

ΟΛΟΚΛΗΡΩΜΕΝΗ ΟΠΤΙΚΗ
(INTEGRATED OPTICS)

ΚΥΛΙΝΔΡΙΚΟΙ ΚΥΜΑΤΟΔΗΓΟΙ

(Θεωρία Οπτικών Ινών)

(Cylindrical Dielectric Waveguides

Step-Index Optical Fibers)

Σημειώσεις

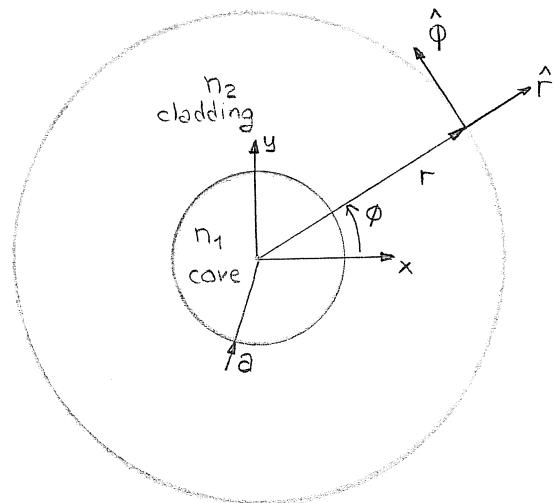
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Cylindrical Dielectric Waveguides:

Cylindrical dielectric waveguides are extensively used in optical communications, especially for long-distance links. In these notes we are going to present the theory of guiding into cylindrical dielectric waveguides (optical fibers). The cross-sectional geometry of the simplest optical fiber (of step-index profile) is shown in the figure.

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It is assumed that the cladding region is infinite and $n_2 < n_1$, where n_2, n_1 are the cladding and core refractive indices.

For a cylindrical dielectric region of refractive index n we have:

$$\vec{\nabla} \times \vec{E} = -j\omega\mu_0 \vec{H}$$

$$\vec{\nabla} \times \vec{H} = j\omega\epsilon n^2 \vec{E}$$

From the Maxwell's equations for constant n we get the following wave equation:

$$\vec{\nabla}^2 \vec{E} + k_0^2 n^2 \vec{E} = 0$$

Now the electric field \vec{E} can be decomposed into a transverse and a longitudinal component. I.e. $\vec{E} = \vec{E}_T + \hat{z} E_z$, where $\vec{E}_T = E_x \hat{x} + E_y \hat{y} = E_r \hat{r} + E_\phi \hat{\phi}$

Then the vector wave equation can be written in the form:

$$\vec{\nabla}^2 \vec{E}_t + k_0^2 n^2 \vec{E}_t = 0 \quad (\text{transverse components}) \quad (1)$$

$$\nabla^2 E_z + k_0^2 n^2 E_z = 0 \quad (\text{longitudinal component})$$

As similar set of equations can be written for the magnetic fields;

$$\vec{\nabla}^2 \vec{H}_t + k_0^2 n^2 \vec{H}_t = 0 \quad (\text{transverse components}) \quad (2)$$

$$\nabla^2 H_z + k_0^2 n^2 H_z = 0 \quad (\text{longitudinal component})$$

Now let's concentrate on the z-components (longitudinal) of the electric & magnetic fields. Since we are seeking for modal solutions the general form of these fields is:

$$E_z = E_z(r, \phi) e^{-j\beta z} \quad (3)$$

$$H_z = H_z(r, \phi) e^{-j\beta z}$$

for forward ($+z$) propagating modes. Of course a time dependence of the $e^{j\omega t}$ is assumed. Of course a similar field form will be valid for the transverse field components. Now let's rework the two curl Maxwell equations in the cylindrical coordinate system.

$$\vec{\nabla} \times \vec{E} = \left(\frac{1}{r} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right) \hat{r} + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right) \hat{\phi} + \left(\frac{1}{r} \frac{\partial (r E_\phi)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} \right) \hat{z}$$

$$= -j\omega \mu_0 \vec{H} = -j\omega \mu_0 [H_r \hat{r} + H_\phi \hat{\phi} + H_z \hat{z}]$$

or

$$\frac{1}{r} \frac{\partial E_z}{\partial \phi} + j\beta E_\phi = -j\omega \mu_0 H_r \quad (4)$$

$$-j\beta E_r - \frac{\partial E_z}{\partial r} = -j\omega \mu_0 H_\phi$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) - \frac{1}{r} \frac{\partial E_r}{\partial \phi} = -j\omega \mu_0 H_z$$

Similarly, for the $\vec{\nabla} \times \vec{H} = +j\omega\epsilon_0 n^2 \vec{E}$ equation we have:

$$\begin{aligned} \frac{1}{r} \frac{\partial H_z}{\partial \phi} + j\beta H_\phi &= +j\omega\epsilon_0 n^2 E_r \\ -j\beta H_r - \frac{\partial H_z}{\partial r} &= +j\omega\epsilon_0 n^2 E_\phi \\ \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) - \frac{1}{r} \frac{\partial H_r}{\partial \phi} &= +j\omega\epsilon_0 n^2 E_z \end{aligned} \quad (5)$$

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From Eqs. (4) & (5) we can express the transverse field components E_r, E_ϕ, H_r, H_ϕ as functions of E_z and H_z only. For example, from the second of Eqs.(4) we can solve for E_r as follows:

$$\begin{aligned} -j\beta E_r &= \frac{\partial E_z}{\partial r} - j\omega\mu_0 H_\phi = \frac{\partial E_z}{\partial r} - j\omega\mu_0 \underbrace{\left\{ \frac{1}{j\beta} \left(j\omega\epsilon_0 n^2 E_r - \frac{1}{r} \frac{\partial H_z}{\partial \phi} \right) \right\}}_{\text{from first of Eqs. (5)}} \\ \Rightarrow E_r &= \frac{-j\beta}{k_0^2 n^2 - \beta^2} \left[\frac{\partial E_z}{\partial r} + \frac{\omega\mu_0}{\beta} \frac{1}{r} \frac{\partial H_z}{\partial \phi} \right] \end{aligned} \quad (6a)$$

Similarly,

$$E_\phi = \frac{-j\beta}{k_0^2 n^2 - \beta^2} \left[\frac{1}{r} \frac{\partial E_z}{\partial \phi} - \frac{\omega\mu_0}{\beta} \frac{\partial H_z}{\partial r} \right] \quad (6b)$$

$$H_r = \frac{-j\beta}{k_0^2 n^2 - \beta^2} \left[\frac{\partial H_z}{\partial r} - \frac{\omega\epsilon_0 n^2}{\beta} \frac{1}{r} \frac{\partial E_z}{\partial \phi} \right] \quad (6c)$$

$$H_\phi = \frac{-j\beta}{k_0^2 n^2 - \beta^2} \left[\frac{1}{r} \frac{\partial H_z}{\partial \phi} + \frac{\omega\epsilon_0 n^2}{\beta} \frac{\partial E_z}{\partial r} \right] \quad (6d)$$

Since E_r, E_ϕ, H_r, H_ϕ are functions of E_z, H_z it is sufficient to solve the wave equations (1) for E_z and (2) for H_z and then use Eqs.(6) for the rest of the field components. Let's start with the solution of E_z :

$$\nabla^2 E_z + k_0^2 n^2 E_z = 0 \Rightarrow$$

$$\frac{\partial^2}{\partial r^2} E_z + \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} - \beta^2 E_z + k_0^2 n^2 E_z = 0$$

Using $E_z(r, \phi) = R(r) \Phi(\phi)$ the above equation results in:

$$\frac{\partial^2 \Phi}{\partial \phi^2} \frac{1}{\Phi} = -\nu^2 \Rightarrow \frac{d^2 \Phi}{d \phi^2} + \nu^2 \Phi = 0 \quad (7a)$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \left(k_0^2 n^2 - \beta^2 - \frac{\nu^2}{r^2} \right) R = 0 \quad (7b)$$

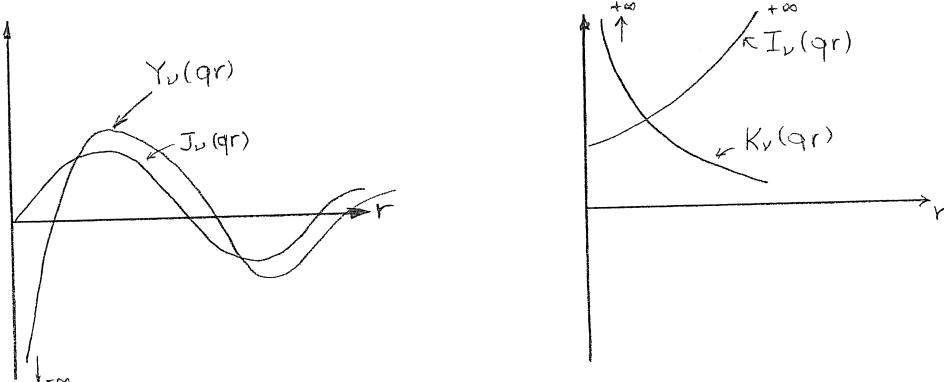
The general solution of (7a) is:

$$\Phi(\phi) = C e^{j\nu\phi} + D e^{-j\nu\phi} \quad (8)$$

The general solution of (7b) is

$$R(r) = \begin{cases} A J_\nu(qr) + B Y_\nu(qr) & \text{if } q^2 = k_0^2 n^2 - \beta^2 > 0 \\ A I_\nu(qr) + B K_\nu(qr) & \text{if } -q^2 = k_0^2 n^2 - \beta^2 < 0 \end{cases} \quad (9)$$

where J_ν and Y_ν are the Bessel functions of the first kind of order ν and I_ν and K_ν are the Bessel functions of the second kind of order ν .



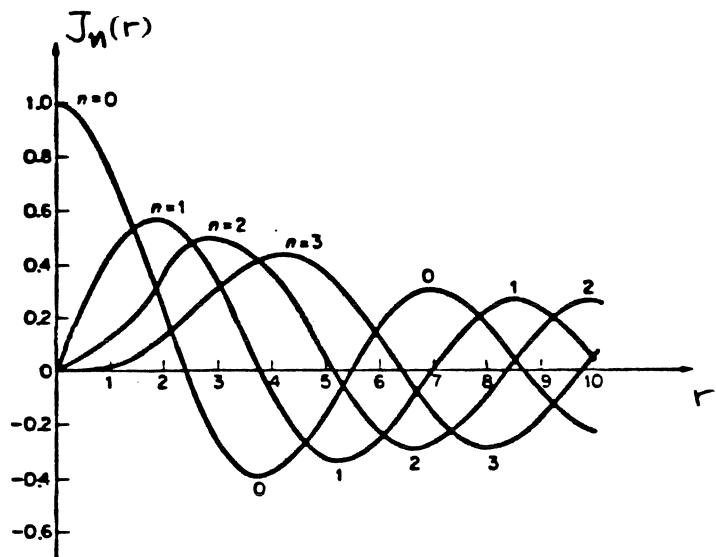
Then the general solution for E_z can be written as:

$$E_z(r, \phi) = [A c_\nu(qr) + B d_\nu(qr)] [C e^{j\nu\phi} + D e^{-j\nu\phi}] \quad (10)$$

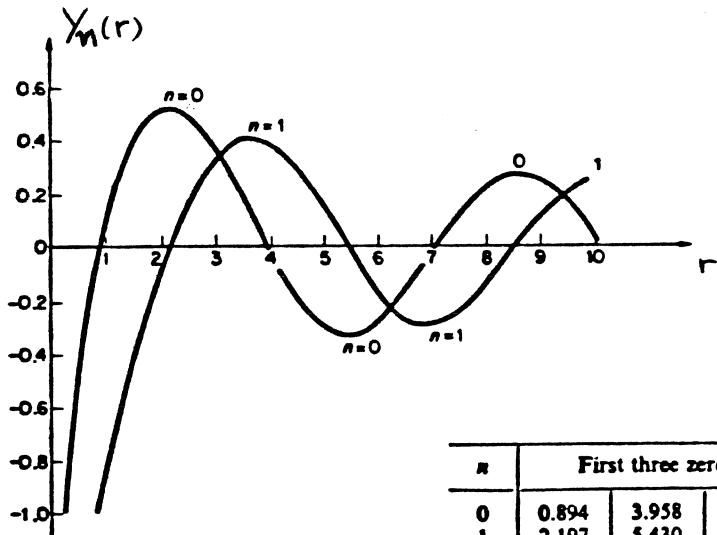
where $c_\nu(qr)$ is either $J_\nu(qr)$ or $I_\nu(qr)$ and $d_\nu(qr)$ is either $Y_\nu(qr)$ or $K_\nu(qr)$. Since $E_z(r, \phi+2\pi) = E_z(r, \phi) \Rightarrow \nu = \text{integer}$,

Oscillatory Bessel Functions

$J_n(r)$, $Y_n(r)$



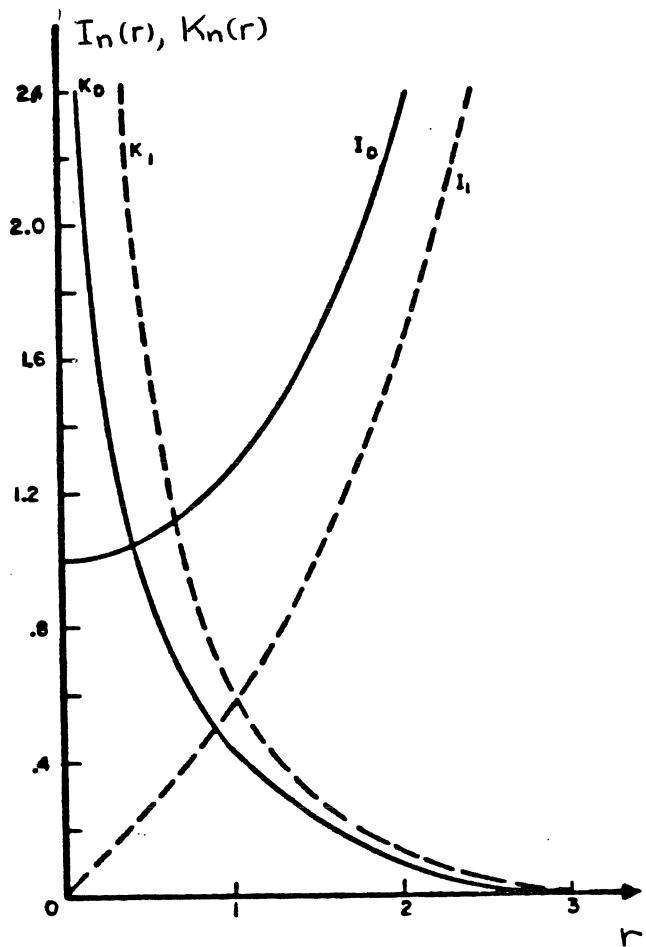
n	First three zeros		
	0	1	2
0	2.405	5.520	8.654
1	0	3.832	7.016
2	0	5.136	8.417
3	0	6.380	9.761



n	First three zeros		
	0	1	2
0	0.894	3.958	7.086
1	2.197	5.430	8.596

Modified Bessel Functions

$I_n(r), K_n(r)$



Now we have to apply these solutions to the step-index fiber. Since we are looking for guided wave solutions $k_0 n_2 < \beta < k_0 n_1$. Also it is reminded that the solution for H_z is similar to E_z [Eq. (10)].

Fiber Core : $r \leq a$

$$E_z = [A_1 J_\nu(\kappa r) + B_1 Y_\nu(\kappa r)] [C_{e_1} e^{j\nu\phi} + D_{e_1} e^{-j\nu\phi}] e^{-j\beta z}$$

$$H_z = [F_1 J_\nu(\kappa r) + G_1 Y_\nu(\kappa r)] [C_{h_1} e^{j\nu\phi} + D_{h_1} e^{-j\nu\phi}] e^{-j\beta z}$$

where $\kappa = (k_0^2 n_1^2 - \beta^2)^{1/2}$. Since the fields have to remain finite at $r=0 \rightarrow B=G=0$, Therefore, E_z, H_z have the form:

$$E_z = A_1 J_\nu(\kappa r) [C_{e_1} e^{j\nu\phi} + D_{e_1} e^{-j\nu\phi}] e^{-j\beta z} \quad (11)$$

$$H_z = F_1 J_\nu(\kappa r) [C_{h_1} e^{j\nu\phi} + D_{h_1} e^{-j\nu\phi}] e^{-j\beta z}$$

Fiber Cladding : $r \geq a$

$$E_z = [A_2 I_\nu(\gamma r) + B_2 K_\nu(\gamma r)] [C_{e_2} e^{j\nu\phi} + D_{e_2} e^{-j\nu\phi}] e^{-j\beta z}$$

$$H_z = [F_2 J_\nu(\gamma r) + G_2 K_\nu(\gamma r)] [C_{h_2} e^{j\nu\phi} + D_{h_2} e^{-j\nu\phi}] e^{-j\beta z}$$

where $\gamma = (\beta^2 - k_0^2 n_2^2)^{1/2}$. Again, in order to have only decaying evanescent field components $A_2 = F_2 = 0$. Therefore, E_z, H_z have the form:

$$E_z = B_2 K_\nu(\gamma r) [C_{e_2} e^{j\nu\phi} + D_{e_2} e^{-j\nu\phi}] e^{-j\beta z} \quad (12)$$

$$H_z = G_2 K_\nu(\gamma r) [C_{h_2} e^{j\nu\phi} + D_{h_2} e^{-j\nu\phi}] e^{-j\beta z}$$

Now we need to satisfy the boundary conditions at $r=a$ interface.

The tangential electric field components are E_ϕ, E_z and the tangential magnetic field components are H_ϕ, H_z .

E_z - continuity:

$$A_1 J_\nu(\zeta\alpha) [C_{e1} e^{j\nu\phi} + D_{e1} \bar{e}^{-j\nu\phi}] = B_2 K_\nu(\gamma\alpha) [C_{e2} e^{j\nu\phi} + D_{e2} \bar{e}^{-j\nu\phi}]$$

which in order to be valid for every ϕ requires:

$$A_1 J_\nu(\zeta\alpha) = B_2 K_\nu(\gamma\alpha) \quad (13)$$

$$C_{e1} = C_{e2} = C_e, \quad D_{e1} = D_{e2} = D_e$$

H_z - continuity:

$$F_1 J_\nu(\zeta\alpha) [C_{h1} e^{j\nu\phi} + D_{h1} \bar{e}^{-j\nu\phi}] = G_2 K_\nu(\gamma\alpha) [C_{h2} e^{j\nu\phi} + D_{h2} \bar{e}^{-j\nu\phi}]$$

which in order to be valid for every ϕ requires:

$$F_1 J_\nu(\zeta\alpha) = G_2 K_\nu(\gamma\alpha) \quad (14)$$

$$C_{h1} = C_{h2} = C_h, \quad D_{h1} = D_{h2} = D_h$$

E_ϕ - continuity:

$$\frac{-j\beta}{k_0^2 n_1^2 - \beta^2} \left\{ \frac{1}{\alpha} A_1 J_\nu(\zeta\alpha) [j\nu C_e e^{j\nu\phi} - j\nu D_e e^{-j\nu\phi}] - \frac{\omega \mu_0}{\beta} F_1 \zeta K_J'(\zeta\alpha) [C_h e^{j\nu\phi} + D_h e^{-j\nu\phi}] \right\} =$$

$$\frac{-j\beta}{k_0^2 n_2^2 - \beta^2} \left\{ \frac{1}{\alpha} B_2 K_\nu(\gamma\alpha) [j\nu C_e e^{j\nu\phi} - j\nu D_e e^{-j\nu\phi}] - \frac{\omega \mu_0}{\beta} G_2 \gamma K'_\nu(\gamma\alpha) [C_h e^{j\nu\phi} + D_h e^{-j\nu\phi}] \right\}$$

The above condition should be valid for every ϕ . This requires:

$$\frac{C_e}{C_h} = - \frac{D_e}{D_h} = \text{constant} = x$$

$$\frac{-j\beta}{k_0^2 n_1^2 - \beta^2} \left\{ \frac{j\nu}{\alpha} A_1 J_\nu(\zeta\alpha) - \frac{\omega \mu_0}{\beta} F_1 \zeta K_J'(\zeta\alpha) \right\} = \frac{-j\beta}{k_0^2 n_2^2 - \beta^2} \left\{ \frac{j\nu}{\alpha} B_2 K_\nu(\gamma\alpha) - \frac{\omega \mu_0}{\beta} G_2 \gamma K'_\nu(\gamma\alpha) \right\}$$

without loss of generality $\alpha = 1$ which implies $C_h = C_e = C$

and $D_h = -D_e = -D$.

Then the last equation becomes:

$$\frac{\beta \nu}{\kappa^2 \alpha} J_\nu(\kappa \alpha) A_1 + \frac{j \omega \mu_0}{\kappa} J'_\nu(\kappa \alpha) F_1 = \frac{\beta \nu}{-\gamma^2 \alpha} K_\nu(\gamma \alpha) B_2 + \frac{j \omega \mu_0}{-\gamma} K'_\nu(\gamma \alpha) G_2 \quad (15)$$

H_Φ -continuity:

$$\begin{aligned} & \frac{-j\beta}{\kappa^2 n_1^2 - \beta^2} \left\{ \frac{1}{\alpha} F_1 J_\nu(\kappa \alpha) [j_\nu C e^{j\nu\phi} + j_\nu D e^{-j\nu\phi}] + \frac{\omega \epsilon_0 n_1^2}{\beta} A_1 \kappa J'_\nu(\kappa \alpha) [C e^{j\nu\phi} + D e^{-j\nu\phi}] \right\} \\ &= \frac{-j\beta}{\kappa^2 n_2^2 - \beta^2} \left\{ \frac{1}{\alpha} G_2 K_\nu(\gamma \alpha) [j_\nu C e^{j\nu\phi} + j_\nu D e^{-j\nu\phi}] + \frac{\omega \epsilon_0 n_2^2}{\beta} B_2 \gamma K'_\nu(\gamma \alpha) [C e^{j\nu\phi} + D e^{-j\nu\phi}] \right\} \end{aligned}$$

which implies that:

$$\frac{\beta \nu}{\kappa^2 \alpha} J_\nu(\kappa \alpha) F_1 - \frac{j \omega \epsilon_0 n_1^2}{\kappa} J'_\nu(\kappa \alpha) A_1 = \frac{\beta \nu}{-\gamma^2 \alpha} K_\nu(\gamma \alpha) G_2 - \frac{j \omega \epsilon_0 n_2^2}{-\gamma} K'_\nu(\gamma \alpha) B_2 \quad (16)$$

Now let's write the four boundary conditions in a matrix form:

$$\underbrace{\begin{bmatrix} J_\nu(\kappa \alpha) & 0 & -K_\nu(\gamma \alpha) & 0 \\ 0 & J_\nu(\kappa \alpha) & 0 & -K_\nu(\gamma \alpha) \\ \frac{\beta \nu}{\kappa^2 \alpha} J_\nu(\kappa \alpha) & \frac{j \omega \mu_0}{\kappa} J'_\nu(\kappa \alpha) & \frac{\beta \nu}{\gamma^2 \alpha} K_\nu(\gamma \alpha) & \frac{j \omega \mu_0}{\gamma} K'_\nu(\gamma \alpha) \\ -\frac{j \omega \epsilon_0 n_1^2}{\kappa} J'_\nu(\kappa \alpha) & \frac{\beta \nu}{\kappa^2 \alpha} J_\nu(\kappa \alpha) & -\frac{j \omega \epsilon_0 n_2^2}{\gamma} K'_\nu(\gamma \alpha) & \frac{\beta \nu}{\gamma^2 \alpha} K_\nu(\gamma \alpha) \end{bmatrix}}_{\tilde{A}(\beta; \nu)} \begin{bmatrix} A_1 \\ F_1 \\ B_2 \\ G_2 \end{bmatrix} = \begin{bmatrix} \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \end{bmatrix} \quad (17)$$

For nontrivial solutions it is required that $\det \{\tilde{A}(\beta; \nu)\} = \emptyset$

Calculation of the determinant yields:

$$\frac{\beta^2 \nu^2}{\alpha^2} \left[\frac{1}{\gamma^2} + \frac{1}{\kappa^2} \right]^2 = \left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)} \right] \left[\frac{k_0^2 n_1^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_2^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)} \right] \quad (18)$$

Of course, Eq.(18) can be solved only numerically in order to find β .

For example, for every ν value m solutions can be found. Then, the guided modes of the optical fiber can be characterized by two integers; ν and m which are needed to represent the transverse confinement of the optical field. The m -value is called the "radial mode number" while ν -value is called the "angular (azimuthal) mode number".

Once β is determined from Eq.(18), three of the four coefficients of E_z and H_z (A_1, F_1, B_2, G_2) can be expressed in terms of the forth.

For example, if A_1 is selected to be the free parameter then:

$$B_2 = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} A_1 \quad (\text{from Eq.(13)}) \quad (19)$$

$$G_2 = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} F_1 \quad (\text{from Eq.(14)}) \quad (20)$$

Now A_1, F_1 can be related either from Eq. (15) or from Eq. (16):

Therefore if Eq. (15) is used (continuity of E_ϕ) then:

$$F_1 = \frac{j \beta \nu}{w \mu_0 \alpha} \left(\frac{1}{\kappa^2} + \frac{1}{\gamma^2} \right) \left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)} \right]^{-1} A_1 \quad (21)$$

If Eq. (16) is used (continuity of H_ϕ) then:

$$F_1 = \frac{j w \alpha \epsilon_0}{\beta \nu} \left[\frac{n_1^2}{\kappa} \frac{J_\nu'(\kappa a)}{J_\nu(\kappa a)} + \frac{n_2^2}{\gamma} \frac{K_\nu'(\gamma a)}{K_\nu(\gamma a)} \right] \left[\frac{1}{\kappa^2} + \frac{1}{\gamma^2} \right]^{-1} A_1 \quad (22)$$

It is worth mentioning that F_1/A_1 is purely imaginary. Also

$G_2/B_2 = F_1/A_1$. Therefore E_z, H_z are out of phase by $\pi/2$.

Mode Characterization:

Let's consider first the case of $\nu = \emptyset$. Then from Eq.(18) we get:

$$\underbrace{\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} \right]}_{A_1} \underbrace{\left[\frac{k_0^2 n_1^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_2^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} \right]}_{A_2} = \emptyset \quad (23)$$

For the case of $\nu = \emptyset$ Eqs. (19) - (22) become:

$$B_2 = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} A_1 \quad (24)$$

$$G_2 = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} F_1 \quad (25)$$

$$\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} \right] F_1 = \emptyset \quad (26)$$

$$\left[\frac{k_0^2 n_1^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_2^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} \right] A_1 = \emptyset \quad (27)$$

Now if $A_1 = \frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} = 0$ from Eq. (23) then

$F_1 \neq 0$, but $A_1 = 0 = B_2$. In this case $E_z = 0$ and these modes are called TE (transverse electric) modes.

On the other hand if $A_2 = \frac{k_0^2 n_1^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_2^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} = 0$ from Eq. (23)

then $F_1 = 0$ and $A_1 \neq 0$ and $G_2 = 0$. In this case $H_z = 0$ and these modes are called TM (transverse magnetic) modes.

Using Bessel function identities:

$$\nu J_\nu(x) - x J'_\nu(x) = x J_{\nu+1}(x)$$

$$(-1)^\nu K'_\nu(x) = (-1)^{\nu+1} K_{\nu+1}(x) + \frac{\nu}{x} (-1)^\nu K_\nu(x)$$

the following eigenvalue equations can be found for TE and TM modes:

$$\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} + \frac{K_1(\gamma a)}{\gamma K_0(\gamma a)} = 0 \quad \text{for } TE_{\phi m} \quad (\nu=0) \text{ modes} \quad (28)$$

$$n_1^2 \frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} + n_2^2 \frac{K_1(\gamma a)}{\gamma K_0(\gamma a)} = 0 \quad \text{for } TM_{\phi m} \quad (\nu=0) \text{ modes} \quad (29)$$

For $\nu \neq 0$ the solutions are called $HE_{\nu m}$ or $EH_{\nu m}$. Following Yariv's approach [A. Yariv, "Optical Electronics in Modern Communications"]

Eq. (18) can be solved for $\frac{J_{\nu}'(\kappa a)}{(\kappa a) J_{\nu}(\kappa a)}$. Then the eigenvalue equations are:

$EH_{\nu m}$ - modes $(\nu \neq 0)$

$$\frac{J_{\nu+1}(\kappa a)}{\kappa a J_{\nu}(\kappa a)} = \frac{n_1^2 + n_2^2}{2n_1^2} \frac{K_{\nu}'(\gamma a)}{\gamma a K_{\nu}(\gamma a)} + \left(\frac{1}{(\kappa a)^2} - R \right) \quad (30)$$

$HE_{\nu m}$ - modes $(\nu \neq 0)$

$$\frac{J_{\nu-1}(\kappa a)}{\kappa a J_{\nu}(\kappa a)} = - \frac{n_1^2 + n_2^2}{2n_1^2} \frac{K_{\nu}'(\gamma a)}{\gamma a K_{\nu}(\gamma a)} + \left(\frac{1}{(\kappa a)^2} - R \right) \quad (31)$$

where $R = \left[\left(\frac{n_1^2 - n_2^2}{2n_1^2} \right)^2 \frac{K_{\nu}'(\gamma a)}{\gamma a K_{\nu}(\gamma a)} + \left(\frac{\nu \beta}{n_1 k_0} \right)^2 \left(\frac{1}{(\kappa a)^2} + \frac{1}{(\gamma a)^2} \right)^2 \right]^{1/2}$

Another point which is worth mentioning is the angular (azimuthal) dependence of the fields.

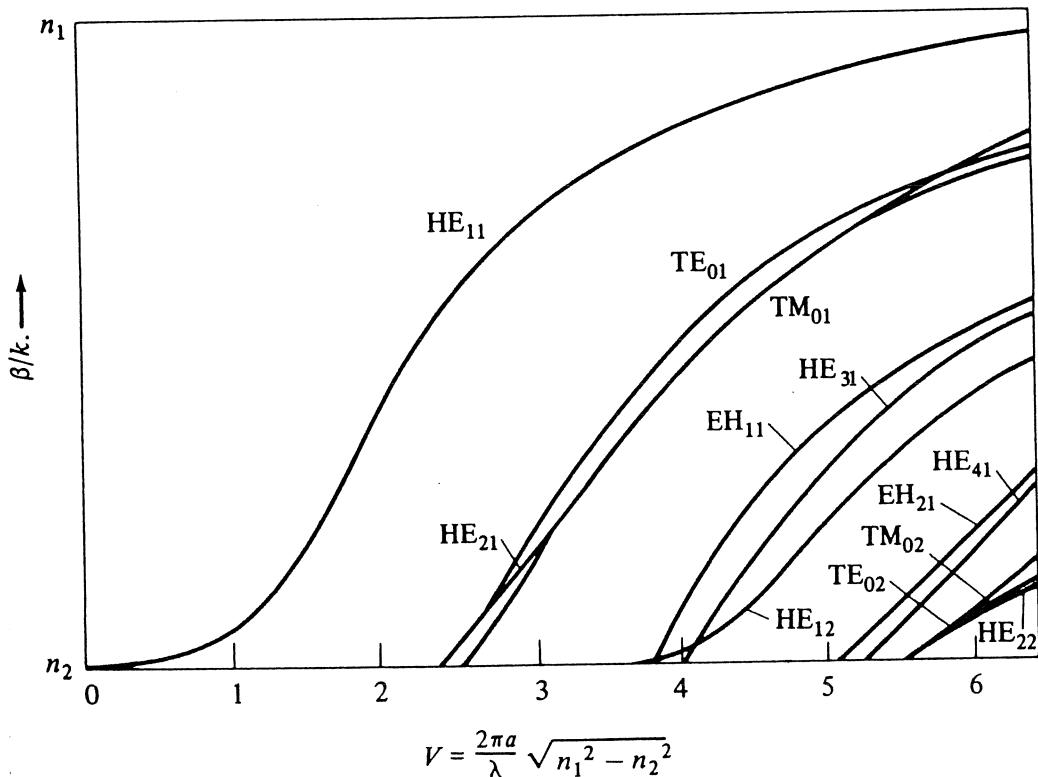
$$E_z = [A c_{\nu}(qr) + B d_{\nu}(qr)] [C e^{j\nu\phi} + D e^{-j\nu\phi}] \quad (32)$$

$$H_z = [F c_{\nu}(qr) + G d_{\nu}(qr)] [C e^{j\nu\phi} - D e^{-j\nu\phi}]$$

Since C, D can be selected independently we have in general two independent solutions. For example, if $C=D=\frac{1}{2}$ the $\Phi(\phi)=\cos(\nu\phi)$ for E_z and $\Phi(\phi)=+j\sin(\nu\phi)$ for H_z . If $-C=+D=\frac{1}{2}$ then $\Phi(\phi)=j\sin(\nu\phi)$ for E_z and $\Phi(\phi)=-\cos(\nu\phi)$ for H_z .

DISPERSION CHARACTERISTICS OF GUIDED MODES IN A STEP-INDEX OPTICAL FIBER

(from A. Yariv, "Optical Electronics in Modern Communications")



Then the 2 independent solutions can be written as:

$$E_z = [A c_\nu(qr) + B d_\nu(qr)] \begin{pmatrix} \cos(\nu\phi) \\ j\sin(\nu\phi) \end{pmatrix} \quad (33)$$

$$H_z = [F c_\nu(qr) + G d_\nu(qr)] \begin{pmatrix} +j\sin(\nu\phi) \\ -\cos(\nu\phi) \end{pmatrix}$$

Weakly-Guided Approximation:

For most optical fibers $n_1 \approx n_2$. Therefore, a common approximation made is that $\frac{n_1^2}{n_2^2} \approx 1$. This is called the weakly guiding approximation. Under this approximation the Eq.(18) can be simplified into the following form:

$$(ka) \frac{J_{l-1}(ka)}{J_l(ka)} = -(\gamma a) \frac{K_{l-1}(\gamma a)}{K_l(\gamma a)} \quad l = \begin{cases} 1 & \text{for TE}_{0m}, \text{TM}_{0m} \\ \nu+1 & \text{for EH}_{\nu m} \\ \nu-1 & \text{for HE}_{\nu m} \end{cases} \quad (34)$$

The significance of the above equation is that more than a single mode can have the same propagation constant β . For example, TE_{0m} and TM_{0m} modes become degenerate. Also $\text{HE}_{\nu+1,m}$ modes and $\text{EH}_{\nu-1,m}$ modes are also degenerate. Since degenerate modes can travel with the same velocity (under the weakly guiding approximation) it is possible to define superpositions of different degenerate modes that can be linearly polarized. These superpositions can define linearly polarized modes with negligible E_z component and are called $\text{LP}_{\nu m}$ modes. Specifically, the $\text{LP}_{\nu m}$ modes can be constructed as follows:

$$\begin{aligned} \text{LP}_{1m} &\rightarrow \text{sum of TE}_{0m}, \text{TM}_{0m} \text{ and } \text{HE}_{2m} \text{ modes} \\ \text{LP}_{\nu m} &\rightarrow \text{sum of } \text{HE}_{\nu+1,m} \text{ and } \text{EH}_{\nu-1,m} \text{ modes } (\nu \geq 2) \\ \text{LP}_{0m} &\rightarrow \text{HE}_{1m} \text{ mode only} \end{aligned} \quad (35)$$

LP-modes:

Since the LP modes are linearly polarized and E_z is negligible they can also be considered solutions of the scalar wave equation:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + (k_0^2 n^2 - \beta^2) \psi = 0 \quad (36)$$

where $\psi = \psi(r, \phi) e^{-j\beta z}$ and ψ can be either E_x or E_y (H_x or H_y).

Equation (36) is similar to (7b) and the general solution can be written as

$$\psi(r, \phi) = [A_{J_\nu}(qr) + B_{d_\nu}(qr)] [C e^{j\nu\phi} + D e^{-j\nu\phi}] e^{-j\beta z} \quad (37)$$

where $q^2 = k_0^2 n^2 - \beta^2$ and $C_\nu = J_\nu$ or I_ν and $d_\nu = Y_\nu$ or K_ν depending on the sign of $k_0^2 n^2 - \beta^2 \gtrless 0$.

The boundary conditions in this case are the continuity of ψ and $\frac{d\psi}{dr}$ at the core/cladding interface.

Fiber Core: ($r \leq \alpha$)

$$\psi(r, \phi) = A_1 J_\nu(\kappa r) [C e^{j\nu\phi} + D e^{-j\nu\phi}] e^{-j\beta z} \quad (38)$$

Fiber Cladding: ($r \geq \alpha$)

$$\psi(r, \phi) = B_2 K_\nu(\gamma r) [C e^{j\nu\phi} + D e^{-j\nu\phi}] e^{-j\beta z} \quad (39)$$

where $\kappa^2 = k_0^2 n_1^2 - \beta^2$, $\gamma^2 = \beta^2 - k_0^2 n_2^2$ and C, D have to be the same as it was shown in the hybrid mode case.

Continuity of ψ at $r = \alpha$

$$A_1 J_\nu(\kappa\alpha) = B_2 K_\nu(\gamma\alpha) \quad (40)$$

Continuity of $\frac{d\psi}{dr}$ at $r = \alpha$

$$A_1 \kappa J'_\nu(\kappa\alpha) = B_2 \gamma K'_\nu(\gamma\alpha) \quad (41)$$

Then the eigenvalue equation becomes:

$$\frac{x J_v'(x\alpha)}{J_v(x\alpha)} = \frac{y K_v'(y\alpha)}{K_v(y\alpha)} \quad (42)$$

which with the Bessel function identities $J_v' = J_{v-1} - \frac{y}{x} J_v$ and $K_v' = -K_{v-1} - \frac{y}{x} K_v$ can be transformed into:

$$\frac{(x\alpha) J_{v-1}(x\alpha)}{J_v(x\alpha)} = - \frac{(y\alpha) K_{v-1}(y\alpha)}{K_v(y\alpha)} \quad (43)$$

Equation (43) is the eigenvalue equation of the $LP_{v,m}$ modes.

(remember that for LP_{0m} , $J_{-1} = (-1)J_1$ and $K_{-1} = K_1$).

After β is specified from Eq. (43) then the $LP_{v,m}$ field is given by:

$$\psi(r, \phi, z) = \begin{cases} A_1 J_v(kr) \begin{pmatrix} \cos(v\phi) \\ \sin(v\phi) \end{pmatrix} e^{-j\beta z} & r \leq a \\ \left(A_1 \frac{J_v(x\alpha)}{K_v(y\alpha)} \right) K_v(yr) \begin{pmatrix} \cos(v\phi) \\ \sin(v\phi) \end{pmatrix} e^{-j\beta z} & r > a \end{cases} \quad (44)$$

Again we observe that we have 2 independent solutions $\begin{pmatrix} \cos(v\phi) \\ \sin(v\phi) \end{pmatrix}$.

Also since ψ represents the transverse electric field could be either x or y polarized. Therefore, each $LP_{v,m}$ can have 4 possible configurations: E_x, E_y and $\cos(v\phi)$ or $\sin(v\phi)$. As an example the possible configurations of LP_{11} are shown on the next page. Of course in the case of $v=0$ there are only 2 configurations since there is no azimuthal dependence.

It is interesting to observe that the total number of modes is the same in both the LP and the hybrid mode representation. For example LP_{21} corresponds to 4 degenerate field configurations. Similarly HE_{31}, EH_{11} correspond to 4 field configurations. (2 for HE_{31} and 2 for EH_{11}).

LINEARLY POLARIZED (LP) MODES IN A STEP-INDEX OPTICAL FIBER

(from J. A. Buck, "Fiber Optics")

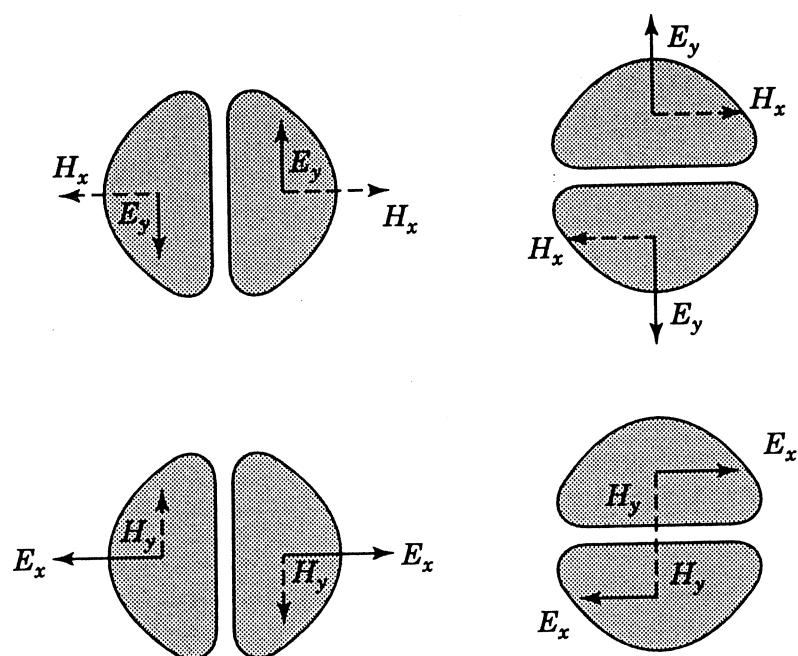
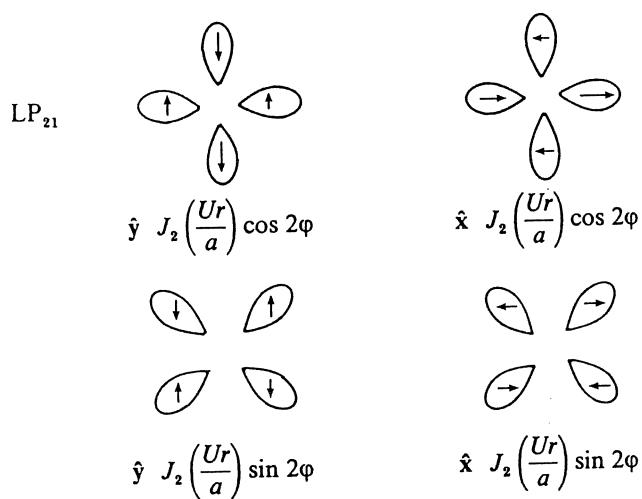
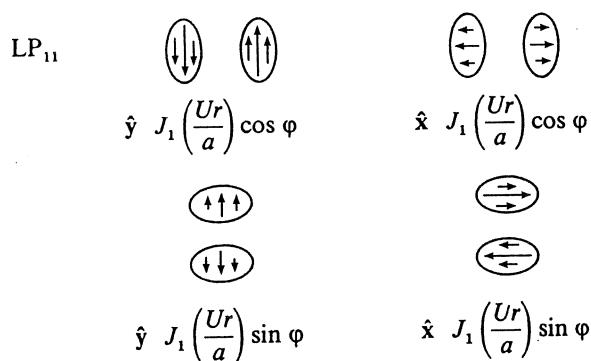
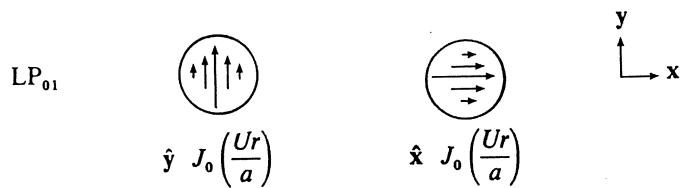


Figure 3.7. LP_{11} field patterns, showing the four possible configurations. (Adapted from ref. 5.)

LINEARLY POLARIZED (LP) MODES IN A STEP-INDEX OPTICAL FIBER

(from A. Ghatak and K. Thyagarajan, "Introduction to Fiber Optics")



LINEARLY POLARIZED (LP) MODES IN A STEP-INDEX OPTICAL FIBER

(from J. A. Buck, "Optical Fibers")

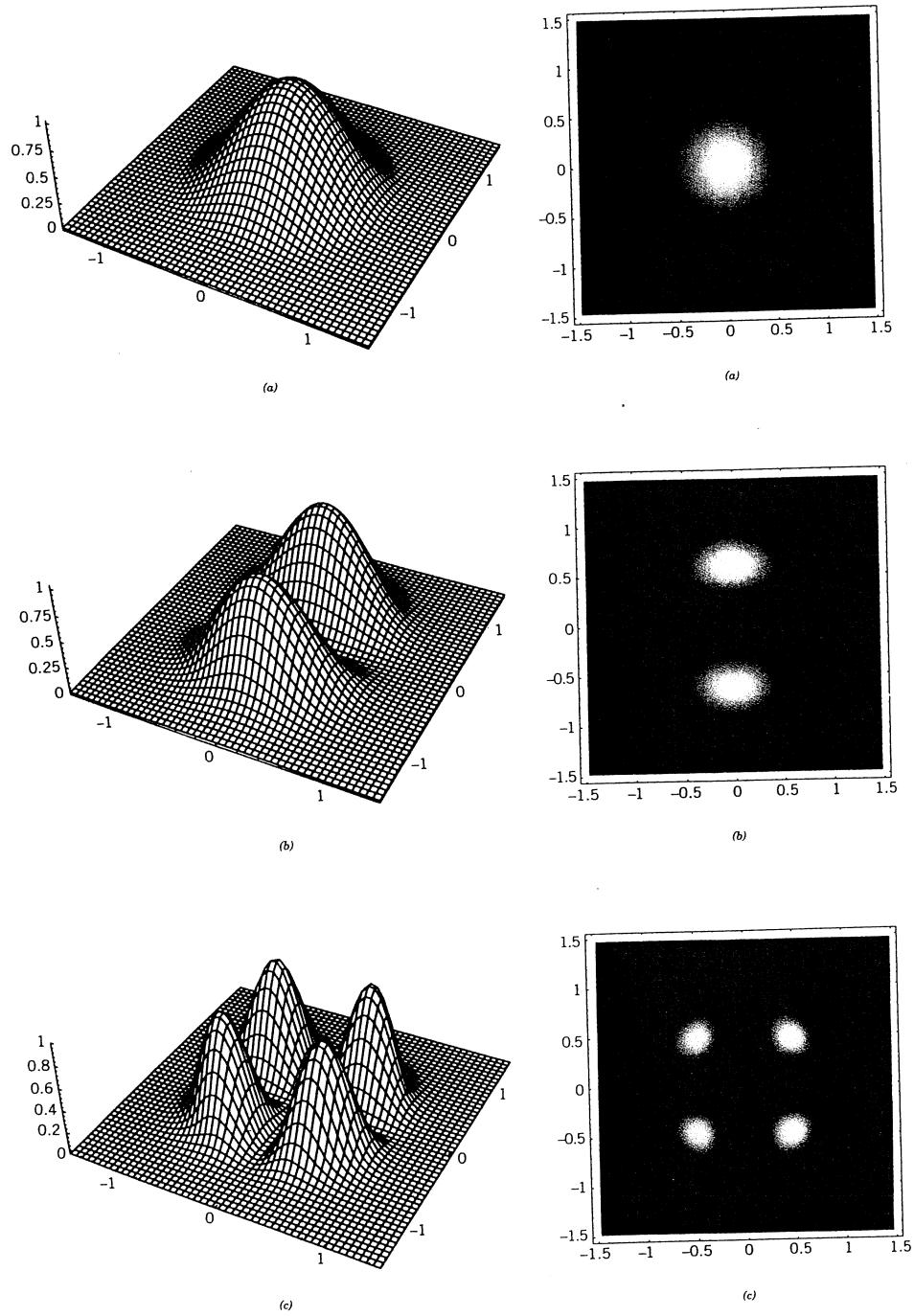
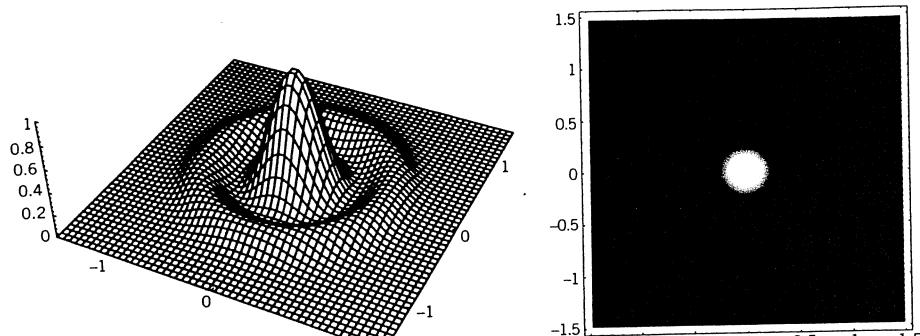


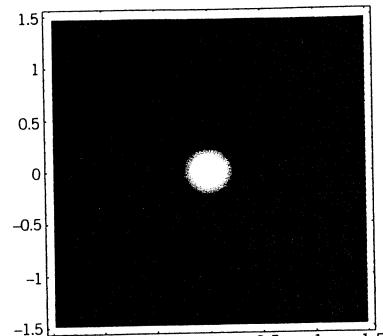
Figure 3.9. Intensity plots for the six LP modes, with $a = 1$. (a) LP_{01} : $u = 2$. (b) LP_{11} : $u = 3$. (c) LP_{21} : $u = 4.5$. (d) LP_{02} : $u = 4.5$. (e) LP_{31} : $u = 5.6$. (f) LP_{12} : $u = 6.3$.

LINEARLY POLARIZED (LP) MODES IN A STEP-INDEX OPTICAL FIBER

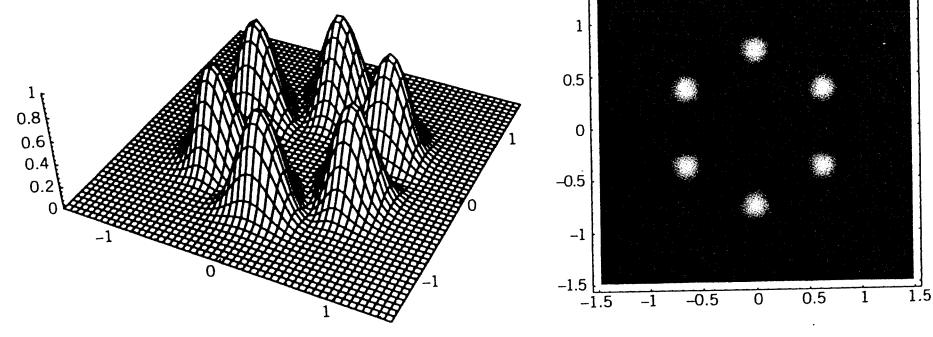
(from J. A. Buck, "Optical Fibers")



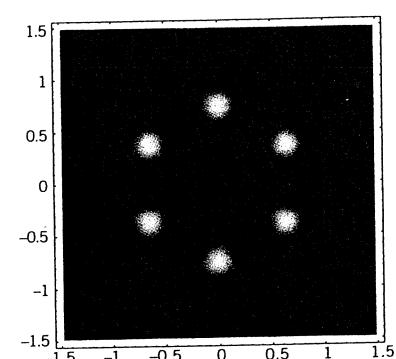
(d)



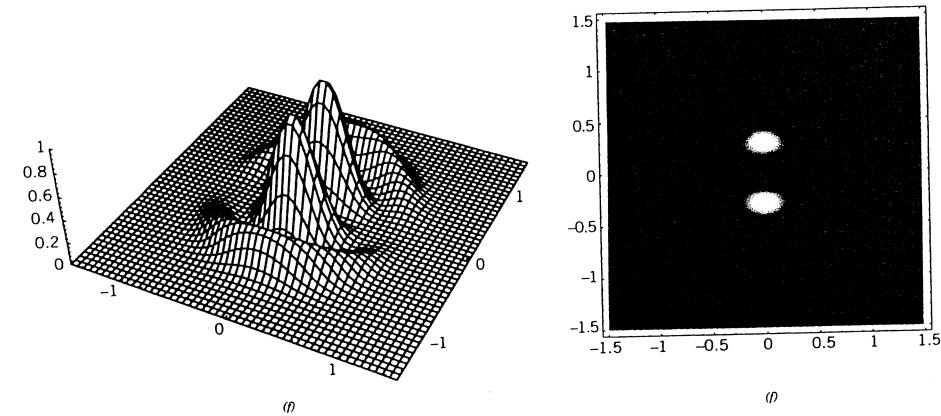
(d)



(e)



(e)



(f)

(f)

LINEARLY POLARIZED (LP) MODES IN A STEP-INDEX OPTICAL FIBER

(from J. A. Buck, "Fiber Optics")

Table 3.2 Cutoff Conditions and Designations of the First 12 LP Modes in a Step Index Fiber

V_c	Bessel Function	l	Degenerate Modes	LP Designation
0	—	0	HE_{11}	LP_{01}
2.405	J_0	1	$TE_{01}, TM_{01}, HE_{21}$	LP_{11}
3.832	J_1	2	EH_{11}, HE_{31}	LP_{21}
3.832	J_{-1}	0	HE_{12}	LP_{02}
5.136	J_2	3	EH_{21}, HE_{41}	LP_{31}
5.520	J_0	1	$TE_{02}, TM_{02}, HE_{22}$	LP_{12}
6.380	J_3	4	EH_{31}, HE_{51}	LP_{41}
7.016	J_1	2	EH_{12}, HE_{32}	LP_{22}
7.016	J_{-1}	0	HE_{13}	LP_{03}
7.588	J_4	5	EH_{41}, HE_{61}	LP_{51}
8.417	J_2	3	EH_{22}, HE_{42}	LP_{32}
8.654	J_0	1	$TE_{03}, TM_{03}, HE_{23}$	LP_{13}

Cutoff Conditions:

The cutoff condition for any given mode can be determined from the eigenvalue equation (18) for $\gamma = (\beta^2 - k_0^2 n_z^2)^{1/2} = \emptyset$. Whenever, $\gamma = 0 \Rightarrow \beta = k_0 n_z$ and $\omega = k_0 (n_1^2 - n_2^2)^{1/2}$. Then we can define the normalized frequency V , $V = k_0 (n_1^2 - n_2^2)^{1/2} \propto$ (similarly to the step-index slab waveguides). However, solutions of Eq.(18) are in general difficult to solve for the cutoff V 's. Therefore, it is common to use Eq. (34) (under the weakly guiding approximation) to find the cutoff conditions. Using Eq. (34) for $\gamma^2 = 0$ it can be shown that the cutoff condition is obtained the solution of the equation:

$$\frac{V J_{l-1}(V)}{J_l(V)} = \emptyset \quad l = \begin{cases} 1 & \text{for } TE_{0m}, TM_{0m} \\ v+1 & \text{for } EH_{vm} \quad (v \geq 1) \\ v-1 & \text{for } HE_{vm} \quad (v \geq 1) \end{cases} \quad (45)$$

The above equation can be obtained by using the following expressions for the Bessel functions:

$$K_v(z) \approx \frac{1}{2} \Gamma(v) \left(\frac{1}{2}z\right)^{-v} \quad (v > 0) \quad z \rightarrow 0$$

$$K_0(z) \approx -\ln z \quad (v=0) \quad z \rightarrow 0$$

$$K_{-v}(z) = K_v(z)$$

$$J_v(z) \approx \frac{1}{2} z^v / \Gamma(v+1) \quad (v > 0) \quad z \rightarrow 0$$

$$J_{-v}(z) = (-1)^v J_v(z)$$

For $V \neq 0$ Eq. (45) reduces to $J_{l-1}(V) = 0$. In the case of $V = 0$ it can be shown that $V = 0$ is a valid solution only for $l = 0 \sim v = 1$ for HE_{11} mode. Consequently, the HE_{11} mode has a zero cutoff normalized frequency. For large V , it can be shown that the number of modes can be approximated by $N \approx \frac{4}{\pi^2} V^2$

Summarizing, under the weakly guiding approximation, we have the following cutoff conditions:

$$J_0(V_m) = 0 \quad \text{for } TE_{0m}, TM_{0m}$$

$$V=0 \quad \text{for } HE_{11}, \quad J_1(V_m)=0 \text{ for } HE_{1m} \quad (m \geq 2) \quad (46)$$

$$J_{v-2}(V_m)=0 \quad \text{for } HE_{vm} \quad (v \geq 2)$$

$$J_v(V_m)=0 \quad \text{for } EH_{vm} \quad (v \geq 1)$$

where $J_v(V_m)$ corresponds to the m -th root of J_v .

Since Eq. (45) is also valid for LP modes then the cutoff conditions for them are:

$$J_1(V_m) = 0 \quad \text{for } LP_{0m} \text{ modes}$$

$$J_0(V_m) = 0 \quad \text{for } LP_{1m} \text{ modes} \quad (47)$$

$$J_{v-1}(V_m) = 0 \quad \text{for } LP_{vm} \quad (v \geq 2) \quad (V_m \neq 0)$$

Obviously, the LP_{01} has $V=0$ as cutoff normalized frequency.

Cutoff frequencies for hybrid and LP modes are summarized on page 21 and 24.

Note: In several texts the cutoff condition for $HE_{vm} \quad (v \geq 2)$ is given

by

$$\left(\frac{n_1^2}{n_2^2} + 1\right) J_{v-1}(V_m) = \frac{V_m}{v-1} J_v(V_m)$$

when $\frac{n_1^2}{n_2^2} \approx 1 \Rightarrow 2J_{v-1}(V_m) = \frac{V_m}{v-1} J_v(V_m)$. But $J_{v-1} + J_{v+1} = \frac{2v}{V} J_v$ or

$$J_{v-2} + J_v = \frac{2(v-1)}{V} J_{v-1} \Rightarrow 2J_{v-1} = \frac{V}{v-1} (J_{v-2} + J_v) \text{ and the previous}$$

equation is equivalent to $\frac{V_m J_{v-2}(V_m)}{v-1} = 0 \Rightarrow J_{v-2}(V_m) = 0$ which is

the same with Eq. (46).

LINEARLY POLARIZED (LP) MODES IN A STEP-INDEX OPTICAL FIBER CUTOFF NORMALIZED FREQUENCIES

(from A. Ghatak and K. Thyagarajan, "Introduction to Fiber Optics")

Table 8.2. *Cutoff frequencies of various LP_{lm} modes in a step index fiber*

$l = 0$ modes ($J_1(V_c) = 0$)		$l = 1$ modes ($J_0(V_c) = 0$)	
Mode	V_c	Mode	V_c
LP ₀₁	0	LP ₁₁	2.4048
LP ₀₂	3.8317	LP ₁₂	5.5201
LP ₀₃	7.0156	LP ₁₃	8.6537
LP ₀₄	10.1735	LP ₁₄	11.7915
$l = 2$ modes ($J_1(V_c) = 0; V_c \neq 0$)		$l = 3$ modes ($J_0(V_c) = 0; V_c \neq 0$)	
Mode	V_c	Mode	V_c
LP ₂₁	3.8317	LP ₃₁	5.1356
LP ₂₂	7.0156	LP ₃₂	8.4172
LP ₂₃	10.1735	LP ₃₃	11.6198
LP ₂₄	13.3237	LP ₃₄	14.7960

Power Considerations: (LP modes)

The Poynting vector along the z -direction is given by: (for a mode)

$$S_z = \frac{1}{2} \operatorname{Re} \{ \vec{E} \times \vec{H}^* \}_z = \frac{1}{2} \operatorname{Re} \{ E_x H_y^* - E_y H_x^* \} = \frac{1}{2} \operatorname{Re} \{ E_r H_\phi^* - E_\phi H_r^* \} \quad (48)$$

Then the power in the core and cladding can be found from:

$$\begin{aligned} P_{\text{core}} &= \int_{\phi=0}^{2\pi} \int_{r=0}^a S_z r dr d\phi \\ P_{\text{clad}} &= \int_{\phi=0}^{2\pi} \int_{r=a}^{\infty} S_z r dr d\phi \end{aligned} \quad (49)$$

For $LP_{\nu m}$ modes (weakly guiding approximation) it can be shown that

$$\begin{aligned} P_{\text{core}} &= \frac{\beta_v}{2\omega\mu_0} \pi a^2 |A_1|^2 [J_\nu^2(\gamma a) - J_{\nu-1}(\gamma a) J_{\nu+1}(\gamma a)] \\ P_{\text{clad}} &= \frac{\beta_v}{2\omega\mu_0} \pi a^2 |A_1|^2 \left[-J_\nu^2(\gamma a) - \left(\frac{\gamma}{\lambda} \right)^2 J_{\nu-1}(\gamma a) J_{\nu+1}(\gamma a) \right] \\ P &= P_{\text{clad}} + P_{\text{core}} = \frac{\beta_v}{2\omega\mu_0} \pi a^2 |A_1|^2 \left(1 + \left(\frac{\gamma}{\lambda} \right)^2 \right) \left(-J_{\nu-1}(\gamma a) J_{\nu+1}(\gamma a) \right) \end{aligned} \quad (50)$$

Then, using Eqs.(50) and (43) (as well as properties of Bessel functions)

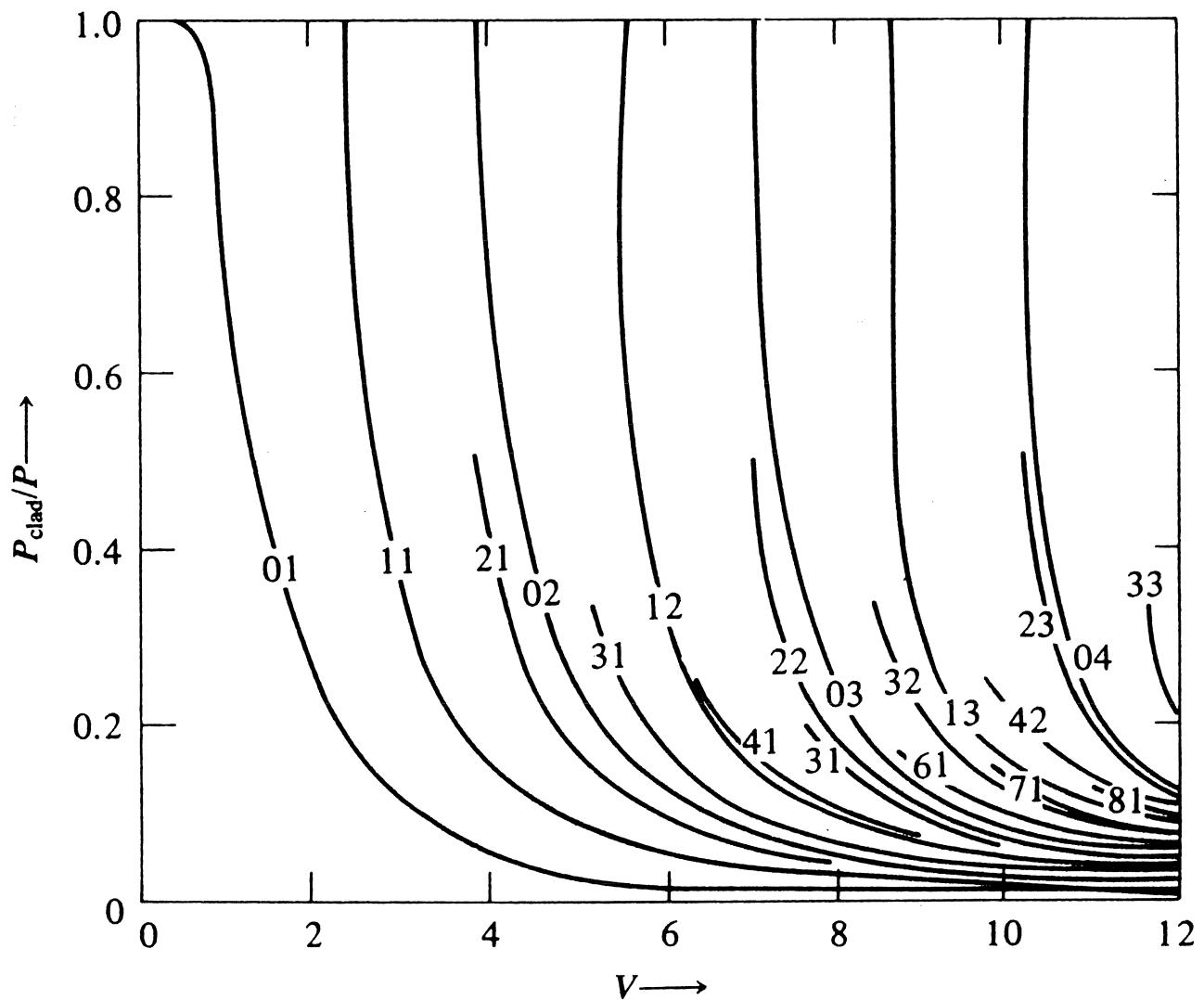
that

$$\begin{aligned} \Gamma_{cl} &= \frac{P_{\text{clad}}}{P} = \frac{1}{V^2} \left[(\gamma a)^2 + \frac{(\gamma a)^2 J_\nu^2(\gamma a)}{J_{\nu-1}(\gamma a) J_{\nu+1}(\gamma a)} \right] = \left(\frac{\gamma a}{V} \right)^2 \left(1 - \frac{K_\nu(\gamma a)}{K_{\nu-1}(\gamma a) K_{\nu+1}(\gamma a)} \right) \\ \Gamma_{co} &= \frac{P_{\text{core}}}{P} = 1 - \Gamma_{cl} = 1 - \left(\frac{\gamma a}{V} \right)^2 \left[1 - \frac{K_\nu(\gamma a)}{K_{\nu-1}(\gamma a) K_{\nu+1}(\gamma a)} \right] \end{aligned} \quad (51)$$

The ratio Γ_{cl} is shown in the next page as function of V . For $\nu=0$ and $\nu=1$ there is no appreciable power in the core at cutoff. For $\nu \geq 2$ the ratio $\frac{P_{\text{clad}}}{P} \rightarrow \frac{1}{V}$ at cutoff. As V increases the power concentrates in the core as expected.

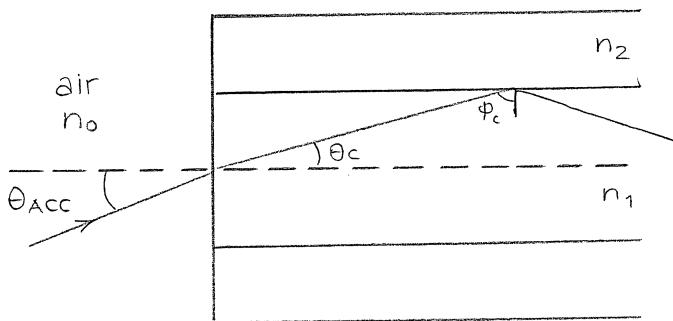
**FRACTIONAL POWER P_{clad}/P
OF GUIDED MODES (LP_{vm}) IN
A STEP-INDEX OPTICAL FIBER**

(from A. Yariv, "Optical Electronics in Modern Communications")



Acceptance Angle & Numerical Aperture of an Optical Fiber

Assume a step-index fiber.



$$\text{From Snell's Law : } n_0 \sin \theta_{ACC} = n_1 \sin \theta_c = n_1 \sin \left(\frac{\pi}{2} - \phi_c \right) =$$

$$= n_1 \cos \phi_c$$

$$n_0 = 1 ; \rightarrow \sin \theta_{ACC} = n_1 \sqrt{1 - \sin^2 \phi_c} = n_1 \sqrt{1 - (n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}$$

$$\text{Maximum acceptance angle : } \theta_{ACC} = \sin^{-1} \left(\sqrt{n_1^2 - n_2^2} \right)$$

$$\text{Numerical Aperture NA} = \sin \theta_{ACC} = \sqrt{n_1^2 - n_2^2}$$