

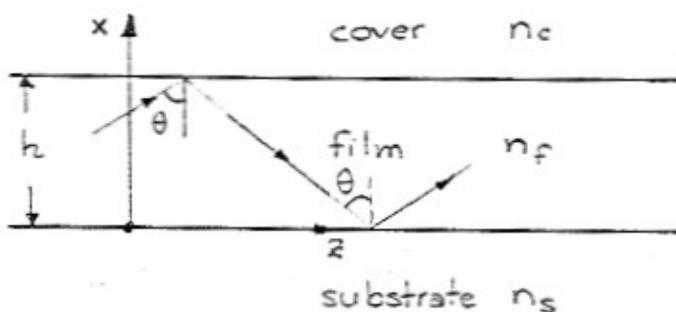
ΟΛΟΚΛΗΡΩΜΕΝΗ ΟΠΤΙΚΗ
(INTEGRATED OPTICS)

ΕΙΣΑΓΩΓΙΚΕΣ ΕΝΝΟΙΕΣ
ΕΠΙΠΕΔΩΝ ΟΠΤΙΚΩΝ ΚΥΜΑΤΑΓΩΓΩΝ
(Fundamentals of Slab Optical Waveguides)

Σημειώσεις

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SLAB PLANAR WAVEGUIDE



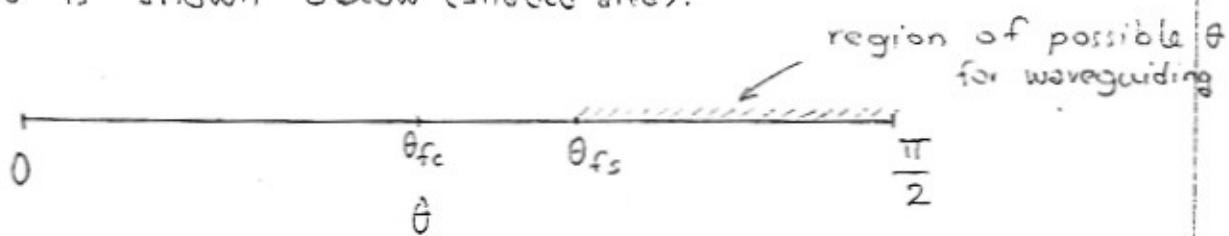
The geometry of a planar slab waveguide structure is shown in the figure above. The basic concept of guiding the light in the film region is the total internal reflection at both boundaries. For this to occur θ should be greater than the critical angles of film-cover and film-substrate.

Thus, $\theta > \max(\theta_{fc}, \theta_{fs})$ where θ_{fc} and θ_{fs} are the critical angles of the film-cover and film-substrate interfaces, respectively. The two critical angles are given by

$$\theta_{fc} = \sin^{-1}\left(\frac{n_c}{n_f}\right), \quad \theta_{fs} = \sin^{-1}\left(\frac{n_s}{n_f}\right)$$

For these angles to be defined it is necessary for n_f to be larger than both n_s, n_c , i.e. $n_f > \max(n_s, n_c)$.

In most practical waveguide configurations $n_s > n_c$. Thus, for the rest of this section the following ordering of the indices is assumed: $n_f > n_s > n_c$. The range of possible zig-zag angles θ is shown below (shaded area):



However, since the angle of incidence is greater than the

critical angle for each reflection there is a corresponding phase shift. In order for a plane wave to exist guided in the structure the beam shown in the figure should be self-consistent. This means that the sum of all the phase-shifts that the wave undergoes should sum up to multiple of 2π .

Considering a constant \bar{z} the sum of the phase shifts going upwards, reflecting, going downwards, and reflecting should sum up to integer multiple of 2π .

Assuming a plane wave of the form $\bar{E} = (\bar{E}_{f_1} e^{-jk_{fx}x} + \bar{E}_{f_2} e^{+jk_{fx}x}) e^{-jk_{fx}\bar{z}}$ the total phase shift for constant \bar{z} is: $-2k_{fx}h$

The phase-shift due to the reflections is $2\phi_{fs} + 2\phi_{fc}$

Thus, the self-consistency condition (or transverse resonance condition) can be written as

$$2k_{fx}h - 2\phi_{fs} - 2\phi_{fc} = 2v\pi \quad v=0, \pm 1, \pm 2, \dots$$

Since $k_{fx} = k_{nf} \cos \theta$ the previous condition can be written in the form:

$$2k_{nf}h \cos \theta - 2\phi_{fs} - 2\phi_{fc} = 2v\pi \quad v=0, 1, 2, \dots$$

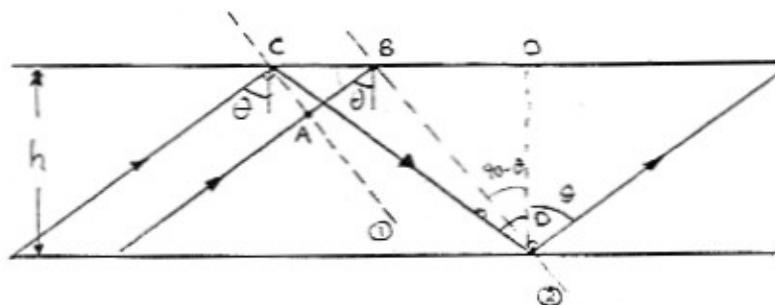
(where v is now restricted to positive integer since for $\alpha < \pi/2$ there are no solutions for $v < 0$). The phase shifts ϕ_{fs}, ϕ_{fc} are:

$$\phi_{fs} = \phi_{fs}(\theta) = \begin{cases} \tan^{-1} \left[(n_f^2 \sin^2 \theta - n_s^2)^{1/2} / n_f \cos \theta \right] & \text{for TE} \\ \tan^{-1} \left[\frac{n_f^2}{n_s^2} (n_f^2 \sin^2 \theta - n_s^2) / n_f \cos \theta \right] & \text{for TM} \end{cases}$$

$$\phi_{fc} = \phi_{fc}(\theta) = \begin{cases} \tan^{-1} \left[(n_f^2 \sin^2 \theta - n_c^2)^{1/2} / n_f \cos \theta \right] & \text{for TE} \\ \tan^{-1} \left[\frac{n_f^2}{n_c^2} (n_c^2 \sin^2 \theta - n_f^2)^{1/2} / n_f \cos \theta \right] & \text{for TM} \end{cases}$$

The above condition is a transcendental equation with respect to the unknown zig-zag angle θ .

Alternate Derivation (Marcuse):



50 SHEETS
100 SHEETS
200 SHEETS

22-141
22-142
22-144

ANSWER

The points (C,A) and (B,D) belong to the same phase fronts ① & ② respectively.

$$\text{Thus } \phi_{DC} = \phi_{BA} + 2v\pi \quad (v = 0, \pm 1, \pm 2, \dots)$$

C, D points are just before and just after reflection respectively. $\phi_{DC} = 2\phi_{fc} + 2\phi_{fs} - k(CD)$

$$\phi_{BA} = -k(AB)$$

$$k = k_0 n_f$$

$$(AB) = (CB) \sin \theta$$

$$(CB) = (OC) - (OB) = h \tan \theta - h \tan(90 - \theta) = h \tan \theta - \frac{h}{\tan \theta}$$

$$\text{Thus, } (AB) = h \left[\tan \theta - \frac{1}{\tan \theta} \right] \sin \theta = h \left(\frac{\sin^2 \theta}{\cos \theta} - \frac{\cos \theta}{\sin^2 \theta} \right) = h \frac{\sin^2 \theta - \cos^2 \theta}{\cos \theta}$$

$$(CD) = \frac{h}{\cos \theta}$$

$$\phi_{DC} - \phi_{BA} = 2v\pi \Rightarrow -k(CD - AB) + 2\phi_{fc} + 2\phi_{fs} = 2v\pi$$

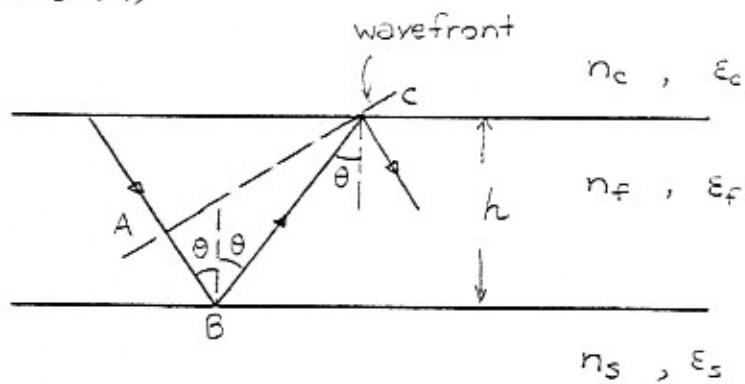
$$-k \left[\frac{h}{\cos \theta} - \frac{h \sin^2 \theta}{\cos \theta} + \frac{h \cos^2 \theta}{\cos \theta} \right] + 2\phi_{fc} + 2\phi_{fs} = 2v\pi$$

$$-k \left[\frac{h}{\cos \theta} \right] [1 - \sin^2 \theta + \cos^2 \theta] + 2\phi_{fc} + 2\phi_{fs} = 2v\pi$$

$$-2kh \cos \theta + 2\phi_{fs} + 2\phi_{fc} = 2v\pi \quad \text{or}$$

$$2k_0 n_f h \cos \theta - 2\phi_{fs} - 2\phi_{fc} = 2v\pi \quad v = 0, 1, 2, \dots$$

(Other derivation)



$$\Phi_{ABC} = \pm 2\pi p \quad p = 0, 1, 2, \dots$$

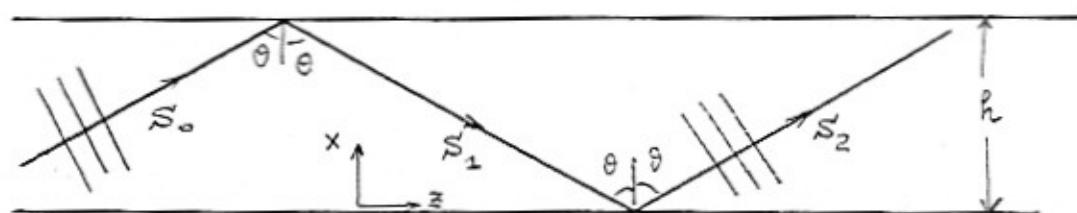
$$\begin{aligned} l_{ABC} &= l_{AC} + l_{BC} = l_{BC} \cos 2\theta + l_{BC} = l_{BC} (1 + \cos 2\theta) = \\ &= \frac{h}{\cos \theta} (1 + \cos 2\theta) = \frac{h 2 \cos^2 \theta}{\cos \theta} = 2h \cos \theta \end{aligned}$$

$$\Phi_{ABC} = -2h \cos \theta (k_0 n_f) + 2\phi_{fc} + 2\phi_{fs} = \pm 2\pi p$$

$$2k_0 n_f h \cos \theta - 2\phi_{fc} - 2\phi_{fs} = \pm 2\pi p = 2\pi v \quad v = 0, 1, 2, \dots$$

- * Points A and C belong to the same wavefront of the plane wave that is directed downwards.

Alternate Derivation:



$$S_0 = A_0 e^{-j k_0 n_f (x \cos \theta + z \sin \theta)} = S_0(x, z)$$

$$S_1 = A_1 e^{-j k_0 n_f (-x \cos \theta + z \sin \theta)} = S_1(x, z)$$

$$S_2 = A_2 e^{-j k_0 n_f (x \cos \theta + z \sin \theta)} = S_2(x, z)$$

$$S_1(x=h, z) = r_{fc} S_0(x=h, z) = e^{j 2 \phi_{fc}} S_0(x=h, z) \quad (1)$$

$$S_2(x=0, z) = r_{fs} S_1(x=0, z) = e^{j 2 \phi_{fs}} S_1(x=0, z) \quad (2)$$

$$(1) \quad A_1 e^{+j h \cos \theta k_0 n_f} = e^{j 2 \phi_{fc}} A_0 e^{-j k_0 n_f h \cos \theta} \quad \left. \right\} \Rightarrow$$

$$(2) \quad A_2 = e^{j 2 \phi_{fs}} A_1$$

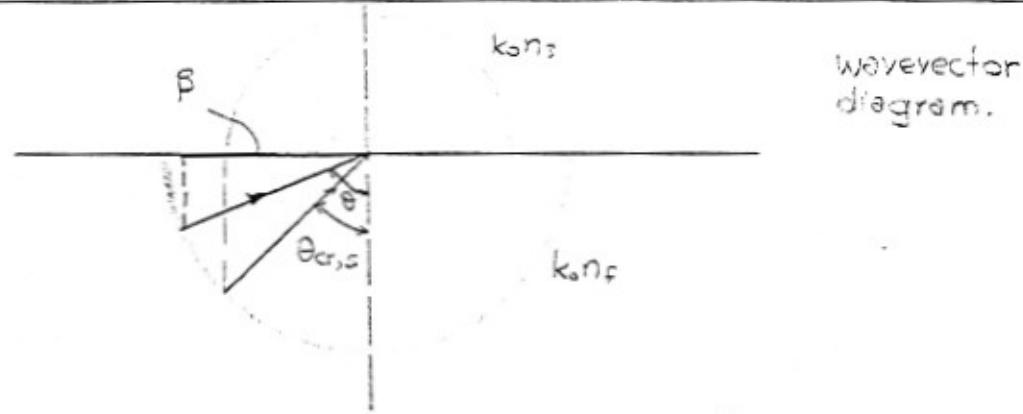
$$A_2 = e^{j 2 \phi_{fs}} e^{j 2 \phi_{fc}} e^{-j 2 k_0 n_f h \cos \theta} A_0$$

$$A_2 = A_0 e^{j [2 \phi_{fc} + 2 \phi_{fs} - 2 k_0 n_f h \cos \theta]}$$

$$\text{For consistency } A_2 = A_0 e^{-j 2 \pi v}$$

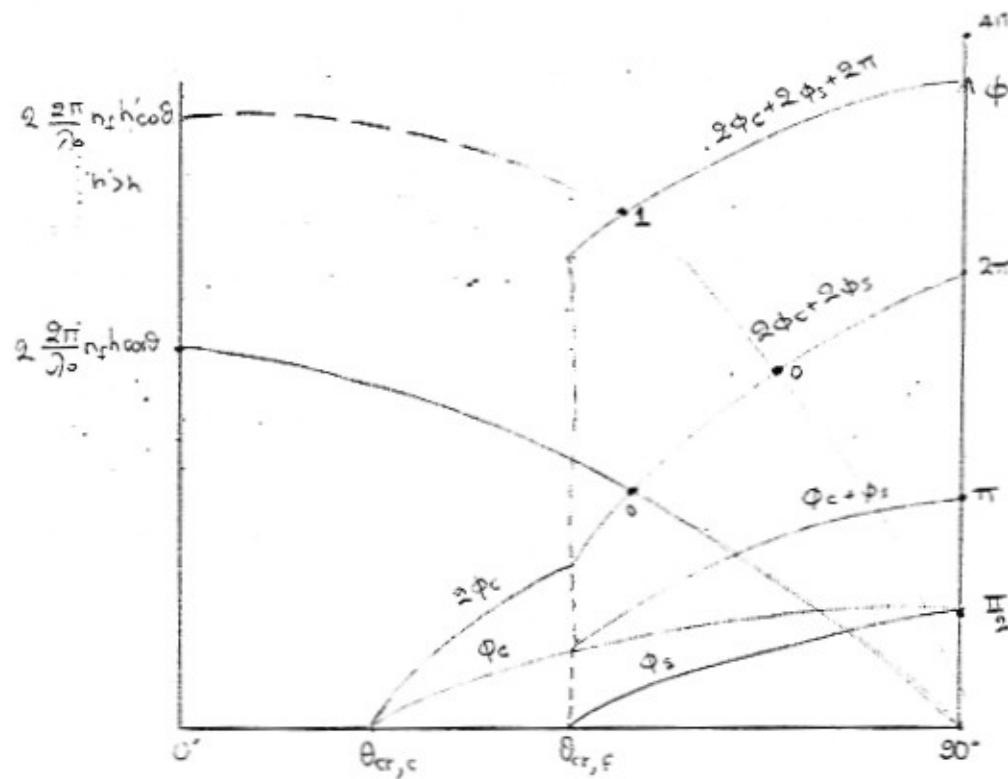
$$-2 \pi v = 2 \phi_{fc} + 2 \phi_{fs} - 2 k_0 n_f h \cos \theta \Rightarrow$$

$$2 k_0 n_f h \cos \theta - 2 \phi_{fc} - 2 \phi_{fs} = 2 \pi v \quad v = 0, 1, 2, \dots$$

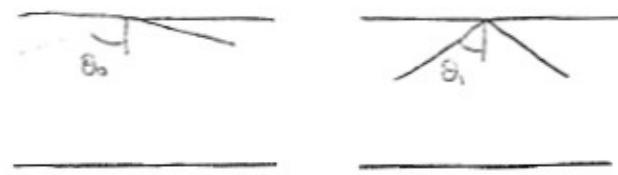


$$k_0 n_s < \beta < k_0 n_f \Rightarrow n_s < \frac{\beta}{k_0} < n_f \Rightarrow$$

$$n_s < N < n_f$$



Talk about the symmetric grating and cutoffs.



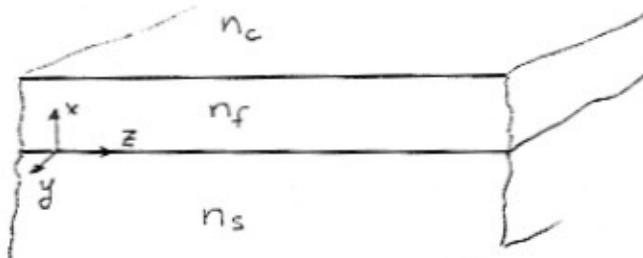
$$\theta_0 > \theta_1 > \theta_{cr,sf}$$

The zig-zag angle θ is characteristic of the type of electro-magnetic field which is guided in the film region. The field that corresponds to the θ that is a solution of the self-consistency condition consists a mode. The angle θ depends on v and v is also characteristic of the mode. For this reason the solutions that correspond to TE polarization are denoted by TE_v and the solutions that correspond to TM polarization are denoted by TM_v . For each mode the effective index N_v and the propagation constant β_v can be defined as follow:

$$N_v = n_f \sin \theta_v \quad \text{and} \quad \beta_v = k_0 n_f \sin \theta_v = k_0 N_v$$

Slab Waveguide:

(Electromagnetic fields solution)



From Maxwell's equations: (time-harmonic form)

$$\vec{\nabla} \times \vec{E} = -j\omega \mu_0 \vec{H}$$

$$\vec{\nabla} \times \vec{H} = j\omega \epsilon \vec{E}$$

$$\vec{\nabla} \cdot \vec{D} = 0 \rightarrow \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -j\omega \mu_0 (\vec{\nabla} \times \vec{H}) \Rightarrow$$

$$\vec{\nabla} (\vec{\nabla} \times \vec{E}) - \vec{\nabla}^2 \vec{E} = -j\omega \mu_0 (j\omega \epsilon \vec{E}) \Rightarrow$$

$$\vec{\nabla}^2 \vec{E} = -\omega^2 \mu_0 \epsilon_0 \epsilon_r \vec{E} \Rightarrow$$

$$\vec{\nabla}^2 \vec{E} + \omega^2 \mu_0 \epsilon_0 \epsilon_r \vec{E} = 0 \quad] \Rightarrow \vec{\nabla}^2 \vec{E} + k_0^2 n^2 \vec{E} = 0$$

$$k_0 = \frac{\omega}{c} = \omega \sqrt{\mu_0 \epsilon_0} \quad n^2 = \epsilon_r \quad] \text{ similarly: } \vec{\nabla}^2 \vec{H} + k_0^2 n^2 \vec{H} = 0$$

Due to the 2-D symmetry no dependence on y-coordinate

is expected $\rightarrow \frac{\partial}{\partial y} = 0$

Assume solutions of the form:

$$\vec{E} = \vec{E}(x) e^{-j\beta z} \quad \text{where } \beta \text{ is the propagation}$$

constant.

Assuming $\vec{E}(x) = E_x(x) \hat{x} + E_y(x) \hat{y} + E_z(x) \hat{z}$ and using the wave equation we get:

$$\hat{x} : \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) E_x(x) e^{-j\beta z} + k_o^2 n^2 E_x(x) e^{-j\beta z} = \emptyset \Rightarrow \\ \left(\frac{\partial^2 E_x}{\partial x^2} + j^2 \beta^2 E_x \right) e^{-j\beta z} + k_o^2 n^2 E_x(x) e^{-j\beta z} = 0 \Rightarrow \\ \frac{\partial^2 E_x}{\partial x^2} + \underbrace{(k_o^2 n^2 - \beta^2)}_{k_x^2} E_x = 0 \sim \frac{\partial^2 E_x}{\partial x^2} - j^2 k_x^2 E_x = \emptyset$$

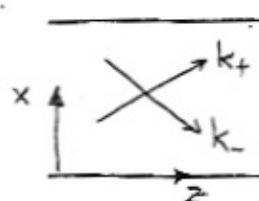
$$\text{Similarly, } \frac{\partial^2 E_y}{\partial x^2} - j^2 k_x^2 E_y = \emptyset$$

$$\frac{\partial^2 E_z}{\partial x^2} - j^2 k_x^2 E_z = \emptyset$$

Thus, $E_x(x) = A_{1x} e^{-j k_x x} + A_{2x} e^{+j k_x x}$ and similarly

for $E_y, E_z \rightarrow$

$$\vec{E} = (A_{1x} \hat{x} + A_{1y} \hat{y} + A_{1z} \hat{z}) e^{-j k_x x} e^{-j \beta z} + \\ (A_{2x} \hat{x} + A_{2y} \hat{y} + A_{2z} \hat{z}) e^{+j k_x x} e^{-j \beta z} = \\ \vec{E}_+ e^{-j \vec{k}_+ \cdot \vec{r}} + \vec{E}_- e^{-j \vec{k}_- \cdot \vec{r}}$$



Similar equations can also be written for the magnetic fields. However, it is common for isotropic materials, to distinguish two different types of modes (acceptable solutions of Maxwell's equations).

- a. TE modes (Transverse Electric field)
- b. TM modes (Transverse Magnetic field)

From $\vec{\nabla} \times \vec{E} = -j\omega\mu_0 \vec{H} \Rightarrow \vec{H} = (H_x(x)\hat{x} + H_y(x)\hat{y} + H_z(x)\hat{z})e^{-j\beta z}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ E_x e^{-j\beta z} & E_y e^{-j\beta z} & E_z e^{-j\beta z} \end{vmatrix} = \hat{x} \left[-\frac{\partial(E_y e^{-j\beta z})}{\partial z} \right] + \hat{y} \left(\frac{\partial(E_x e^{-j\beta z})}{\partial z} - \frac{\partial(E_z e^{-j\beta z})}{\partial x} \right) + \hat{z} \left(\frac{\partial(E_y e^{-j\beta z})}{\partial x} \right)$$

$$j\beta E_y(x) e^{-j\beta z} = -j\omega\mu_0 H_x(x) e^{-j\beta z}$$

$$\left(-j\beta E_x(x) - \frac{\partial E_z}{\partial x} \right) e^{-j\beta z} = -j\omega\mu_0 H_y(x) e^{-j\beta z}$$

$$\frac{\partial E_y}{\partial x} e^{-j\beta z} = -j\omega\mu_0 H_z(x) e^{-j\beta z}$$

$$\boxed{\begin{aligned} \beta E_y &= -\omega\mu_0 H_x \\ j\beta E_x + \frac{\partial E_z}{\partial x} &= j\omega\mu_0 H_y \\ \frac{\partial E_y}{\partial x} &= -j\omega\mu_0 H_z \end{aligned}}$$

From $\vec{\nabla} \times \vec{H} = +j\omega\epsilon \vec{E} \Rightarrow$

$$j\beta H_y = j\omega\epsilon E_x \Rightarrow$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \Rightarrow$$

$$\beta H_y = \omega\epsilon E_x$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y$$

$$\frac{\partial H_y}{\partial x} = j\omega\epsilon E_z$$

Note that only (E_y, H_x, H_z) and (H_y, E_x, E_z) are coupled each other.

This can also be shown if we write the above equations in the form:

$$\frac{\partial}{\partial x} \begin{bmatrix} E_y \\ H_z \\ H_y \\ E_z \end{bmatrix} = \begin{bmatrix} 0 & -j\omega\mu_0 & 0 & 0 \\ (j\omega\epsilon + j\frac{\beta^2}{\omega\mu_0}) & 0 & 0 & 0 \\ 0 & 0 & 0 & j\omega\epsilon \\ 0 & 0 & j\omega\mu_0 - j\frac{\beta^2}{\omega\mu_0} & 0 \end{bmatrix} \begin{bmatrix} E_y \\ H_z \\ H_y \\ E_z \end{bmatrix}$$

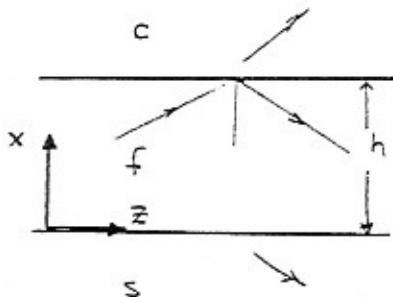
$$\begin{bmatrix} E_x \\ H_x \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\omega\epsilon} & 0 \\ 0 & -\frac{\beta}{\omega\mu_0} \end{bmatrix} \begin{bmatrix} H_y \\ E_y \end{bmatrix}$$

The (E_y, H_x, H_z) & (H_y, E_x, E_z) fields are not coupled:

TE modes: E_y, H_x, H_z

TM modes: H_y, E_x, E_z

For this isotropic region problem the TE & TM modes are uncoupled. This is not true in the case of anisotropic regions in which case all fields are coupled and the modes are neither TE nor TM. In the latter case the modes are hybrid.



Solutions of Maxwell's equations:

$$\text{Cover: } \vec{E}_L = \vec{E}_{C1} e^{-j(k_{Cx}x + k_{Cz}z)} + \vec{E}_{C2} e^{-j(-k_{Cx}x + k_{Cz}z)} \quad (1)$$

$$\text{Film: } \vec{E}_L = \vec{E}_{F1} e^{-j(k_{Fx}x + k_{Fz}z)} + \vec{E}_{F2} e^{-j(-k_{Fx}x + k_{Fz}z)} \quad (2)$$

$$\text{Substrate: } \vec{E}_L = \vec{E}_{S1} e^{-j(k_{Sx}x + k_{Sz}z)} + \vec{E}_{S2} e^{-j(-k_{Sx}x + k_{Sz}z)} \quad (3)$$

From phase-matching conditions along the boundaries ($x=0, x=h$) we get:

$$k_{Cz} = k_{Fz} = k_{Sz} = \beta = \text{propagation constant}$$

In addition,

$$k_{Cx}^2 + \beta^2 = k_0^2 n_c^2 \quad (4)$$

$$k_{Fx}^2 + \beta^2 = k_0^2 n_f^2 \quad (5)$$

$$k_{Sx}^2 + \beta^2 = k_0^2 n_s^2 \quad (6)$$

For guided modes total internal reflection should occur at the two boundaries: $k_0 n_c < k_0 n_s < \beta < k_0 n_f$ ($n_c < n_s < n_f$)

Then, k_{Cx}, k_{Sx} are imaginary

$$k_{Cx}^2 = k_0^2 n_c^2 - \beta^2 < 0 \Rightarrow k_{Cx} = \pm j (\beta^2 - k_0^2 n_c^2)^{1/2}$$

$$k_{Fx}^2 = k_0^2 n_f^2 - \beta^2 > 0 \Rightarrow k_{Fx} = \pm (k_0^2 n_f^2 - \beta^2)^{1/2}$$

$$k_{Sx}^2 = k_0^2 n_s^2 - \beta^2 < 0 \Rightarrow k_{Sx} = \pm j (\beta^2 - k_0^2 n_s^2)^{1/2}$$

The signs have to be chosen such that we have exponentially decaying fields in the cover & the substrate region.

$$e^{-jk_{Cx}x} = e^{-j(\pm j)[\pm (\beta^2 - k_0^2 n_c^2)^{1/2}]x} = e^{\mp j(\beta^2 - k_0^2 n_c^2)^{1/2}x} \quad \text{in order to decay}$$

$-j(\beta^2 - k_0^2 n_c^2)^{1/2}$ has to be used. Then $\vec{E}_{C2} = 0$. Similarly, $\vec{E}_{S1} = 0$.

New expressions of the electric fields:

$$\text{Cover: } \vec{E} = \vec{E}_c e^{-\gamma_c(x-h)} e^{-j\beta z} \quad \gamma_c = (\beta^2 - k_0^2 n_c^2)^{1/2} \quad (10)$$

$$\text{Film: } \vec{H} = \vec{E}_{f1} e^{jk_{fx}x} e^{-j\beta z} + \vec{E}_{f2} e^{-jk_{fx}x} e^{-j\beta z} \quad k_{fx} = (k_0^2 n_f^2 - \beta^2)^{1/2} \quad (11)$$

$$\text{Substrate: } \vec{H} = \vec{E}_s e^{\gamma_s x} e^{-j\beta z} \quad \gamma_s = (\beta^2 - k_0^2 n_s^2)^{1/2} \quad (12)$$

The constants can be found using the boundary conditions.

These are: 1. Continuity of tangential E components

2. Continuity of tangential H components

$$[\vec{E}(\text{cover})]_t \Big|_{x=h} = [\vec{E}(\text{film})]_t \Big|_{x=h} \quad \} \quad \text{cover-film interface}$$

$$[\vec{H}(\text{cover})]_t \Big|_{x=h} = [\vec{H}(\text{film})]_t \Big|_{x=h} \quad \}$$

$$[\vec{E}(\text{substrate})]_t \Big|_{x=0} = [\vec{E}(\text{film})]_t \Big|_{x=0} \quad \} \quad \text{substrate-film interface}$$

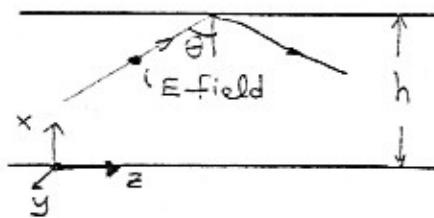
$$[\vec{H}(\text{substrate})]_t \Big|_{x=0} = [\vec{H}(\text{film})]_t \Big|_{x=0} \quad \}$$

Now we have to distinguish between different polarizations:

A. TE polarization

B. TM polarization

A. TE - polarization (TE modes)



$$\beta = k_0 n_f \sin \theta$$

Cover: $\vec{E} = \hat{y} E_c e^{-j\gamma_c(x-h)} e^{-j\beta z}$

Film: $\vec{E} = \hat{y} (E_{f1} e^{jk_{fx}x} + E_{f2} e^{-jk_{fx}x}) e^{-j\beta z}$

Substrate: $\vec{E} = \hat{y} E_s e^{\gamma_s x} e^{-j\beta z}$

$$\vec{\nabla} \times \vec{E} = -j\omega \mu_0 \vec{H} \Rightarrow \vec{H} = -\frac{1}{j\omega\mu_0} \vec{\nabla} \times \vec{E} = \frac{1}{\omega\mu_0} \vec{\nabla} \times \vec{E}$$

$$\vec{H} = \frac{j}{\omega\mu_0} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \emptyset & E_y & \emptyset \end{vmatrix} = \frac{j}{\omega\mu_0} \left[\hat{x} \left(-\frac{\partial E_y}{\partial z} \right) + \hat{y} \left(\emptyset \right) + \hat{z} \left(\frac{\partial E_y}{\partial x} \right) \right]$$

Magnetic Field Expressions

Cover: $\vec{H} = \left\{ \left[-\frac{\beta}{\omega\mu_0} E_c e^{-j\gamma_c(x-h)} \right] \hat{x} + \left(-\frac{j}{\omega\mu_0} \gamma_c E_c e^{-j\gamma_c(x-h)} \right) \hat{z} \right\} e^{-j\beta z}$

Film: $\vec{H} = \left\{ \left[-\frac{\beta}{\omega\mu_0} (E_{f1} e^{jk_{fx}x} + E_{f2} e^{-jk_{fx}x}) \right] \hat{x} + \left[-\frac{k_{fx}}{\omega\mu_0} E_{f1} e^{jk_{fx}x} + \frac{k_{fx}}{\omega\mu_0} E_{f2} e^{-jk_{fx}x} \right] \hat{z} \right\} e^{-j\beta z}$

Substrate: $\vec{H} = \left\{ \left[-\frac{\beta}{\omega\mu_0} E_s e^{\gamma_s x} \right] \hat{x} + \left[\frac{j}{\omega\mu_0} \gamma_s E_s e^{\gamma_s x} \right] \hat{z} \right\} e^{-j\beta z}$

Applying the boundary conditions for the tangential fields at the interface we get:

$$E_c = E_{f_1} e^{j k_{fx} h} + E_{f_2} e^{-j k_{fx} h} \quad (\text{bc1})$$

$$-\frac{j}{\gamma_s \gamma_c} \gamma_c E_c = -\frac{k_{fx}}{\gamma_s \gamma_c} E_{f_1} e^{j k_{fx} h} + \frac{k_{fx}}{\gamma_s \gamma_c} E_{f_2} e^{-j k_{fx} h} \quad (\text{bc2})$$

$$E_s = E_{f_1} + E_{f_2} \quad (\text{bc3})$$

$$\frac{j}{\gamma_s \gamma_c} \gamma_s E_s = -\frac{k_{fx}}{\gamma_s \gamma_c} E_{f_1} + \frac{k_{fx}}{\gamma_s \gamma_c} E_{f_2} \quad (\text{bc4})$$

$$\underbrace{\begin{bmatrix} -1 & e^{j k_{fx} h} & e^{-j k_{fx} h} & 0 \\ +j \gamma_c & -k_{fx} e^{j k_{fx} h} & +k_{fx} e^{-j k_{fx} h} & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -k_{fx} & +k_{fx} & -j \gamma_s \end{bmatrix}}_A \begin{bmatrix} E_c \\ E_{f_1} \\ E_{f_2} \\ E_s \end{bmatrix} = 0 \quad (*)$$

For nontrivial solutions: $\det A = 0$

$$\det A = 0 \Rightarrow$$

$$\tan(k_{fx} h) = \frac{k_{fx}(\gamma_s + \gamma_c)}{k_{fx}^2 - \gamma_s \gamma_c} = \frac{\frac{\gamma_s}{k_{fx}} + \frac{\gamma_c}{k_{fx}}}{1 - \frac{\gamma_s}{k_{fx}} \frac{\gamma_c}{k_{fx}}}$$

$$k_{fx} h \pm \pi = \tan^{-1}\left(\frac{\gamma_s}{k_{fx}}\right) + \tan^{-1}\left(\frac{\gamma_c}{k_{fx}}\right)$$

$$\text{But } k_{fx} = (k_0^2 n_f^2 - \beta^2)^{1/2} = k_0 n_f \cos \theta$$

$$\gamma_s = (\beta^2 - k_0^2 n_s^2)^{1/2} = k_0 [n_f^2 \sin^2 \theta - n_s^2]^{1/2}$$

$$\gamma_c = (\beta^2 - k_0^2 n_c^2)^{1/2} = k_0 [n_f^2 \sin^2 \theta - n_c^2]^{1/2}$$

$$\frac{2\pi}{\beta_0} n_f h \cos \theta - \tan^{-1}\left(\frac{(n_f^2 \sin^2 \theta - n_s^2)^{1/2}}{n_f \cos \theta}\right) - \tan^{-1}\left(\frac{(n_f^2 \sin^2 \theta - n_c^2)^{1/2}}{n_f \cos \theta}\right) = \pi$$

Same as before.

Conclusion:

For TE_r modes we find discrete set of acceptable β_r 's which are solutions of the dispersion equation. These β_r 's can specify θ_r , (zig-zag angle), N_r (effective index) and $E_r(x)$ mode profile.

Example:

$$E_r = \begin{cases} E_{c_r} e^{-\gamma_{c_r} x - d} e^{-j\beta_r z} \\ (E_{f_1r} e^{-j k_{fr} x} + E_{f_2r} e^{+j k_{fr} x}) e^{-j\beta_r z} \\ E_{s_r} e^{\gamma_{s_r} x} e^{-j\beta_r z} \end{cases}$$

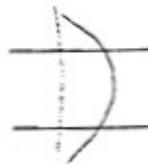
$$\vec{E}_r = E_r(x) \hat{y} e^{-j\beta_r z}$$

$$\beta_r = k_0 N_r = k_0 n_f \sin \theta_r$$

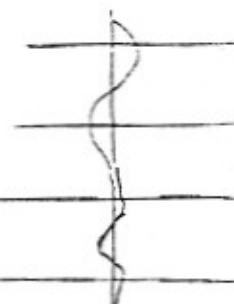
Types of Modes:

TE modes are characterized by v . $TE_v \quad v=0, 1, 2, \dots$

$v=0 \quad TE_0$

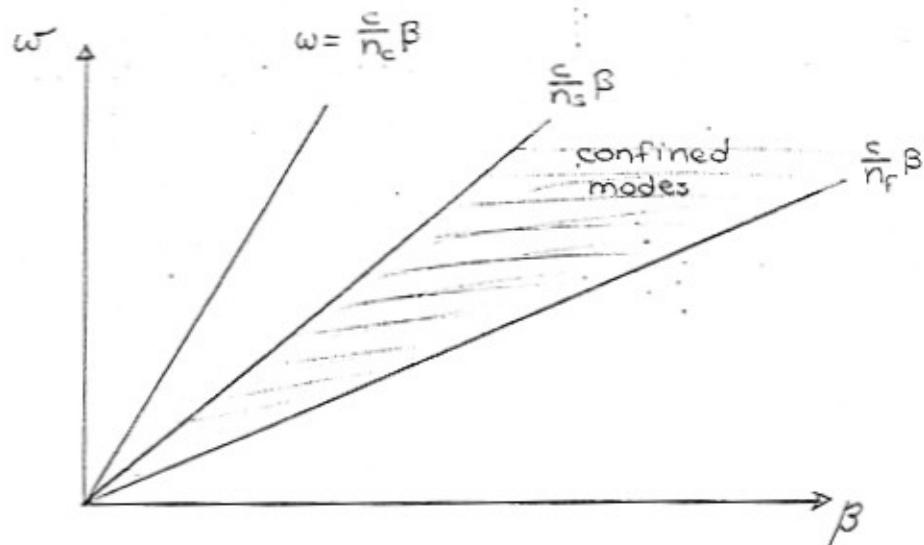


$v=1 \quad TE_1$



$v=2 \quad TE_2$

$$k_{onc} < k_{ons} < \beta < k_{inf} \quad k_o = \frac{\omega}{c}$$

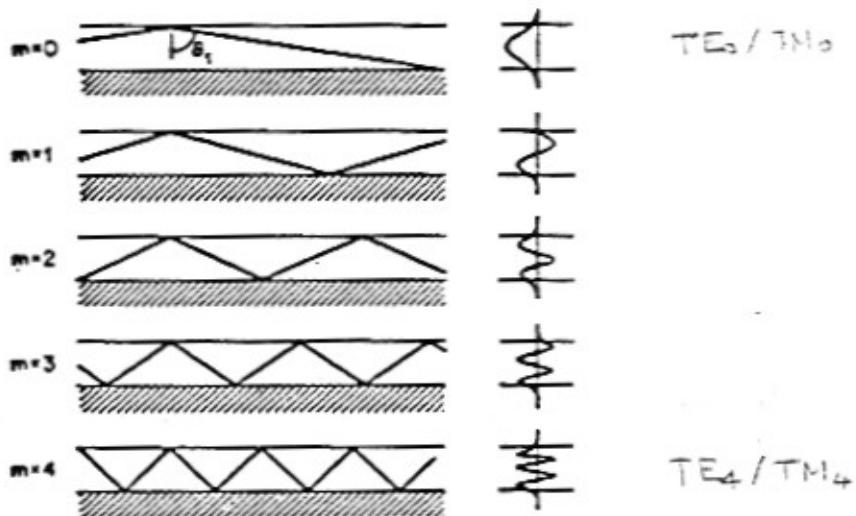


Transparencies.

$$\vec{E}_r = \hat{y} E_{rf} \cos(k_{fxr} x - \phi_{sr}) e^{-j\beta_r z}$$

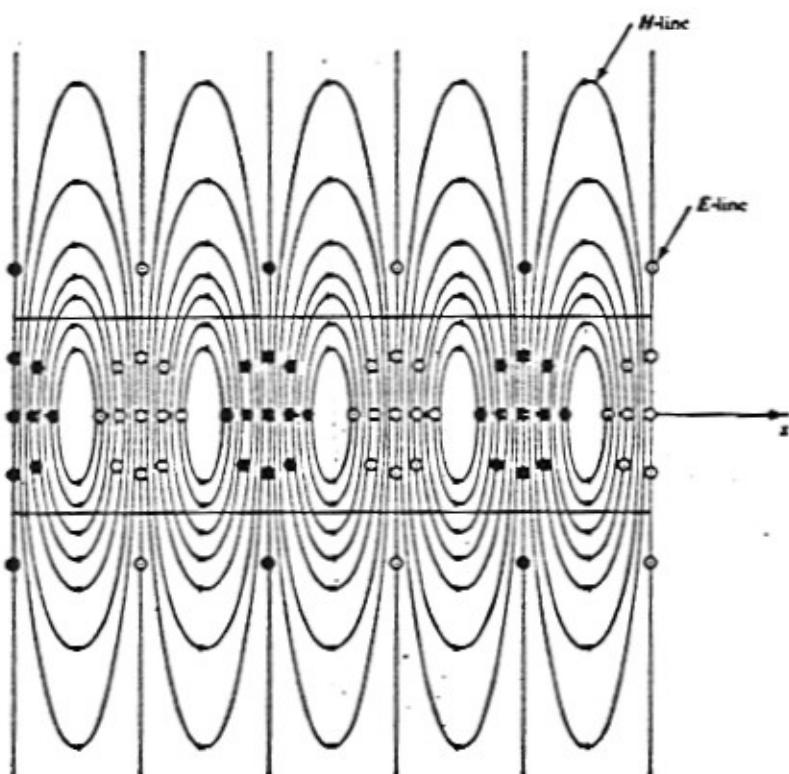
$$\theta_r \rightarrow \delta_r \rightarrow k_{fxr}, \gamma_{cr}, \gamma_{sr}$$

WAVEGUIDE MODES OF A SLAB WAVEGUIDE



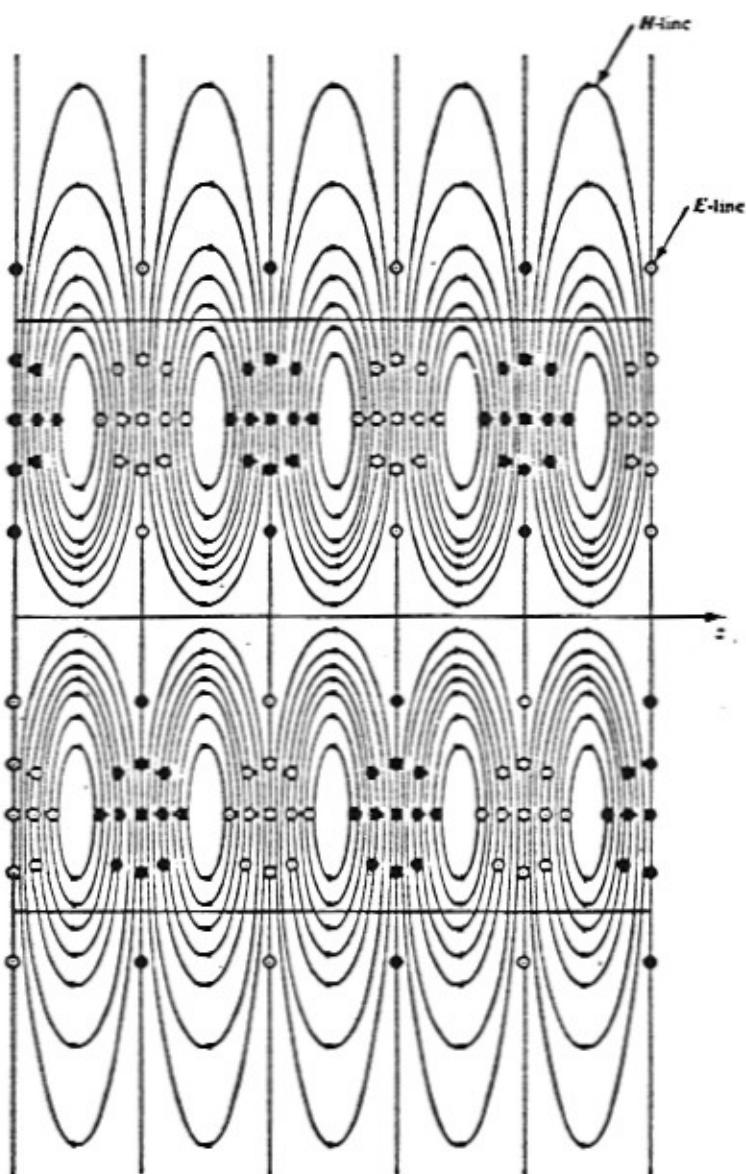
Zigzag ray paths and field distributions for a film waveguide that propagates five waveguide modes. The $m = 4$ mode has a zigzag angle close to the critical angle. P.K. Tien, Rev. Mod. Phys. 49, 361 (1977).

TE_0 MODE IN SYMMETRIC SLAB WAVEGUIDE

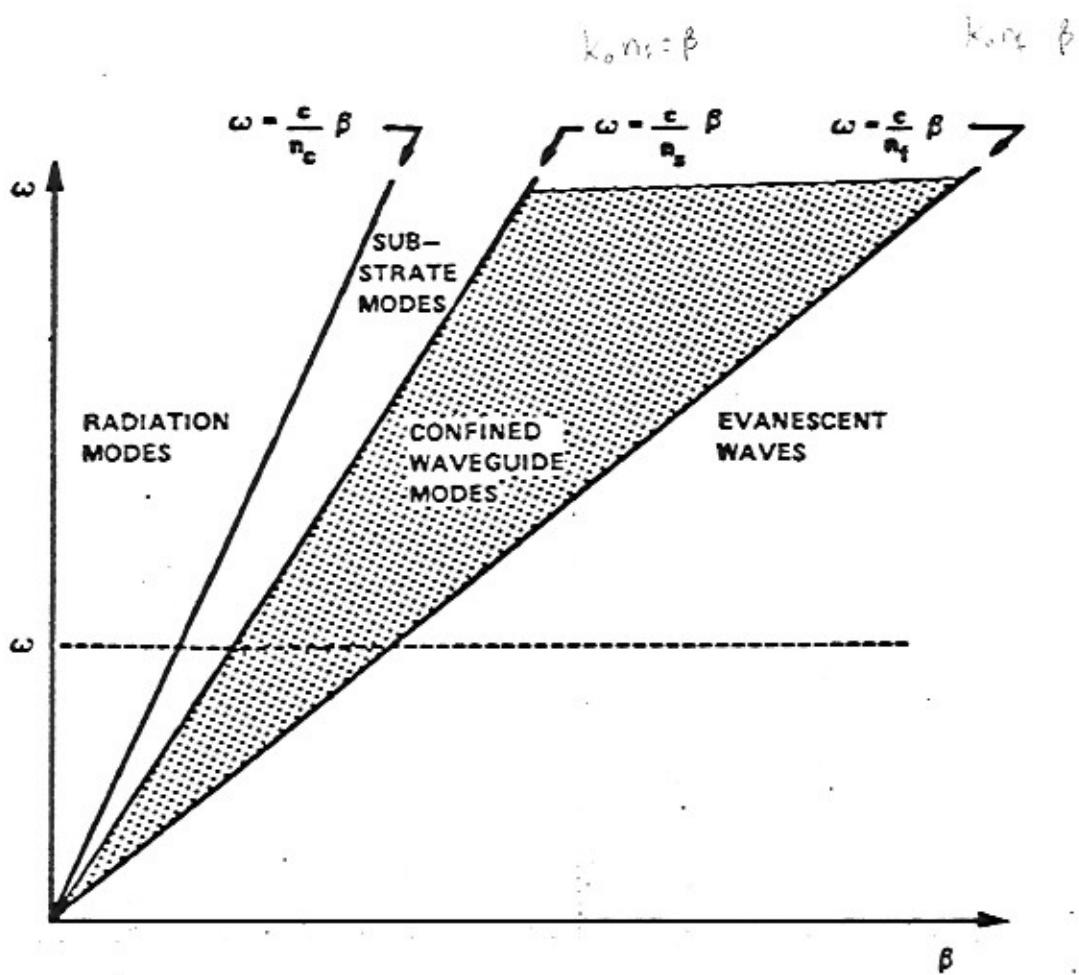


H.A. Haus, Waves and Fields in Optoelectronics.
Englewood Cliffs: Prentice-Hall, 1984.

TE₁ MODE IN SYMMETRIC SLAB WAVEGUIDE

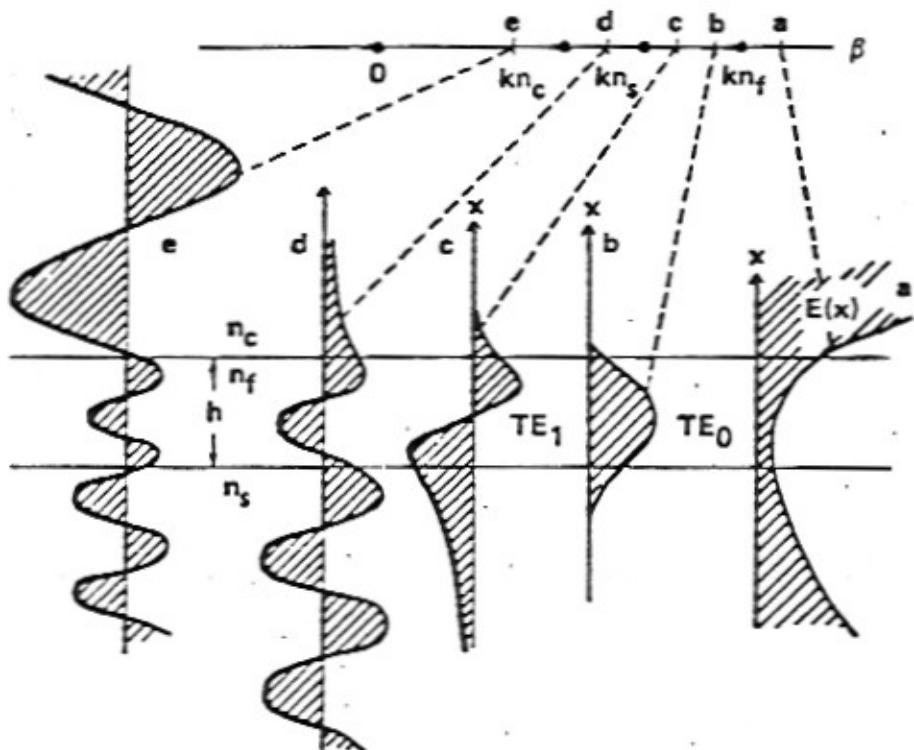


H.A. Haus, Waves and Fields in Optoelectronics,
Englewood Cliffs: Prentice-Hall, 1984



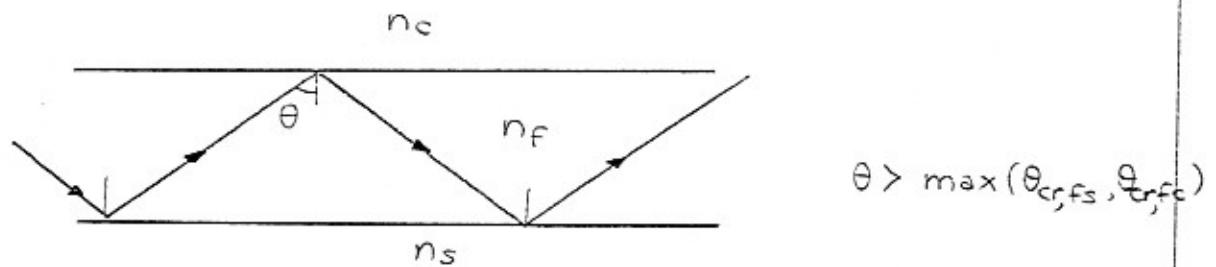
$\omega-\beta$ diagram showing the locations of various modes in a slab waveguide.

<u>β</u>	<u>Type of Mode</u>	<u>χ_c</u>	<u>χ_f</u>	<u>χ_s</u>
$0 \rightarrow k_{n_c}$	radiation	real	real	real
$k_{n_c} \rightarrow k_{n_s}$	substrate	imag.	real	real
$k_{n_s} \rightarrow k_{n_f}$	guided	imag.	real	imag.
$k_{n_f} \rightarrow \infty$	evanescent	imag.	imag.	imag.

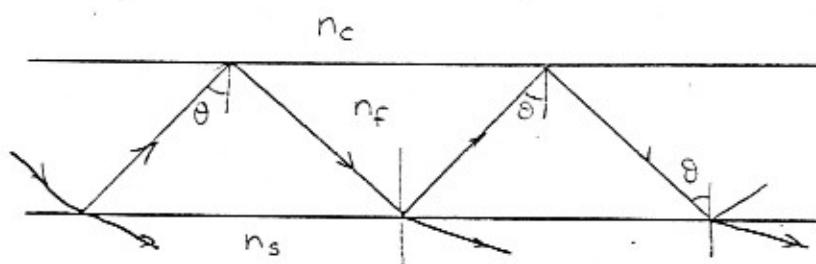


Electric field distributions for various values of the propagation constant (for a given angular frequency). H. F. Taylor and A. Yariv, Proc. IEEE 62, 1044 (1974).

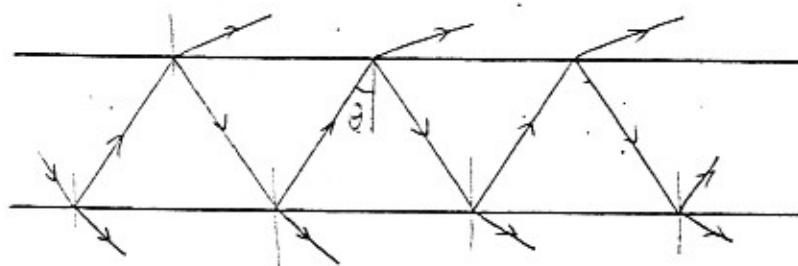
Guided modes:



Substrate modes:



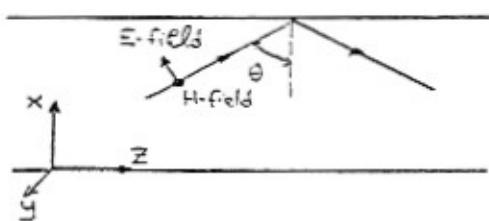
Radiation modes:



$$\theta < \min(\theta_{cr,fs}, \theta_{cr,fc})$$

Can we have cover modes? Yes if $n_f > n_c > n_s$
then light leaks first in the cover region and then in the substrate.

B. TM-polarization (TM modes)



$$\beta = k_0 n_s \sin \theta$$

In the TM case the following equations hold:

Cover: $\vec{H} = \hat{y} H_c e^{-j\gamma_c(x-h)} e^{-j\beta z}$

Film: $\vec{H} = \hat{y} (H_{f_1} e^{jk_{fx}x} + H_{f_2} e^{-jk_{fx}x}) e^{-j\beta z}$

Substrate: $\vec{H} = \hat{y} H_s e^{\delta s x} e^{-j\beta z}$

$$\nabla \times \vec{H} = +j\omega \epsilon_0 n^2 \vec{E} \Rightarrow \vec{E} = -\frac{j}{\omega \epsilon_0 n^2} \left[\hat{x} \left(-\frac{\partial H_y}{\partial z} \right) + \hat{z} \left(\frac{\partial H_y}{\partial x} \right) \right]$$

Thus the tangential \vec{E} components (E_z) are:

Cover: $E_z = \frac{j}{\omega \epsilon_0 n_c^2} H_c \gamma_c e^{-j\gamma_c(x-h)} e^{-j\beta z}$

Film: $E_z = \frac{j}{\omega \epsilon_0 n_f^2} [H_{f_1} k_{fx} e^{jk_{fx}x} - H_{f_2} k_{fx} e^{-jk_{fx}x}] e^{-j\beta z}$

Substrate: $E_z = \frac{-j}{\omega \epsilon_0 n_s^2} H_s \gamma_s e^{\delta s x} e^{-j\beta z}$

Boundary Conditions:

$$x=0 \quad \frac{-j}{\omega \epsilon_0 n_s^2} \cdot H_s \gamma_s = \frac{j}{\omega \epsilon_0 n_f^2} [H_{f_1} k_{fx} - H_{f_2} k_{fx}] \quad (bc1)$$

$$H_s = H_{f_1} + H_{f_2} \quad (bc2)$$

$$x=h \quad \frac{j}{\omega \epsilon_0 n_c^2} H_c \gamma_c = \frac{j}{\omega \epsilon_0 n_f^2} [H_{f_1} k_{fx} e^{jk_{fx}h} - H_{f_2} k_{fx} e^{-jk_{fx}h}] \quad (bc3)$$

$$H_c = H_{f_1} e^{jk_{fx}h} + H_{f_2} e^{-jk_{fx}h} \quad (bc4)$$

In a matrix form:

$$\begin{bmatrix} +j\gamma_s/n_s^2 & k_{fx}/n_f^2 - k_{fv}/n_s^2 & \emptyset \\ -1 & 1 & 1 & \emptyset \\ -\emptyset & k_{fx}e^{jk_f h}/n_f^2 - k_{fv}e^{-jk_f h}/n_s^2 - j\gamma_c/n_c^2 \\ \emptyset & e^{jk_f h} & e^{-jk_f h} & -1 \end{bmatrix} \begin{bmatrix} H_s \\ H_{f_1} \\ H_{f_2} \\ H_c \end{bmatrix} = \emptyset$$

For nontrivial solution $\det A = 0$

$$\det A = e^{jk_f h} \left[-\left(\frac{k_f^2}{n_f^2} - \frac{\gamma_s \gamma_c}{n_s^2 n_c^2} \right) + j \frac{\gamma_s}{n_s^2} \frac{k_{fx}}{n_f^2} + j \frac{\gamma_c}{n_c^2} \frac{k_{fx}}{n_f^2} \right] + e^{-jk_f h} \left[\left(\frac{k_f^2}{n_f^2} - \frac{\gamma_s \gamma_c}{n_s^2 n_c^2} \right) + j \frac{\gamma_s}{n_s^2} \frac{k_{fx}}{n_f^2} + j \frac{\gamma_c}{n_c^2} \frac{k_{fx}}{n_f^2} \right] = 0 \Rightarrow$$

$$= \tan(k_{fx} h) = \frac{\frac{k_{fx}}{n_f^2} \left(\frac{\gamma_s}{n_s^2} + \frac{\gamma_c}{n_c^2} \right)}{\frac{k_{fx}^2}{n_f^4} - \frac{\gamma_s^2}{n_s^2} \frac{\gamma_c^2}{n_c^2}} = \frac{\frac{\gamma_s/n_s^2}{k_{fx}/n_f^2} + \frac{\gamma_c/n_c^2}{k_{fx}/n_f^2}}{1 - \frac{\gamma_s/n_s^2}{k_{fx}/n_f^2} \frac{\gamma_c/n_c^2}{k_{fx}/n_f^2}}$$

$$k_{fx} h \pm v\pi = \tan^{-1}\left(\frac{\gamma_s/n_s^2}{k_{fx}/n_f^2}\right) \pm \tan^{-1}\left(\frac{\gamma_c/n_c^2}{k_{fx}/n_f^2}\right)$$

(*)

But

$$k_{fx} = (k_o^2 n_f^2 - \beta^2)^{1/2} = k_o n_f \cos \theta$$

$$\gamma_c = (\beta^2 - k_o^2 n_c^2)^{1/2} = k_o [n_f^2 \sin^2 \theta - n_c^2]^{1/2}$$

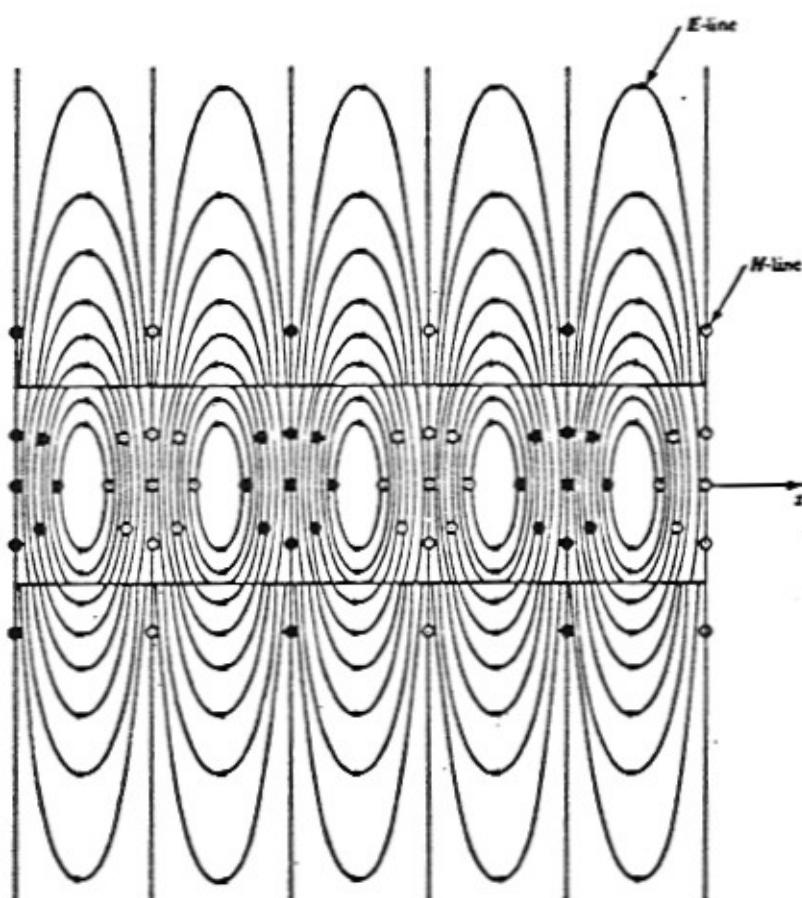
$$\gamma_s = (\beta^2 - k_o^2 n_s^2)^{1/2} = k_o [n_f^2 \sin^2 \theta - n_s^2]^{1/2}$$

and Eq. (*) becomes :

$$\frac{2\pi}{\lambda_o} n_f h \cos \theta - \tan^{-1}\left(\frac{n_f^2}{n_s^2} \frac{(n_f^2 \sin^2 \theta - n_s^2)^{1/2}}{n_f \cos \theta}\right) - \tan^{-1}\left(\frac{n_f^2}{n_c^2} \frac{(n_f^2 \sin^2 \theta - n_c^2)^{1/2}}{n_f \cos \theta}\right) = v\pi$$

$$\frac{2\pi}{\lambda_o} n_f h \cos \theta - \tan^{-1}(\beta_{s,TM}) - \tan^{-1}(\beta_{c,TM}) = v\pi$$

TM₀ MODE IN SYMMETRIC SLAB WAVEGUIDE



H.A. Haus, Waves and Fields in Optoelectronics.
Englewood Cliffs: Prentice-Hall, 1984.

Cutoff conditions:

$$\frac{2\pi}{\lambda_0} n_f h \cos \theta - \tan^{-1} \left[\frac{(n_f^2 \sin^2 \theta - n_s^2)^{1/2}}{n_f \cos \theta} \right] - \tan^{-1} \left[\frac{(n_f^2 \sin^2 \theta - n_s^2)^{1/2}}{n_f \cos \theta} \right] = v\pi \quad v = 0, 1, 2, \dots$$

Cutoff when $\theta = \max(\theta_{cr,fs}, \theta_{cr,fc}) = \theta_{cr,fs}$

$$\theta_{cr,fs} = \sin^{-1} \left(\frac{n_s}{n_f} \right) \approx n_f \sin \theta = n_s$$

Then the above condition becomes

$$\frac{2\pi}{\lambda_0} n_f h \cos \theta_{cr,fs} - \tan \left[\frac{(n_s^2 - n_c^2)^{1/2}}{(n_f^2 - n_s^2)^{1/2}} \right] = v\pi$$

$$n_f \cos \theta_{cr,fs} = n_f \sqrt{1 - \sin^2 \theta_{cr,fs}} = n_f \sqrt{1 - \frac{n_s^2}{n_f^2}} = \sqrt{n_f^2 - n_s^2}$$

$$\text{If } \lambda_0 \text{ constant: } h_{cut,v} = \frac{v\pi + \tan^{-1} \left[\frac{(n_s^2 - n_c^2)^{1/2}}{(n_f^2 - n_s^2)^{1/2}} \right]}{\frac{2\pi}{\lambda_0} \sqrt{n_f^2 - n_s^2}}$$

Similarly if h is constant and λ_0 varies

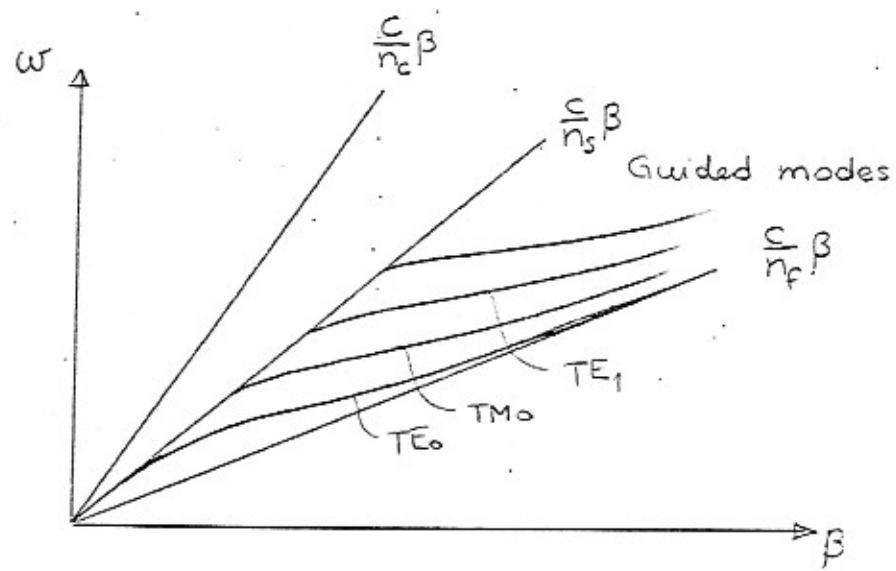
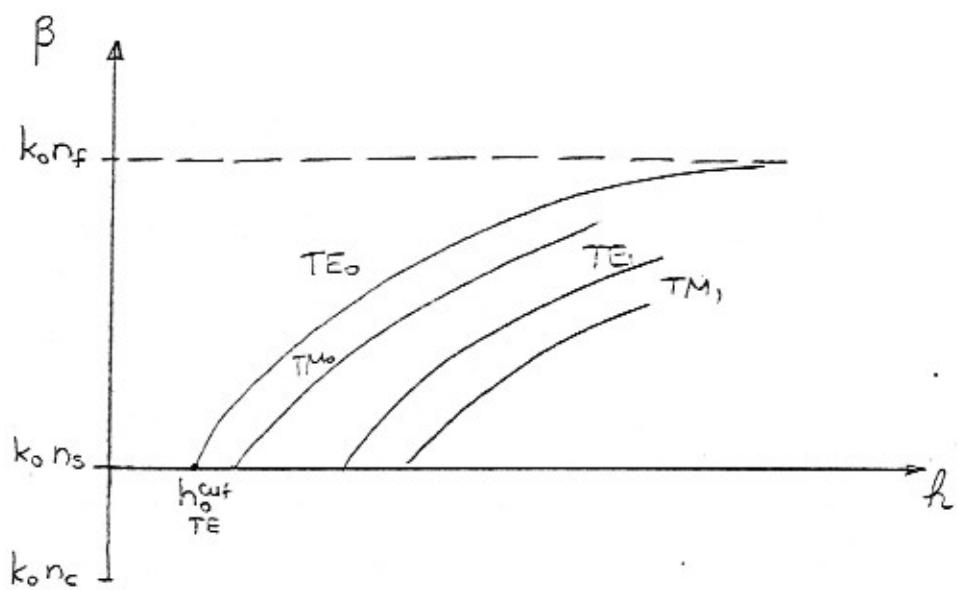
$$\text{then: } \lambda_{0,cut,v} = \frac{2\pi h \sqrt{n_f^2 - n_s^2}}{v\pi + \tan^{-1} \left[\frac{(n_s^2 - n_c^2)^{1/2}}{(n_f^2 - n_s^2)^{1/2}} \right]}$$

What happens in the case that $n_s = n_c$ (symmetric waveguide)?

$$h_{out,v} = \frac{v\pi}{\frac{2\pi}{\lambda_0} \sqrt{n_f^2 - n_s^2}} \quad \begin{array}{l} \text{zero cutoff thickness} \\ \text{for } v=0 \end{array}$$

$$\lambda_{0,out,v} = \frac{2\pi h \sqrt{n_f^2 - n_s^2}}{v\pi} \quad \lambda_{0,out,0} = \infty$$

No cutoff thickness or wavelength for the symmetric waveguide for the TE₀ mode.



Normalized Frequency:

$$V = \frac{2\pi}{\lambda_0} h \sqrt{n_f^2 - n_s^2} \quad (\text{dimensionless}) = k_0 h \sqrt{n_f^2 - n_s^2}$$

$$\alpha_{TE} = \frac{n_s^2 - n_e^2}{n_f^2 - n_s^2} \quad \text{asymmetry parameter for TE-modes}$$

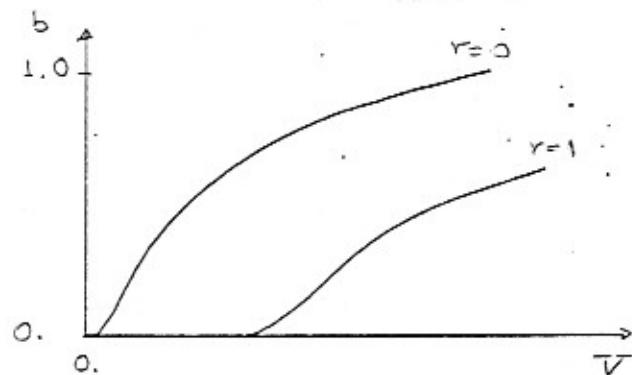
$$\alpha_{TM} = \frac{n_e^4}{n_e^2} \frac{n_s^2 - n_e^2}{n_f^2 - n_s^2} \quad \text{asymmetry parameter for TM-modes}$$

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2} \quad \text{normalized guided index}$$

$$N = n_s \sin\theta \quad n_s < N < n_f$$

Normalized dispersion relation:

$$\text{for TE} \quad V \sqrt{1-b} - \tan^{-1}\left(\sqrt{\frac{b}{1-b}}\right) - \tan^{-1}\left(\sqrt{\frac{b+r}{1-b}}\right) = r\pi$$



$$\text{Cutoff frequency } V_0 = \tan^{-1} \sqrt{a}$$

$$V_r = \tan^{-1} \sqrt{a} + r\pi$$

Cutoff thickness, wavelength etc can be found.

$$\text{Total number of modes} \approx 1 + \frac{1}{\pi} [V - \tan^{-1} \sqrt{a}]$$

Relations between $E_c, E_{f_1}, E_{f_2}, E_s$ coefficients: (TE-modes)

Boundary conditions:

$$E_c = E_{f_1} e^{jk_{fx}h} + E_{f_2} e^{-jk_{fx}h} \quad (1)$$

$$-j\gamma_s E_c = -k_{fx} E_{f_1} e^{jk_{fx}h} + k_{fx} E_{f_2} e^{-jk_{fx}h} \quad (2)$$

$$E_s = E_{f_1} + E_{f_2} \quad (3)$$

$$j\gamma_s E_s = -k_{fx} E_{f_1} + k_{fx} E_{f_2} \quad (4)$$

For β 's that represent guided modes: $\det(\tilde{A}(\beta)) = 0$ and equations

(1) - (4) become dependent.

From Eqs. (3) & (4):

$$\begin{bmatrix} 1 & 1 \\ -k_{fx} & k_{fx} \end{bmatrix} \begin{bmatrix} E_{f_1} \\ E_{f_2} \end{bmatrix} = \begin{bmatrix} 1 \\ j\gamma_s \end{bmatrix} E_s \Rightarrow \begin{bmatrix} E_{f_1} \\ E_{f_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -k_{fx} & k_{fx} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ j\gamma_s \end{bmatrix} E_s$$

$$\Rightarrow \begin{bmatrix} E_{f_1} \\ E_{f_2} \end{bmatrix} = \frac{1}{2k_{fx}} \begin{bmatrix} k_{fx} & -1 \\ k_{fx} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ j\gamma_s \end{bmatrix} E_s = \begin{bmatrix} \frac{1}{2}(1-j\frac{\gamma_s}{k_{fx}}) E_s \\ \frac{1}{2}(1+j\frac{\gamma_s}{k_{fx}}) E_s \end{bmatrix} \Rightarrow$$

$$E_{f_1} = \frac{1}{2} \sqrt{1 + (\frac{\gamma_s}{k_{fx}})^2} E_s e^{-j\tan^{-1}(\gamma_s/k_{fx})} \quad \phi_{f_1} = \tan^{-1}(\frac{\gamma_s}{k_{fx}})$$

$$E_{f_2} = \frac{1}{2} \sqrt{1 + (\frac{\gamma_s}{k_{fx}})^2} E_s e^{+j\tan^{-1}(\gamma_s/k_{fx})}$$

Let's define the free parameter as $E_f = \frac{1}{2} \sqrt{1 + (\frac{\gamma_s}{k_{fx}})^2} E_s$. Then,

$$E_{f_1} = E_f e^{-j\phi_f} \quad \text{and} \quad E_{f_2} = E_f e^{+j\phi_f}$$

Then from Eq. (2) ~ $E_s = E_{f_1} + E_{f_2} = 2E_f \cos(\phi_f)$ and

$$\begin{aligned} \text{from Eq. (1): } E_c &= E_{f_1} e^{jk_{fx}h} + E_{f_2} e^{-jk_{fx}h} = \\ &= E_f e^{j(k_{fx}h - \phi_f)} + E_f e^{-j(k_{fx}h - \phi_f)} \Rightarrow \\ \Rightarrow E_c &= 2E_f \cos(k_{fx}h - \phi_f) \end{aligned}$$

Therefore all coefficients can be expressed as functions of the free parameter E_f and constants that can depend on β .

Power Considerations:

TE-modes:

$$P = \int_{-\infty}^{+\infty} S_z dx$$

$$\vec{S} = \frac{1}{2} \operatorname{Re} [\vec{E} \times \vec{H}^*]$$

$$\vec{S} = \frac{1}{2} \operatorname{Re} \left[\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \hat{E}_x & \hat{E}_y & \hat{E}_z \\ H_x^* & 0 & H_z^* \end{vmatrix} \right] \sim S_z = \frac{1}{2} \operatorname{Re} [-E_y H_x^*]$$

$$\text{If } E_{f_1} = |E_f| e^{-j\phi_f} \quad \phi_s = \tan^{-1}(\gamma_s/k_{fx})$$

$$E_{f_2} = |E_f| e^{j\phi_s} \quad \text{where } |E_f| \text{ is the free-parameter.}$$

After a fair amount of algebraic manipulations

$$P = \frac{N}{4} \sqrt{\frac{\epsilon_0}{\mu_0}} |E_f|^2 h_{\text{eff}}$$

$$h_{\text{eff}} = \frac{1}{\gamma_c} + \frac{1}{\gamma_s} + \frac{1}{\gamma_i}$$

per unit length along
the y direction

Proof:

Let's compute the $-E_y H_x^*$ product in the cover, film and substrate.

Cover: $E_y = E_c e^{-\gamma_c(x-d)} e^{-j\beta z}$
 $H_x = -\frac{\beta}{\omega \mu_0} E_c e^{-\gamma_c(x-d)} e^{-j\beta z}$

$$-E_y H_x^* = \frac{\beta}{\omega \mu_0} |E_c|^2 e^{-2\gamma_c(x-d)}$$

$$\int_{-\infty}^{\infty} (-E_y H_x^*) dx = \frac{\beta}{\omega \mu_0} |E_c|^2 \int_{-\infty}^{\infty} e^{-2\gamma_c(x-d)} dx = \frac{\beta}{\omega \mu_0} |E_c|^2 \frac{1}{2\gamma_c}$$

$$= N \sqrt{\frac{\epsilon_0}{\mu_0}} |E_c|^2 \frac{1}{2\gamma_c}$$

But from the BC we have

$$E_{f_1} = E_f e^{-j\phi_s}$$

$$E_s = 2E_f \cos \phi_s$$

$$E_{f_2} = E_f e^{+j\phi_s}$$

$$E_c = 2E_f \cos(k_{fx}h - \phi_s)$$

$$\phi_s = \tan^{-1}(\beta_s/k_{fx})$$

Thus,

$$\frac{1}{2} \int_{-\infty}^{\infty} (-E_y H_x^*) dx = \frac{1}{2} N \sqrt{\frac{\epsilon_0}{\mu_0}} 4 |E_f|^2 \cos^2(k_{fx}h - \phi_s) \frac{1}{2\gamma_c} \quad (P1)$$

Film: $E_y = 2E_f \cos(k_{fx}x - \phi_s) e^{-j\beta z}$

$$H_x = -\frac{\beta}{\omega \mu_0} 2E_f \cos(k_{fx}x - \phi_s) e^{-j\beta z}$$

$$\begin{aligned} \frac{1}{2} \int_0^h (-E_y H_x^*) dx &= \frac{1}{2} \frac{\beta}{\omega \mu_0} 4 E_f^2 \int_0^h \cos^2(k_{fx}x - \phi_s) dx = \\ &= \frac{1}{2} N \sqrt{\frac{\epsilon_0}{\mu_0}} 4 E_f^2 \frac{1}{2} \left[h + \frac{\sin(2k_{fx}h - 2\phi_s) + \sin 2\phi_s}{2k_{fx}} \right] \quad (P2) \end{aligned}$$

Substrate:

$$E_y = 2E_f \cos \phi_s e^{\beta_s x} e^{-j\beta z}$$

$$H_x = -\frac{\beta}{\omega \mu_0} 2E_f \cos \phi_s e^{\beta_s x} e^{-j\beta z}$$

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^0 (-E_y H_x^*) dx &= \frac{1}{2} \frac{\beta}{\omega \mu_0} 4 E_f^2 \cos^2 \phi_s \int_{-\infty}^0 e^{2\beta_s x} dx = \\ &= \frac{1}{2} N \sqrt{\frac{\epsilon_0}{\mu_0}} 4 E_f^2 \cos^2 \phi_s \frac{1}{2\gamma_s} \quad (P3) \end{aligned}$$

Total Power:

$$P = \frac{N}{4} \sqrt{\frac{\epsilon_0}{\mu_0}} (4 E_f^2) \left[\frac{\cos^2(k_{fx}h - \phi_s)}{\gamma_c} + h + \frac{\sin(2k_{fx}h - 2\phi_s) + \sin 2\phi_s}{2k_{fx}} + \frac{\cos^2 \phi_s}{\gamma_s} \right]$$

$$\begin{aligned}
 P &= \frac{N}{4} \sqrt{\frac{\epsilon_0}{\mu_0}} (4 \epsilon_f^2) \left[\frac{1}{\delta_c} + \frac{1}{\delta_s} + h - \frac{\sin^2(k_{fx} h - \phi_s)}{\delta_c} - \frac{\sin^2 \phi_s}{\delta_s} + \right. \\
 &\quad \left. + \frac{\cos(k_{fx} h - \phi_s) \sin(k_{fx} h - \phi_s) + \cos \phi_s \sin \phi_s}{k_{fx}} \right] = \\
 &= \frac{N}{4} \sqrt{\frac{\epsilon_0}{\mu_0}} (4 \epsilon_f^2) \left[h + \frac{1}{\delta_c} + \frac{1}{\delta_s} + \frac{\cos(k_{fx} h - \phi_s) \sin(k_{fx} h - \phi_s)}{\delta_c} \left[-\tan(k_{fx} h - \phi_s) + \frac{\delta_c}{k_{fx}} \right] + \right. \\
 &\quad \left. + \frac{\cos \phi_s \sin \phi_s}{\delta_s} \left[-\tan \phi_s + \frac{\delta_s}{k_{fx}} \right] \right] \\
 &= \frac{N}{4} \sqrt{\frac{\epsilon_0}{\mu_0}} (4 \epsilon_f^2) \underbrace{\left[h + \frac{1}{\delta_c} + \frac{1}{\delta_s} \right]}_{h_{eff}}
 \end{aligned}$$

For TM modes:

$$\begin{aligned}
 P &= \int_{-\infty}^{+\infty} S_z(x) dx \\
 S_z &= \frac{1}{2} \operatorname{Re} \begin{bmatrix} \hat{E}_x & \hat{H}_y & \hat{H}_z \\ 0 & 0 & 0 \\ 0 & H_y & 0 \end{bmatrix} \Rightarrow S_z = \frac{1}{2} \operatorname{Re} [E_x H_y^*]
 \end{aligned}$$

$$P = \frac{N}{4} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{n_f^2} (4 H_f^2) h_{eff}$$

$$h_{eff} = h + \frac{1}{\delta_c q_c} + \frac{1}{\delta_s q_s}$$

$$q_c = \frac{N^2}{n_c^2} + \frac{N^2}{n_f^2} - 1 \quad q_s = \frac{N^2}{n_s^2} + \frac{N^2}{n_f^2} - 1$$

where $H_{f1} = H_f e^{-j\phi_s}$

$H_{f2} = H_f e^{+j\phi_s}$

$$E_x = - \frac{j}{\omega \epsilon_0 n_c^2} \left(- \frac{\partial H_y}{\partial z} \right) = + \frac{j}{\omega \epsilon_0 n_c^2} \frac{\partial H_y}{\partial z}$$

$$E_x = \begin{cases} \frac{j}{\omega \epsilon_0 n_c^2} H_c e^{-\gamma_c(x-h)} (-j\beta) e^{-j\beta z} = \frac{\beta}{\omega \epsilon_0 n_c^2} H_c e^{-\gamma_c(x-h)} e^{-j\beta z} \\ \frac{\beta}{\omega \epsilon_0 n_f^2} [H_{f1} e^{jk_f x} + H_{f2} e^{-jk_f x}] e^{-j\beta z} \\ \frac{\beta}{\omega \epsilon_0 n_s^2} H_s e^{\gamma_s x} e^{-j\beta z} \end{cases}$$

$$\frac{1}{2} \int_h^\infty E_x H_y^* dx = \frac{1}{2} \int_h^\infty \frac{\beta}{\omega \epsilon_0 n_c^2} H_c e^{-\gamma_c(x-h)} H_c^* e^{-\gamma_c(x-h)} dx$$

From the boundary conditions we have:

$$\begin{aligned} j \frac{\gamma_s}{n_s^2} H_s &= - \frac{k_f x}{n_f^2} H_{f1} + \frac{k_f x}{n_f^2} H_{f2} \\ H_s &= H_{f1} + H_{f2} \end{aligned} \quad \Rightarrow$$

$$\begin{bmatrix} -\frac{k_f x}{n_f^2} & \frac{k_f x}{n_f^2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} H_{f1} \\ H_{f2} \end{bmatrix} = \begin{bmatrix} j \frac{\gamma_s}{n_s^2} \\ 1 \end{bmatrix} H_s$$

$$\begin{bmatrix} H_{f1} \\ H_{f2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{k_f x}{n_f^2} \\ -1 & -\frac{k_f x}{n_f^2} \end{bmatrix} \begin{bmatrix} j \frac{\gamma_s}{n_s^2} \\ 1 \end{bmatrix} \frac{H_s}{-\left(\frac{k_f x}{n_f^2} 2\right)} =$$

$$= \begin{bmatrix} -\frac{k_f x}{n_f^2} + j \frac{\gamma_s}{n_s^2} \\ -\frac{k_f x}{n_f^2} - j \frac{\gamma_s}{n_s^2} \end{bmatrix} \frac{H_s}{\frac{k_f x}{n_f^2} (-2)}$$

$$= \begin{bmatrix} \frac{1}{2} \left(1 - j \frac{\gamma_s / n_s^2}{k_f x / n_f^2} \right) \\ \frac{1}{2} \left(1 + j \frac{\gamma_s / n_s^2}{k_f x / n_f^2} \right) \end{bmatrix} H_s$$

Thus $H_{f1} = H_f e^{-j\phi_s}$

$$H_{f2} = H_f e^{+j\phi_s} \quad \phi_s = \tan^{-1} \left(\frac{\gamma_s / n_s^2}{k_f x / n_f^2} \right)$$

$$H_s = 2 H_f \cos \phi_s$$

$$H_c = 2 H_f \cos(k_{fx} h - \phi_s)$$

Cover:

$$\begin{aligned} & \frac{1}{2} \int_h^\infty \frac{\beta}{\omega \epsilon_0 n_c^2} |H_c|^2 e^{-2\gamma_c(x-h)} dx = \\ &= \frac{1}{2} \frac{\beta |H_c|^2}{\omega \epsilon_0 n_c^2} \int_h^\infty e^{-2\gamma_c(x-h)} dx = \\ &= \frac{1}{2} \frac{\beta 4 H_f^2 \cos^2(k_{fx} h - \phi_s)}{\omega \epsilon_0 n_c^2} \frac{1}{2\gamma_c} = \\ &= \frac{1}{2} 4 H_f^2 (k_0 N) \frac{1}{\omega \epsilon_0 n_c^2} \frac{1}{2\gamma_c} \cos^2(k_{fx} h - \phi_s) \\ &= \frac{1}{2} 4 H_f^2 (k_0 N) \frac{1}{\frac{\omega \epsilon_0 \mu_0 \sqrt{\epsilon_0}}{n_c^2}} \frac{1}{n_c^2} \frac{1}{2\gamma_c} \cos^2(k_{fx} h - \phi_s) \\ &= \frac{1}{2} 4 H_f^2 N \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{n_c^2} \frac{1}{2\gamma_c} \cos^2(k_{fx} h - \phi_s) = \\ &= \frac{N}{4} \sqrt{\frac{\mu_0}{\epsilon_0}} (4 H_f^2) \left[\frac{1}{n_c^2 \gamma_c} \cos^2(k_{fx} h - \phi_s) \right] \quad (a1) \end{aligned}$$

Film:

$$E_x = \frac{2\beta}{\omega \epsilon_0 n_f^2} H_f \cos(k_{fx} x - \phi_s) e^{-j\beta z}$$

$$H_y = 2 H_f \cos(k_{fx} x - \phi_s) e^{-j\beta z}$$

$$\begin{aligned} \frac{1}{2} \int_0^h E_x H_y^* dx &= \frac{1}{2} \frac{\beta}{\omega \epsilon_0 n_f^2} 4 H_f^2 \int_0^h \cos^2(k_{fx} x - \phi_s) dx = \\ &= \frac{1}{2} N \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{n_f^2} 4 H_f^2 \frac{1}{2} \left[h + \frac{\sin(2k_{fx} h - 2\phi_s) + \sin 2\phi_s}{2k_{fx}} \right] \\ &= \frac{N}{4} \sqrt{\frac{\mu_0}{\epsilon_0}} 4 H_f^2 \left[\frac{h}{n_f^2} + \frac{\sin(2k_{fx} h - 2\phi_s) + \sin 2\phi_s}{2k_{fx} n_f^2} \right] \quad (a2) \end{aligned}$$

Substrate:

$$E_x = \frac{\beta}{\omega \epsilon_0 n_s^2} 2 H_f \cos \phi_s e^{\gamma_s x} e^{-j\beta z}$$

$$H_y = 2 H_f \cos \phi_s e^{\gamma_s x} e^{-j\beta z}$$

$$\frac{1}{2} \int_{-\infty}^{\infty} (E_x H_y) dx = \frac{N}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} 4 H_f^2 \left(\frac{\cos^2 \phi_s}{n_s^2 \gamma_s} \right) \quad (13)$$

Adding (1) + (2) + (3) we get:

$$\begin{aligned} P &= \frac{N}{4} \sqrt{\frac{\mu_0}{\epsilon_0}} 4 H_f^2 \left[\underbrace{\frac{\cos^2(k_{fx} h - \phi_s)}{n_c^2 \gamma_c} + \frac{h}{n_f^2} + \frac{\sin(2k_{fx} h - 2\phi_s) + \sin 2\phi_s}{2k_{fx} n_f^2}}_{A} \right. \\ &\quad \left. + \frac{\cos^2 \phi_s}{n_s^2 \gamma_s} \right] = \\ &= \frac{N}{4} \sqrt{\frac{\mu_0}{\epsilon_0}} 4 H_f^2 \frac{1}{n_f^2} \left[h + \frac{n_f^2}{n_c^2 \gamma_c} \cdot \frac{1}{1 + \tan^2(k_{fx} h - \phi_s)} + \right. \\ &\quad \left. + \frac{\tan(k_{fx} h - \phi_s)}{1 + \tan^2(k_{fx} h - \phi_s)} \frac{1}{k_{fx}} + \frac{\tan \phi_s}{1 + \tan^2 \phi_s} \frac{1}{k_{fx}} + \frac{1}{1 + \tan^2 \phi_s} \frac{n_f^2}{n_s^2 \gamma_s} \right] \end{aligned}$$

Let's call the inside the parenthesis part h_{eff} .

$$\tan(k_{fx} h - \phi_s) = \tan \phi_e = \frac{n_f^2}{n_c^2} \frac{\gamma_c}{k_{fx}}$$

$$\tan \phi_s = \frac{n_f^2}{n_s^2} \frac{\gamma_s}{k_{fx}}$$

$$h_{eff} = h + \underbrace{\frac{1}{1 + \tan^2(k_{fx} h - \phi_s)}}_A \left[\frac{n_f^2}{n_c^2 \gamma_c} + \frac{\tan(k_{fx} h - \phi_s)}{k_{fx}} \right] + \underbrace{\frac{1}{1 + \tan^2 \phi_s}}_B \left[\frac{n_f^2}{n_s^2 \gamma_s} + \frac{\tan \phi_s}{k_{fx}} \right]$$

$$A = \frac{1}{1 + \frac{n_f^4}{n_c^4} \frac{\gamma_c^2}{k_{fx}^2}} \left[\frac{n_f^2}{n_c^2 \gamma_c} + \frac{n_f^2}{n_c^2} \frac{\gamma_c}{k_{fx}^2} \right] =$$

$$= \frac{1}{1 + \frac{n_f^4}{n_c^4} \frac{N^2 - n_c^2}{n_f^2 - N^2}} \left[\frac{n_f^2}{n_c^2 \gamma_c} + \frac{1}{\gamma_c} \frac{n_f^2}{n_c^2} \cdot \frac{N^2 - n_c^2}{n_f^2 - N^2} \right]$$

$$\begin{aligned}
 &= \frac{1}{\gamma_c} \frac{n_c^4 (n_f^2 - N^2)}{n_c^4 (n_f^2 - N^2) + n_f^4 (N^2 - n_c^2)} \cdot \frac{n_f^2 n_c^2 (n_f^2 - N^2) + n_f^2 n_c^2 (N^2 - n_c^2)}{\cancel{n_c^4 (n_f^2 - N^2)}} = \\
 &= \frac{1}{\gamma_c} \frac{1}{-n_c^2 n_f^2 (n_f^2 - n_c^2) + N^2 (n_f^2 - n_c^2) (n_f^2 + n_c^2)} (n_f^4 n_c^2 - \cancel{n_f^2 n_c^2 N^2} + \cancel{n_f^2 n_c^2 N^2} - n_f^2 n_c^4) \\
 &= \frac{1}{\gamma_c} \cdot \frac{1}{N^2 (n_f^2 - n_c^2) (n_f^2 + n_c^2) - n_c^2 n_f^2 (n_f^2 - n_c^2)} = \\
 &= \frac{1}{\gamma_c} \cdot \frac{1}{\frac{N^2}{n_c^2} + \frac{N^2}{n_f^2} - 1} = \frac{1}{\gamma_c q_c}.
 \end{aligned}$$

Similarly $B = \frac{1}{\gamma_s q_s}$ $\therefore q_s = \frac{N^2}{n_s^2} + \frac{N^2}{n_f^2} - 1$

So $h_{\text{eff}} = h + \frac{1}{\gamma_s q_s} + \frac{1}{\gamma_c q_c}$
