

Basic Electromagnetics Review

Integrated Optics

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Maxwell Equations

Differential Form

$$\begin{aligned}\vec{\nabla} \times \vec{\mathcal{E}} &= -\frac{\partial \vec{\mathcal{B}}}{\partial t}, \\ \vec{\nabla} \times \vec{\mathcal{H}} &= \vec{J} + \frac{\partial \vec{\mathcal{D}}}{\partial t}, \\ \vec{\nabla} \cdot \vec{\mathcal{D}} &= \rho, \\ \vec{\nabla} \cdot \vec{\mathcal{B}} &= 0,\end{aligned}$$

Integral Form

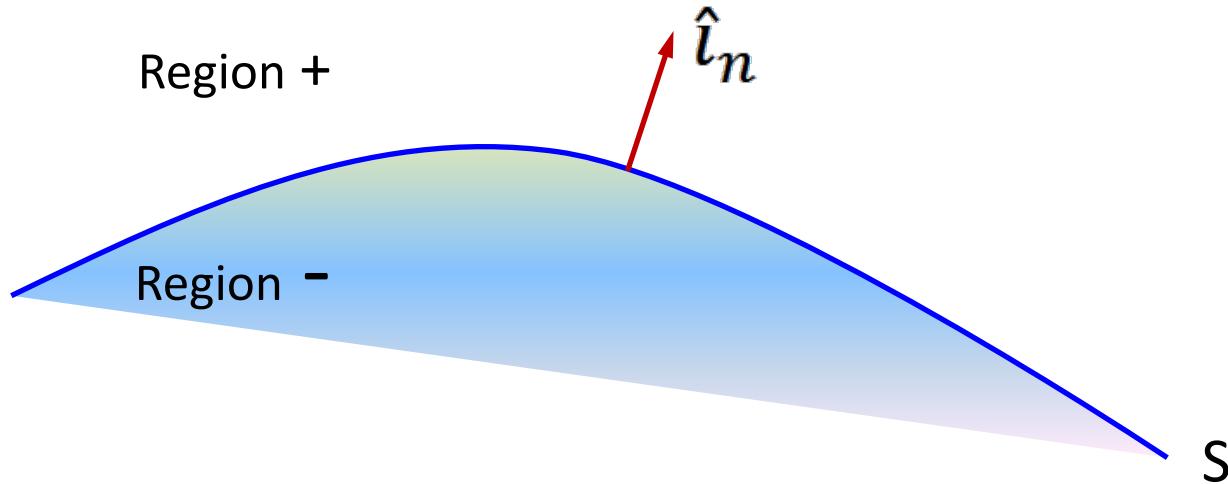
$$\oint_C \vec{\mathcal{E}} \cdot d\vec{\ell} = -\frac{d}{dt} \iint_S \vec{\mathcal{B}} \cdot d\vec{S}, \quad (\text{Faraday Law}),$$

$$\oint_C \vec{\mathcal{H}} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} + \frac{d}{dt} \iint_S \vec{\mathcal{D}} \cdot d\vec{S}, \quad (\text{Ampere Law}),$$

$$\iint_S \vec{\mathcal{D}} \cdot d\vec{S} = \iiint_V \rho dV, \quad (\text{Gauss Law}),$$

$$\iint_S \vec{\mathcal{B}} \cdot d\vec{S} = 0, \quad (\text{Absence of Magnetic Monopoles}),$$

Boundary Conditions



$$\begin{aligned}\hat{i}_n \times (\vec{\mathcal{E}}_+ - \vec{\mathcal{E}}_-)_S &= 0, \\ \hat{i}_n \times (\vec{\mathcal{H}}_+ - \vec{\mathcal{H}}_-)_S &= \vec{\mathcal{K}}, \\ \hat{i}_n \cdot (\vec{\mathcal{D}}_+ - \vec{\mathcal{D}}_-)_S &= \sigma, \\ \hat{i}_n \cdot (\vec{\mathcal{B}}_+ - \vec{\mathcal{B}}_-)_S &= 0,\end{aligned}$$

Maxwell Equations

Time Harmonic Form

$$\begin{aligned}
 \vec{\nabla} \times \vec{E}(\vec{r}, \omega) &= -j\omega \vec{B}(\vec{r}, \omega), \\
 \vec{\nabla} \times \vec{H}(\vec{r}, \omega) &= \vec{J}(\vec{r}, \omega) + j\omega \vec{D}(\vec{r}, \omega), \\
 \vec{\nabla} \cdot \vec{D}(\vec{r}, \omega) &= \rho(\vec{r}, \omega), \\
 \vec{\nabla} \cdot \vec{B}(\vec{r}, \omega) &= 0.
 \end{aligned}$$

Constitutive Relations

$$\vec{P}(\vec{r}, t) = \epsilon_0 \int_0^\infty G_e(\tau) \vec{\mathcal{E}}(\vec{r}, t - \tau) d\tau \iff \vec{P}(\vec{r}, \omega) = \epsilon_0 \chi_e(\omega) \vec{E}(\vec{r}, \omega),$$

$$\vec{M}(\vec{r}, t) = \int_0^\infty G_m(\tau) \vec{\mathcal{H}}(\vec{r}, t - \tau) d\tau \iff \vec{M}(\vec{r}, \omega) = \chi_m(\omega) \vec{H}(\vec{r}, \omega),$$

$$\vec{J}(\vec{r}, t) = \int_0^\infty G_c(\tau) \vec{\mathcal{E}}(\vec{r}, t - \tau) d\tau \iff \vec{J}(\vec{r}, \omega) = \sigma(\omega) \vec{E}(\vec{r}, \omega),$$

$$\vec{D}(\vec{r}, \omega) = \epsilon_0 \vec{E}(\vec{r}, \omega) + \vec{P}(\vec{r}, \omega) = \epsilon_0 [1 + \chi_e(\omega)] \vec{E}(\vec{r}, \omega) = \epsilon(\omega) \vec{E}(\vec{r}, \omega),$$

$$\vec{B}(\vec{r}, \omega) = \mu_0 [\vec{H}(\vec{r}, \omega) + \vec{M}(\vec{r}, \omega)] = \mu_0 [1 + \chi_m(\omega)] \vec{H}(\vec{r}, \omega) = \mu(\omega) \vec{H}(\vec{r}, \omega),$$

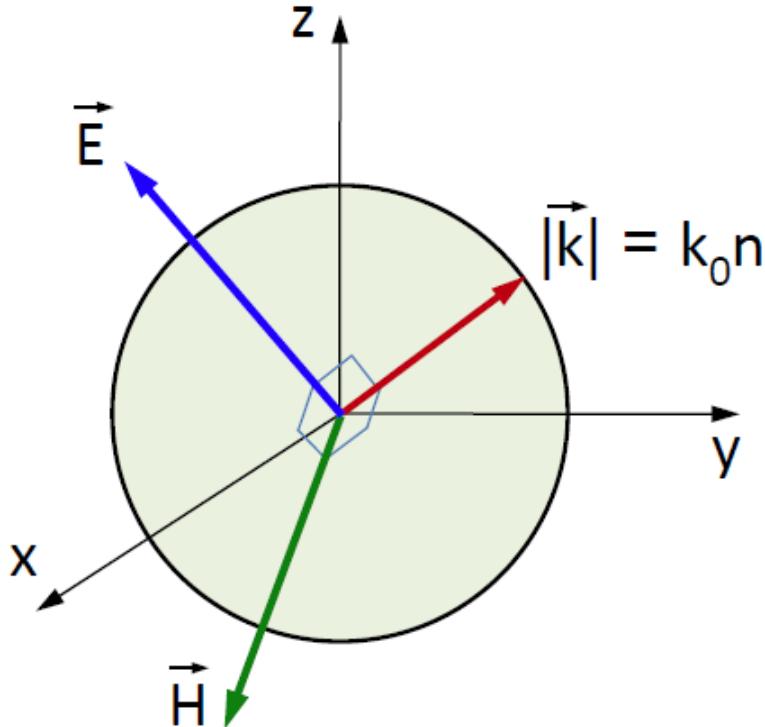
$$\vec{J}(\vec{r}, \omega) = \sigma(\omega) \vec{E}(\vec{r}, \omega),$$

Maxwell Equations

Plane Wave Solutions – Isotropic Case

$$\vec{E}(\vec{r}, \omega) = \vec{E}_0 \exp(-j\vec{k} \cdot \vec{r})$$

$$\vec{\mathcal{E}}(\vec{r}, t) = \text{Re}\{\vec{E}_0 \exp(-j\vec{k} \cdot \vec{r}) \exp(j\omega t)\}$$



$$\vec{k} \times \vec{E} = \omega \mu_0 \vec{H},$$

$$\vec{k} \times \vec{H} = -\omega \epsilon_0 \epsilon_r \vec{E},$$

$$\vec{k} \cdot \vec{E} = 0,$$

$$\vec{k} \cdot \vec{H} = 0.$$

$$[\vec{k} \cdot \vec{k} - \omega^2 \epsilon_0 \mu_0 n^2] \vec{E} = 0$$

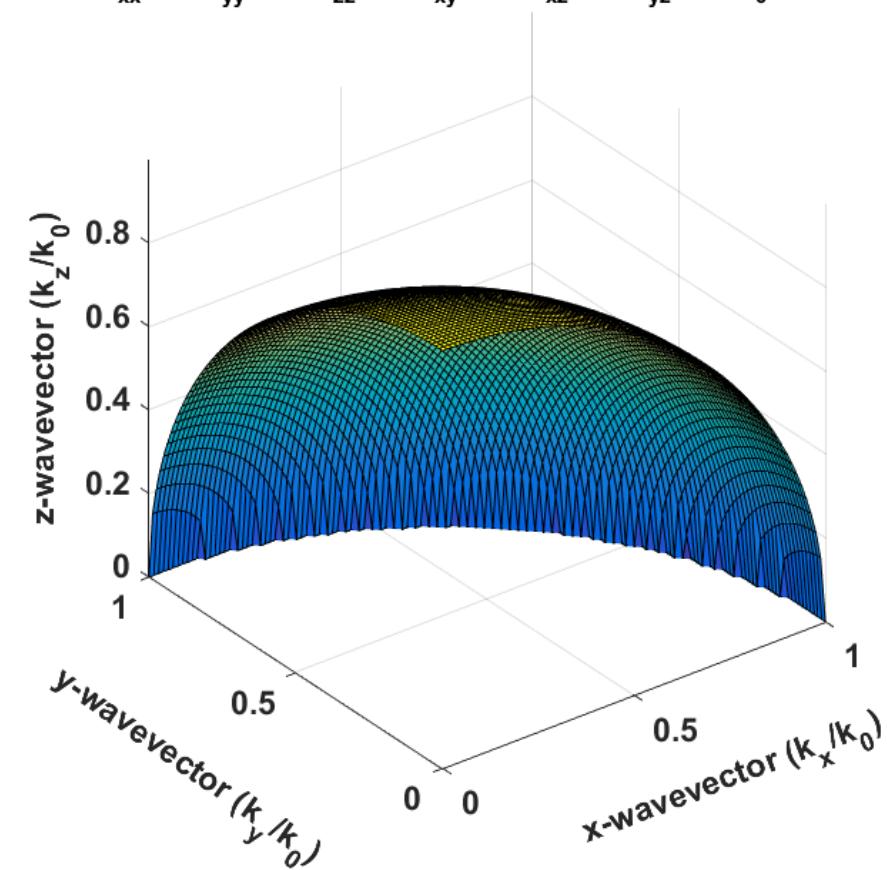
$$\vec{k} \cdot \vec{k} - \omega^2 \epsilon_0 \mu_0 n^2 = \vec{k} \cdot \vec{k} - k_0^2 n^2 = 0$$

Maxwell Equations

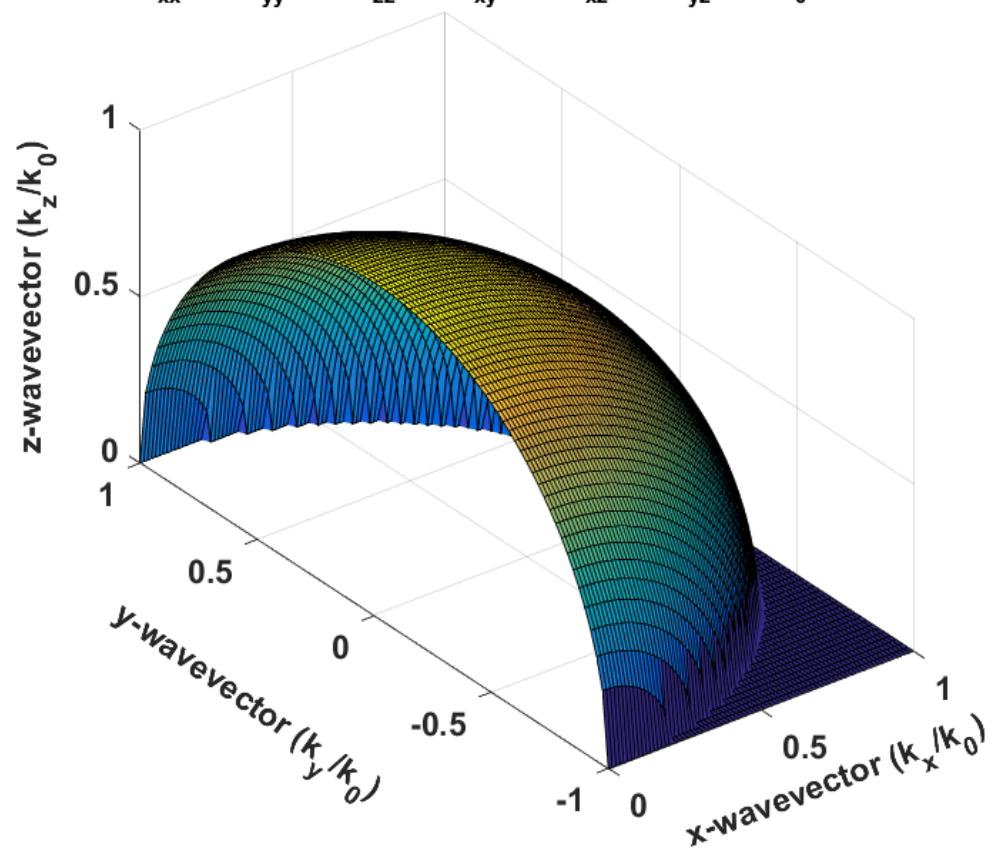
Plane Wave Solutions – Isotropic Case

Wavevector Surfaces (Isotropic)

$$\epsilon_{xx} = 1, \epsilon_{yy} = 1, \epsilon_{zz} = 1, \epsilon_{xy} = 0, \epsilon_{xz} = 0, \epsilon_{yz} = 0, \lambda_0 = 1 \mu\text{m}$$



$$\epsilon_{xx} = 1, \epsilon_{yy} = 1, \epsilon_{zz} = 1, \epsilon_{xy} = 0, \epsilon_{xz} = 0, \epsilon_{yz} = 0, \lambda_0 = 1 \mu\text{m}$$



Maxwell Equations

Plane Wave Solutions – Anisotropic Case

$$\begin{aligned}
 \vec{k} \times \vec{E} &= \omega \mu_0 \vec{H}, \\
 \vec{k} \times \vec{H} &= -\omega \epsilon_0 [\tilde{\epsilon}_r] \vec{E}, \\
 \vec{k} \cdot \vec{D} &= \epsilon_0 \vec{k} \cdot [\tilde{\epsilon}_r] \vec{E} = 0, \\
 \vec{k} \cdot \vec{H} &= 0.
 \end{aligned}$$

$$\vec{k}(\vec{k} \cdot \vec{E}) - (\vec{k} \cdot \vec{k})\vec{E} = -k_0^2[\tilde{\epsilon}_r]\vec{E}$$

$$\begin{bmatrix}
 k_0^2 \epsilon_{r,xx} - (k_y^2 + k_z^2) & k_x k_y + k_0^2 \epsilon_{r,xy} & k_x k_z + k_0^2 \epsilon_{r,xz} \\
 k_y k_x + k_0^2 \epsilon_{r,yx} & k_0^2 \epsilon_{r,yy} - (k_x^2 + k_z^2) & k_y k_z + k_0^2 \epsilon_{r,yz} \\
 k_z k_x + k_0^2 \epsilon_{r,zx} & k_z k_y + k_0^2 \epsilon_{r,zy} & k_0^2 \epsilon_{r,zz} - (k_x^2 + k_y^2)
 \end{bmatrix}
 \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \implies$$

$$\implies \left[\tilde{\mathcal{A}}(k_x, k_y, k_z) \right] \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0.$$

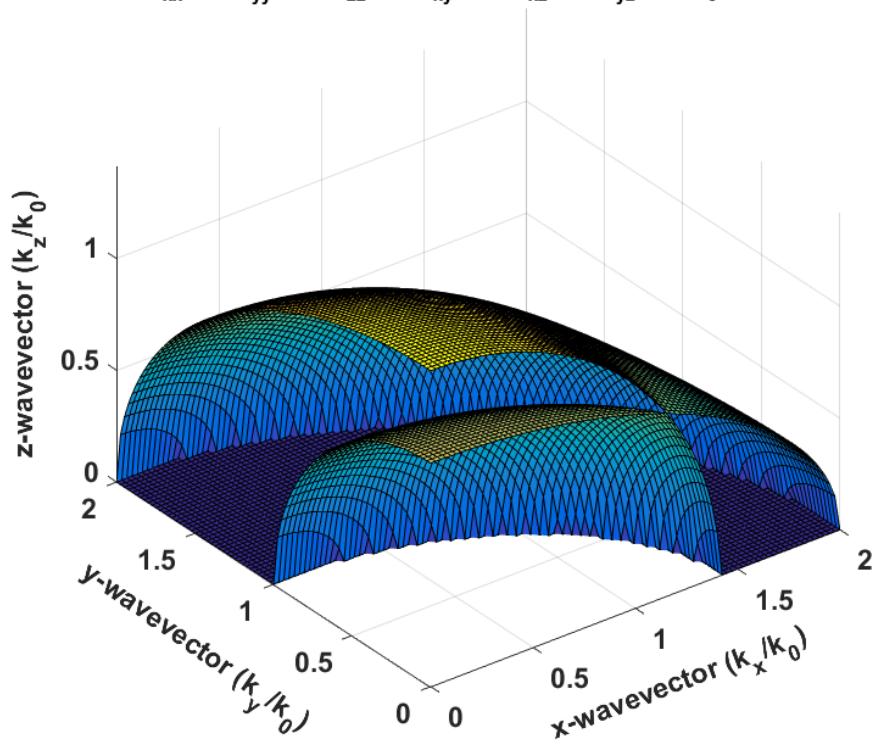
$$\det[\tilde{\mathcal{A}}(k_x, k_y, k_z)] = \det[k_0^2[\tilde{\epsilon}_r] - k^2 \tilde{I} + \vec{k}\vec{k}] = 0$$

Maxwell Equations

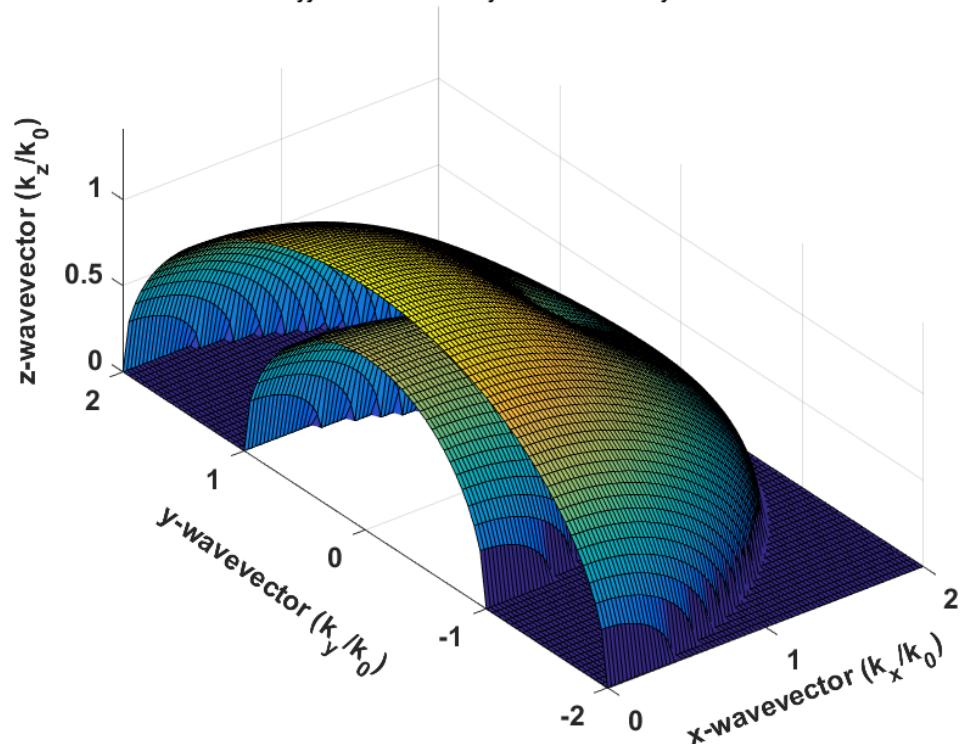
Plane Wave Solutions – Anisotropic Case

Wavevector Surfaces (Biaxial)

$$\epsilon_{xx} = 1, \epsilon_{yy} = 2, \epsilon_{zz} = 4, \epsilon_{xy} = 0, \epsilon_{xz} = 0, \epsilon_{yz} = 0, \lambda_0 = 1 \mu\text{m}$$



$$\epsilon_{xx} = 1, \epsilon_{yy} = 2, \epsilon_{zz} = 4, \epsilon_{xy} = 0, \epsilon_{xz} = 0, \epsilon_{yz} = 0, \lambda_0 = 1 \mu\text{m}$$



Maxwell Equations

Plane Wave Solutions – Anisotropic Case

Propagation In a Biaxial Material

$$\vec{k} = k_0 n(a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) = k_0 n(\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z})$$

$$\det \left[\tilde{\mathcal{A}}(k_x, k_y, k_z) \right] = \det \left[k_0^2 \tilde{\varepsilon}_r - k^2 \tilde{I} + \vec{k} \vec{k} \right] = 0$$

$$An^4 + Bn^2 + C = 0, \quad \text{where,}$$

$$\begin{aligned}
A &= a_x^4 \varepsilon_{r,xx} + a_y^4 \varepsilon_{r,yy} + a_z^4 \varepsilon_{r,zz} + a_x^2 a_y^2 \varepsilon_{r,xx} + a_x^2 a_z^2 \varepsilon_{r,xx} + a_x^2 a_y^2 \varepsilon_{r,yy} + a_y^2 a_z^2 \varepsilon_{r,yy} \\
&\quad a_x^2 a_z^2 \varepsilon_{r,zz} + a_y^2 a_z^2 \varepsilon_{r,zz} + 2a_x a_y^3 \varepsilon_{r,xy} + 2a_x^3 a_y \varepsilon_{r,xy} + 2a_x a_z^3 \varepsilon_{r,xz} + 2a_x^3 a_z \varepsilon_{r,xz} \\
&\quad + 2a_y a_z^3 \varepsilon_{r,yz} + 2a_y^3 a_z \varepsilon_{r,yz} + 2a_x a_y a_z^2 \varepsilon_{r,xy} + 2a_x a_y^2 a_z \varepsilon_{r,xz} + 2a_x^2 a_y a_z \varepsilon_{r,yz}, \\
B &= -a_x^2 \varepsilon_{r,xx} \varepsilon_{r,yy} + a_x^2 \varepsilon_{r,xz}^2 + a_y^2 \varepsilon_{r,xy}^2 + a_y^2 \varepsilon_{r,yz}^2 + a_z^2 \varepsilon_{r,xz}^2 + a_z^2 \varepsilon_{r,yz}^2 + a_x^2 \varepsilon_{r,xy}^2 - a_y^2 \varepsilon_{r,xx} \varepsilon_{r,yy} \\
&\quad - a_x^2 \varepsilon_{r,xx} \varepsilon_{r,zz} - a_z^2 \varepsilon_{r,xx} \varepsilon_{r,zz} - a_y^2 \varepsilon_{r,yy} \varepsilon_{r,zz} - a_z^2 \varepsilon_{r,yy} \varepsilon_{r,zz} + 2a_x a_y \varepsilon_{r,xz} \varepsilon_{r,yz} - 2a_x a_y \varepsilon_{r,xy} \varepsilon_{r,zz} \\
&\quad + 2a_x a_z \varepsilon_{r,xy} \varepsilon_{r,yz} - 2a_x a_z \varepsilon_{r,xz} \varepsilon_{r,yy} + 2a_y a_z \varepsilon_{r,xy} \varepsilon_{r,xz} - 2a_y a_z \varepsilon_{r,xx} \varepsilon_{r,yz}, \\
C &= \varepsilon_{r,xx} \varepsilon_{r,yy} \varepsilon_{r,zz} - \varepsilon_{r,zz} \varepsilon_{r,xy}^2 + 2\varepsilon_{r,xy} \varepsilon_{r,xz} \varepsilon_{r,yz} - \varepsilon_{r,yy} \varepsilon_{r,xz}^2 - \varepsilon_{r,xx} \varepsilon_{r,yz}^2.
\end{aligned}$$

Maxwell Equations

Plane Wave Solutions – Anisotropic Case

Propagation In a Biaxial Material

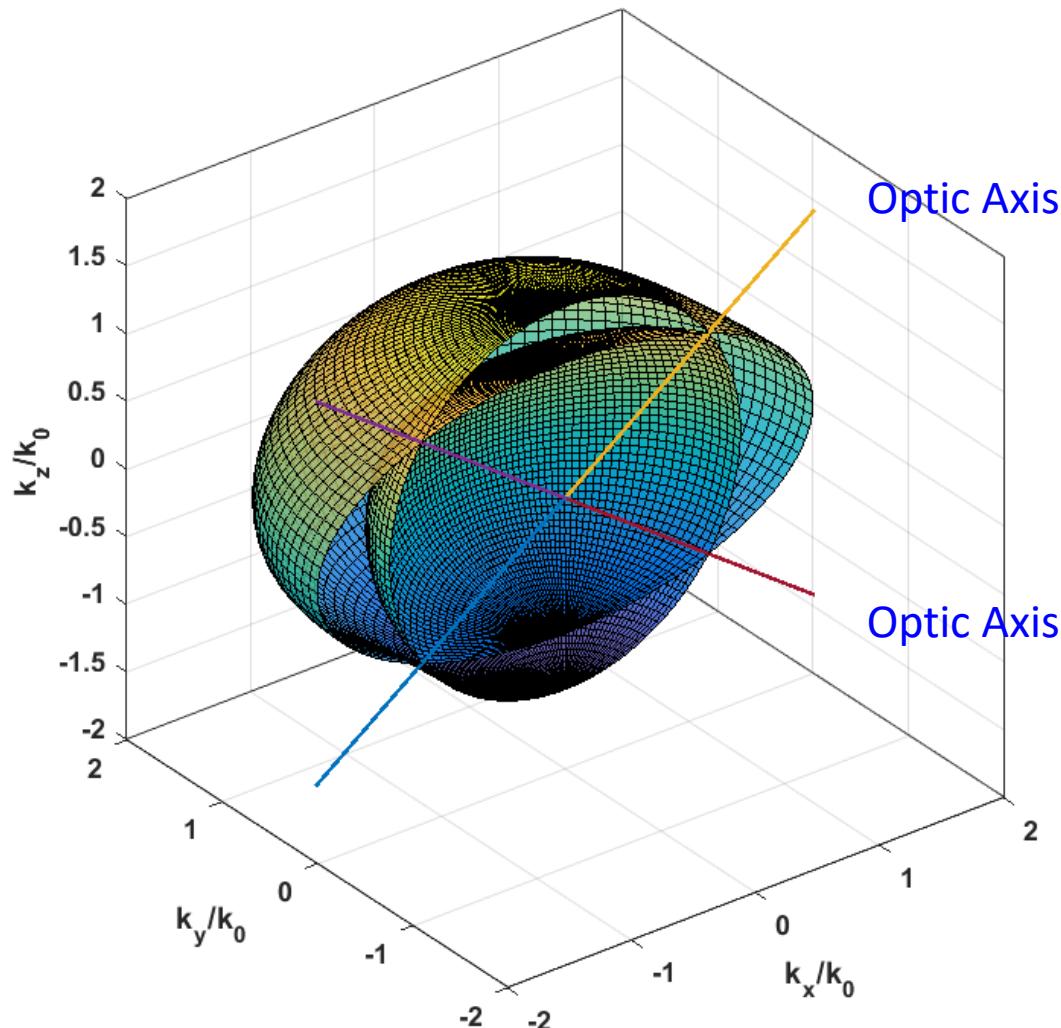
$$\begin{aligned} a_x^2 n^2 (n^2 - \varepsilon_{r,yy}) (n^2 - \varepsilon_{r,zz}) &+ a_y^2 n^2 (n^2 - \varepsilon_{r,xx}) (n^2 - \varepsilon_{r,zz}) + a_z^2 n^2 (n^2 - \varepsilon_{r,xx}) (n^2 - \varepsilon_{r,yy}) \\ &= (n^2 - \varepsilon_{r,xx}) (n^2 - \varepsilon_{r,yy}) (n^2 - \varepsilon_{r,zz}) \\ (\varepsilon_{r,xy} = \varepsilon_{r,xz} = \varepsilon_{r,yz} = 0) \end{aligned}$$

$$\boxed{\frac{a_x^2}{n^2 - \varepsilon_{r,xx}} + \frac{a_y^2}{n^2 - \varepsilon_{r,yy}} + \frac{a_z^2}{n^2 - \varepsilon_{r,zz}} = \frac{1}{n^2}}$$

Maxwell Equations

Plane Wave Solutions – Anisotropic Case

Wavevector Surfaces (Biaxial)

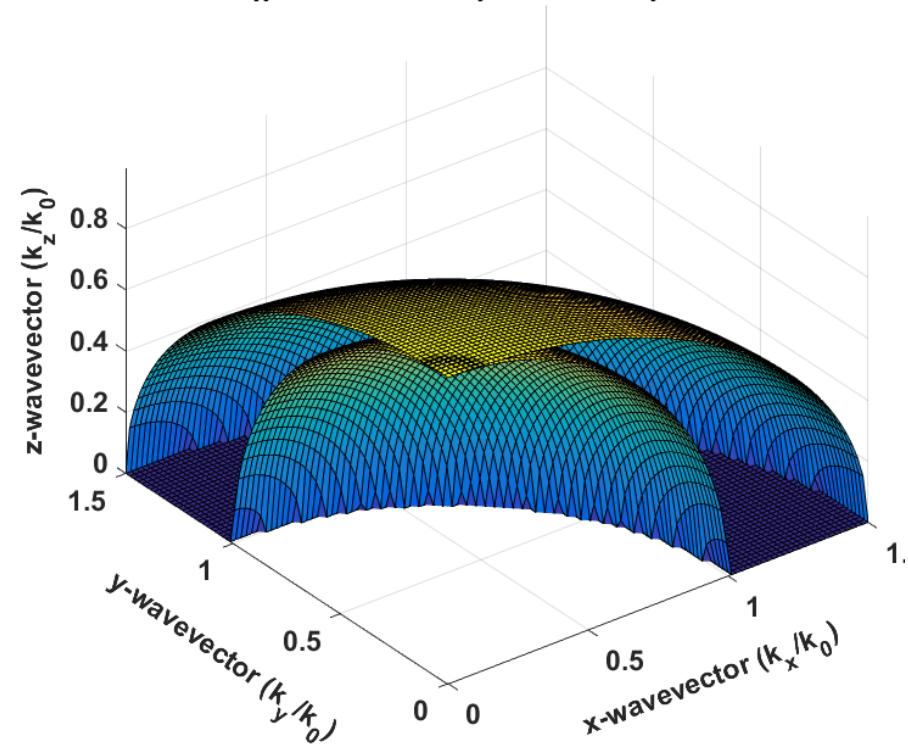


Maxwell Equations

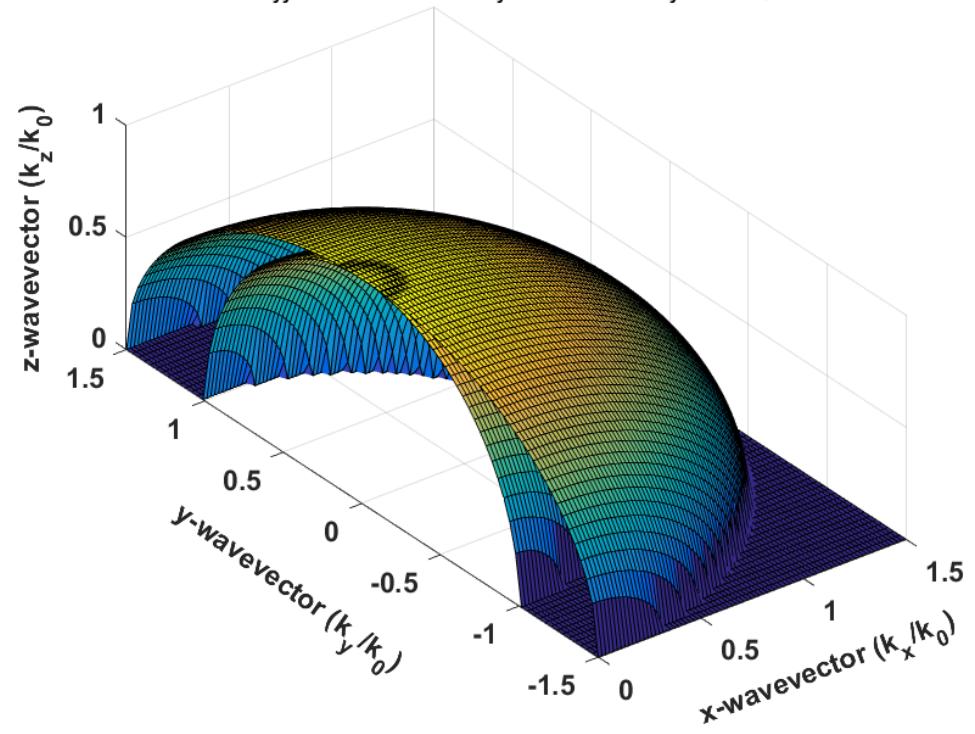
Plane Wave Solutions – Uniaxial Anisotropic Case

Wavevector Surfaces (Uniaxial)

$$\epsilon_{xx} = 1, \epsilon_{yy} = 1, \epsilon_{zz} = 2.25, \epsilon_{xy} = 0, \epsilon_{xz} = 0, \epsilon_{yz} = 0, \lambda_0 = 1 \mu\text{m}$$



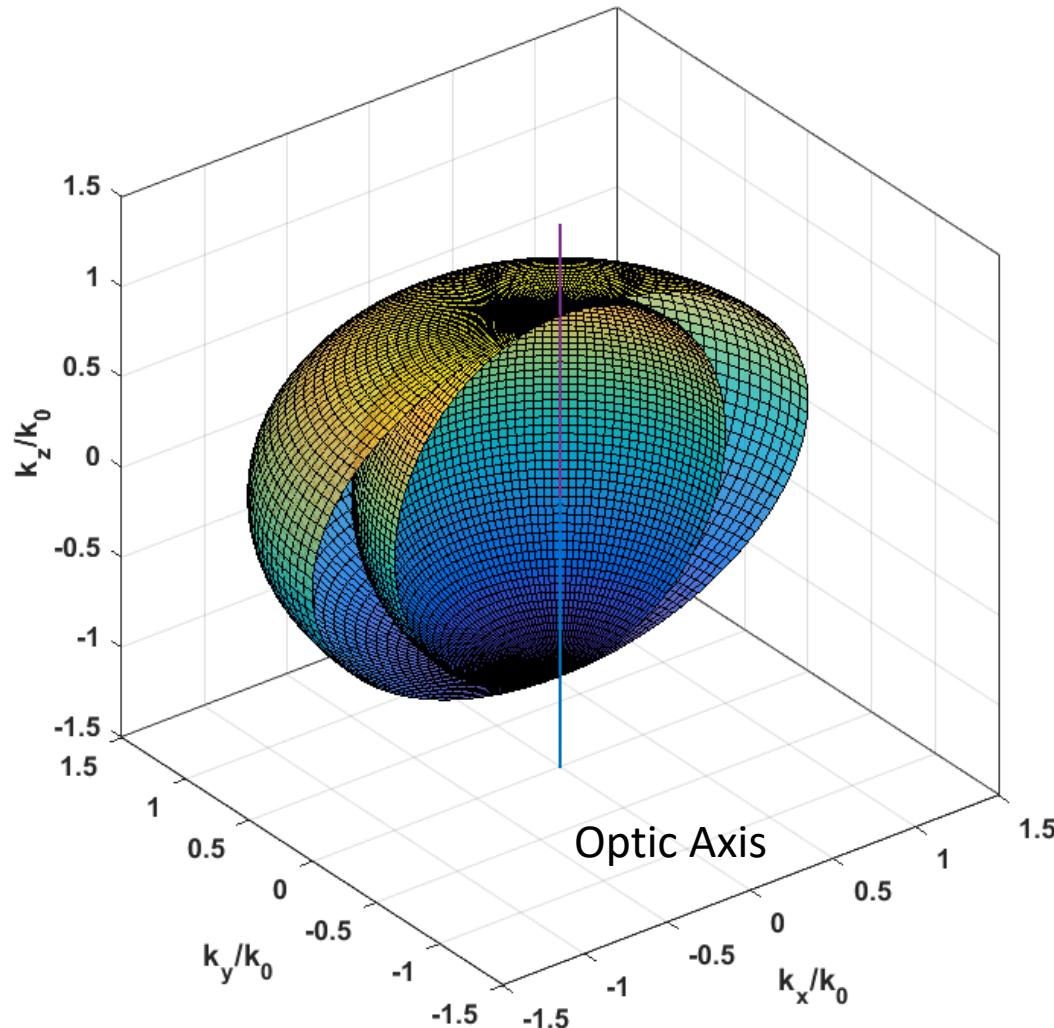
$$\epsilon_{xx} = 1, \epsilon_{yy} = 1, \epsilon_{zz} = 2.25, \epsilon_{xy} = 0, \epsilon_{xz} = 0, \epsilon_{yz} = 0, \lambda_0 = 1 \mu\text{m}$$



Maxwell Equations

Plane Wave Solutions – Uniaxial Anisotropic Case

Wavevector Surfaces (Uniaxial)



Maxwell Equations

Plane Wave Solutions – Uniaxial Anisotropic Case

Wavevector Surfaces (Uniaxial)

For principal axis system:

$$\tilde{\epsilon} = \epsilon_0 \text{diag}[\epsilon_O, \epsilon_O, \epsilon_E] = \epsilon_0 \text{diag}[n_O^2, n_O^2, n_E^2]$$

For optic axis direction:

$$\hat{c} = c_x \hat{x} + c_y \hat{y} + c_z \hat{z}$$

Plane wave solutions:

$$\begin{aligned}
 (\vec{k}_t \times \hat{c}) E_c + k_c (\hat{c} \times \vec{E}_t) &= \omega \mu_0 \vec{H}_t, \\
 \vec{k}_t \times \vec{E}_t &= \omega \mu_0 H_c \hat{c}, \\
 (\vec{k}_t \times \hat{c}) H_c + k_c (\hat{c} \times \vec{H}_t) &= -\omega \epsilon_0 n_O^2 \vec{E}_t, \\
 \vec{k}_t \times \vec{H}_t &= -\omega \epsilon_0 n_E^2 E_c \hat{c}, \\
 n_O^2 \vec{k}_t \cdot \vec{E}_t + n_E^2 k_c E_c &= 0, \\
 \vec{k}_t \cdot \vec{H}_t + k_c H_c &= 0.
 \end{aligned}$$

Maxwell Equations

Plane Wave Solutions – Uniaxial Anisotropic Case

Wavevector Surfaces (Uniaxial)

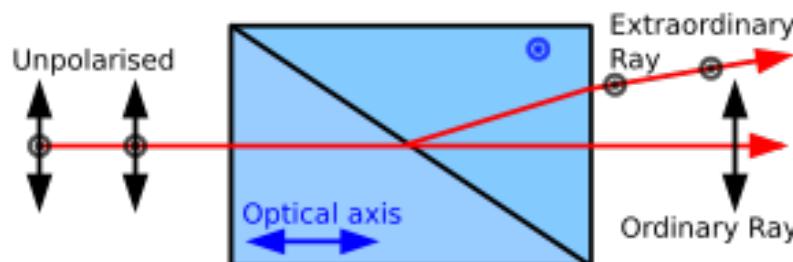
Dispersion Equation:

$$[\vec{k} \cdot \vec{k} - k_0^2 n_O^2][n_O^2 \vec{k} \cdot \vec{k} + (n_E^2 - n_O^2)(\vec{k} \cdot \hat{c})^2 - k_0^2 n_O^2 n_E^2] = 0$$

$$\vec{k} \cdot \vec{k} - k_0^2 n_O^2 = 0 \implies \vec{E} \cdot \hat{c} = 0, \quad \text{ordinary wave,}$$

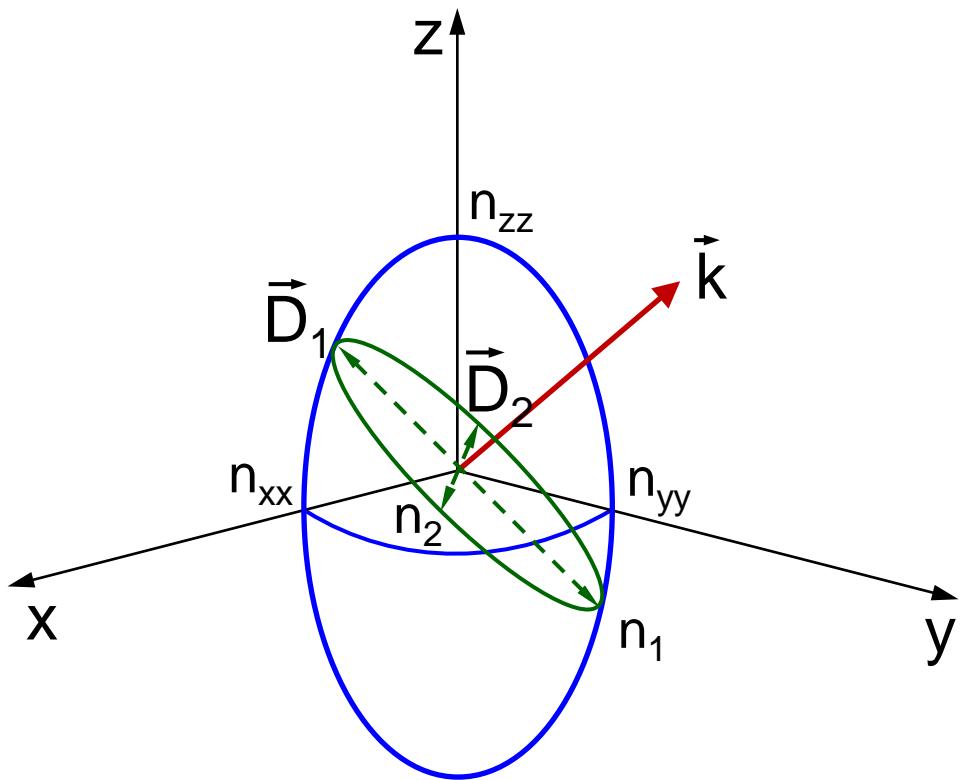
$$n_O^2 \vec{k} \cdot \vec{k} + (n_E^2 - n_O^2)(\vec{k} \cdot \hat{c})^2 - k_0^2 n_O^2 n_E^2 = 0, \implies \vec{H} \cdot \hat{c} = 0 \quad \text{extraordinary wave.}$$

Rochon Prism
for $n_E - n_O > 0$



https://en.wikipedia.org/wiki/Rochon_prism

Index Ellipsoid

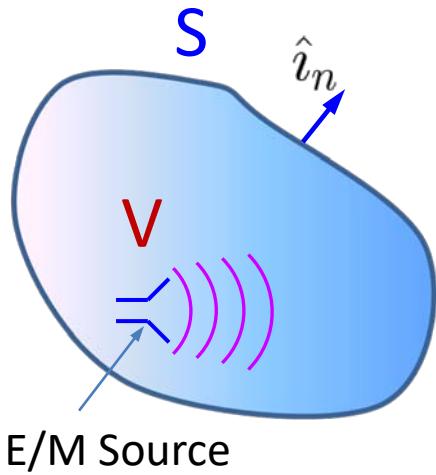


$$\frac{x^2}{n_{xx}^2} + \frac{y^2}{n_{yy}^2} + \frac{z^2}{n_{zz}^2} = 1$$

$$\begin{bmatrix} x & y & z \end{bmatrix}^T [\mathcal{A}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}^T \begin{bmatrix} \frac{1}{n_{xx}^2} & \frac{1}{n_{xy}^2} & \frac{1}{n_{xz}^2} \\ \frac{1}{n_{yx}^2} & \frac{1}{n_{yy}^2} & \frac{1}{n_{yz}^2} \\ \frac{1}{n_{zx}^2} & \frac{1}{n_{zy}^2} & \frac{1}{n_{zz}^2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$$

Poynting's Theorem

$$-\oint_S (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) \cdot d\vec{S} = \iiint_V \vec{\mathcal{E}} \cdot \vec{\mathcal{J}} dV + \iiint_V \frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} \vec{\mathcal{E}} \cdot \vec{\mathcal{E}} + \frac{\mu_0}{2} \vec{\mathcal{H}} \cdot \vec{\mathcal{H}} \right) dV + \\ \iiint_V \vec{\mathcal{E}} \cdot \frac{\partial \vec{\mathcal{P}}}{\partial t} dV + \iiint_V \mu_0 \vec{\mathcal{H}} \cdot \frac{\partial \vec{\mathcal{M}}}{\partial t} dV,$$



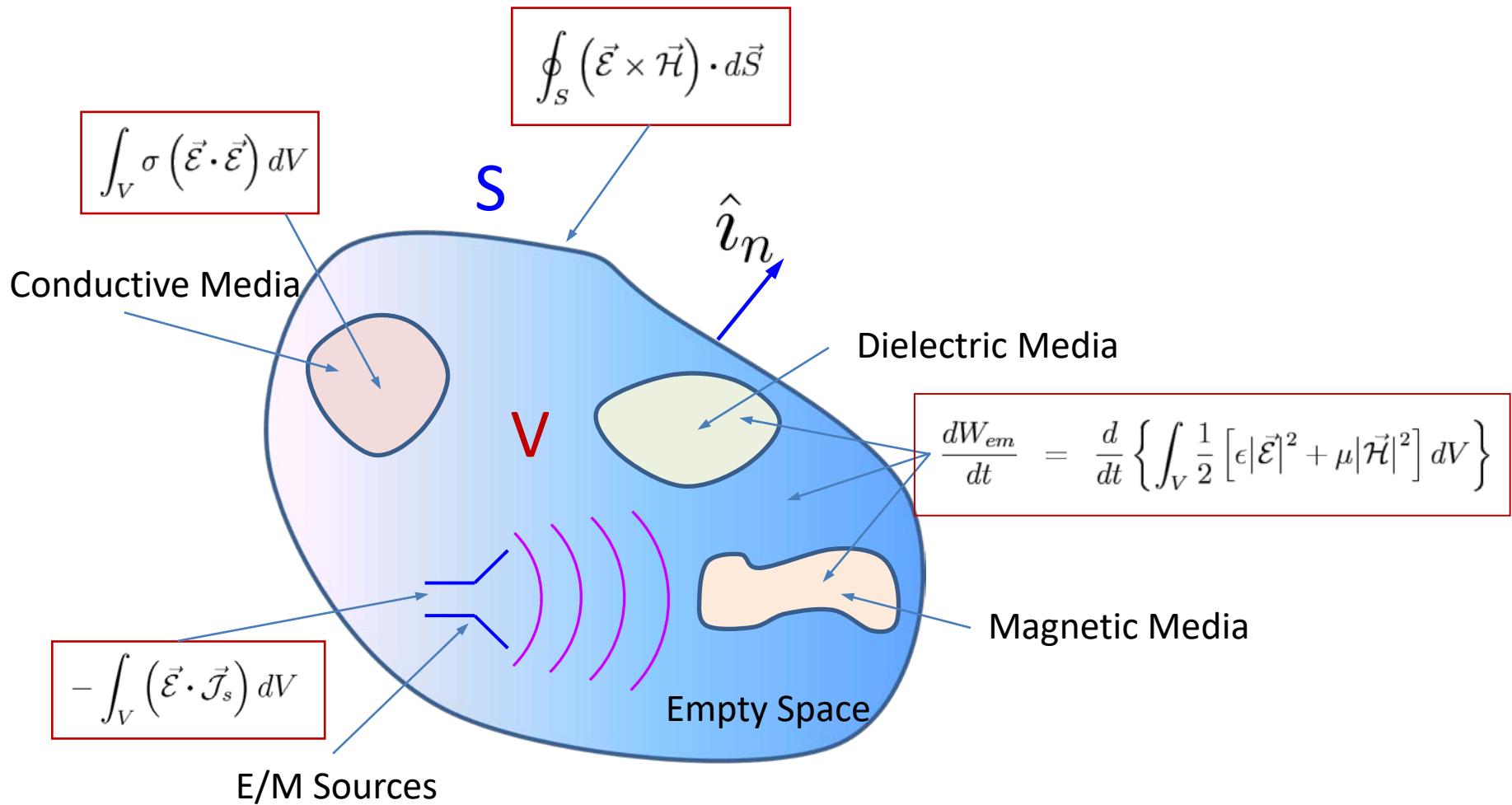
For Dispersionless Isotropic Media

E/M Source

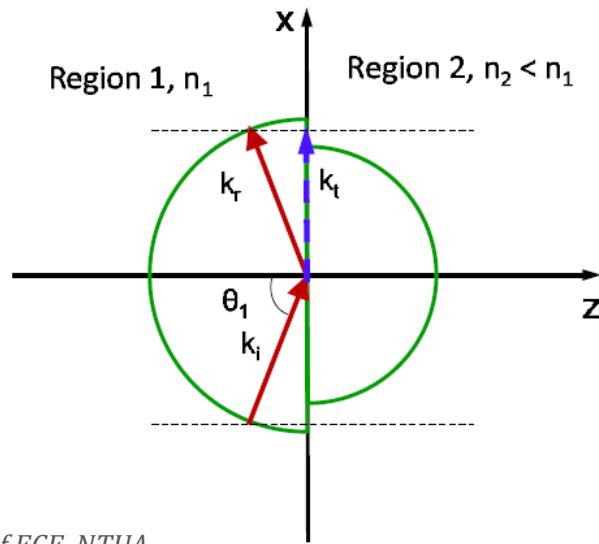
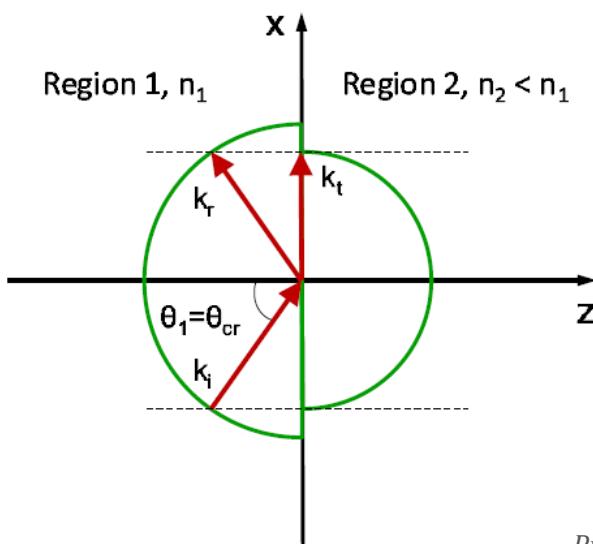
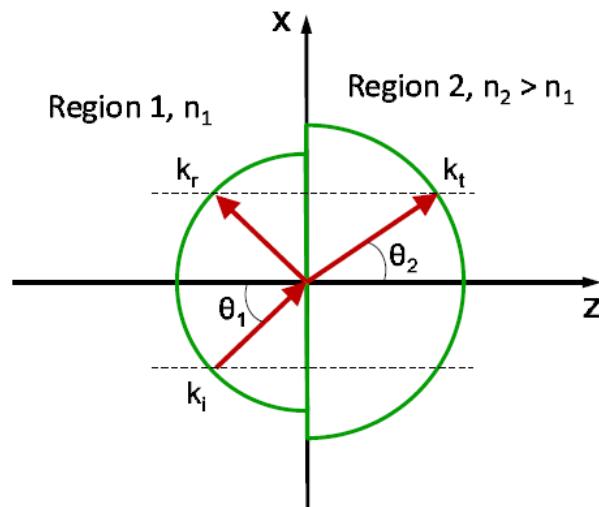
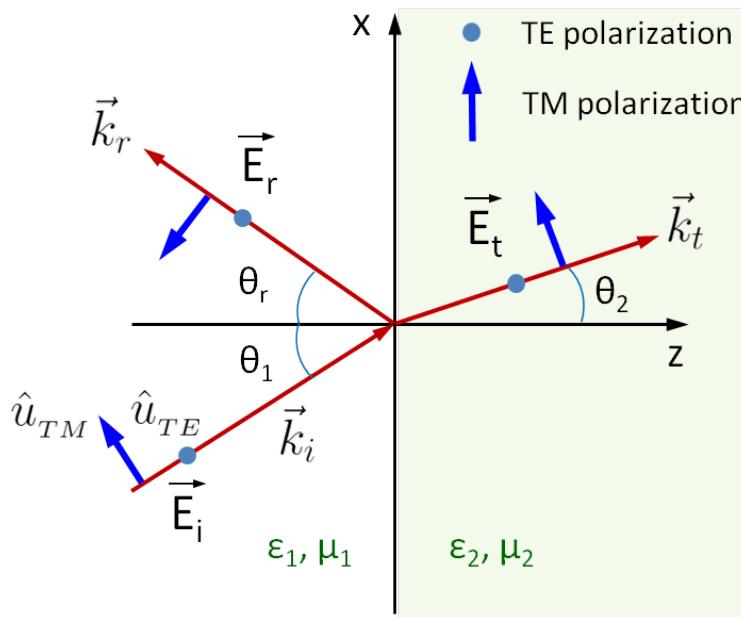
$$w_{em} = w_e + w_m = \frac{1}{2} (\vec{\mathcal{E}} \cdot \vec{\mathcal{D}} + \vec{\mathcal{H}} \cdot \vec{\mathcal{B}}) = \frac{1}{2} (\epsilon |\vec{\mathcal{E}}|^2 + \mu |\vec{\mathcal{H}}|^2)$$

$$-\oint_S (\vec{\mathcal{E}} \times \vec{\mathcal{H}}) \cdot d\vec{S} = \iiint_V \left[\vec{\mathcal{E}} \cdot \vec{\mathcal{J}} + \frac{\partial w_{em}}{\partial t} \right] dV$$

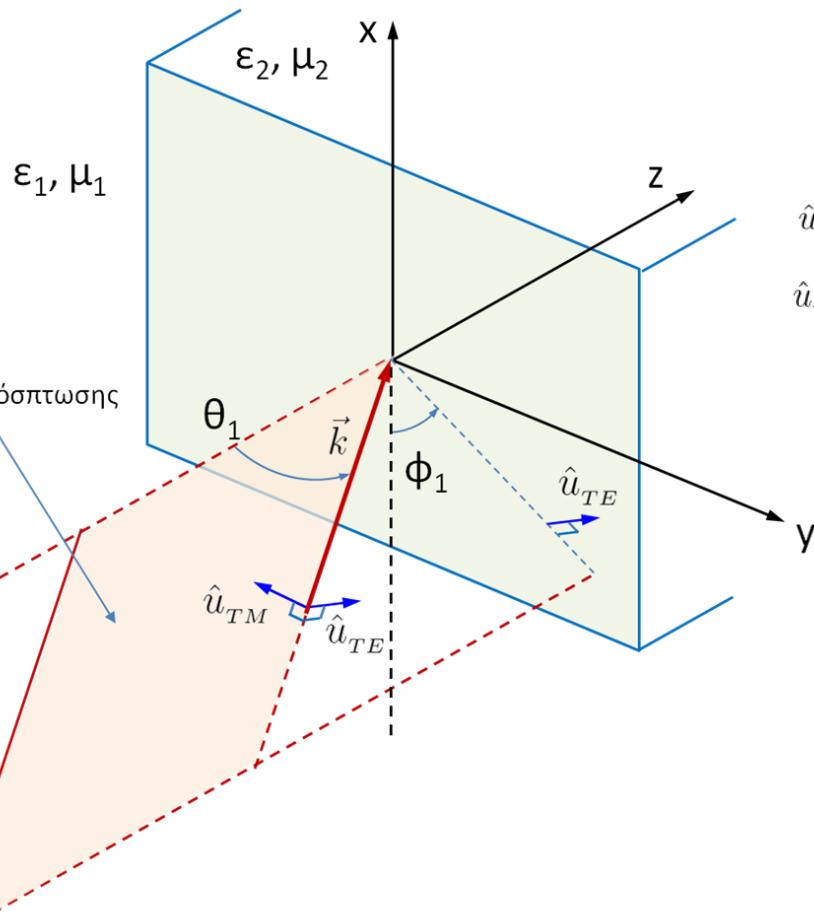
Poynting's Theorem



Planar Interface - Wavevector Diagrams



Plane Wave Reflection and Transmission

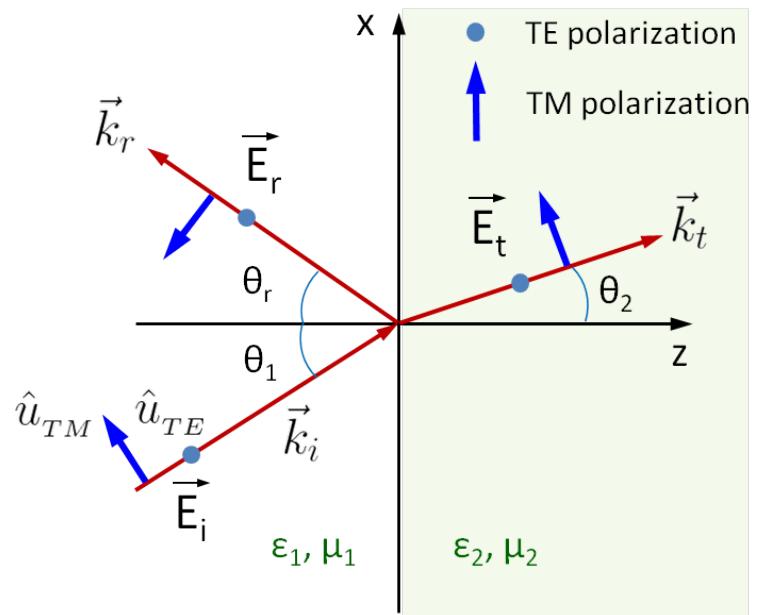


$$\vec{E} = \vec{E}_0 \exp(-j\vec{k} \cdot \vec{r}) = [E_{TE} \hat{u}_{TE} + E_{TM} \hat{u}_{TM}] \exp(-j\vec{k} \cdot \vec{r})$$

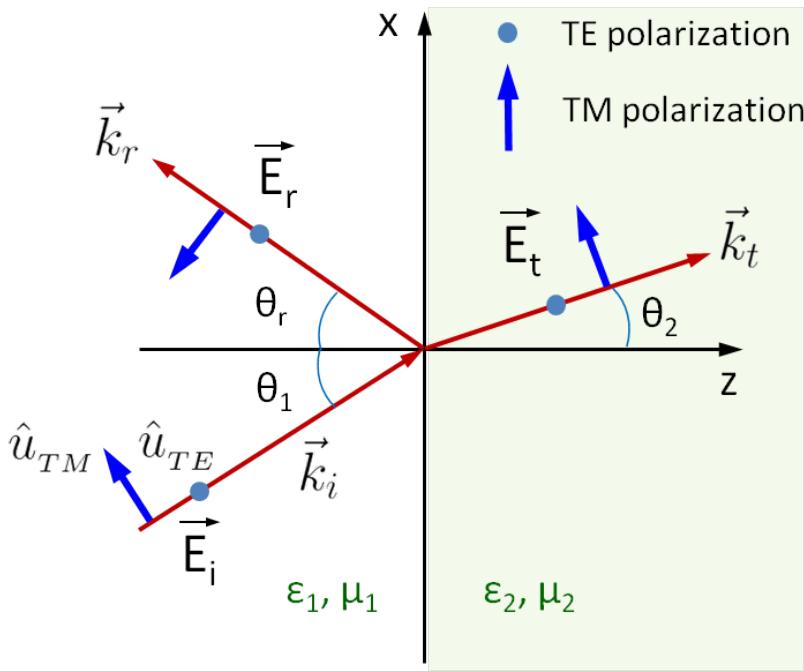
$$\hat{u}_{TE} = \sin \phi_1 \hat{i}_x + \cos \phi_1 \hat{i}_y$$

$$\hat{u}_{TM} = \cos \phi_1 \cos \theta_1 \hat{i}_x - \sin \phi_1 \cos \theta_1 \hat{i}_y - \sin \theta_1 \hat{i}_z$$

$$\begin{aligned}\vec{E} &= \vec{E}_0 \exp(-j\vec{k} \cdot \vec{r}) = [E_{TE} \hat{u}_{TE} + E_{TM} \hat{u}_{TM}] \exp(-j\vec{k} \cdot \vec{r}) \\ \hat{u}_{TE} &= \hat{i}_y \\ \hat{u}_{TM} &= \cos \theta_1 \hat{i}_x - \sin \theta_1 \hat{i}_z\end{aligned}$$



Fresnel Equations

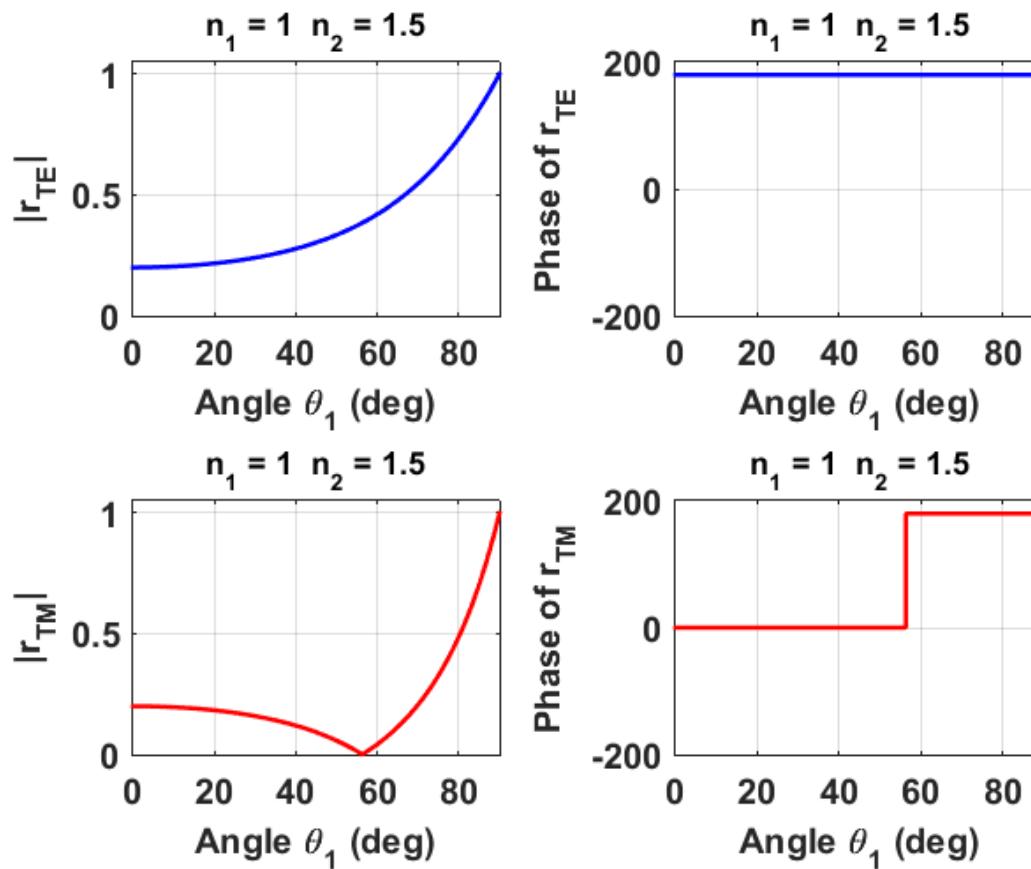


$$\begin{aligned}
 r_{TE} = r_{\perp} &= \frac{E_r}{E_i} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2}, \\
 t_{TE} = t_{\perp} &= \frac{E_t}{E_i} = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2}, \\
 r_{TM} = r_{\parallel} &= \frac{E_r}{E_i} = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2}, \\
 t_{TM} = t_{\parallel} &= \frac{E_t}{E_i} = \frac{2n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_2}.
 \end{aligned}$$

$$\begin{aligned}
 \frac{P_r}{P_i} + \frac{P_t}{P_i} &= |r_{TE}|^2 + |t_{TE}|^2 \frac{\text{Re}\{n_2 \cos \theta_2\}}{n_1 \cos \theta_1} = 1, && \text{TE Polarization} \\
 \frac{P_r}{P_i} + \frac{P_t}{P_i} &= |r_{TM}|^2 + |t_{TM}|^2 \frac{\text{Re}\{n_2^* \cos \theta_2\}}{n_1 \cos \theta_1} = 1, && \text{TM Polarization}
 \end{aligned}$$

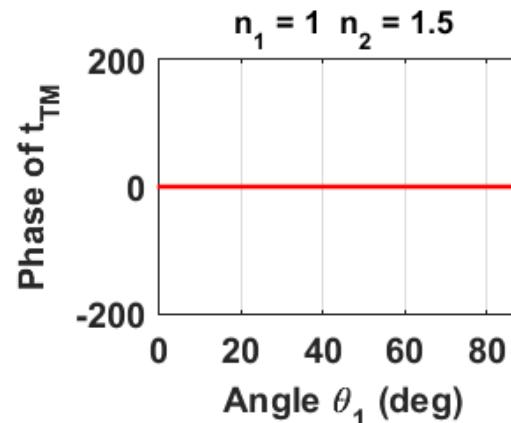
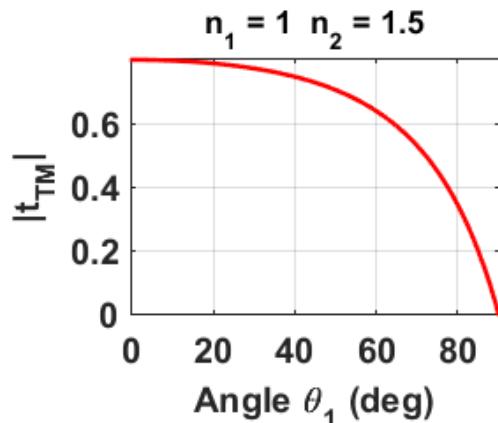
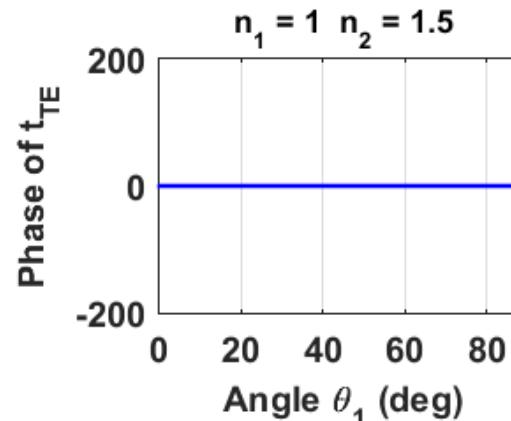
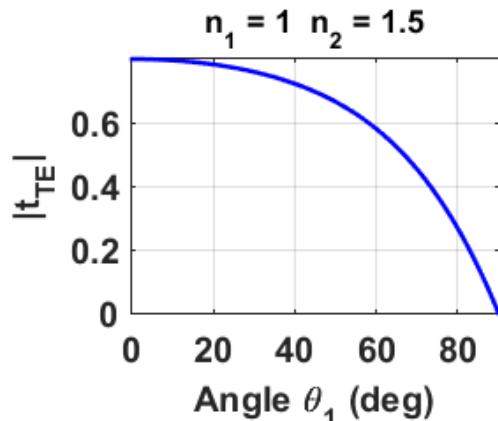
Example of Fresnel Equations

$$n_1 = 1, n_2 = 1.5$$



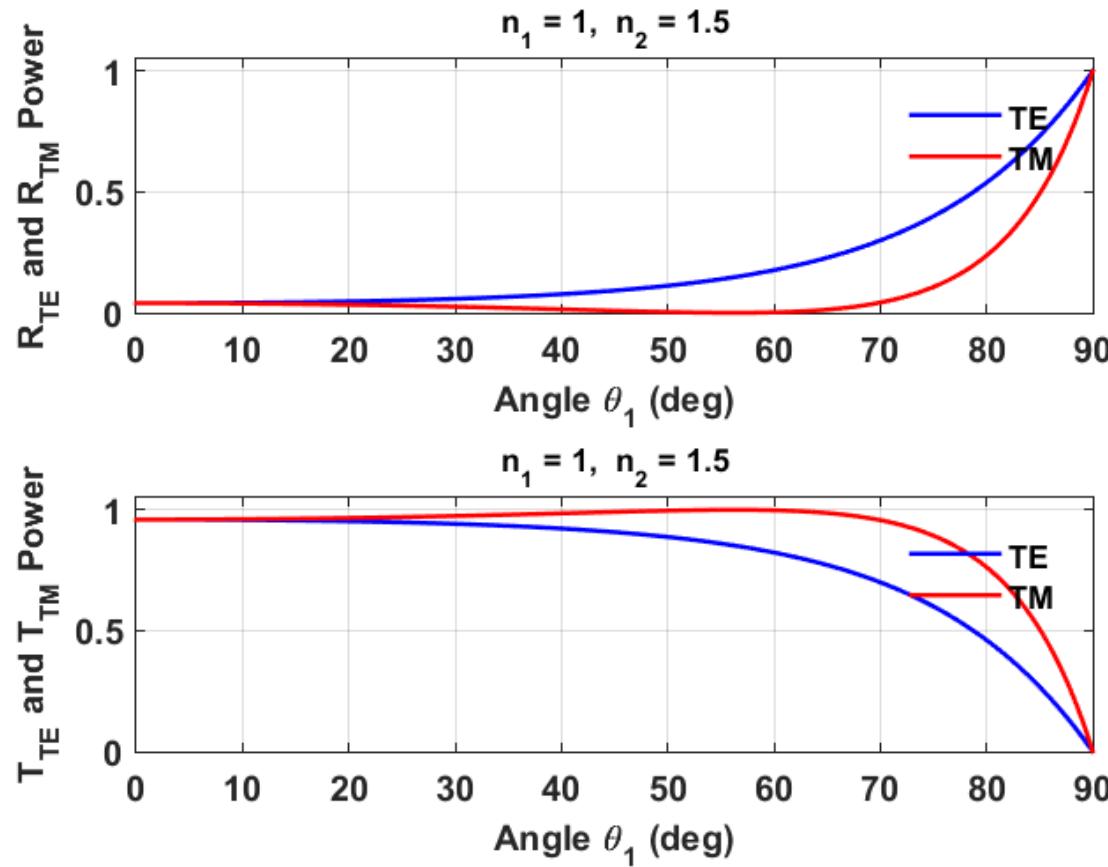
Example of Fresnel Equations

$$n_1 = 1, n_2 = 1.5$$



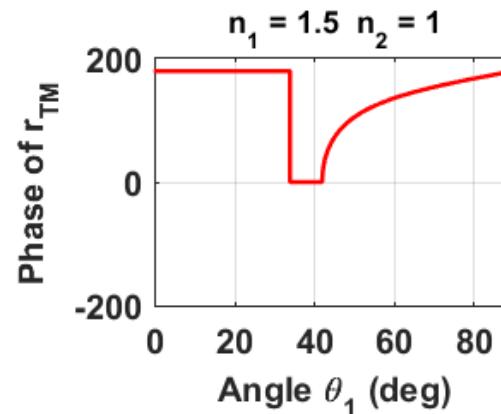
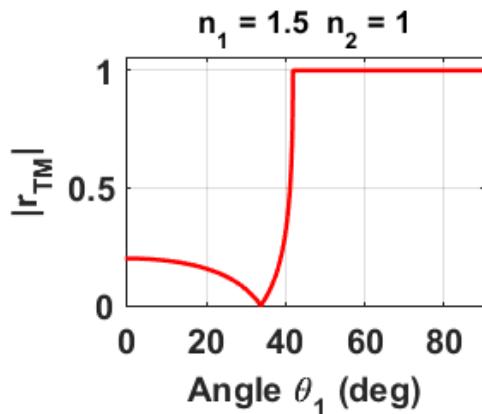
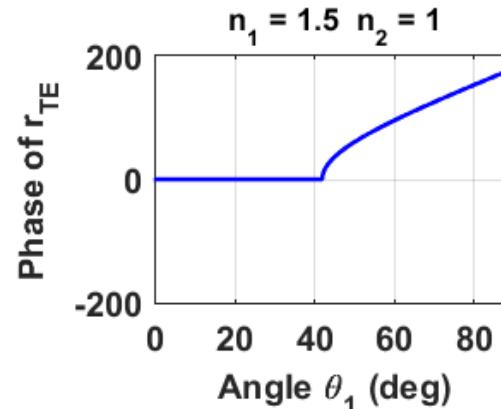
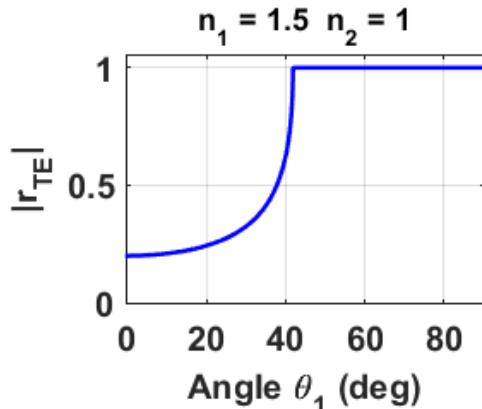
Example of Fresnel Equations

$$n_1 = 1, n_2 = 1.5$$



Example of Fresnel Equations

$$n_1 = 1.5, n_2 = 1$$

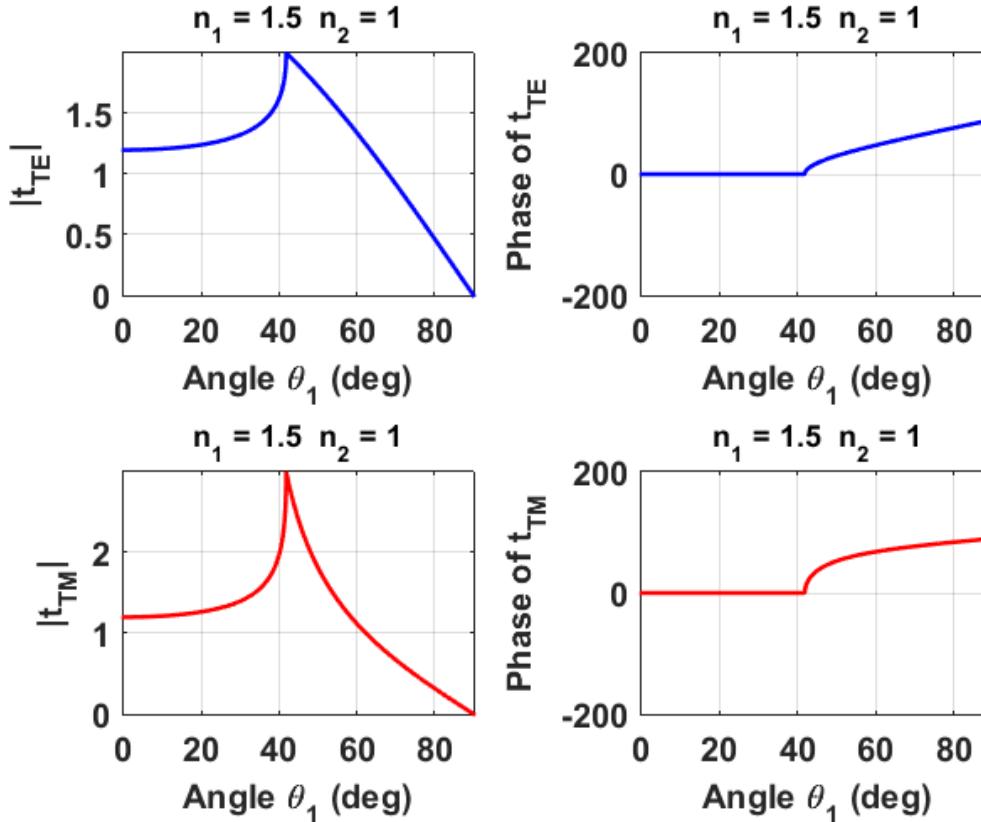


$$r_{TE} = \frac{E_r}{E_i} = 1e^{j2\phi_{TE}(\theta_1)} = 1 \exp \left[j2 \tan^{-1} \left\{ \frac{\sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}}{n_1 \cos \theta_1} \right\} \right],$$

$$r_{TM} = \frac{E_r}{E_i} = 1e^{j2\phi_{TM}(\theta_1)} = 1 \exp \left[j2 \tan^{-1} \left\{ \frac{n_1^2}{n_2^2} \frac{\sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}}{n_1 \cos \theta_1} \right\} \right]$$

Example of Fresnel Equations

$$n_1 = 1.5, n_2 = 1$$

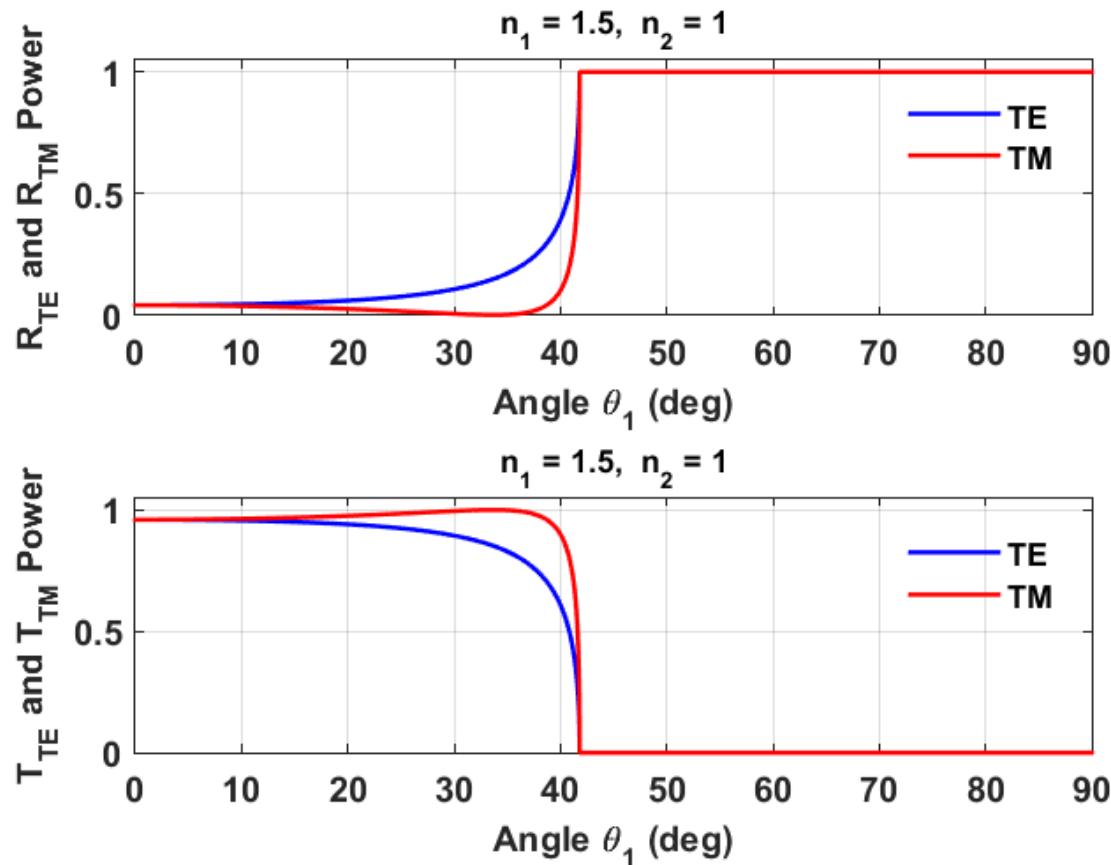


$$t_{TE} = \frac{E_t}{E_i} = |t_{TE}| e^{j\phi_{TE}(\theta_1)} = \frac{2n_1 \cos \theta_1}{\sqrt{n_1^2 - n_2^2}} \exp \left[j \tan^{-1} \left\{ \frac{\sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}}{n_1 \cos \theta_1} \right\} \right],$$

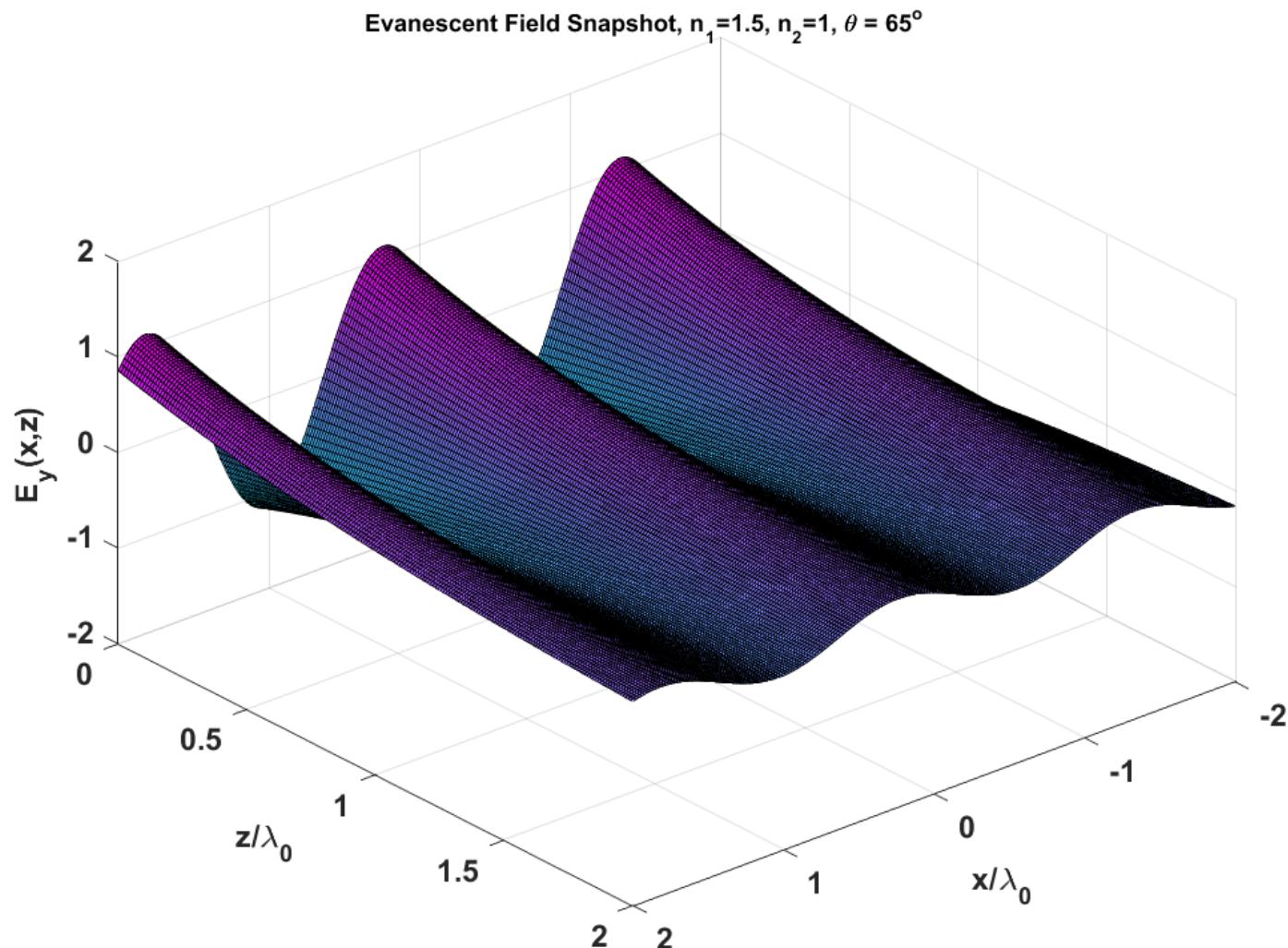
$$t_{TM} = \frac{E_t}{E_i} = |t_{TM}| e^{j\phi_{TM}(\theta_1)} = \frac{2n_1 n_2 \cos \theta_1}{\sqrt{n_2^4 \cos^2 \theta_1 + n_1^4 \sin^2 \theta_1 - n_1^2 n_2^2}} \exp \left[j \tan^{-1} \left\{ \frac{n_1^2 \sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}}{n_2^2 n_1 \cos \theta_1} \right\} \right]$$

Example of Fresnel Equations

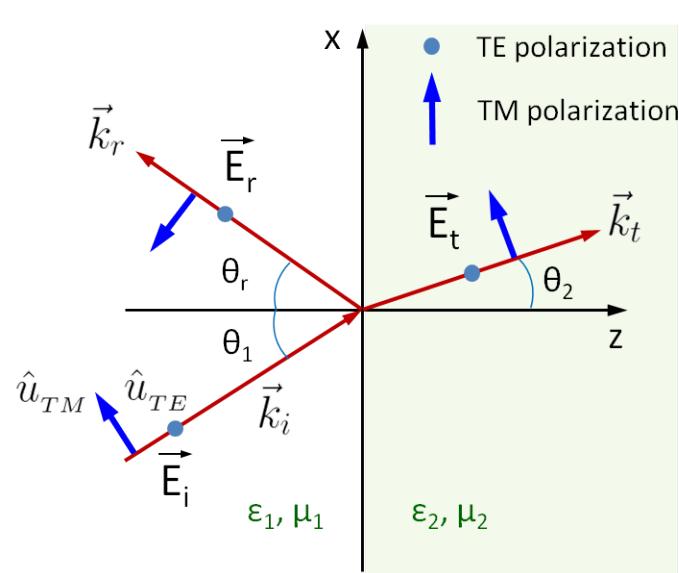
$$n_1 = 1.5, n_2 = 1$$



Example of Evanescent Field



Fresnel Equations Generalization



$$\vec{E}_i = E_{TE} \hat{u}_{TE} \exp(-j\vec{k}_i \cdot \vec{r}) + E_{TM} \hat{u}_{TM} \exp(-j\vec{k}_i \cdot \vec{r}), \quad \text{where}$$

$$\hat{u}_{TE} = \hat{y},$$

$$\hat{u}_{TM} = \cos \theta_1 \hat{x} - \sin \theta_1 \hat{z},$$

$$\vec{k}_i = k_0 \sqrt{\epsilon_r \mu_r} (\sin \theta_1 \hat{x} + \cos \theta_1 \hat{z}),$$

$$\vec{E}_r = E_{TE} r_{TE} \hat{u}_{TE}^r \exp(-j\vec{k}_r \cdot \vec{r}) + E_{TM} r_{TM} \hat{u}_{TM}^r \exp(-j\vec{k}_r \cdot \vec{r}), \quad \text{where}$$

$$\hat{u}_{TE}^r = \hat{u}_{TM}^r = \hat{y},$$

$$\hat{u}_{TM}^r = -\cos \theta_1 \hat{x} - \sin \theta_1 \hat{z},$$

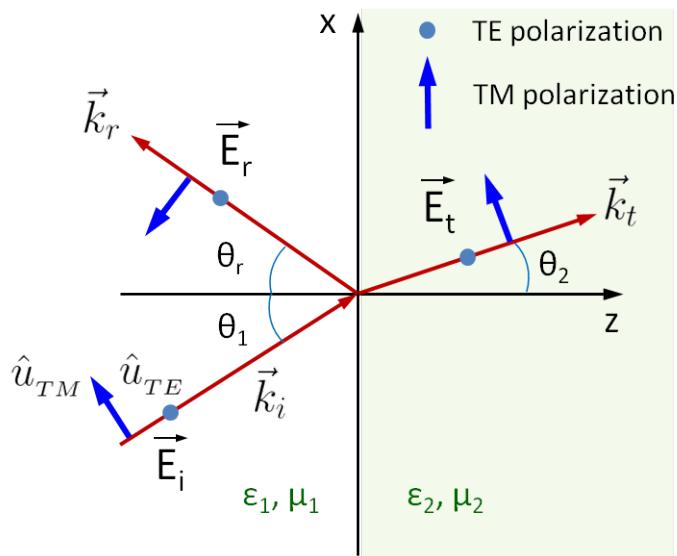
$$\vec{k}_r = k_0 \sqrt{\epsilon_r \mu_r} (\sin \theta_1 \hat{x} - \cos \theta_1 \hat{z}),$$

$$\vec{E}_1 = E_{TE} \hat{u}_{TE} \exp(-j\vec{k}_i \cdot \vec{r}) + E_{TM} \hat{u}_{TM} \exp(-j\vec{k}_i \cdot \vec{r}) +$$

$$E_{TE} r_{TE} \hat{u}_{TE} \exp(-j\vec{k}_r \cdot \vec{r}) + E_{TM} r_{TM} \hat{u}_{TM}^r \exp(-j\vec{k}_r \cdot \vec{r})$$

$$\vec{E}_2 = t_{TE} E_{TE} \hat{u}_{TE} \exp(-j\vec{k}_t \cdot \vec{r}) + t_{TM} E_{TM} \hat{u}_{TM}^t \exp(-j\vec{k}_t \cdot \vec{r})$$

Fresnel Equations Generalization



$$r_{TE} = \frac{Z_2 \cos \theta_1 - Z_1 \cos \theta_2}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2}$$

$$t_{TE} = \frac{2Z_2 \cos \theta_1}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2}$$

$$r_{TM} = \frac{Z_1 \cos \theta_1 - Z_2 \cos \theta_2}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2}$$

$$t_{TM} = \frac{2Z_2 \cos \theta_1}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2}$$

$$\frac{P_r}{P_i} = \frac{|r_{TE}|^2 |E_{TE}|^2 + |r_{TM}|^2 |E_{TM}|^2}{|E_{TE}|^2 + |E_{TM}|^2}$$

$$\frac{P_t}{P_i} = \frac{Z_1}{\cos \theta_1} \frac{|t_{TE}|^2 |E_{TE}|^2 \operatorname{Re}\{(\cos \theta_2)^*/Z_2^*\} + |t_{TM}|^2 |E_{TM}|^2 \operatorname{Re}\{\cos \theta_2/Z_2^*\}}{|E_{TE}|^2 + |E_{TM}|^2}$$

Goos-Hänchen Shift*

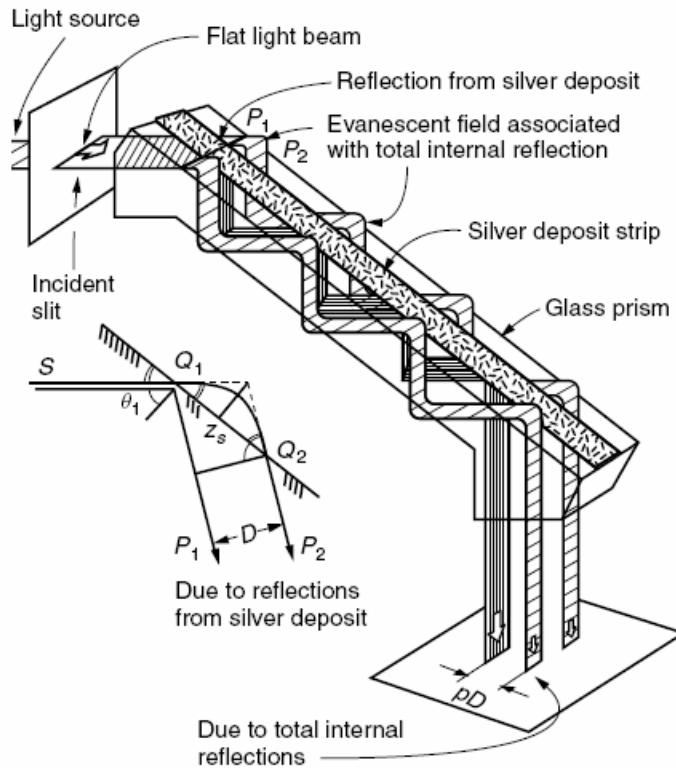


Figure 2.18 Goos and Hänchen experiment.

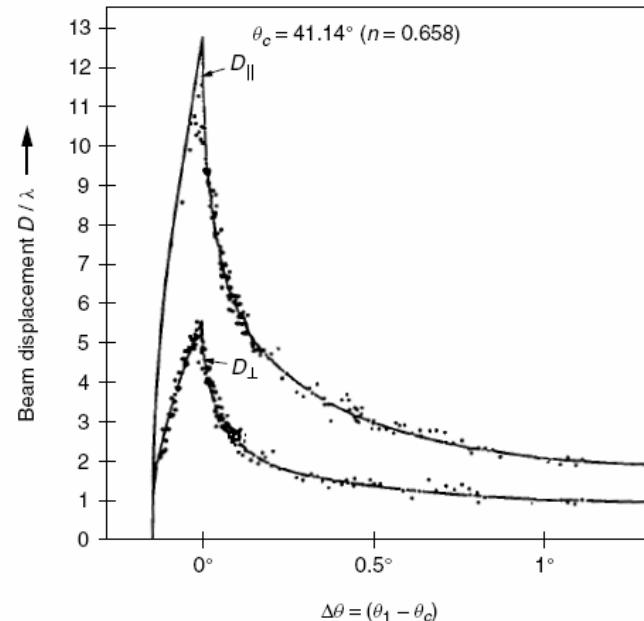
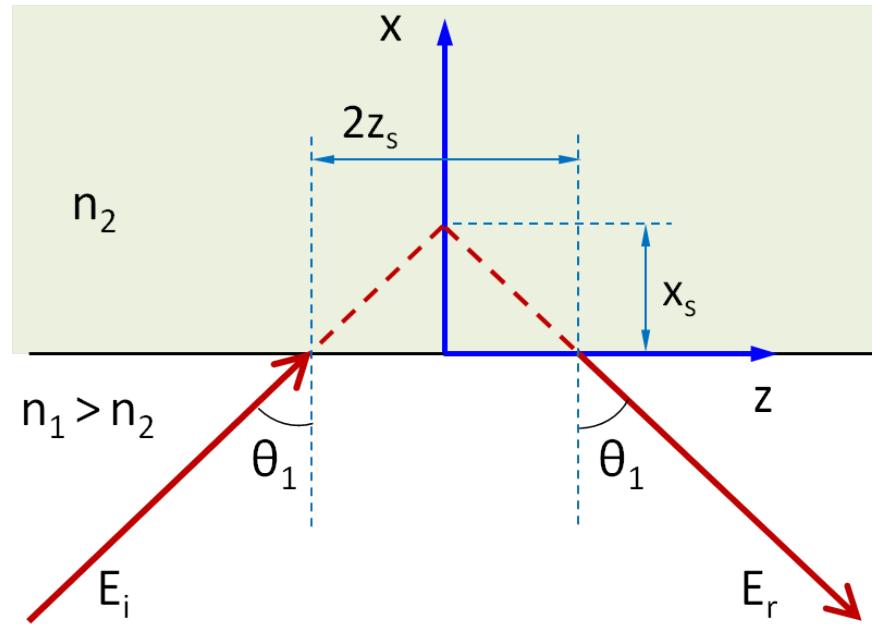


Figure 2.19 The beam displacement of the Goos-Hänchen shift versus the angle difference $(\theta_1 - \theta_c)$ in degrees. (After H. K. V. Lotsch [7] and H. Wolter [8].)

* K. Iizuka, Elements of Photonics, Vol. 1, John Wiley & Sons, Inc., New York. 2002

- V. F. Goos and H. Hänchen, "Ein neuer und fundamentaler Versuch zur Totalreflexion," *Ann. Phys.* 6. Folge, Band 1, Heft 7/8, 333–346 (1947).
 H. K. V. Lotsch, "Beam displacement at total reflection: The Goos–Hänchen effect, II, III, IV" *Optik* 32, 189–204, 299–319, 553–569 (1971).

Goos-Hänchen Shift Basics



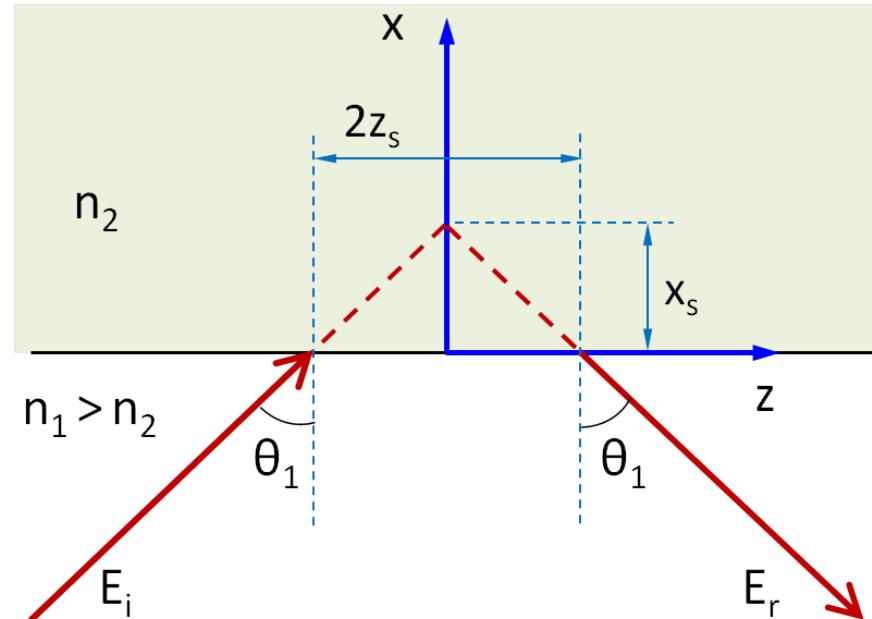
$$\vec{F}_i(x = 0, z) = \vec{F}_0 [e^{-j(k_z + \Delta k_z)z} + e^{-j(k_z - \Delta k_z)z}] = 2\vec{F}_0 \cos(\Delta k_z z) e^{-jk_z z}$$

$$\vec{F}_r(x = 0, z) = \vec{F}_0 [e^{j2\phi(k_z + \Delta k_z)} e^{-j(k_z + \Delta k_z)z} + e^{j2\phi(k_z - \Delta k_z)} e^{-j(k_z - \Delta k_z)z}]$$

$$\vec{F}_r(x = 0, z) = 2\vec{F}_0 \cos[\Delta k_z(z - 2z_s)] e^{-jk_z z} e^{j2\phi(k_z)},$$

$$z_s = \left. \frac{d\phi}{dk_z} \right|_{k_z}.$$

Goos-Hänchen Shift Basics



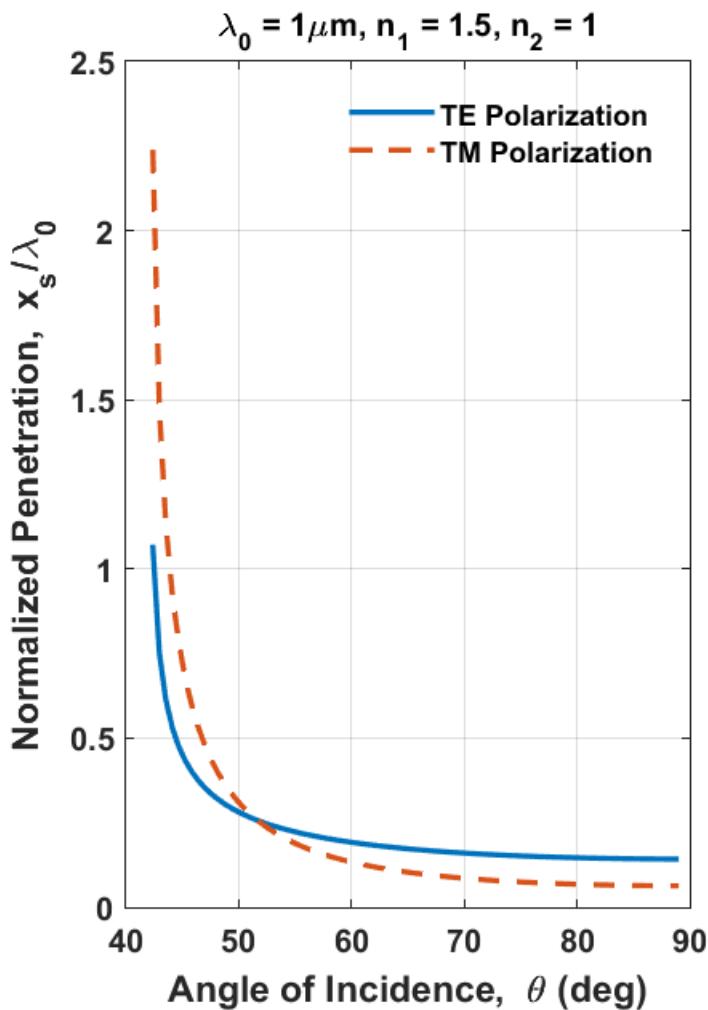
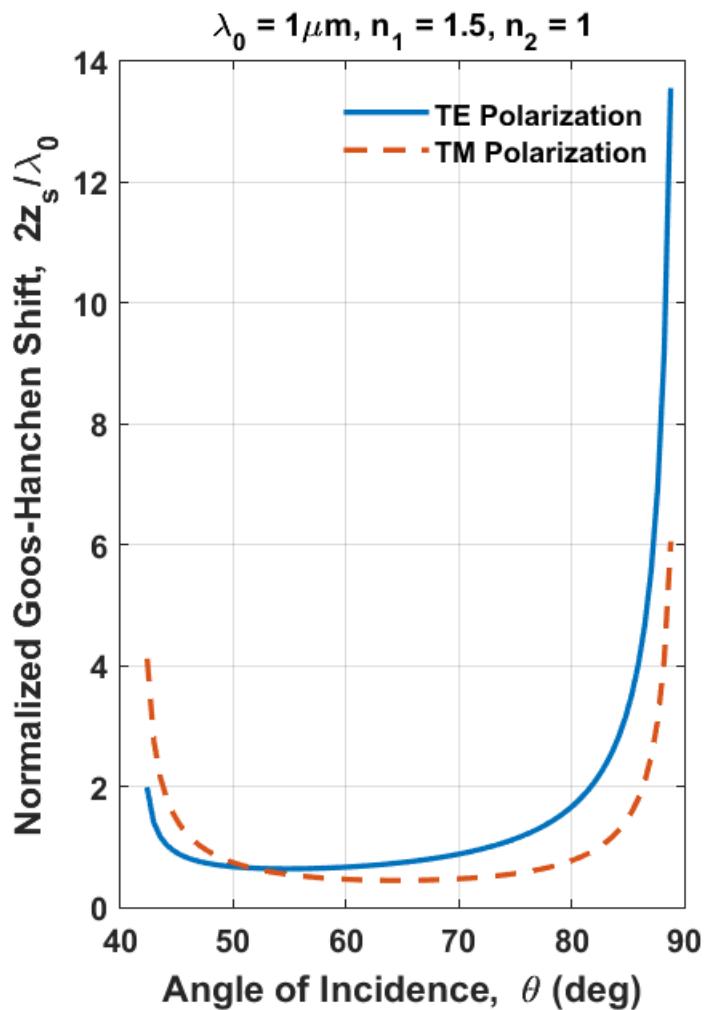
$$2z_{s_{TE}} = \frac{2\lambda_0 \tan \theta_1}{2\pi(n_1^2 \sin^2 \theta_1 - n_2^2)^{1/2}},$$

$$x_{s_{TE}} = \frac{\lambda_0}{2\pi(n_1^2 \sin^2 \theta_1 - n_2^2)^{1/2}},$$

$$2z_{s_{TM}} = \frac{(n_1/n_2)^2(n_1^2 - n_2^2)}{n_1^2[\cos^2 \theta_1 + (n_1/n_2)^4 \sin^2 \theta_1] - (n_1/n_2)^4 n_2^2} \frac{2\lambda_0 \tan \theta_1}{2\pi(n_1^2 \sin^2 \theta_1 - n_2^2)^{1/2}}$$

$$x_{s_{TM}} = \frac{(n_1/n_2)^2(n_1^2 - n_2^2)}{n_1^2[\cos^2 \theta_1 + (n_1/n_2)^4 \sin^2 \theta_1] - (n_1/n_2)^4 n_2^2} \frac{\lambda_0}{2\pi(n_1^2 \sin^2 \theta_1 - n_2^2)^{1/2}}$$

Goos-Hänchen Shifts Example



Goos-Hänchen Shift Generalization

$$\vec{F}_i(x = 0, y, z) = \vec{F}_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{F}(k_y, k_z) e^{-j(k_y y + k_z z)} dk_y dk_z$$

$$\vec{F}_r(x = 0, y, z) = \vec{F}_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{F}(k_y, k_z) e^{j2\phi(k_y, k_z)} e^{-j(k_y y + k_z z)} dk_y dk_z$$

A 2D Taylor expansion of $\phi(k_y, k_z)$ around a central point (k_{y0}, k_{z0}) gives

$$\phi(k_y, k_z) \simeq \phi(k_{y0}, k_{z0}) + \frac{\partial \phi}{\partial k_y} \Big|_{k_{y0}} (k_y - k_{y0}) + \frac{\partial \phi}{\partial k_z} \Big|_{k_{z0}} (k_z - k_{z0}) = \phi(k_{y0}, k_{z0}) + (\vec{k}_{\parallel} - \vec{k}_{\parallel 0}) \cdot \vec{\nabla} \phi \Big|_{\vec{k}_{\parallel 0}}$$

$$\begin{aligned} \vec{F}_r(x = 0, y, z) &= \vec{F}_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{F}(k_y, k_z) \exp[-j\vec{k}_{\parallel} \cdot (\vec{r}_{\parallel} - 2\vec{\nabla} \phi \Big|_{\vec{k}_{\parallel 0}})] dk_y dk_z \\ &= e^{j2[\phi(k_{y0}, k_{z0}) - \vec{k}_{\parallel 0} \cdot \vec{\nabla} \phi \Big|_{\vec{k}_{\parallel 0}}]} \vec{F}_i(x = 0, \vec{r}_{\parallel} - 2\vec{\nabla} \phi \Big|_{\vec{k}_{\parallel 0}}), \end{aligned}$$

$$\vec{d}_{GH} = 2\vec{\nabla} \phi \Big|_{\vec{k}_{\parallel 0}} = 2 \left\{ \frac{\partial \phi}{\partial k_y} \Big|_{k_{y0}} \hat{y} + \frac{\partial \phi}{\partial k_z} \Big|_{k_{z0}} \hat{z} \right\}$$