

Dispersion in Dielectric Optical Waveguides

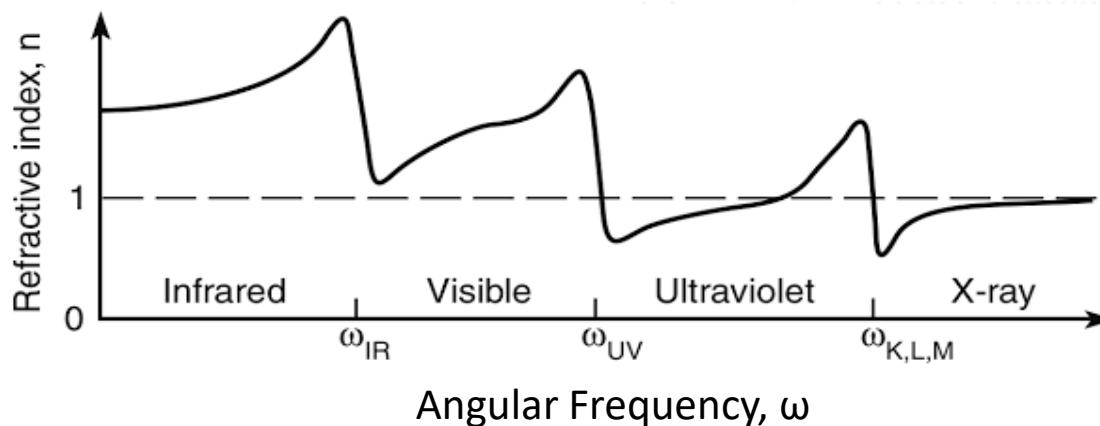
Integrated Optics

Prof. Elias N. Glytsis

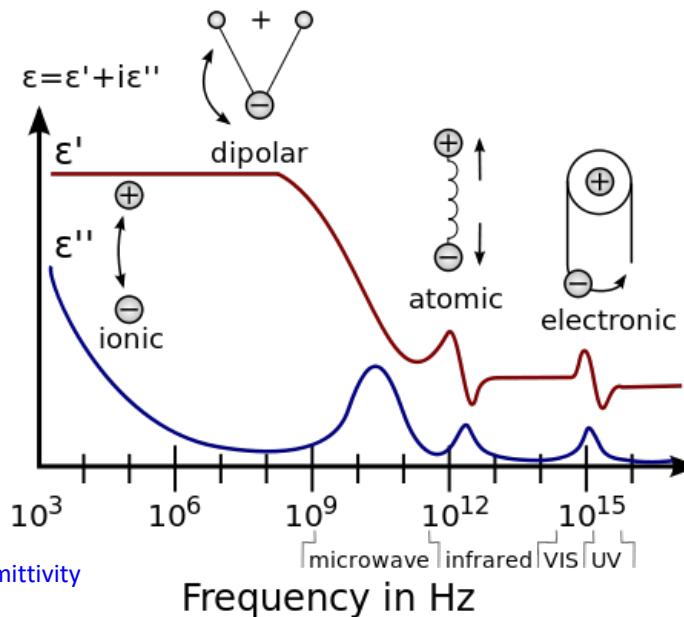


*School of Electrical & Computer Engineering
National Technical University of Athens*

Typical Dispersion of a Dielectric Material



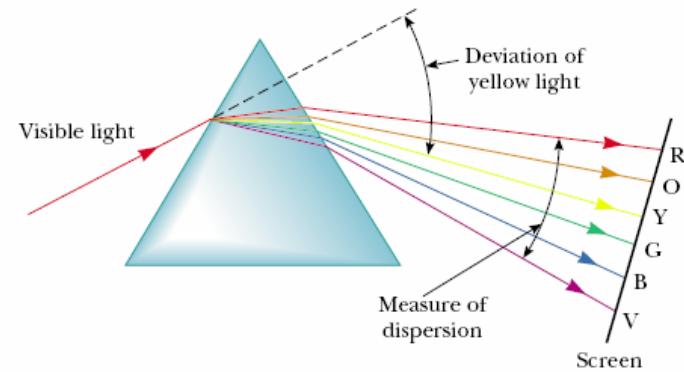
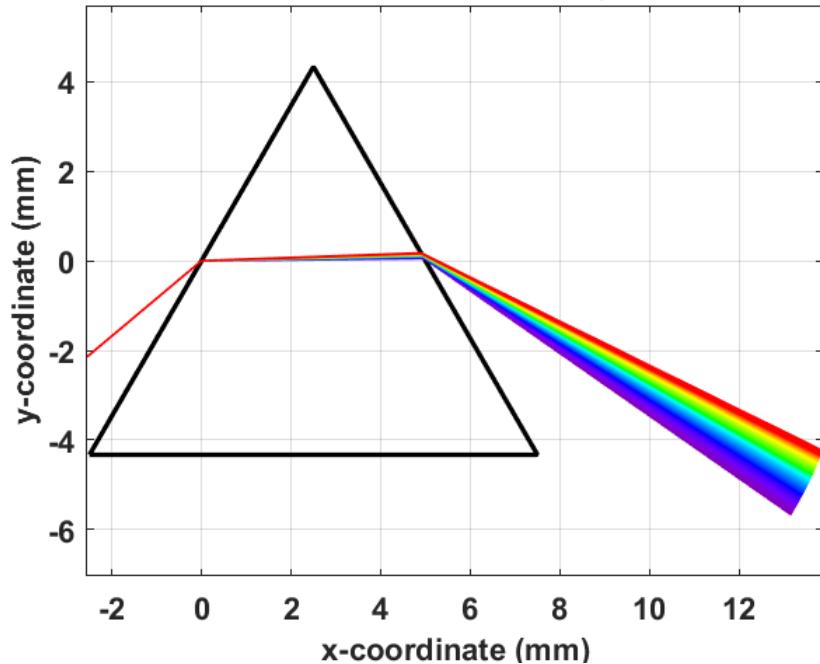
http://1.bp.blogspot.com/-kcHWNW81dl4/UR_qgF5B9dI/AAAAAAAACNg/h5oijsp_gTs/s640/figure+3.png



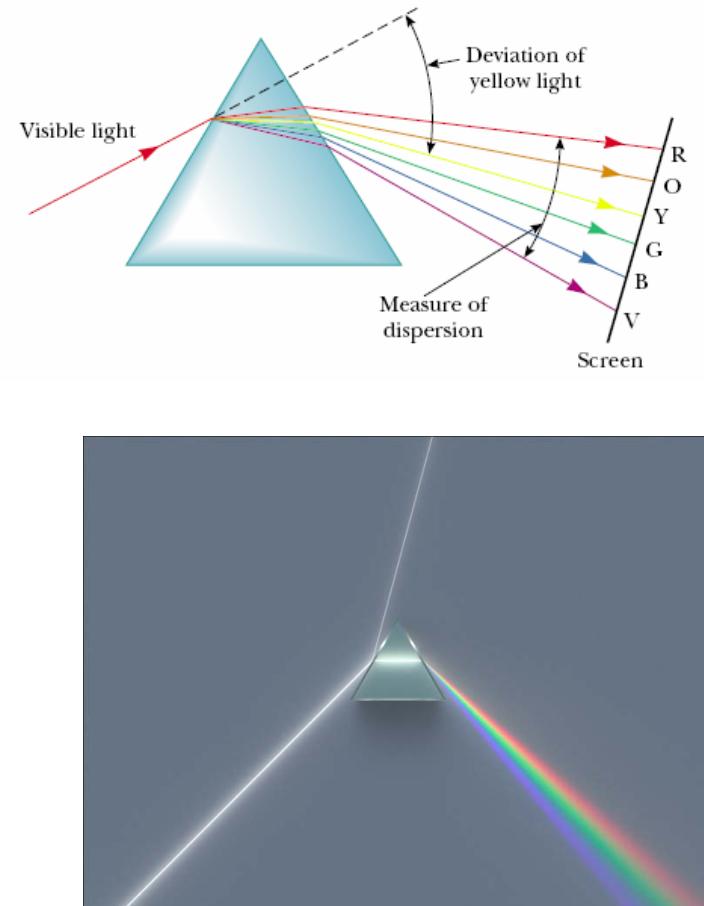
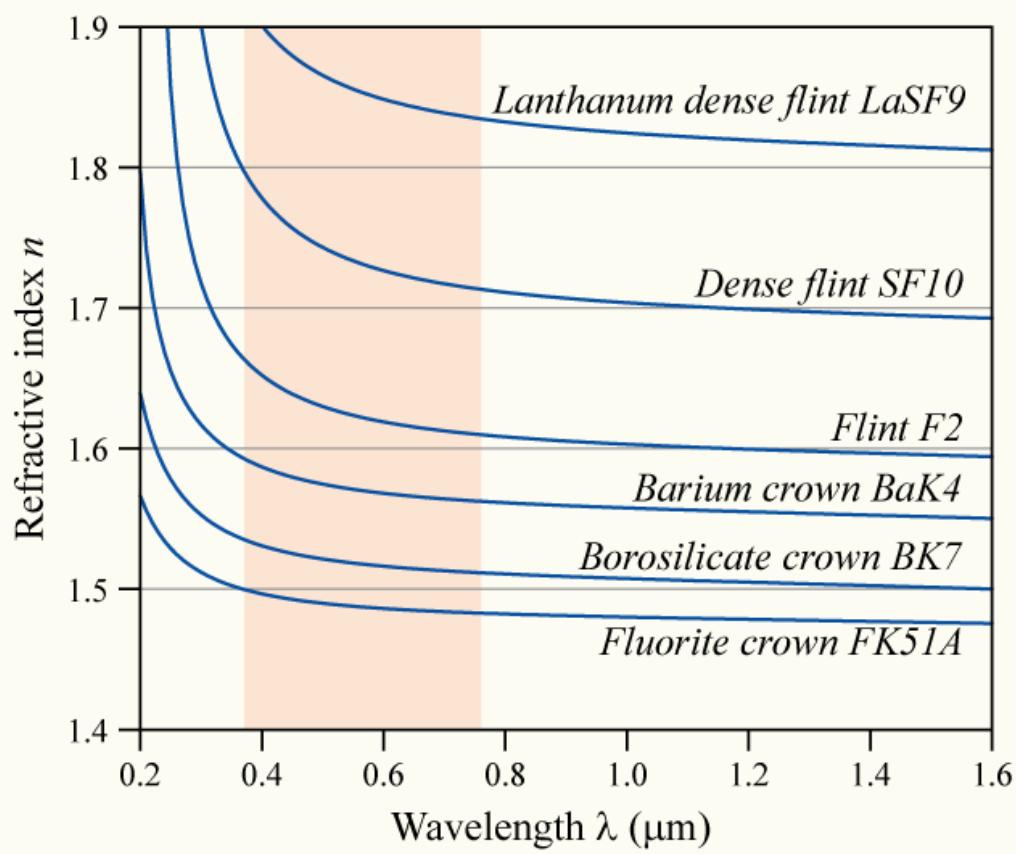
<https://en.wikipedia.org/wiki/Permittivity>

Prism Dispersion – N-SF11

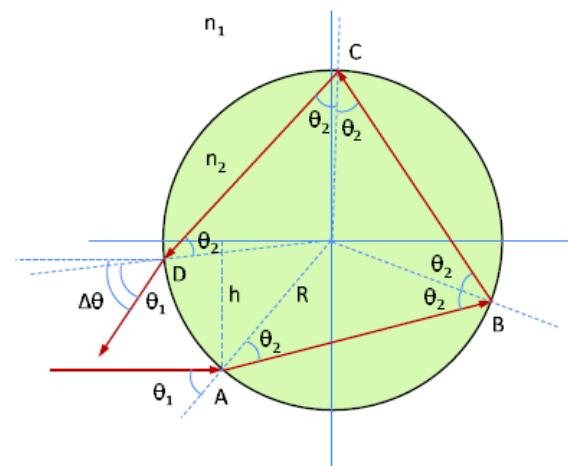
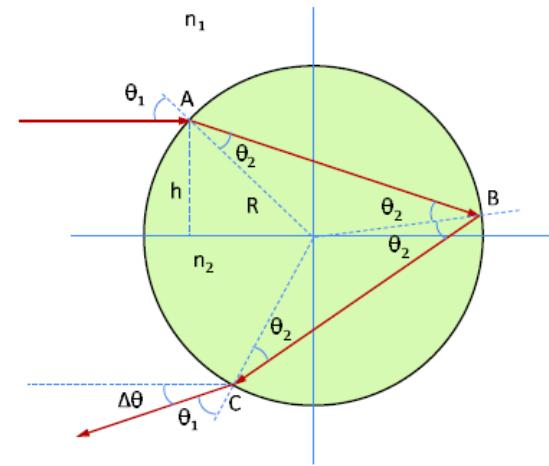
Dispersive Prism: N-SF11 high dispersive glass (Schott)
 $\sigma = 60^\circ$, $\alpha_1 = 70^\circ$, $L = 10\text{mm}$, $n_0 = 1$, $n_{p,\text{max}} = 1.8454$



Normal Dispersion

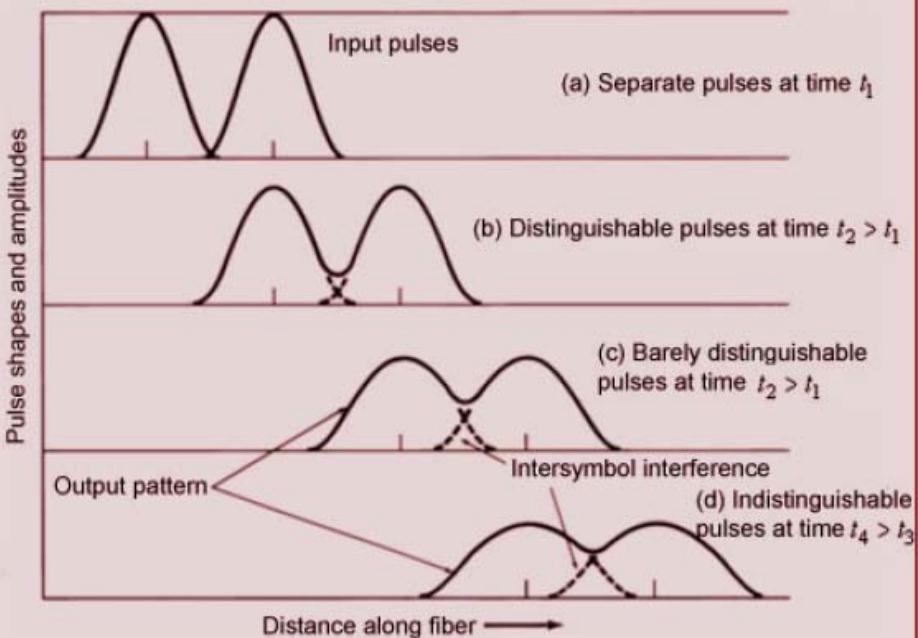


Rainbow (water dispersion)

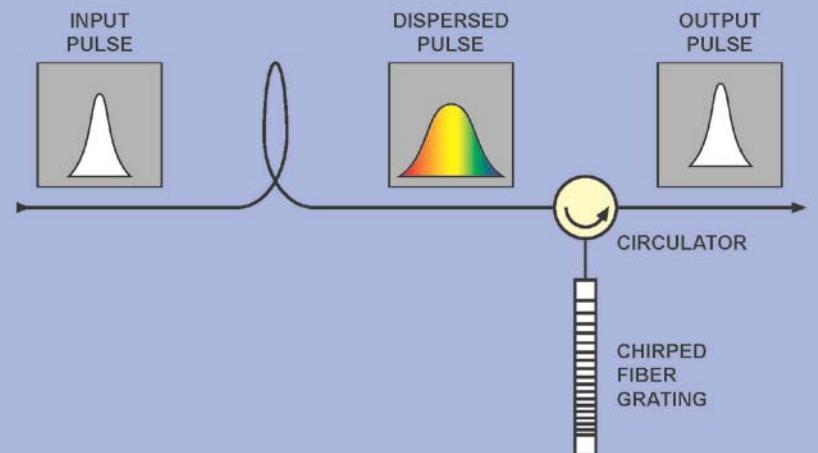


Optical Pulse Dispersion & Compensation

Pulse Broadening due to Dispersion



DISPERSION COMPENSATION
WITH CHIRPED FIBER GRATING



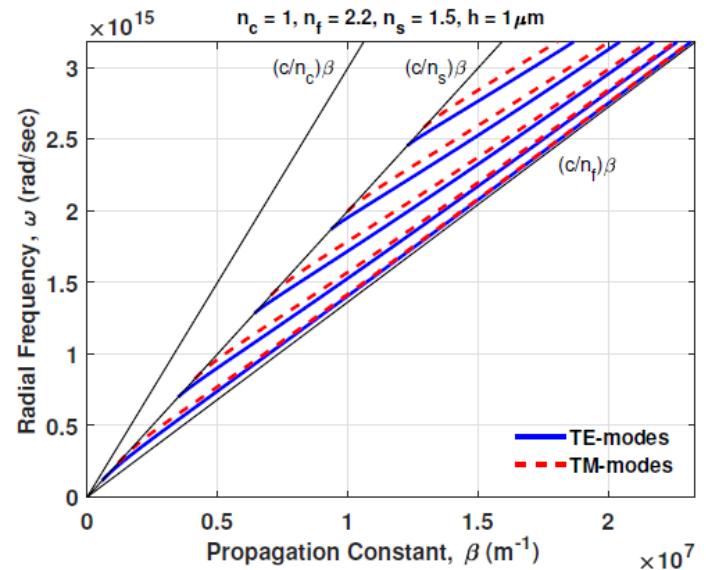
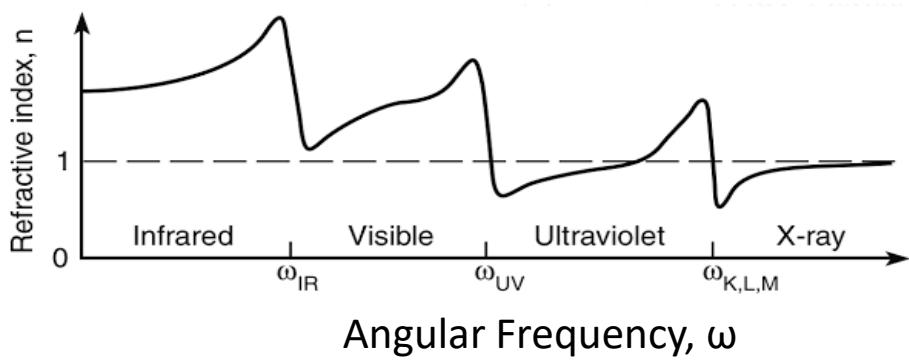
<https://nptel.ac.in/content/storage2/courses/117101054/downloads/lect5.pdf>

Dispersion

Material Dispersion: Lights of different wavelengths travel at different velocities within a given medium

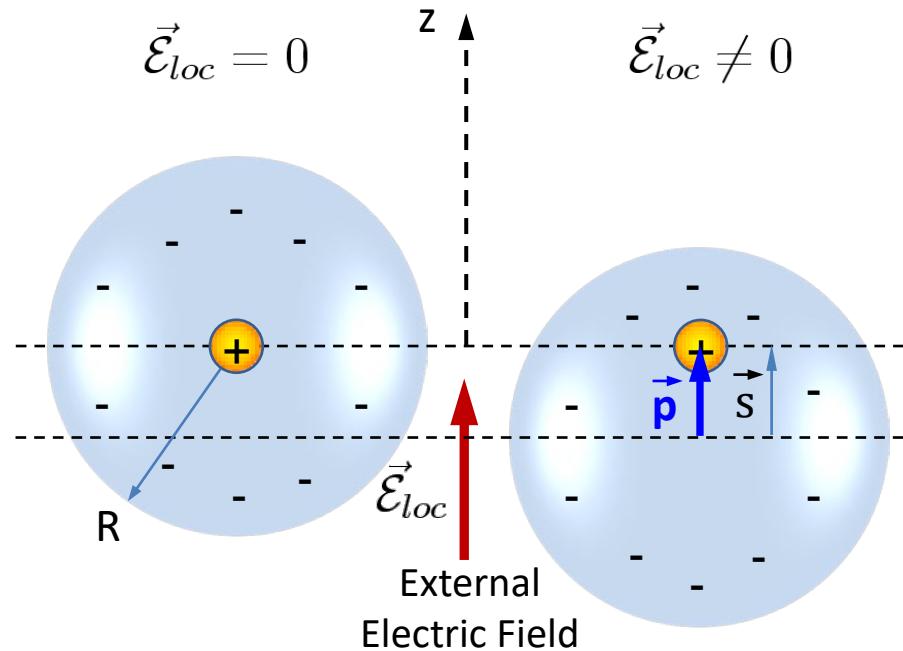
Modal Dispersion: Exists in waveguides with multiple modes. Each mode propagates with different velocity. Occurs even if the materials are dispersionless.

Waveguide Dispersion: Exists in waveguides with even with a single mode. This dispersion occurs since modal β depends on frequency.



Classical Electron Oscillator Model

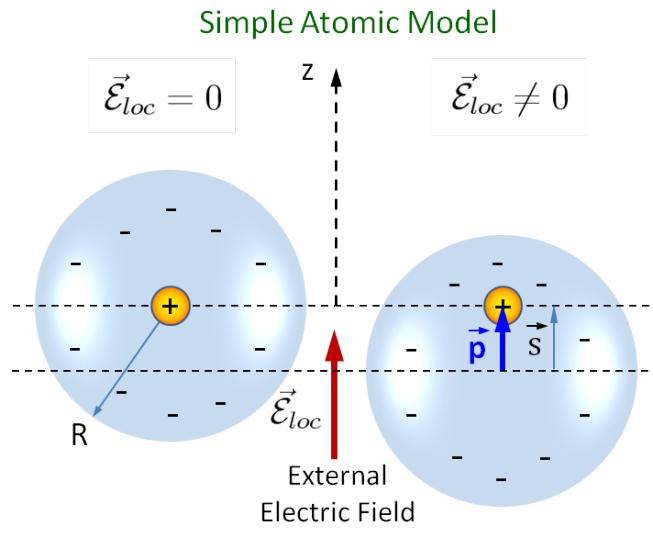
Simple Atomic Model



Equation of motion for an electron

$$m_e \frac{d^2[\vec{s}(t)]}{dt^2} = \underbrace{-e\vec{\mathcal{E}}(t)}_{\text{Lorentz Force}} - \underbrace{K\vec{s}(t)}_{\text{Restoring Force}} - \underbrace{\beta \frac{d[\vec{s}(t)]}{dt}}_{\text{Damping}}$$

Classical Electron Oscillator Model



$$\frac{d^2[\vec{s}(t)]}{dt^2} + \sigma \frac{d[\vec{s}(t)]}{dt} + \omega_0^2 \vec{s}(t) = -\frac{e}{m_e} \vec{\mathcal{E}}(t)$$

$$\sigma = \beta/m_e$$

$$\omega_0 = \sqrt{K/m_e}$$

Fourier Transform Pairs

$$\begin{aligned} \vec{s}(t) &\xrightarrow{\mathcal{F}} \vec{S}(\omega) \\ \vec{\mathcal{E}}(t) &\xrightarrow{\mathcal{F}} \vec{E}(\omega) \end{aligned}$$

$$\text{for } \omega \simeq \omega_0 \implies \vec{S}(\omega) \simeq \frac{-e/m_e}{2\omega_0(\omega_0 - \omega) + j\omega_0\sigma} \vec{E}(\omega)$$

$$\vec{S}(\omega) = \frac{-e/m_e}{(\omega_0^2 - \omega^2) + j\omega\sigma} \vec{E}(\omega)$$

Electric Dipole Moment

$$\vec{p}(t) = e [\vec{s}(t)] \xrightarrow{\mathcal{F}} \vec{p}(\omega) = e \vec{S}(\omega) = -\frac{e^2/m_e}{2\omega_0(\omega_0 - \omega) + j\omega_0\sigma} \vec{E}(\omega)$$

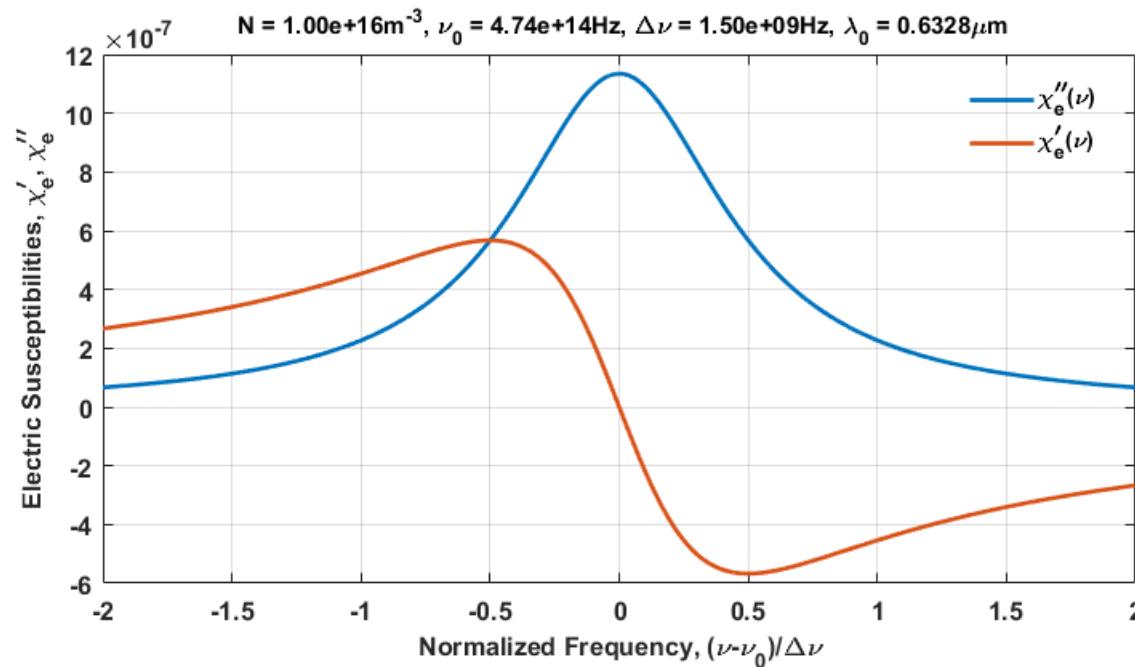
Macroscopic Polarization

$$\vec{P}(t) = \vec{p}(t)N \xrightarrow{\mathcal{F}} \vec{P}(\omega) = \vec{p}(\omega)N = \frac{-j(Ne^2/\omega_0\sigma m_e)}{1 + j\frac{2(\omega - \omega_0)}{\sigma}} \vec{E}(\omega)$$

Classical Electron Oscillator Model

$$\vec{P}(\omega) = \epsilon_0 [\chi'(\omega) - j\chi''(\omega)] \vec{E}(\omega) = \frac{-j(Ne^2/\omega_0\sigma m_e)}{1 + j\frac{2(\omega - \omega_0)}{\sigma}} \vec{E}(\omega)$$

$$\begin{aligned}\chi_e''(\nu) &= \frac{Ne^2}{16\pi^2 m_e \nu_0 \epsilon_0} \frac{\Delta\nu}{(\nu - \nu_0)^2 + (\Delta\nu/2)^2} \\ \chi_e'(\nu) &= \frac{2(\nu_0 - \nu)}{\Delta\nu} \chi_e''(\nu) = \frac{Ne^2}{8\pi^2 m_e \nu_0 \epsilon_0} \frac{\nu_0 - \nu}{(\nu - \nu_0)^2 + (\Delta\nu/2)^2}\end{aligned}$$



Normal Dispersion - Sellmeier Formula

$$n^2(\lambda) = 1 + \sum_i \frac{B_i \lambda^2}{\lambda^2 - C_i}$$

Table of coefficients of Sellmeier equation

Material	B₁	B₂	B₃	C₁	C₂	C₃
<u>borosilicate crown glass</u> (known as <i>BK7</i>)	1.03961212	0.231792344	1.01046945	6.00069867×10 ⁻³ μm ²	2.00179144×10 ⁻² μm ²	1.03560653×10 ² μm ²
sapphire (for <u>ordinary wave</u>)	1.43134930	0.65054713	5.3414021	5.2799261×10 ⁻³ μm ²	1.42382647×10 ⁻² μm ²	3.25017834×10 ² μm ²
sapphire (for <u>extraordinary wave</u>)	1.5039759	0.55069141	6.5927379	5.48041129×10 ⁻³ μm ²	1.47994281×10 ⁻² μm ²	4.0289514×10 ² μm ²
<u>fused silica</u>	0.696166300	0.407942600	0.897479400	4.67914826×10 ⁻³ μm ²	1.35120631×10 ⁻² μm ²	97.9340025 μm ²

Group Index and Group Delay

For propagation of optical pulses that contain a spread of wavelength in a dispersive material the group velocity instead of the phase velocity is of interest. The group velocity is defined as:

$$v_g = \frac{d\omega}{dk} \quad \text{where } k = \frac{\omega}{c} n(\omega)$$

$$\text{Then } \frac{dk}{d\omega} = \frac{d}{d\omega} \left(\frac{\omega}{c} n(\omega) \right) = \frac{n(\omega)}{c} + \frac{\omega}{c} \frac{dn}{d\omega} = \frac{1}{c} \left(n + \omega \frac{dn}{d\omega} \right)$$

If λ instead of ω is used then: (λ = freespace wavelength)

$$\frac{dn}{d\lambda} = \frac{dn}{d\omega} \frac{d\omega}{d\lambda} = \frac{dn}{d\omega} \frac{d}{d\lambda} \left(\frac{2\pi c}{\lambda} \right) = \frac{dn}{d\omega} \left(-\frac{1}{\lambda^2} 2\pi c \right) \Rightarrow$$

$$\frac{dn}{d\omega} = -\frac{\lambda^2}{2\pi c} \frac{dn}{d\lambda} \rightsquigarrow \frac{1}{c} \left(n + \omega \frac{dn}{d\omega} \right) = \frac{1}{c} \left(n + \frac{2\pi c}{\lambda} \left(-\frac{\lambda^2}{2\pi c} \frac{dn}{d\lambda} \right) \right) = \frac{1}{c} \left(n - \lambda \frac{dn}{d\lambda} \right).$$

Group Index and Group Delay

Then the group velocity can be written as:

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}} = \frac{c}{n - 2 \frac{dn}{dk}} = \frac{c}{N_g}$$

where N_g is the group index.

The group delay τ_g is defined as the time it takes for a pulse of light to travel a unit distance. Therefore,

$$\tau_g = \frac{1}{v_g} = \frac{dk}{d\omega}$$

Group Velocity Dispersion (GVD)

A narrow optical pulse that contains a spread of wavelengths $\Delta\lambda$ can propagate in a dispersive medium of length L . After propagation the pulse widens by $\Delta\tau$:

$$\Delta\tau = L \Delta\left(\frac{1}{v_g}\right) = L \frac{d}{d\lambda}\left(\frac{1}{v_g}\right) \Delta\lambda$$

The group velocity dispersion $D(\lambda)$ (or material dispersion) is defined

$$D(\lambda) = \frac{\Delta\tau}{L \Delta\lambda} = \frac{d}{d\lambda}\left(\frac{1}{v_g}\right)$$

However $v_g = \frac{c}{N_g} \Rightarrow \frac{1}{v_g} = \frac{1}{c} N_g \approx \frac{d}{d\lambda}\left(\frac{1}{v_g}\right) = \frac{1}{c} \frac{dN_g}{d\lambda} = \frac{1}{c} \frac{d}{d\lambda}\left(n - \lambda \frac{dn}{d\lambda}\right)$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{1}{v_g}\right) = \frac{1}{c} \left(\frac{dn}{d\lambda} - \frac{dn}{d\lambda} - \lambda \frac{d^2n}{d\lambda^2} \right) = -\frac{\lambda}{c} \frac{d^2n}{d\lambda^2}$$

Group Velocity Dispersion (GVD)

An equivalent expression can be found if $v_g = \frac{dw}{dk}$ is used.

$$\frac{d}{d\lambda}\left(\frac{1}{v_g}\right) = \frac{d}{dw}\left(\frac{1}{v_g}\right) \frac{dw}{d\lambda} = \frac{d}{dw}\left(\frac{dk}{dw}\right) \frac{dw}{d\lambda} = \frac{d^2k}{dw^2} \left(-\frac{2\pi c}{\lambda^2}\right)$$

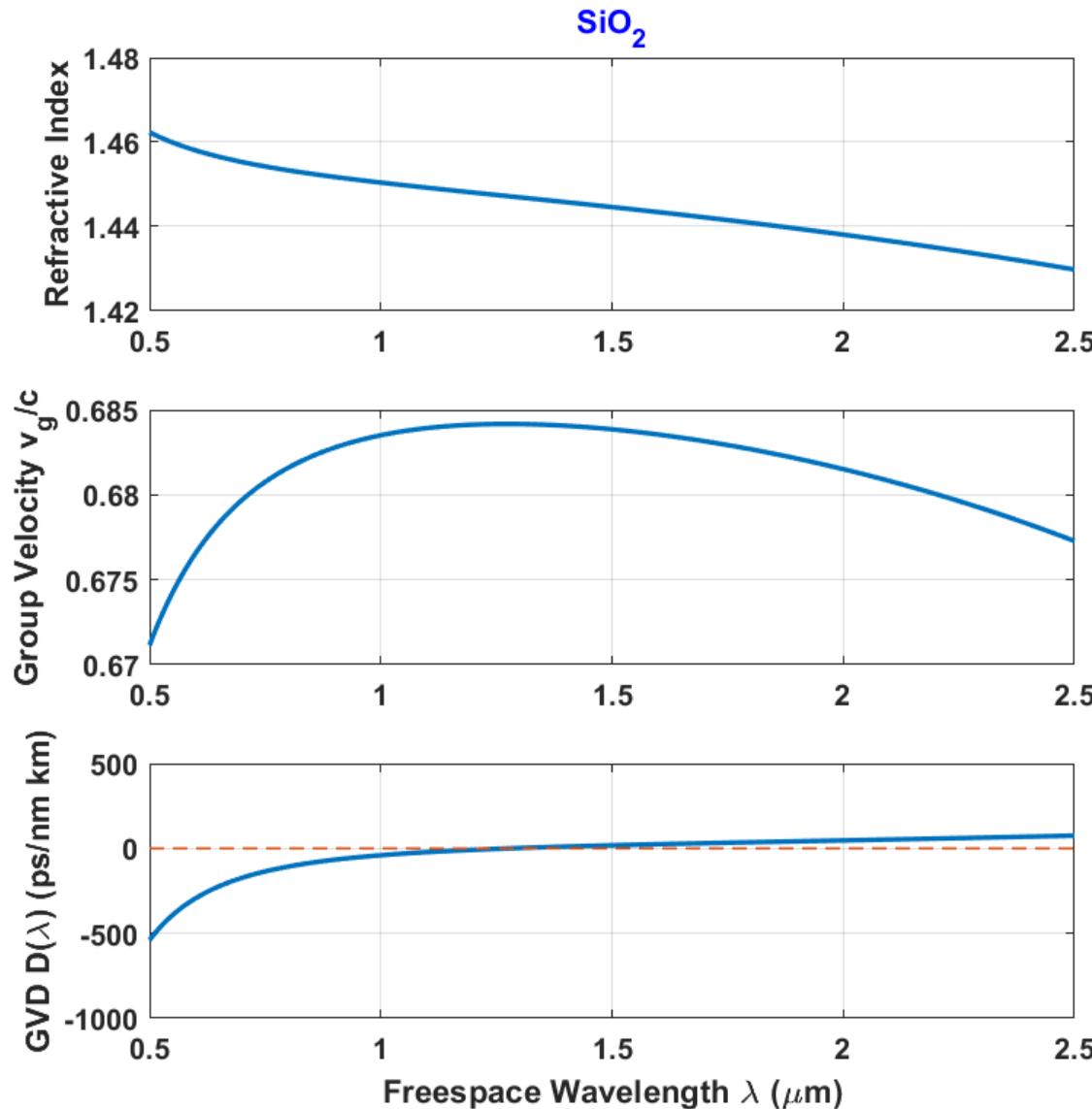
Therefore,

$$D(\lambda) = -\frac{\lambda}{c} \frac{d^2n}{d\lambda^2} = -\left(\frac{2\pi c}{\lambda^2}\right) \frac{d^2k}{dw^2}$$

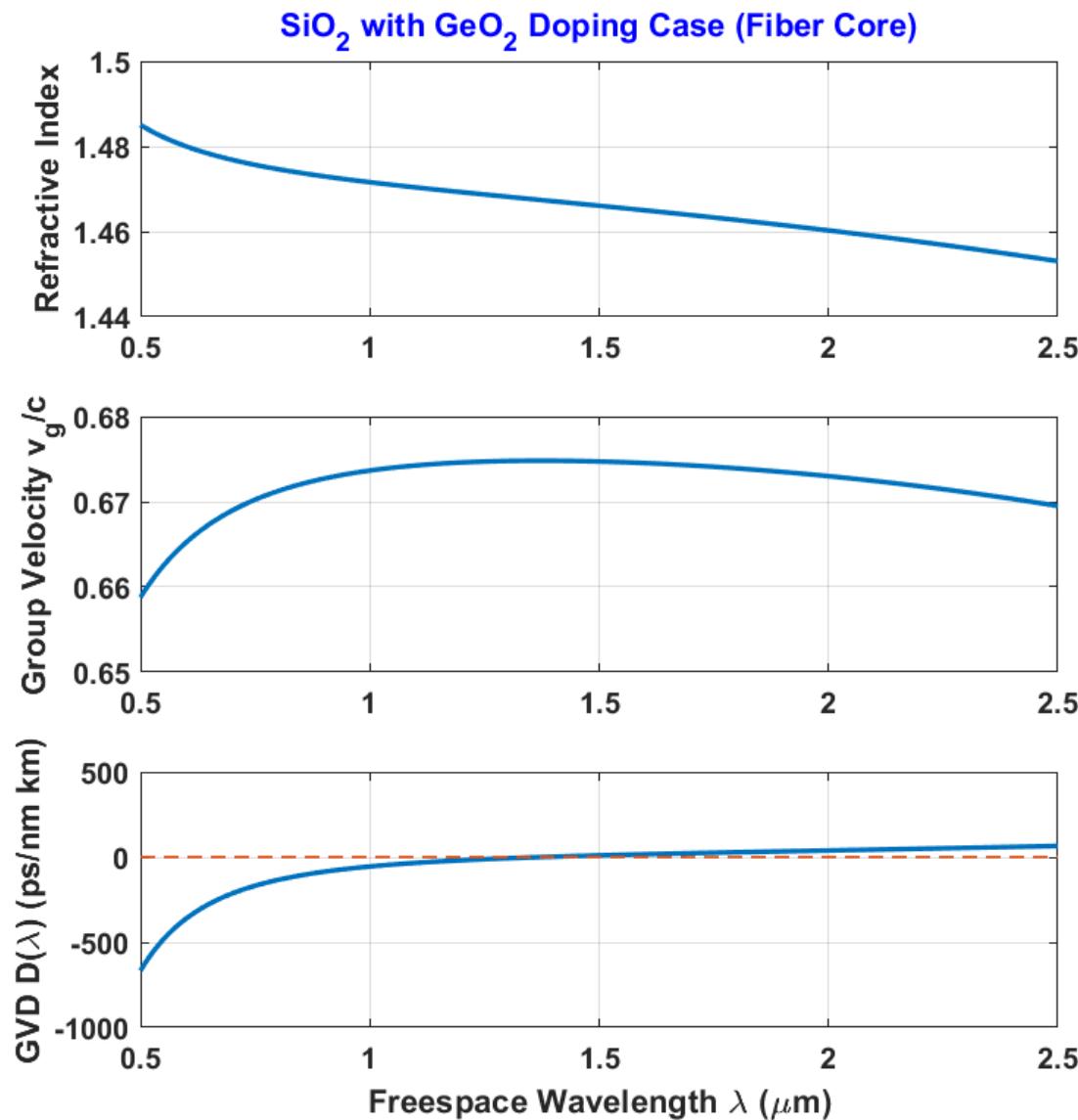
Later k can be replaced by β when the waveguide effects are also being considered.

Using the A and G_k parameters for the sapphire (Al_2O_3) we obtain the results shown on page 7. The same results for SiO_2 and SiO_2 doped with GeO_2 (fiber core) are shown on pages 8 and 9 respectively.

Group Velocity Dispersion (GVD)



Group Velocity Dispersion (GVD)



Group Velocity Dispersion (GVD)

The GVD (D) passes through zero at the zero material dispersion wavelength (let's define it as $\lambda_{D=0}$). It is straightforward to understand that $\lambda_{D=0}$ occurs at an extremum of v_g (in the graphs of pages 7-9 at the maximum). This is because $D = -\frac{c}{\lambda} \frac{d^2n}{d\lambda^2} = 0 \Rightarrow \frac{d^2n}{d\lambda^2} = 0 \Rightarrow \frac{dN_g}{d\lambda} = 0 \Rightarrow N_g(\lambda_{D=0})$ an extremum $\Rightarrow v_g(\lambda_{D=0}) = \frac{c}{N_g(\lambda_{D=0})}$ at an extremum.

Therefore, for $\lambda < \lambda_{D=0}$ $\frac{dv_g}{d\lambda} > 0 \sim$ Longer wavelengths travel faster than shorter wavelengths. But since $D = \frac{d}{d\lambda} \left(\frac{1}{v_g} \right) = -\frac{1}{v_g^2} \frac{dv_g}{d\lambda}$ $D < 0$ when $\frac{dv_g}{d\lambda} > 0$.

Similarly, when $\lambda > \lambda_{D=0}$ $\frac{dv_g}{d\lambda} < 0 \sim$ shorter wavelengths travel faster than longer wavelengths. Again if $\frac{dv_g}{d\lambda} < 0 \sim D > 0$.

Waveguide Dispersion

Whenever light is guided in a waveguide the modal (or intermodal) and the waveguide (or intramodal) dispersion occur in addition to the material dispersion. In a single-mode waveguide the modal dispersion does not exist. However, the waveguide dispersion, even if it is not as large as the material dispersion, is still important since it can shift the dispersion characteristics of the waveguide.

It can be shown [A. Yariv, "Optical Electronics in Modern Communications] that the Group Velocity Dispersion D parameter can be written in the following form:

$$\begin{aligned} D &= -\frac{\lambda}{c} \left[\left(\frac{d^2 n}{d \lambda^2} \right)_m + \left(\frac{d^2 N_{eff}}{d \lambda^2} \right)_w \right] = \\ &= -\frac{2\pi c}{\lambda^2} \left[\left(\frac{d^2 k}{d \omega^2} \right)_m + \left(\frac{d^2 \beta}{d \omega^2} \right)_w \right] = \frac{d}{d \lambda} \left(\frac{1}{v_g} \right) = -\frac{1}{v_g^2} \frac{d v_g}{d \lambda} \end{aligned}$$

Waveguide Dispersion

The material term can be calculated using the Sellmeir coefficients as it was done previously. The waveguide term though requires the use of the transcendental eigenvalue equation: Let $f(\omega, \beta) = 0$ this eigenvalue equation. For example, for the HE₁₁ mode

$$f(\omega, \beta) = \kappa \alpha \frac{J_1(\kappa \alpha)}{J_0(\kappa \alpha)} - \gamma \alpha \frac{K_1(\gamma \alpha)}{K_0(\gamma \alpha)} = 0$$

$$\text{where } \kappa = \sqrt{\frac{\omega^2}{c^2} n_1^2 - \beta^2}, \quad \gamma = \sqrt{\beta^2 - \left(\frac{\omega}{c} n_2\right)^2}$$

Waveguide Dispersion

$$\frac{\partial f}{\partial \omega} d\omega + \frac{\partial f}{\partial \beta} d\beta = 0 \Rightarrow \frac{d\omega}{d\beta} = - \frac{\partial f / \partial \beta}{\partial f / \partial \omega} = v_g$$

The term $\frac{d^2\beta}{d\omega^2}$ that is needed can be obtained from:

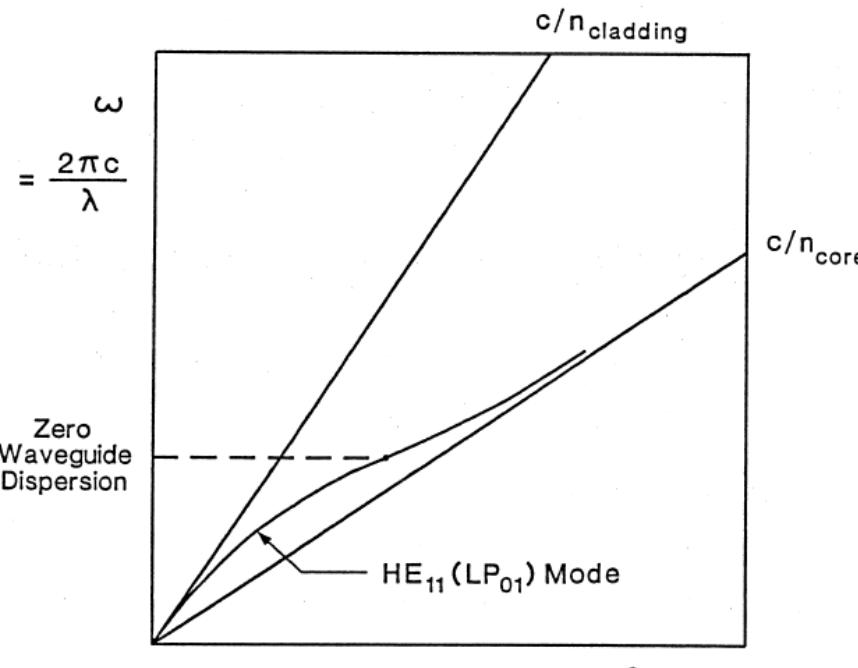
$$\begin{aligned} \frac{d^2\beta}{d\omega^2} &= \frac{d}{d\omega} \left(\frac{d\beta}{d\omega} \right) = \frac{d}{d\omega} \left(\frac{1}{v_g} \right) = \frac{d}{d\omega} \left(- \frac{\partial f / \partial \beta}{\partial f / \partial \omega} \right) = \\ &= - \frac{\frac{d}{d\omega} \left(\frac{\partial f}{\partial \omega} \right) \frac{\partial f}{\partial \beta} - \frac{d}{d\omega} \left(\frac{\partial f}{\partial \beta} \right) \frac{\partial f}{\partial \omega}}{\left(\frac{\partial f}{\partial \beta} \right)^2} \end{aligned}$$

where $\frac{d}{d\omega} \left(\frac{\partial f}{\partial \omega} \right) = \frac{\partial}{\partial \beta} \left(\frac{\partial f}{\partial \omega} \right) \frac{d\beta}{d\omega} + \frac{\partial}{\partial \omega} \left(\frac{\partial f}{\partial \omega} \right) = \frac{\partial^2 f}{\partial \beta \partial \omega} \frac{1}{v_g} + \frac{\partial^2 f}{\partial \omega^2}$ and

$$\frac{d}{d\omega} \left(\frac{\partial f}{\partial \beta} \right) = \frac{\partial}{\partial \beta} \left(\frac{\partial f}{\partial \beta} \right) \frac{d\beta}{d\omega} + \frac{\partial}{\partial \omega} \left(\frac{\partial f}{\partial \beta} \right) = \frac{\partial^2 f}{\partial \beta^2} \frac{1}{v_g} + \frac{\partial^2 f}{\partial \omega \partial \beta}$$

It is worth mentioning that in all the above expressions ω, β should always satisfy $f(\omega, \beta) = 0$, i.e. their corresponding eigenvalue equation. Also it is obvious that waveguide dispersion calculations are tedious and can be performed strictly numerically.

Waveguide Dispersion



At the inflection point $\frac{du_3}{d\omega} = 0$. But

$$\begin{aligned}
 D &= \frac{d}{d\lambda} \left(\frac{1}{u_3} \right) = \frac{d}{d\omega} \left(\frac{1}{u_3} \right) \frac{d\omega}{d\lambda} = \frac{d}{d\omega} \left(\frac{1}{u_3} \right) \left(-\frac{2\pi c}{\lambda^2} \right) \\
 &= \frac{d}{d\omega} \left(\frac{d\beta}{d\omega} \right) \left(-\frac{2\pi c}{\lambda^2} \right) = \frac{d^2\beta}{d\omega^2} \left(-\frac{2\pi c}{\lambda^2} \right) \\
 &= -\frac{1}{u_3^2} \frac{du_3}{d\omega} \frac{d\omega}{d\lambda} = -\frac{1}{u_3^2} \frac{du_3}{d\omega} \left(\frac{2\pi c}{\lambda^2} \right) = +\frac{du_3}{d\omega} \left(\dots \right)
 \end{aligned}$$

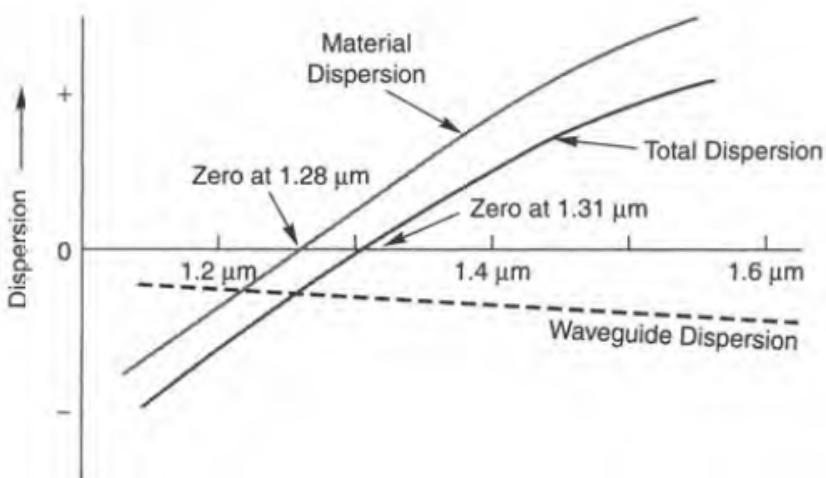
Therefore, $du_3/d\omega = 0 \rightsquigarrow D = 0$.

Waveguide Dispersion

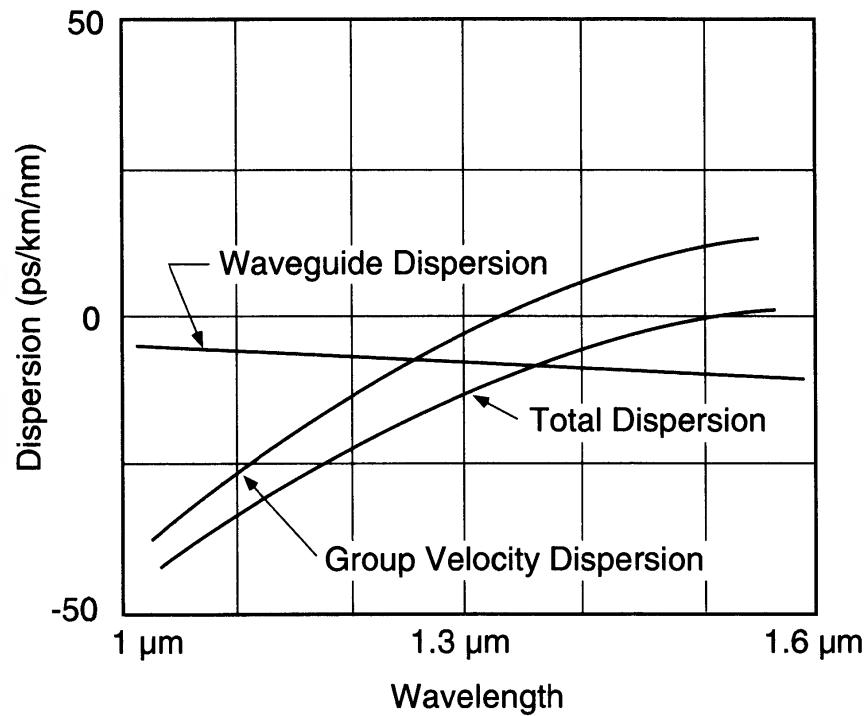
When $\frac{du_g}{d\lambda} > 0 \Rightarrow \frac{du_g}{dw} < 0 \Rightarrow$ longer wavelengths travel

faster than shorter wavelengths (frequencies below the dashed line

(in the figure of page 13). When $\frac{du_g}{d\lambda} < 0 = \frac{du_g}{dw} > 0 \Rightarrow$ shorter wavelengths travel faster than longer wavelengths.

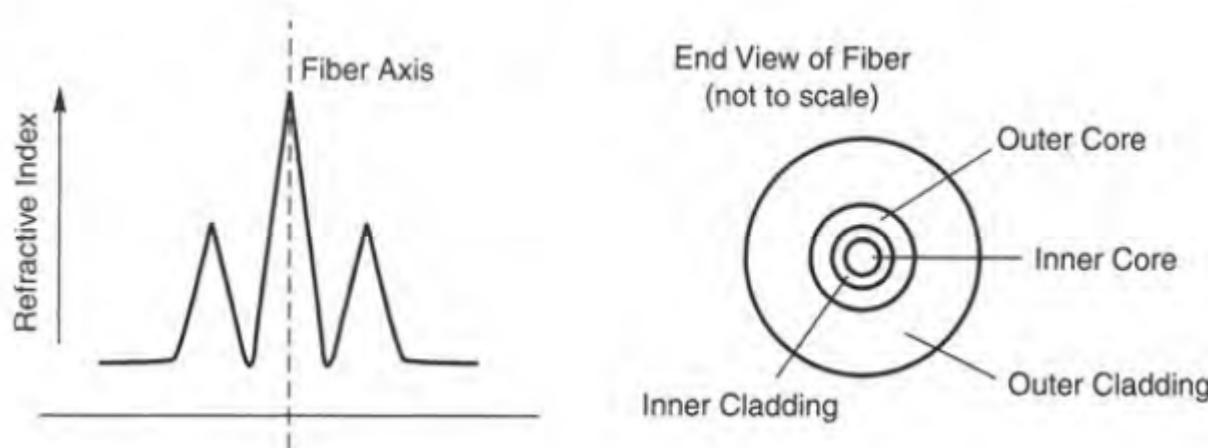


E. Hecht, "Understanding Fiber Optics", LaserLight Press, 5th Ed., 2015

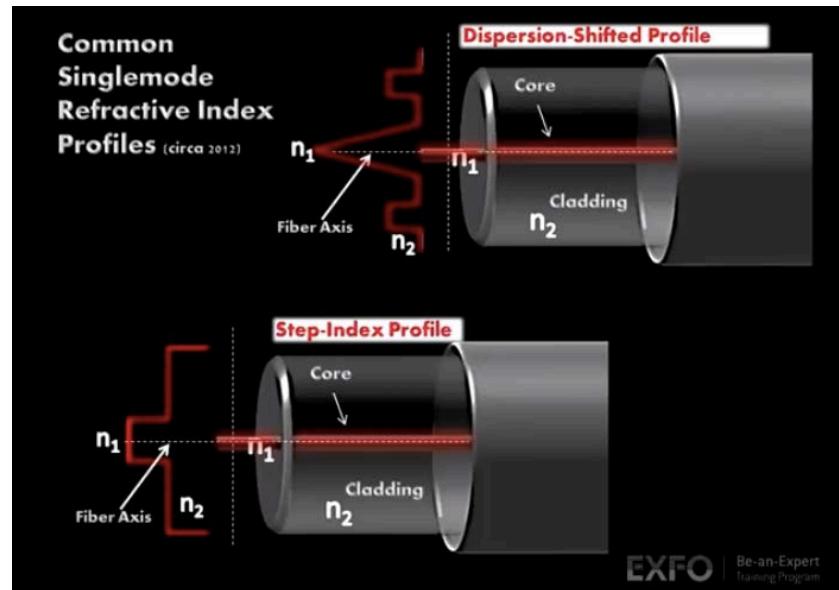


C. R. Pollock, "Fundamentals of Optoelectronics", Irwin, 1995

Dispersion Shifted Fiber



E. Hecht, "Understanding Fiber Optics", LaserLight Press, 5th Ed., 2015



<https://wiki.metropolia.fi/display/Physics/%28M%29+Dispersion+in+Fiber+Optics>

Optical Pulse Propagation

Let's assume a Gaussian pulse (in time) at some point $z = 0$ inside a waveguide (single-mode). The electric field of the pulse can be expressed as:

$$E(x, y, z=0, t) = E_0 \mathcal{E}(x, y) e^{-\frac{1}{2}(\frac{t}{\tau})^2} e^{j\omega_0 t} (e^{-j\beta(\omega_0)(z=0)})$$

The frequency spectrum of $E(x, y, z=0, t)$ is given by

$$\begin{aligned}\tilde{E}(x, y, z=0, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_0 \mathcal{E}(x, y) e^{-\frac{1}{2}(\frac{t}{\tau})^2} e^{j\omega_0 t} e^{-j\omega t} dt = \\ &= \frac{E_0 \mathcal{E}(x, y)}{\sqrt{2\pi}} \tau e^{-\frac{1}{2}(\omega - \omega_0)^2 \tau^2}\end{aligned}$$

In the above it is assumed that $\mathcal{E}(x, y)$ corresponds to one waveguide mode. Now each spectral component of \tilde{E} is propagated by distance z assuming a propagation constant $\beta(\omega)$:

Optical Pulse Propagation

$$E(x, y, z, t) = \int_{-\infty}^{+\infty} \tilde{E}(x, y, z=0, \omega) e^{-j\beta(\omega)z} e^{j\omega t} d\omega$$

usually it is assumed that $\beta(\omega)$ is a smooth function of ω and can be expanded in a Taylor series with only the first three terms:

$$\begin{aligned}\beta(\omega) &\simeq \beta(\omega_0) + \underbrace{(\omega - \omega_0) \frac{d\beta}{d\omega} \Big|_{\omega_0}}_{\beta_1} + \underbrace{\frac{1}{2} (\omega - \omega_0)^2 \frac{d^2\beta}{d\omega^2} \Big|_{\omega_0}}_{\beta_2} = \\ &= \beta(\omega_0) + (\omega - \omega_0) \beta_1 + \frac{1}{2} (\omega - \omega_0)^2 \beta_2\end{aligned}$$

where $\beta_1 = \frac{1}{c} \left(\frac{d\omega}{d\beta} \right) \Big|_{\omega_0} = \frac{1}{v_g(\omega_0)}$ and β_2 is related to the dispersion parameter $D = -\frac{2\pi c}{\beta^2} \left(\frac{d^2\beta}{d\omega^2} \right)$ at $\lambda_0 \leftrightarrow \omega_0$

Optical Pulse Propagation

$$E(x, y, z, t) = \frac{E_0 \mathcal{E}(x, y) [1 - j \frac{\Delta\tau/\tau}{\tau}]^{1/2}}{[1 + (\Delta\tau/\tau)^2]^{1/2}} \exp \left\{ - \frac{(t - \tau_{go})^2}{2(\tau^2 + (\Delta\tau)^2)} \right\} \\ \exp \left\{ \frac{j(\Delta\tau/\tau)(t - \tau_{go})^2}{2[\tau^2 + (\Delta\tau)^2]} \right\} \exp \left\{ j(\omega_0 t - \beta_0 z) \right\}$$

where $\Delta\tau = \beta_2 z / \tau$, $\beta_0 = \beta(\omega_0)$, $\tau_{go} = \beta_1 z$.

The first term corresponds to the pulse

amplitude which has been decreased by $[1 + (\Delta\tau/\tau)^2]^{1/4}$ and depends on β_2 and the distance z traveled.

The second term corresponds to the pulse envelope. The peak of the pulse reaches z at time $\tau_{go} = \beta_1 z = z/v_{go}$. This reveals that the pulse peak (envelope) propagates with the group velocity. Furthermore, the pulse half width has been increased from τ to τ' given by

$$\tau' = [\tau^2 + \Delta\tau^2]^{1/2} = [\tau^2 + (\beta_2 z / \tau)^2]^{1/2} \quad \text{Therefore, the pulse has been broadened.}$$

Optical Pulse Propagation

The third term represents a frequency modulation. The instantaneous frequency of the pulse is

$$\omega' = \omega_0 + \frac{\Delta\tau}{\tau} \left(\frac{t - \tau_0}{\tau^2 + \Delta\tau^2} \right)$$

When $\beta_2 > 0$ ($D = -\left(\frac{2\pi c}{\lambda^2}\right)\beta_2 < 0$, $\frac{d\omega_0}{d\lambda} > 0$) the frequency at a fixed λ increases linearly with time (positive linear chirp). If $\beta_2 < 0$ ($D > 0$, $\frac{d\omega_0}{d\lambda} < 0$) the frequency decreases linearly with time.

Optical Pulse Propagation

