{E}_{ys,i}^{I}(x)e^{+j\beta_{si}^I z} + \sum_{i=1}^{R_1} r_{ri}\mathcal{E}_{yr,i}^{I}(x)e^{+j\beta_{ri}^I z}$$

$$\begin{aligned} H_x^{(1)} = & -\frac{\beta_{inc}^I}{\omega\mu_0} \mathcal{E}_{y,inc}^{I}(x)e^{-j\beta_0^I z} + \sum_{i=1}^{G_1} \frac{\beta_{gi}^I}{\omega\mu_0} r_{gi}\mathcal{E}_{yg,i}^{I}(x)e^{+j\beta_{gi}^I z} + \\ & + \sum_{i=1}^{S_1} \frac{\beta_{si}^I}{\omega\mu_0} r_{si}\mathcal{E}_{ys,i}^{I}(x)e^{+j\beta_{si}^I z} + \sum_{i=1}^{R_1} \frac{\beta_{ri}^I}{\omega\mu_0} r_{ri}\mathcal{E}_{yr,i}^{I}(x)e^{+j\beta_{ri}^I z} \end{aligned}$$

$$E_y^{(2)} = \sum_{i=1}^{G_2} t_{gi}\mathcal{E}_{yg,i}^{II}(x)e^{-j\beta_{gi}^{II} z} + \sum_{i=1}^{S_2} t_{si}\mathcal{E}_{ys,i}^{II}(x)e^{-j\beta_{si}^{II} z} + \sum_{i=1}^{R_2} t_{ri}\mathcal{E}_{yr,i}^{II}(x)e^{-j\beta_{ri}^{II} z}$$

$$H_x^{(2)} = -\sum_{i=1}^{G_2} \frac{\beta_{gi}^{II}}{\omega\mu_0} t_{gi}\mathcal{E}_{yg,i}^{II}(x)e^{-j\beta_{gi}^{II} z} - \sum_{i=1}^{S_2} \frac{\beta_{si}^{II}}{\omega\mu_0} t_{si}\mathcal{E}_{ys,i}^{II}(x)e^{-j\beta_{si}^{II} z} - \sum_{i=1}^{R_2} \frac{\beta_{ri}^{II}}{\omega\mu_0} t_{ri}\mathcal{E}_{yr,i}^{II}(x)e^{-j\beta_{ri}^{II} z}$$

Formulation 2 with Substrate & Radiation Modes

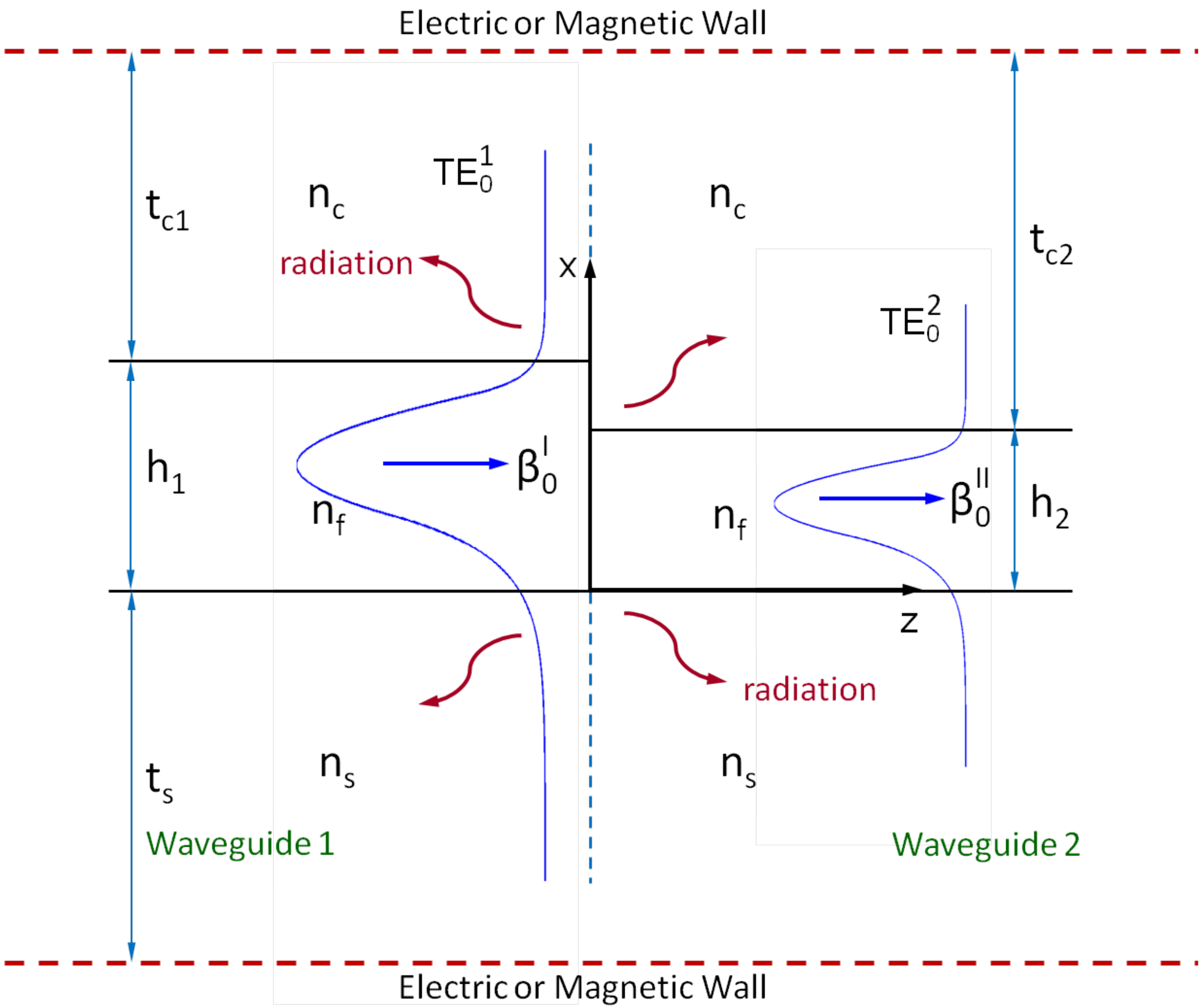
Boundary Conditions

$$\mathcal{E}^{I}_{y,inc}(x) + \sum_{i=1}^{G_1} r_{gi}\mathcal{E}^{I}_{yg,i}(x) + \sum_{i=1}^{S_1} r_{si}\mathcal{E}^{I}_{ys,i}(x) + \sum_{i=1}^{R_1} r_{ri}\mathcal{E}^{I}_{yr,i}(x) = \sum_{i=1}^{G_2} t_{gi}\mathcal{E}^{II}_{yg,i}(x) + \sum_{i=1}^{S_2} t_{si}\mathcal{E}^{II}_{ys,i}(x) + \sum_{i=1}^{R_2} t_{ri}\mathcal{E}^{II}_{yr,i}(x),$$

$$-\frac{\beta^{I}_{inc}}{\omega\mu_0}\mathcal{E}^{I}_{y,inc}(x) + \sum_{i=1}^{G_1}\frac{\beta^{I}_{gi}}{\omega\mu_0} r_{gi}\mathcal{E}^{I}_{yg,i}(x) + \sum_{i=1}^{S_1}\frac{\beta^{I}_{si}}{\omega\mu_0} r_{si}\mathcal{E}^{I}_{ys,i}(x) + \sum_{i=1}^{R_1}\frac{\beta^{I}_{ri}}{\omega\mu_0} r_{ri}\mathcal{E}^{I}_{yr,i}(x) = -\sum_{i=1}^{G_2}\frac{\beta^{II}_{gi}}{\omega\mu_0} t_{gi}\mathcal{E}^{II}_{yg,i}(x) - \sum_{i=1}^{S_2}\frac{\beta^{II}_{si}}{\omega\mu_0} t_{si}\mathcal{E}^{II}_{ys,i}(x) - \sum_{i=1}^{R_2}\frac{\beta^{II}_{ri}}{\omega\mu_0} t_{ri}\mathcal{E}^{II}_{yr,i}(x).$$

Example Discontinuity in a Slab Waveguide

Formulation 2 with Substrate & Radiation Modes



Example Discontinuity in a Slab Waveguide

Formulation 2 with Substrate & Radiation Modes

Boundary Conditions after Projections

$$\left[\begin{array}{c}\tilde{C}^{gg}_{inc}\\ \hdashline \tilde{C}^{gs}_{inc}\\ \hdashline \tilde{C}^{gr}_{inc}\end{array}\right]+\left[\begin{array}{c:c:c}\tilde{C}^{gg} & \tilde{C}^{sg} & \tilde{C}^{rg}\\ \hdashline \tilde{C}^{gs} & \tilde{C}^{ss} & \tilde{C}^{rs}\\ \hdashline \tilde{C}^{gr} & \tilde{C}^{sr} & \tilde{C}^{rr}\end{array}\right]\left[\begin{array}{c}r_g\\ \hdashline r_s\\ \hdashline r_r\end{array}\right]=\left[\begin{array}{c:c:c}\tilde{I} & \tilde{0} & \tilde{0}\\ \hdashline \tilde{0} & \tilde{I} & \tilde{0}\\ \hdashline \tilde{0} & \tilde{0} & \tilde{I}\end{array}\right]\left[\begin{array}{c}t_g\\ \hdashline t_s\\ \hdashline t_r\end{array}\right],$$

$$\tilde{C}_{inc}+\tilde{C}\tilde{R}=\tilde{I}\tilde{T},$$

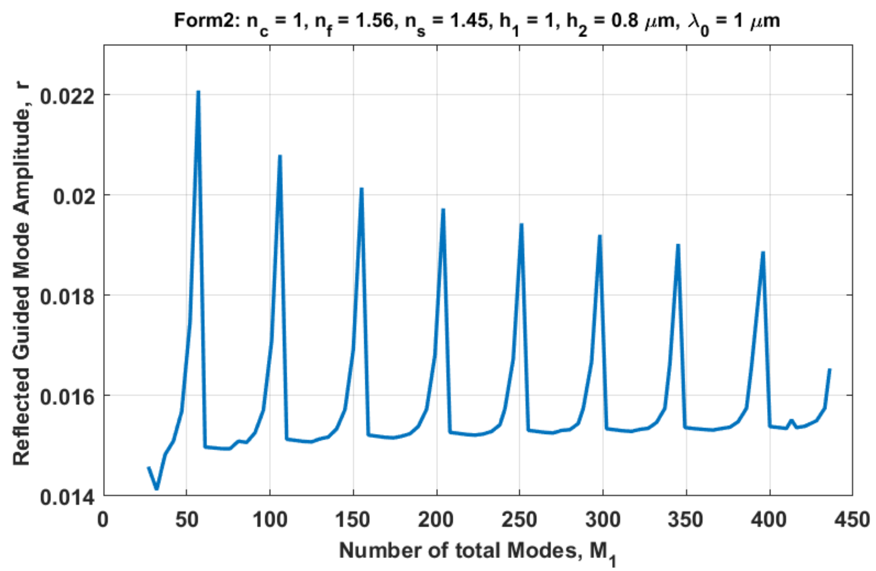
$$-\left[\begin{array}{c}\tilde{\beta}^{I}_{inc}\\ \hdashline \tilde{0}\\ \hdashline \tilde{0}\end{array}\right]+\left[\begin{array}{c:c:c}\tilde{\beta}^{I}_{g} & \tilde{0} & \tilde{0}\\ \hdashline \tilde{0} & \tilde{\beta}^{I}_{s} & \tilde{0}\\ \hdashline \tilde{0} & \tilde{0} & \tilde{\beta}^{I}_{r}\end{array}\right]\left[\begin{array}{c}r_g\\ \hdashline r_s\\ \hdashline r_r\end{array}\right]=-\left[\begin{array}{c:c:c}\tilde{D}^{gg} & \tilde{D}^{sg} & \tilde{D}^{rg}\\ \hdashline \tilde{D}^{gs} & \tilde{D}^{ss} & \tilde{D}^{rs}\\ \hdashline \tilde{D}^{gr} & \tilde{D}^{sr} & \tilde{D}^{rr}\end{array}\right]\left[\begin{array}{c:c:c}\tilde{\beta}^{II}_{g} & \tilde{0} & \tilde{0}\\ \hdashline \tilde{0} & \tilde{\beta}^{II}_{s} & \tilde{0}\\ \hdashline \tilde{0} & \tilde{0} & \tilde{\beta}^{II}_{r}\end{array}\right]\left[\begin{array}{c}t_g\\ \hdashline t_s\\ \hdashline t_r\end{array}\right],$$

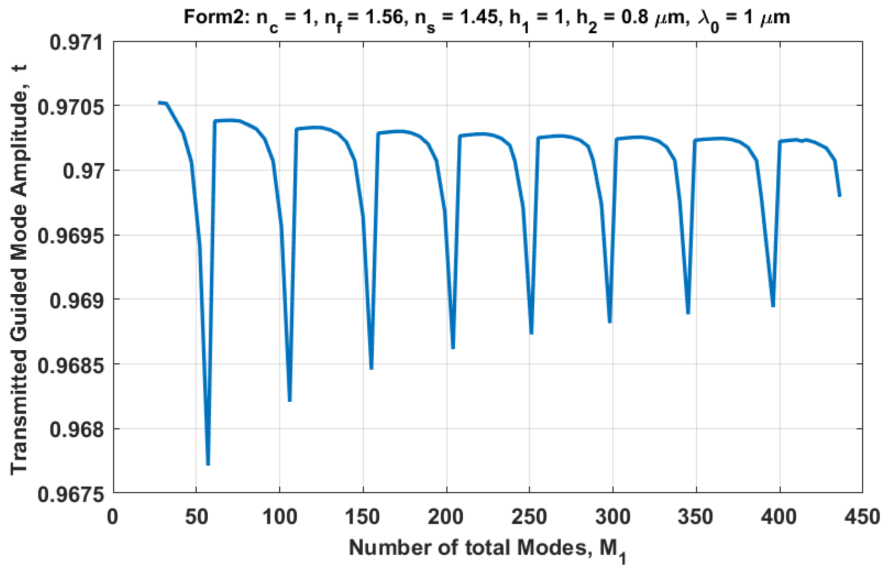
$$-\tilde{b}+\tilde{B}^{I}\tilde{R}=-\tilde{D}\tilde{B}^{II}\tilde{T}.$$

$$\boxed{\left[\begin{array}{c:c}\tilde{C} & -\tilde{I}\\ \hdashline \tilde{B}^{I} & \tilde{D}\tilde{B}^{II}\end{array}\right]\left[\begin{array}{c}\tilde{R}\\ \hdashline \tilde{T}\end{array}\right]=\left[\begin{array}{c}-\tilde{C}_{inc}\\ \hdashline \tilde{b}\end{array}\right]}$$

Example Discontinuity in a Slab Waveguide

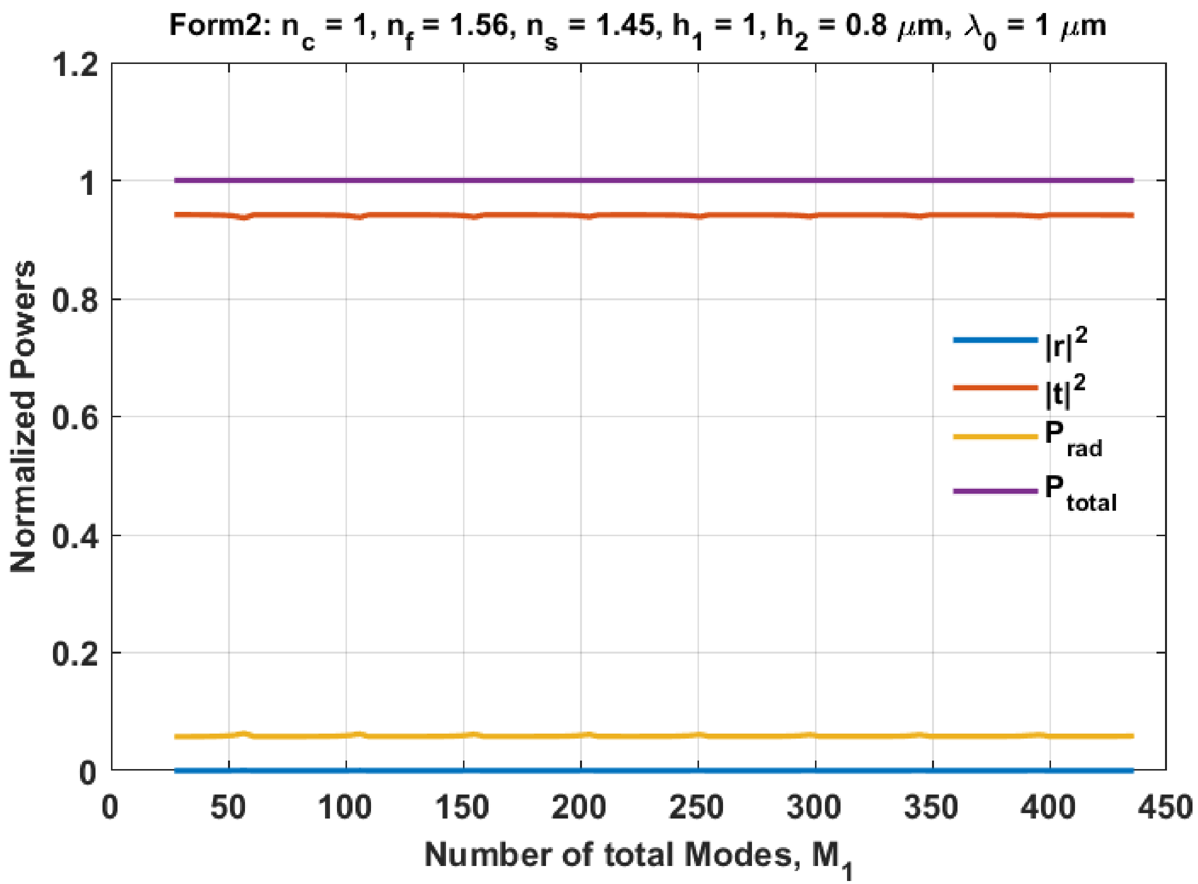
Convergence (Formulation 2) with Substrate and Radiation Modes





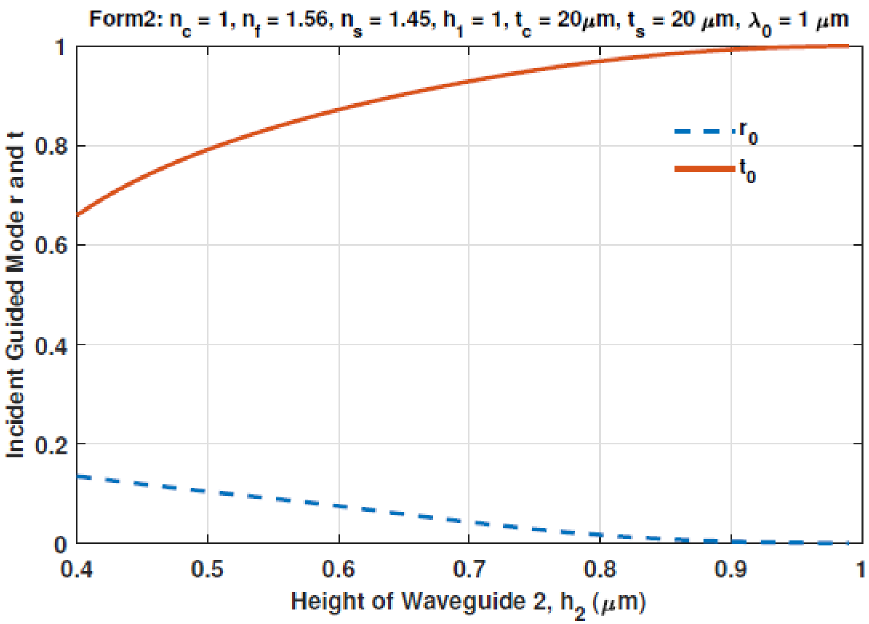
Example Discontinuity in a Slab Waveguide

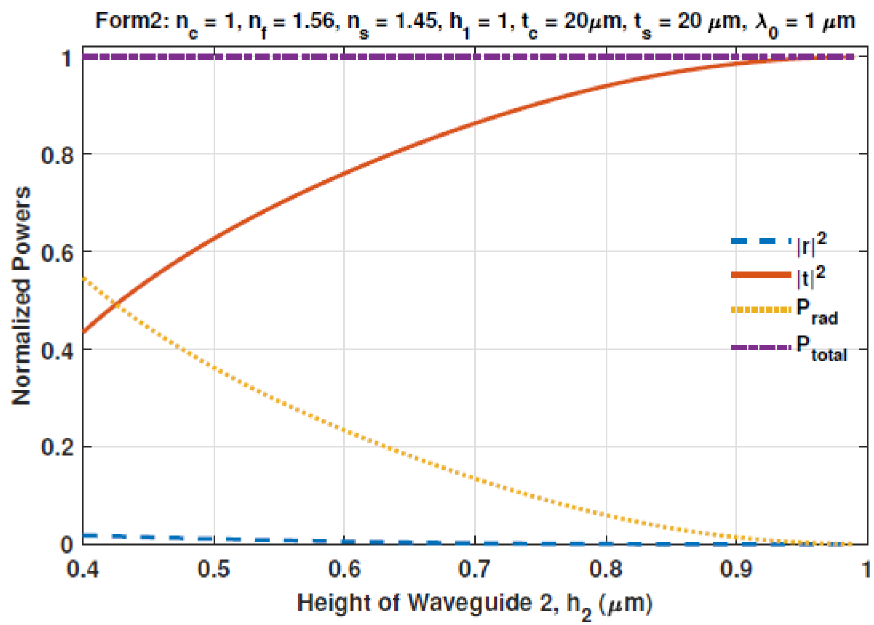
Convergence (Formulation 2) with Substrate and Radiation Modes



Example Discontinuity in a Slab Waveguide

Formulation 2 with Substrate and Radiation Modes (100 modes)





Example Discontinuity in a Slab Waveguide

Formulation 1 with Substrate and Radiation Modes

Boundary Conditions after Projections

$$\left[\begin{array}{c} \tilde{C}_{inc}^{gg} \\ \hdashline \tilde{C}_{inc}^{gs} \\ \hdashline \tilde{C}_{inc}^{gr} \end{array}\right] + \left[\begin{array}{c:c:c} \tilde{C}^{gg} & \tilde{C}^{sg} & \tilde{C}^{rg} \\ \hdashline \tilde{C}^{gs} & \tilde{C}^{ss} & \tilde{C}^{rs} \\ \hdashline \tilde{C}^{gr} & \tilde{C}^{sr} & \tilde{C}^{rr} \end{array}\right] \left[\begin{array}{c} r_g \\ \hdashline r_s \\ \hdashline r_r \end{array}\right] = \left[\begin{array}{c:c:c} \tilde{I} & \tilde{0} & \tilde{0} \\ \hdashline \tilde{0} & \tilde{I} & \tilde{0} \\ \hdashline \tilde{0} & \tilde{0} & \tilde{I} \end{array}\right] \left[\begin{array}{c} t_g \\ \hdashline t_s \\ \hdashline t_r \end{array}\right],$$

$$\tilde{C}_{inc} + \tilde{C}\tilde{R} = \tilde{I}\tilde{T},$$

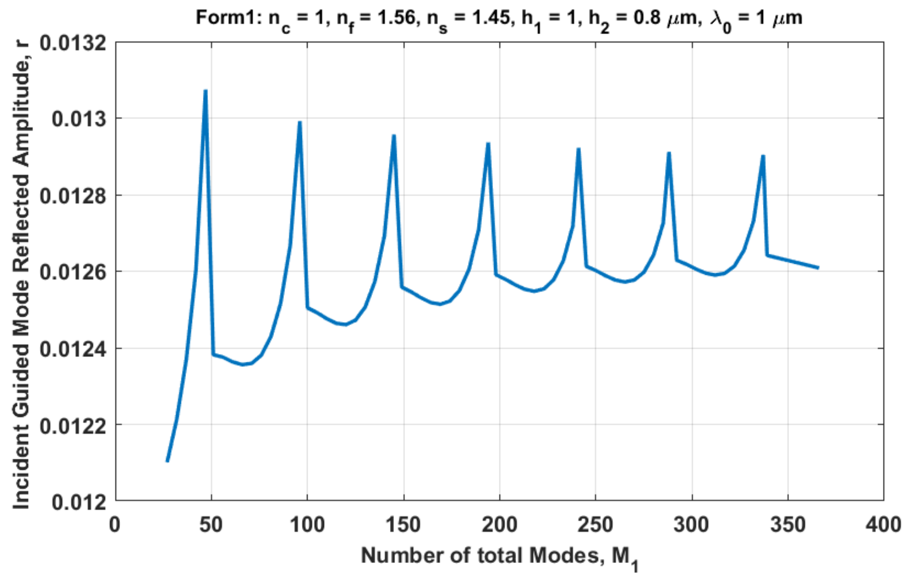
$$-\beta_{inc}^{I} \left[\begin{array}{c} \tilde{C}_{inc}^{gg} \\ \hdashline \tilde{C}_{inc}^{gs} \\ \hdashline \tilde{C}_{inc}^{gr} \end{array}\right] + \left[\begin{array}{c:c:c} \tilde{C}^{gg} & \tilde{C}^{sg} & \tilde{C}^{rg} \\ \hdashline \tilde{C}^{gs} & \tilde{C}^{ss} & \tilde{C}^{rs} \\ \hdashline \tilde{C}^{gr} & \tilde{C}^{sr} & \tilde{C}^{rr} \end{array}\right] \left[\begin{array}{c:c:c} \tilde{\beta}_g^{I} & \tilde{0} & \tilde{0} \\ \hdashline \tilde{0} & \tilde{\beta}_s^{I} & \tilde{0} \\ \hdashline \tilde{0} & \tilde{0} & \tilde{\beta}_r^{I} \end{array}\right] \left[\begin{array}{c} r_g \\ \hdashline r_s \\ \hdashline r_r \end{array}\right] = - \left[\begin{array}{c:c:c} \tilde{\beta}_g^{II} & \tilde{0} & \tilde{0} \\ \hdashline \tilde{0} & \tilde{\beta}_s^{II} & \tilde{0} \\ \hdashline \tilde{0} & \tilde{0} & \tilde{\beta}_r^{II} \end{array}\right] \left[\begin{array}{c} t_g \\ \hdashline t_s \\ \hdashline t_r \end{array}\right],$$

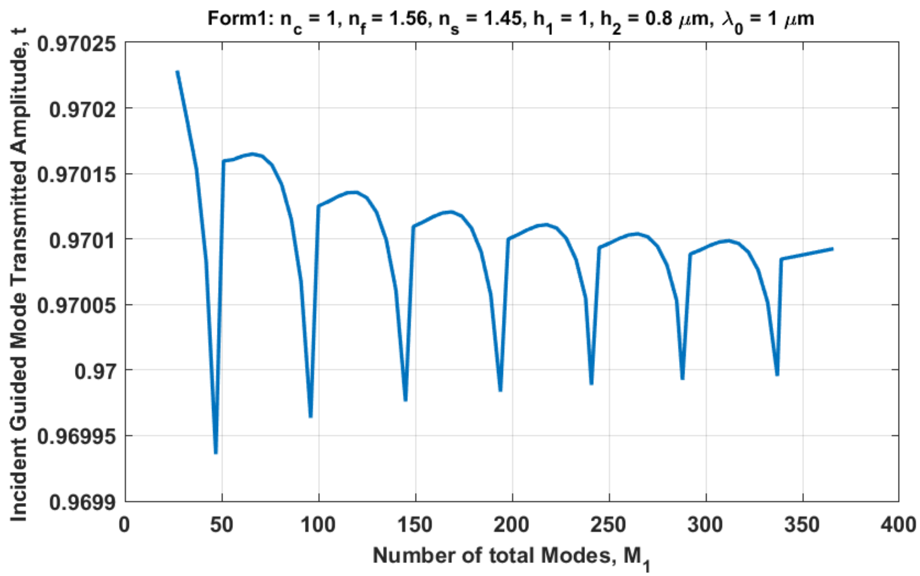
$$-\beta_{inc}^{I}\tilde{C}_{inc} + \tilde{C}\tilde{B}^{I}\tilde{R} = -\tilde{B}^{II}\tilde{T}.$$

$$\boxed{\left[\begin{array}{c:c} \tilde{C} & -\tilde{I} \\ \hdashline \tilde{C}\tilde{B}^{I} & \tilde{B}^{II} \end{array}\right] \left[\begin{array}{c} \tilde{R} \\ \hdashline \tilde{T} \end{array}\right] = \left[\begin{array}{c} -\tilde{C}_{inc} \\ \hdashline \beta_{inc}^{I}\tilde{C}_{inc} \end{array}\right]}$$

Example Discontinuity in a Slab Waveguide

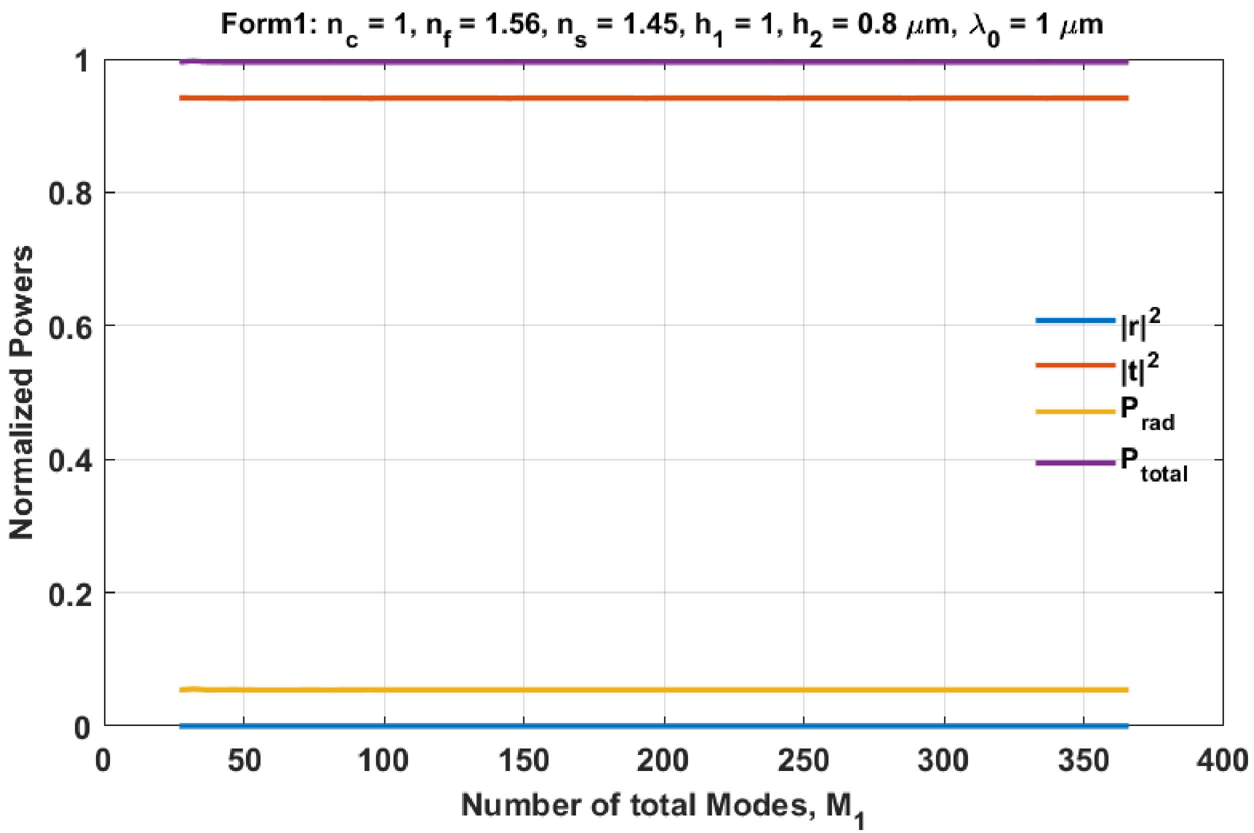
Convergence (Formulation 1) with Substrate and Radiation Modes





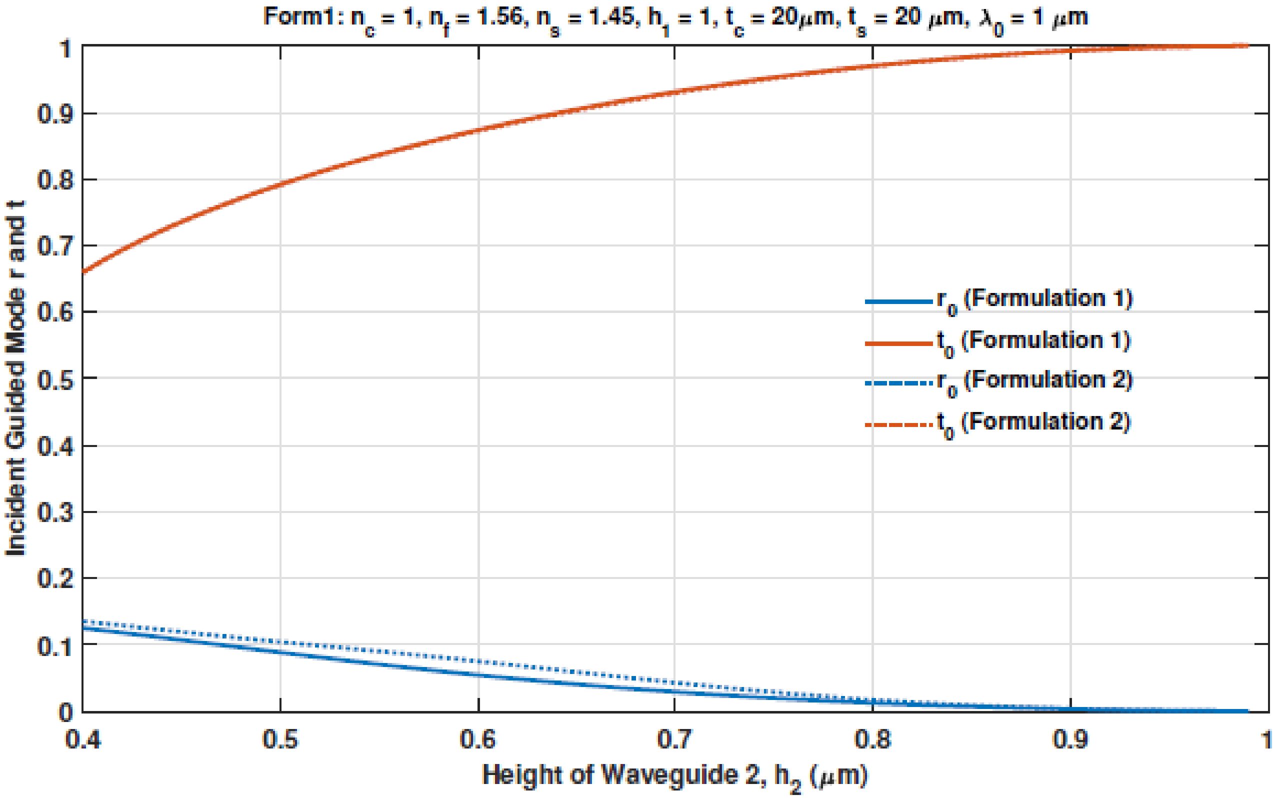
Example Discontinuity in a Slab Waveguide

Convergence (Formulation 1) with Substrate and Radiation Modes



Example Discontinuity in a Slab Waveguide

Formulation 1 and 2 with Substrate and Radiation Modes (100 modes)



Example Discontinuity in a Slab Waveguide

Formulation 1 and 2 with Substrate and Radiation Modes (100 modes)