

Diffraktion Gratings Coupled-Mode Application

Integrated Optics

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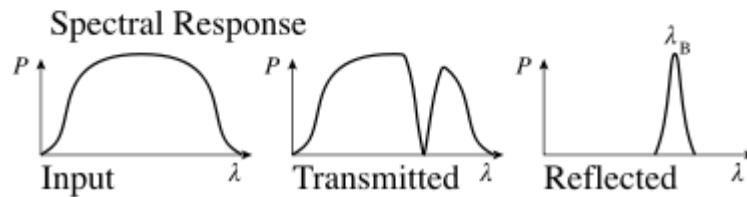
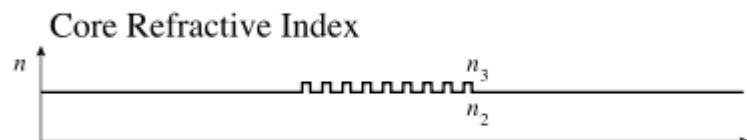
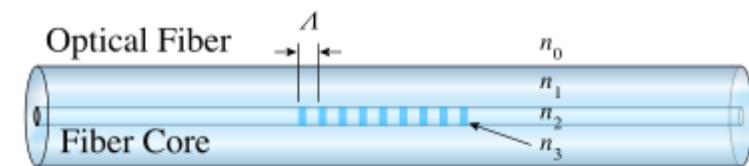
Coupled Mode Theory

$$\begin{aligned}\frac{a_m}{dz} &= -j \sum_n C_{nm} a_n(z) e^{j(\beta_m - \beta_n)z} \\ C_{nm} &= C_{nm}^t + C_{nm}^z \\ C_{nm}^t &= \frac{\omega}{4P_m} \frac{\beta_m}{|\beta_m|} \iint_S (\epsilon' - \epsilon) (\vec{\mathcal{E}}_{tm}^* \cdot \vec{\mathcal{E}}_{tn}) dx dy \\ C_{nm}^z &= \frac{\omega}{4P_m} \frac{\beta_m}{|\beta_m|} \iint_S (\epsilon' - \epsilon) \frac{\epsilon}{\epsilon'} \mathcal{E}_{zm}^* \mathcal{E}_{zn} dx dy\end{aligned}$$

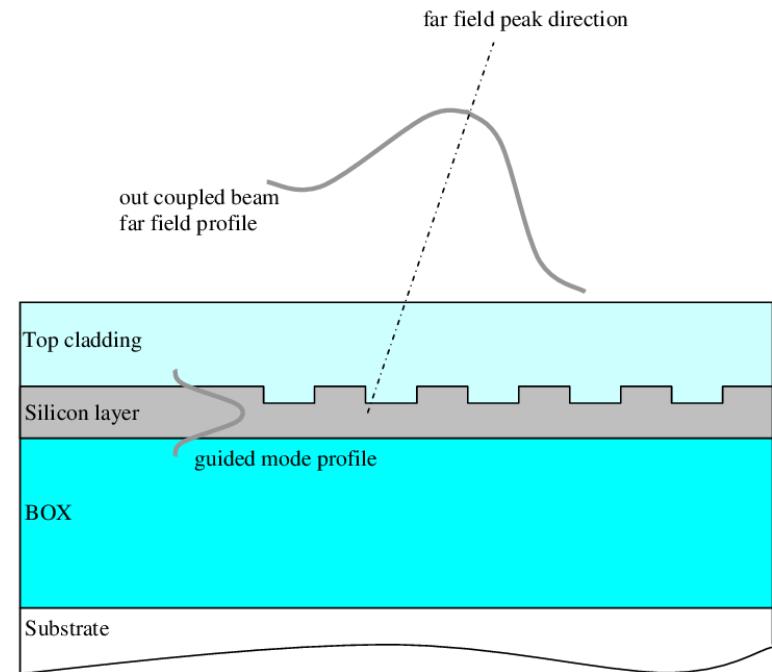
Usually all modes (guided) are normalized to unit power $P_m = 1W$

Usually perturbation is small ($\epsilon/\epsilon' \approx 1$)

Waveguide Gratings

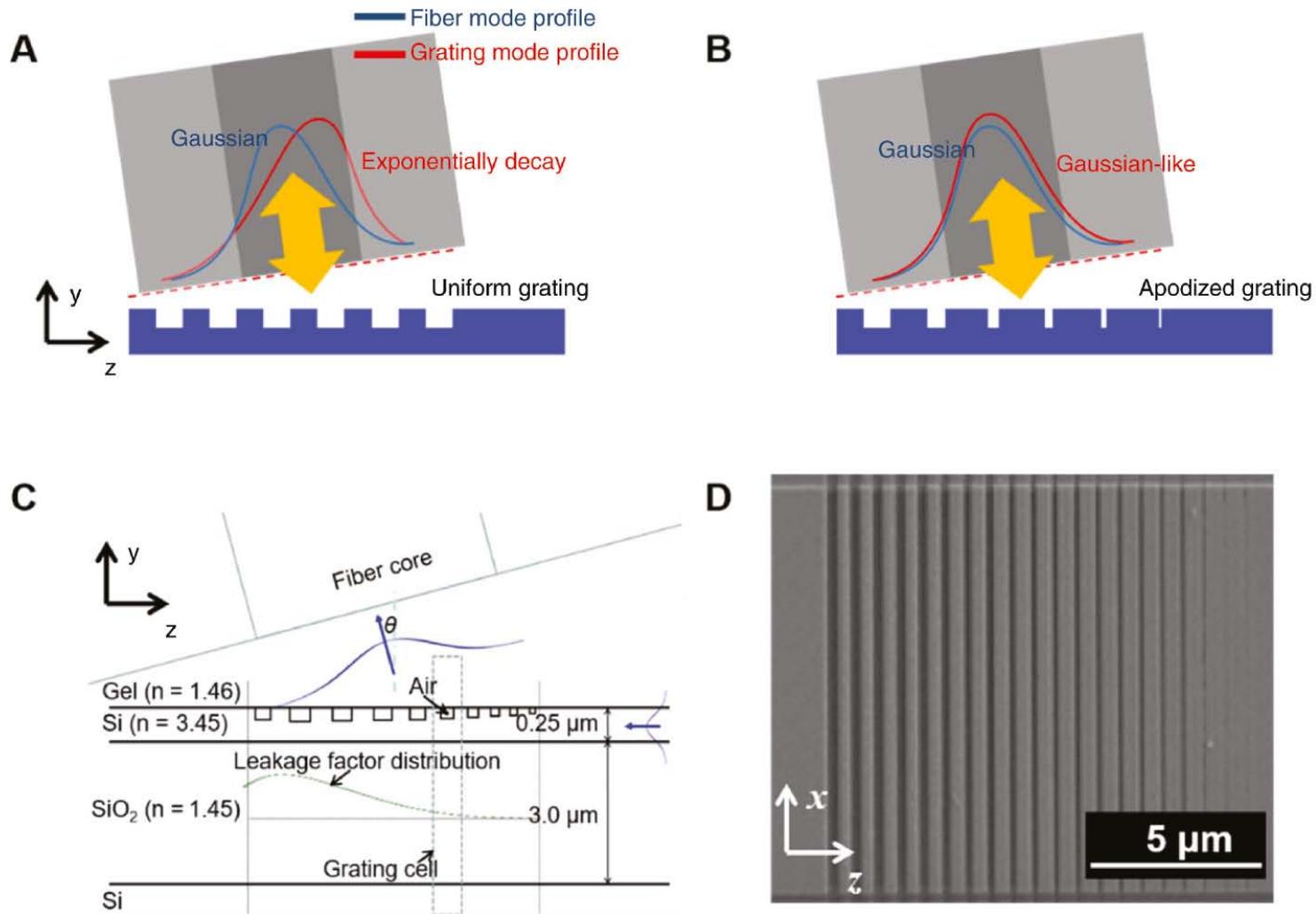


https://en.wikipedia.org/wiki/Fiber_Bragg_grating



https://www.researchgate.net/profile/Regis_Orobchouk/publication/249337957/figure/fig1/AS:668990158417927@1536511091203/Illustration-of-a-grating-coupler-layers-with-the-guided-mode-profile-and-the-out-coupled.png

Waveguide Gratings



<https://www.degruyter.com/view/journals/nanoph/7/12/article-p1845.xml?language=en>

Codirectional Coupling



$$\begin{array}{c} \xrightarrow{\beta_d} \\ \xrightarrow{K} \\ \xleftarrow{\beta_i} \end{array}$$

$$\beta_i = \beta_d + K, \quad K = 2\pi/\Lambda$$

Retain one incident and one diffracted guided mode only

$$\frac{d\alpha_1}{dz} = -j C_{11} \alpha_1 - j C_{21} \alpha_2 e^{+j(\beta_1 - \beta_2)z}$$

$$\frac{d\alpha_2}{dz} = -j C_{22} \alpha_2 - j C_{12} \alpha_1 e^{-j(\beta_1 - \beta_2)z}$$

Let's assume that $\epsilon' = \epsilon + \Delta\epsilon$ and $\Delta\epsilon = \epsilon_0 \sum_h \Delta\epsilon_h e^{jhKz}$

$$\begin{aligned} \text{Then } C_{11} &= \frac{\omega}{4} \epsilon_0 \iint_S \left(\sum_h \Delta\epsilon_h e^{jhKz} \right) \vec{e}_1 \cdot \vec{e}_1^* ds = \\ &= \sum_h \left\{ \underbrace{\frac{\omega\epsilon_0}{4} \iint_S \Delta\epsilon_h \vec{e}_1 \cdot \vec{e}_1^* ds}_{K_{11h}} \right\} e^{jhKz} = \sum_h k_{11h} e^{jhKz} \end{aligned}$$

Codirectional Coupling

$$\text{Similarly } C_{22} = \sum_h \left\{ \frac{\omega \epsilon_0}{4} \underbrace{\iint_S \Delta \epsilon_h \vec{e}_2 \cdot \vec{e}_2^* dS}_{\kappa_{22h}} \right\} e^{jhKz} = \sum_h \kappa_{22h} e^{jhKz}$$

The cross-coupling coefficients are :

$$C_{21} = \frac{\omega \epsilon_0}{4} \iint_S \sum_h \Delta \epsilon_h \vec{e}_2 \cdot \vec{e}_1^* dS e^{jhKz} = \sum_h \kappa_{21h} e^{jhKz}$$

$$C_{12} = \sum_h \kappa_{12h} e^{jhKz} \quad \kappa_{12h} = \frac{\omega \epsilon_0}{4} \iint_S \Delta \epsilon_h \vec{e}_1 \cdot \vec{e}_2^* dS$$

$$\frac{d\alpha_1}{dz} = -j \sum_h \kappa_{11h} e^{jhKz} \alpha_1(z) - j \sum_h \kappa_{21h} e^{j(\beta_1 - \beta_2 + hK)z} \alpha_2(z)$$

$$\frac{d\alpha_2}{dz} = -j \sum_h \kappa_{22h} e^{jhKz} \alpha_2(z) - j \sum_h \kappa_{12h} e^{-j(\beta_1 - \beta_2 - hK)z} \alpha_1(z)$$

Phase Matching Condition

$$\beta_1 = \beta_2 + K \quad (\text{or in general } \beta_1 = \beta_2 + qK \text{ for higher})$$

Codirectional Coupling

$$K_{ii\phi} = \frac{\omega\epsilon_0}{4} \iint_S \Delta\epsilon_\phi \vec{e}_i \cdot \vec{e}_i^* dS = 0 \quad \text{if } \Delta\epsilon_\phi = \Delta\epsilon_{avg} = 0.$$

For the K_{21h} $h = -1 \rightarrow K_{21,-1} = \frac{\omega\epsilon_0}{4} \iint \Delta\epsilon_{-1} \vec{e}_2 \cdot \vec{e}_1^* dS$

For the K_{12h} $h = +1 \rightarrow K_{12,+1} = \frac{\omega\epsilon_0}{4} \iint \Delta\epsilon_{+1} \vec{e}_1 \cdot \vec{e}_2^* dS$

For real $\Delta\epsilon$ changes $\Delta\epsilon_h = \Delta\epsilon_{-h}^* \rightarrow K_{21,-1} = K_{12,+1}^*$

Simplifying the notation, $K_{12} = K_{21,-1}$, $K_{21} = K_{12,+1}$. Then the coupled-mode equations become:

$$\frac{da_1}{dz} = -j K_{12} a_2 e^{j(\beta_1 - \beta_2 - K)z}$$

$$\frac{da_2}{dz} = -j K_{21} a_1 e^{-j(\beta_1 - \beta_2 - K)z}$$

Define $\delta = \beta_1 - \beta_2 - K$ (deviation from the Bragg condition).

$K_{12} = K_{21}^* \rightarrow$ and usually K 's are real coupling coefficients.

Codirectional Coupling

$$\frac{d\alpha_1}{dz} = -j K_{12} \alpha_2 e^{j\delta z}$$

$$\frac{d\alpha_2}{dz} = -j K_{21} \alpha_1 e^{-j\delta z}$$

The solution of the coupled-mode equations for $\alpha_1(0)=1$ and $\alpha_2(0)=0$ is given by:

$$\alpha_1(z) = -\frac{2B}{K_{21}} e^{+j\frac{\delta}{2}z} \left[S \cos(\frac{\delta}{2}z) - j \frac{\delta}{2} \sin(\frac{\delta}{2}z) \right]$$

$$S = \left[\left(\frac{\delta}{2} \right)^2 + K_{12} K_{21} \right]^{1/2} \quad K_{12} K_{21} = K_{12} K_{12}^* = |K_{12}|^2 = |K_{21}|^2$$

$$\alpha_2(z) = e^{-j\frac{\delta}{2}z} \frac{2jB}{2jB} \sin(\frac{\delta}{2}z)$$

where B is a constant.

Let the power in mode 1 be P_0 at $z=0$. Then $P_0 = |\alpha_1(0)|^2 =$

$$= \frac{4|B|^2}{|K_{21}|^2} S^2$$

Codirectional Coupling

$$P_2(z) = |\alpha_2(z)|^2 = \frac{P_0 |K_{21}|^2}{S^2} \sin^2(Sz)$$

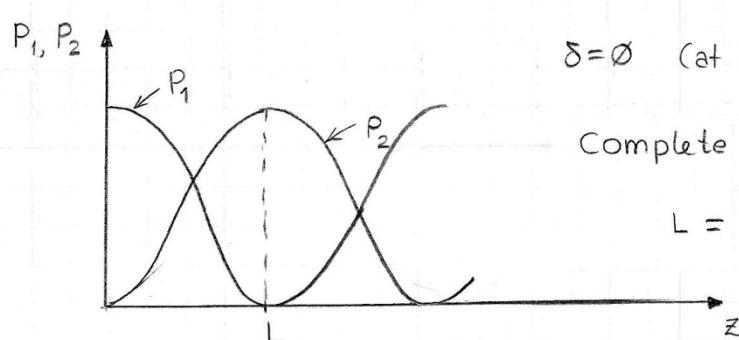
$$P_1(z) = |\alpha_1(z)|^2 = P_0 \left[\cos^2(Sz) + \frac{\delta^2}{4S^2} \sin^2(Sz) \right]$$

When the Bragg condition is satisfied then $\delta=0$ and the above equations become:

$$P_2(z) = P_0 \sin^2(|K_{12}|z)$$

$$P_1(z) = P_0 \cos^2(|K_{12}|z)$$

$$\delta=0 = \beta_1 - \beta_2 - K$$



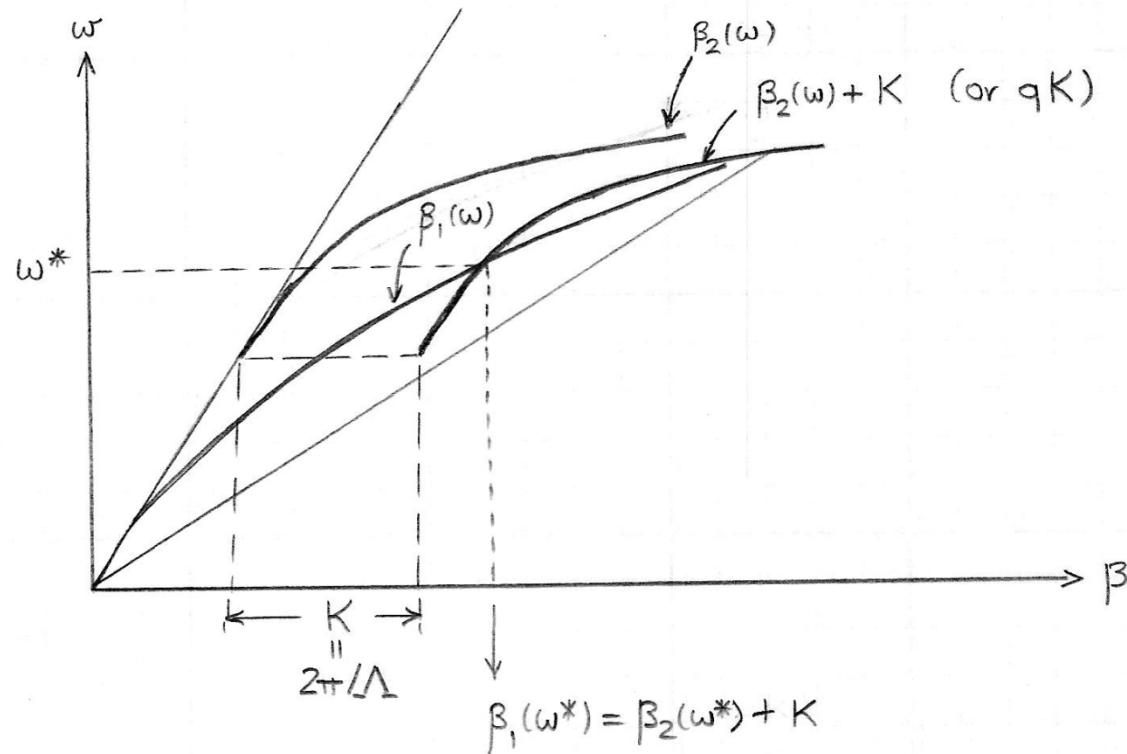
$$\delta=0 \quad (\text{at Bragg condition})$$

Complete coupling is achieved for $|K_{12}|L = \pi/2$ (or $(2m+1)\pi/2$) \Rightarrow

$$L = \frac{\pi}{2} \frac{1}{|K_{12}|}$$

Codirectional Coupling

The phase-matching (Bragg) condition can also be presented in the ω - β plane. $\delta=0$ when $\beta_1 = \beta_2 + K \Rightarrow \beta_1(\omega^*) = \beta_2(\omega^*) + K$ where ω^* is the frequency that satisfies the Bragg condition. This can be presented qualitatively in the following diagram.



Codirectional Coupling

The solutions for $\alpha_1(z)$ and $\alpha_2(z)$ can be expressed as

$$\alpha_1(z) = A e^{-j\gamma_A z} \rightarrow E_1(z) = \alpha_1(z) e^{-j\beta_1 z} = A e^{-j(\gamma_A + \beta_1)z}$$

$$\alpha_2(z) = B e^{-j\gamma_B z} \rightarrow E_2(z) = \alpha_2(z) e^{-j\beta_2 z} = B e^{-j(\gamma_B + \beta_2)z}$$

But $\gamma_B = \gamma_A + \delta$ and $\gamma_A = -\frac{\delta}{2} \pm S$. Then $\gamma_B = \frac{\delta}{2} \pm S$ and the solution of the perturbed is comprised of $E_1(z)$ and $E_2(z)$.

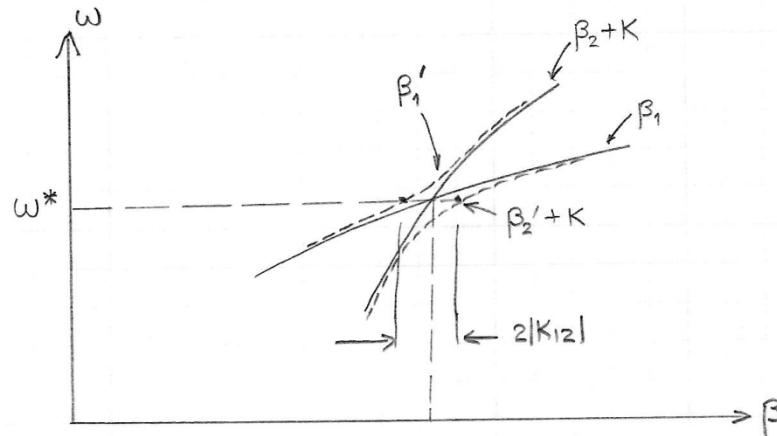
$$E(z) = A e^{-j \underbrace{[-\frac{\delta}{2} \pm S + \beta_1]}_{\beta'_1} z} + B e^{-j \underbrace{[\frac{\delta}{2} \pm S + \beta_2]}_{\beta'_2} z}$$

where $\beta'_1 = -\frac{\delta}{2} \pm S + \beta_1 = -\frac{\beta_1 - \beta_2 - K}{2} + \beta_1 \pm S = \frac{\beta_1 + \beta_2 + K}{2} \pm S$

and $\beta'_2 = +\frac{\delta}{2} \pm S + \beta_2 = \frac{\beta_1 - \beta_2 - K}{2} + \beta_2 \pm S = \frac{\beta_1 + \beta_2 - K}{2} \pm S$

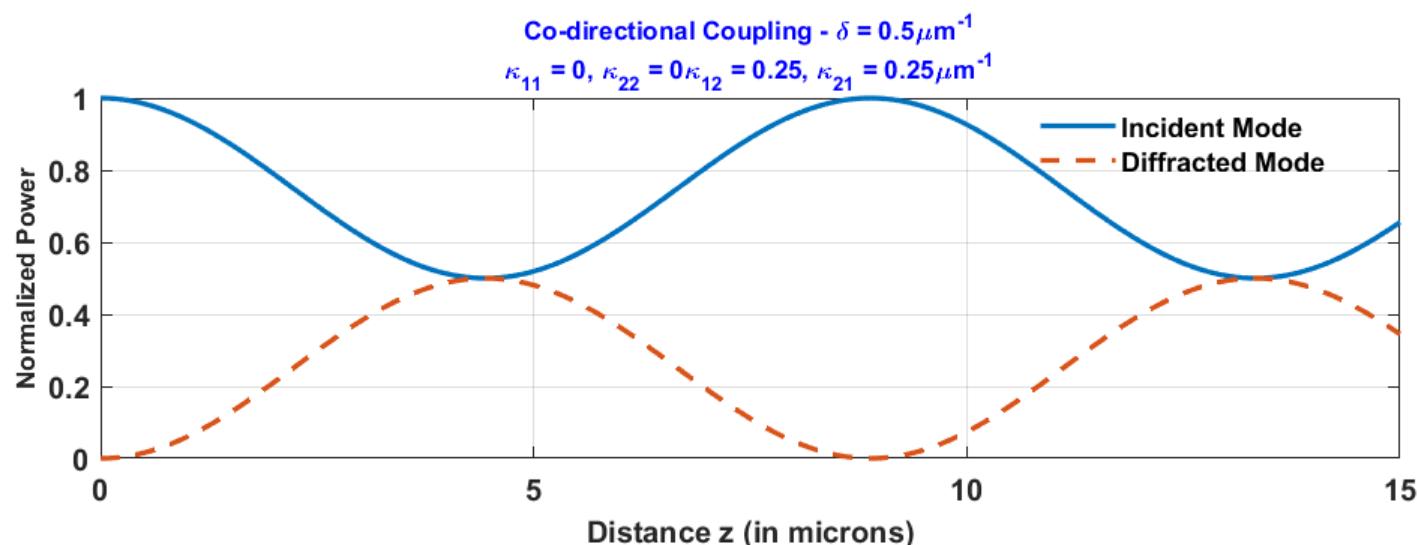
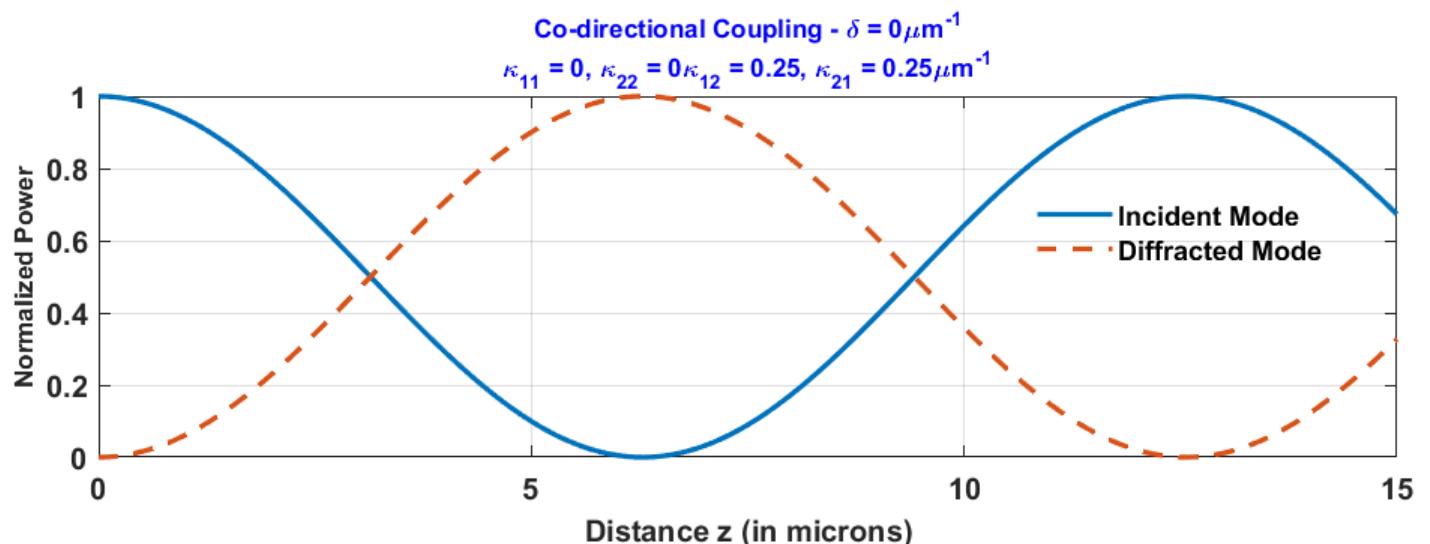
Codirectional Coupling

The β'_1 and β'_2 are the perturbed propagation constants due to the presence of the grating. When there is no grating it is straightforward to show that $\beta'_1 \rightarrow \beta_1$ and $\beta'_2 \rightarrow \beta_2$ ($K_{12}=0$). Also when δ is large again $\beta'_1 \rightarrow \beta_1$ and $\beta'_2 \rightarrow \beta_2$. Only when $\delta \approx 0$ (near the Bragg condition) we have strong interaction between the two modes.

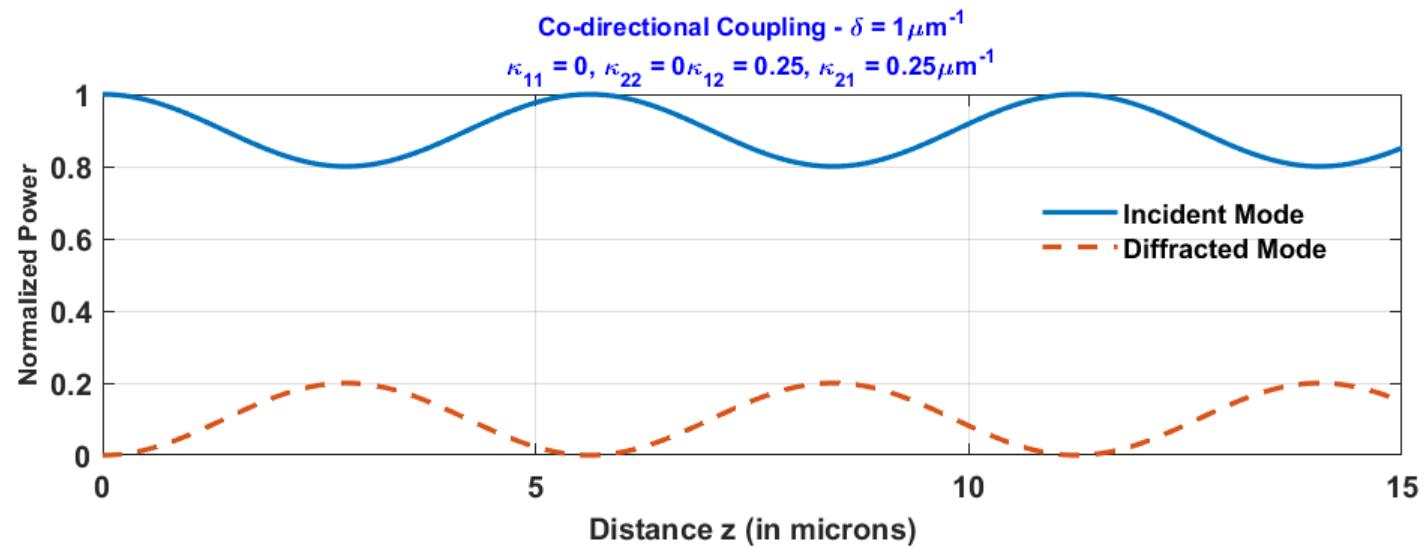


It is interesting to note that at $\omega = \omega^*$, $\beta'_1(\omega^*) - \beta'_2(\omega^*) = K \pm 2|K_{12}|$ or $\beta'_1(\omega^*) - \beta'_2(\omega^*) - K = \pm 2|K_{12}|$ which means that the gap due to the periodic perturbation depends on the coupling coefficient.

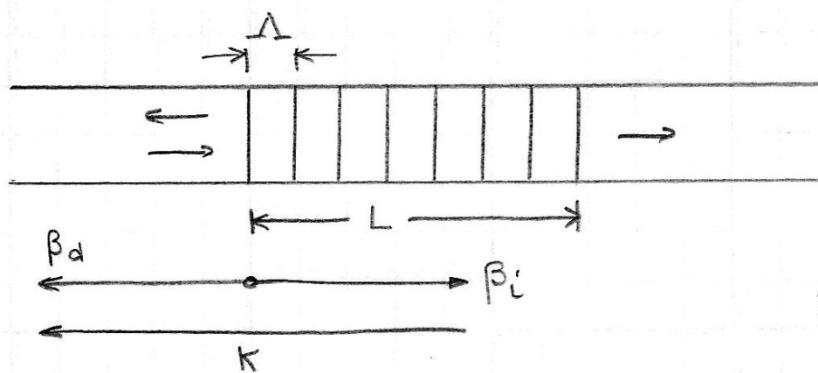
Codirectional Coupling



Codirectional Coupling



Contradirectional Coupling



$$K = \beta_i + \beta_d = \beta_i - (-\beta_d)$$

As in the previous case let's assume $\beta_i = \beta_1$, and $\beta_d = -\beta_2$ and consider only these two modes in the coupled-mode equations.

$$\frac{d\alpha_1}{dz} = -j C_{11} \alpha_1 - j C_{21} \alpha_2 e^{j(\beta_1 - \beta_2)z}$$

$$\frac{d\alpha_2}{dz} = -j C_{22} \alpha_2 - j C_{12} \alpha_1 e^{-j(\beta_1 - \beta_2)z}$$

$$C_{11} = \sum_h k_{11h} e^{jhKz} \quad \text{as previously but } C_{22} = - \sum_h k_{22h} e^{jhKz}$$

because β_2 is backward propagating mode. If it is the same mode that is coupled backwards as the incident one then $C_{22} = -C_{11}$, since $k_{11h} = k_{22h}$.

Contradirectional Coupling

The cross-coupling coefficients are:

$$C_{21} = \sum_h \kappa_{21h} e^{jhKz} \quad \text{and}$$

$$C_{12} = - \sum_h \kappa_{12h} e^{jhKz} \quad ('-' \text{ sign since } z \text{ is backward propagating}).$$

Again if mode 2 is the backward propagating version of mode 1
then $C_{21} = -C_{12}$. The κ_{ijh} are defined as in the codirectional version.

Then the coupled-mode equations are written in the form:

$$\frac{d\alpha_1}{dz} = -j \sum_h \kappa_{11h} e^{jhKz} \alpha_1(z) - j \sum_h \kappa_{21h} e^{j(\beta_1 - \beta_2 + hK)z} \alpha_2(z)$$

$$\frac{d\alpha_2}{dz} = +j \sum_h \kappa_{22h} e^{jhKz} \alpha_2(z) + j \sum_h \kappa_{12h} e^{-j(\beta_1 - \beta_2 - hK)z} \alpha_1(z)$$

The phase matching condition is $\beta_1 - \beta_2 - K = 0$ or

$\beta_1 + \beta_{|z|} - K = 0$ (since $\beta_2 = -\beta_{|z|}$ for mode 2 which is backward propagating mode).

Contradirectional Coupling

Again, for real $\Delta\epsilon$, $\kappa_{21,-1} = \kappa_{12,1}^*$ and $K_{12} = K_{21,-1}$ and $K_{21} = \kappa_{12,1}$. Also we can assume again that $\Delta n_{avg} = 0 \Rightarrow \kappa_{ii,0} = 0$ ($i=1,2$). Then the coupled mode equations take the following form:

$$\frac{d\alpha_1}{dz} = -j K_{12} \alpha_2 e^{j \underbrace{(\beta_1 - \beta_2 - K)}_{\delta} z}$$

$$\frac{d\alpha_2}{dz} = +j K_{21} \alpha_1 e^{-j \underbrace{(\beta_1 - \beta_2 - K)}_{\delta} z}$$

Contradirectional Coupling

The solutions of the above equations are:

$$\alpha_2(z) = \beta'_1 e^{-j\frac{\delta}{2}z} \sinh(Q(z-L))$$

$$\alpha_1(z) = \frac{1}{j\kappa_{21}} \cdot e^{j\frac{\delta}{2}z} \beta'_1 \left[Q \cosh(Q(z-L)) - j \frac{\delta}{2} \sinh(Q(z-L)) \right]$$

$$\text{where } \delta = \beta_1 - \beta_2 - K = \beta_1 + \beta_{121} - K, \quad Q = \left[K_{12} K_{21} - \left(\frac{\delta}{2}\right)^2 \right]^{1/2} = \left[|K_{21}|^2 - \left(\frac{\delta}{2}\right)^2 \right]^{1/2}$$

In terms of powers, assuming that the incident power in mode 1 is P_0 , we have:

$$P_1(z) = |\alpha_1(z)|^2 = P_0 \frac{Q^2 \cosh^2[Q(z-L)] + \left(\frac{\delta}{2}\right)^2 \sinh^2[Q(z-L)]}{Q^2 \cosh^2(QL) + \left(\frac{\delta}{2}\right)^2 \sinh^2(QL)}$$

$$P_2(z) = |\alpha_2(z)|^2 = P_0 \frac{|K_{21}|^2 \sinh^2(Q(z-L))}{Q^2 \cosh^2(QL) + \left(\frac{\delta}{2}\right)^2 \sinh^2(QL)}$$

Contradirectional Coupling

When the Bragg condition is satisfied $\delta = 0$ and the previous equations reduce to:

$$P_1(z) = P_0 \frac{\cosh^2(Q(z-L))}{\cosh^2(QL)}$$

$$Q = |K_{21}|$$

$$P_2(z) = P_0 \frac{\sinh^2(Q(z-L))}{\cosh^2(QL)}$$

It is interesting to note that $P_1(z) - P_2(z) = P_0 / \cosh^2(QL) = \text{constant}$

This means that $P_1(z+\Delta z) - P_2(z+\Delta z) = P_1(z) - P_2(z) \Rightarrow$

$$\underbrace{P_2(z+\Delta z) - P_2(z)}_{\text{power coupled into the backward mode}} = \underbrace{P_1(z) - P_1(z+\Delta z)}_{\text{power lost from the forward mode}}$$

power coupled
into the backward
mode

power lost from
the forward mode

Also $P_2(0) + P_1(L) = P_0$ in order to satisfy the power conservation.

Contradirectional Coupling

Another useful parameter is the grating reflectivity r which is defined

as $r = \frac{P_2(0)}{P_1(0)} = \frac{P_0 \tanh^2(QL)}{P_0} = \tanh^2(QL)$ (at Bragg).

In general

$$r = \frac{|K_{21}|^2 \sinh^2(QL)}{Q^2 \cosh^2(QL) + (\frac{\delta}{2})^2 \sinh^2(QL)}$$

It is worth mentioning that away from the Bragg condition it is possible to have $Q = \pm jS$ when $Q^2 = |K_{21}|^2 - (\frac{\delta}{2})^2 < 0$. In this case

$$r = \frac{|K_{21}|^2 \sin^2(SL)}{S^2 \cos^2(SL) + (\frac{\delta}{2})^2 \sin^2(SL)}$$

$$S = \left[\left(\frac{\delta}{2} \right)^2 - |K_{21}|^2 \right]^{1/2}$$

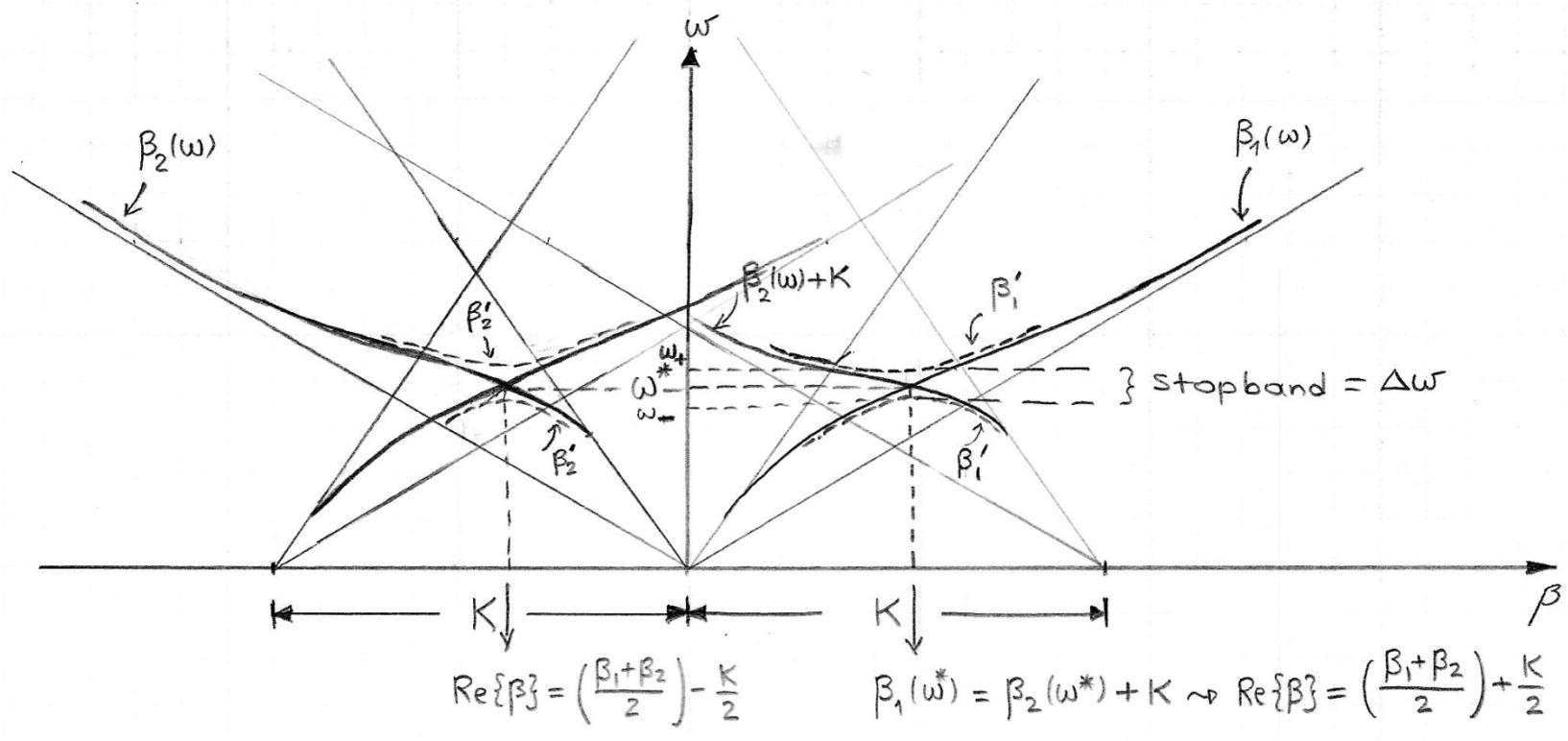
for $\left| \frac{\delta}{2} \right| > |K_{21}|$

In this case the first null of r is at $SL = \pi \Rightarrow$

$$\left(\frac{\delta}{2} \right)^2 - |K_{21}|^2 = \pi^2 / L^2 \Rightarrow \frac{\delta}{2} = \pm \sqrt{\frac{\pi^2}{L^2} + |K_{21}|^2}$$

Contradirectional Coupling

Now let's represent the phase-matching condition in an ω - β diagram.



Contradirectional Coupling

$$E_1(z) = A e^{-j \beta'_1 z} = A e^{-j (\gamma_A + \beta_1) z}$$

$$E_2(z) = B e^{-j \beta'_2 z} = B e^{-j (\gamma_B + \beta_2) z}$$

$$\beta'_1 = -\frac{\delta}{2} \pm jQ + \beta_1 = \frac{\beta_1 + \beta_2}{2} + \frac{\kappa}{2} \pm jQ \quad (\text{when } Q^2 \geq 0)$$

$$\beta_2' = +\frac{\kappa}{2} + \beta_2 \pm jQ = \frac{\beta_1 + \beta_2}{2} - \frac{\kappa}{2} \pm jQ \quad (\text{when } Q^2 \geq 0)$$

The 4 actual solutions come from the $\pm jQ$ terms. However, the

above form of β' is valid near the Bragg condition regime where

$Q^2 = |K_{12}|^2 - \left(\frac{\delta}{2}\right)^2 \gg 0$. We observe that in this regime the solutions

become complex and for this reason a stopband is shown.

The stopband limits in ω can be found as follows:

$$Q^2 = 0 \Rightarrow |\alpha| = \pm |\kappa_{12}| \Rightarrow \delta = \pm n |\kappa_{12}|$$

$$\beta_1(\omega) \approx \beta_1(\omega^*) + \left(\frac{d\omega}{d\beta_1} \right)^{-1} |_{\omega=\omega^*} (\omega - \omega^*) = \beta_1(\omega^*) + \frac{1}{u g_1} (\omega - \omega^*)$$

$$\beta_2(w) \simeq \beta_2(w^*) + \left(\frac{d\beta_2}{dw}\right)_{w^*}^{-1} (w - w^*) = \beta_2(w^*) + \frac{1}{u g_2} (w - w^*)$$

where u_{g1}, u_{g2} are the group velocities of the unperturbed modes

$$\text{at } w=w^*. \text{ Then } \delta = \beta_1(w) - \beta_2(w) - K = \underbrace{\beta_1(w^*) - \beta_2(w^*) - K}_{\text{ }} + \left(\frac{1}{u_{j_1}} - \frac{1}{u_{j_2}} \right) (w - w^*)$$

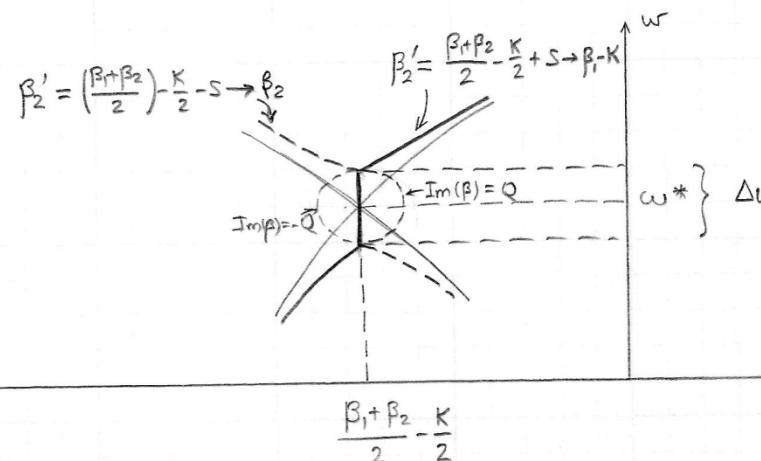
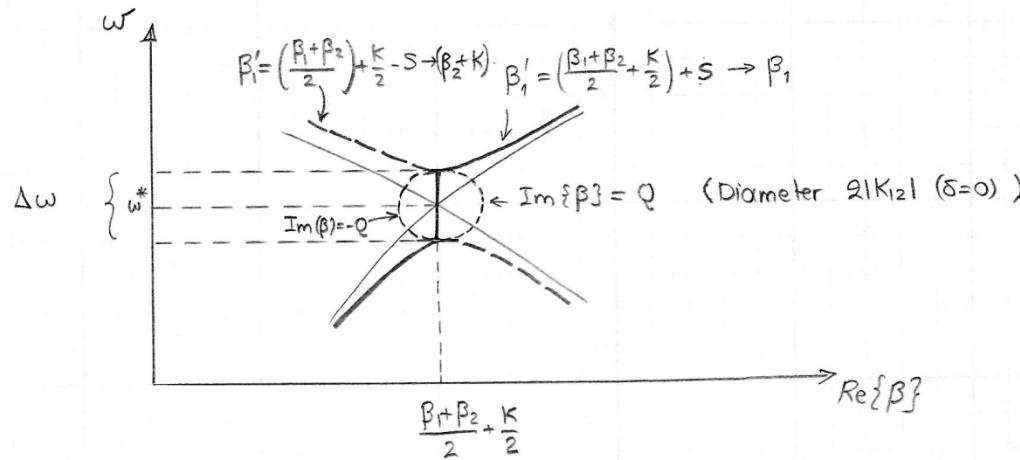
Contradirectional Coupling

$$\sim \delta = \left(\frac{1}{u_{g1}} - \frac{1}{u_{g2}} \right) (\omega - \omega^*) \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \omega - \omega^* = \frac{\pm 2 |K_{12}|}{\left(\frac{1}{u_{g1}} - \frac{1}{u_{g2}} \right)}$$

$$\text{Then } \Delta\omega = 2 |\omega - \omega^*| = \frac{4 |K_{12}|}{\left| \frac{1}{u_{g1}} - \frac{1}{u_{g2}} \right|}$$

Contradirectional Coupling

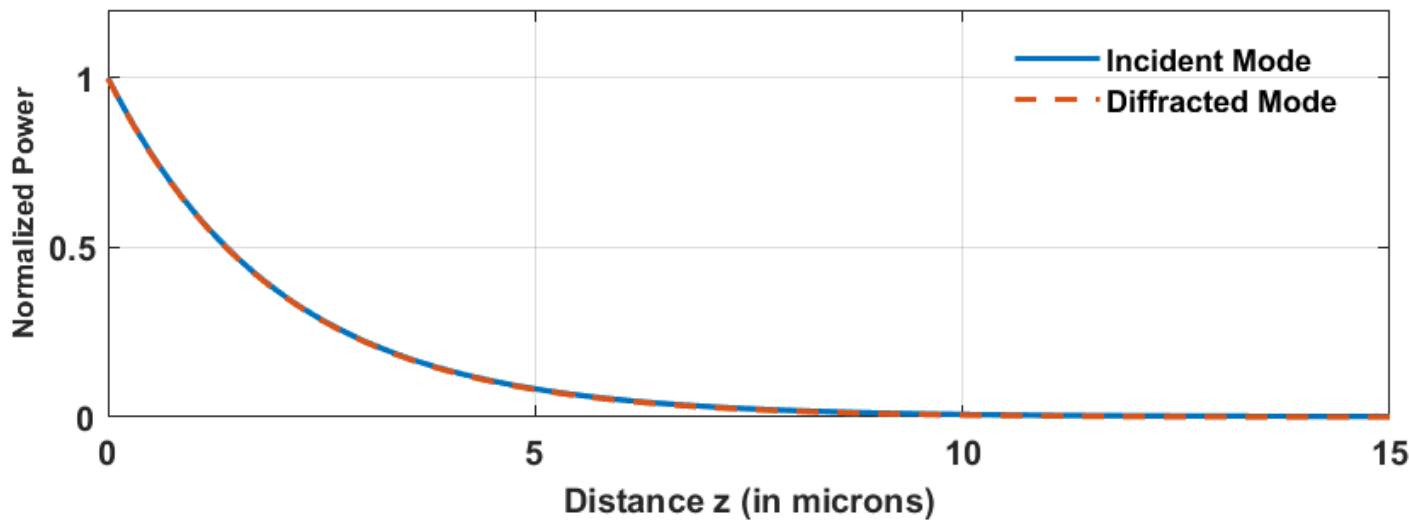
Now let's zoom near the regime where the Bragg condition is satisfied ($Q^2 \geq 0$).



Contradirectional Coupling

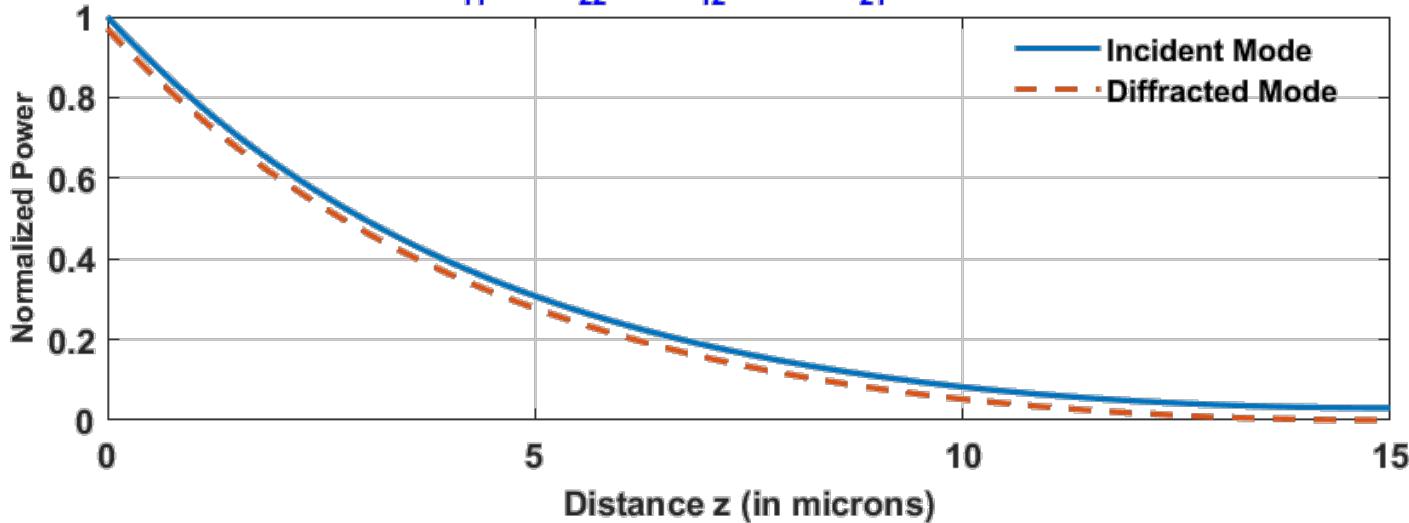
Contra-directional Coupling - $\delta = 0 \mu\text{m}^{-1}$

$$\kappa_{11} = 0, \kappa_{22} = 0, \kappa_{12} = 0.25, \kappa_{21} = 0.25 \mu\text{m}^{-1}$$

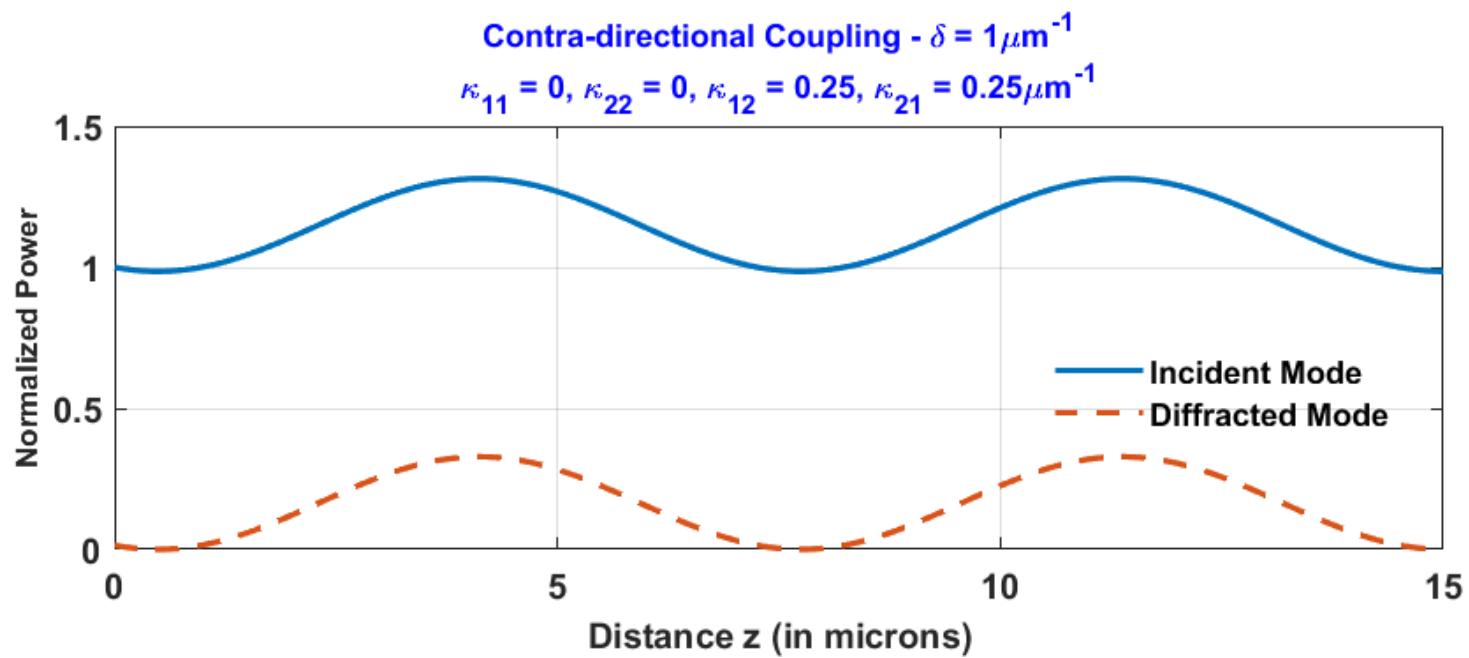


Contra-directional Coupling - $\delta = 0.45 \mu\text{m}^{-1}$

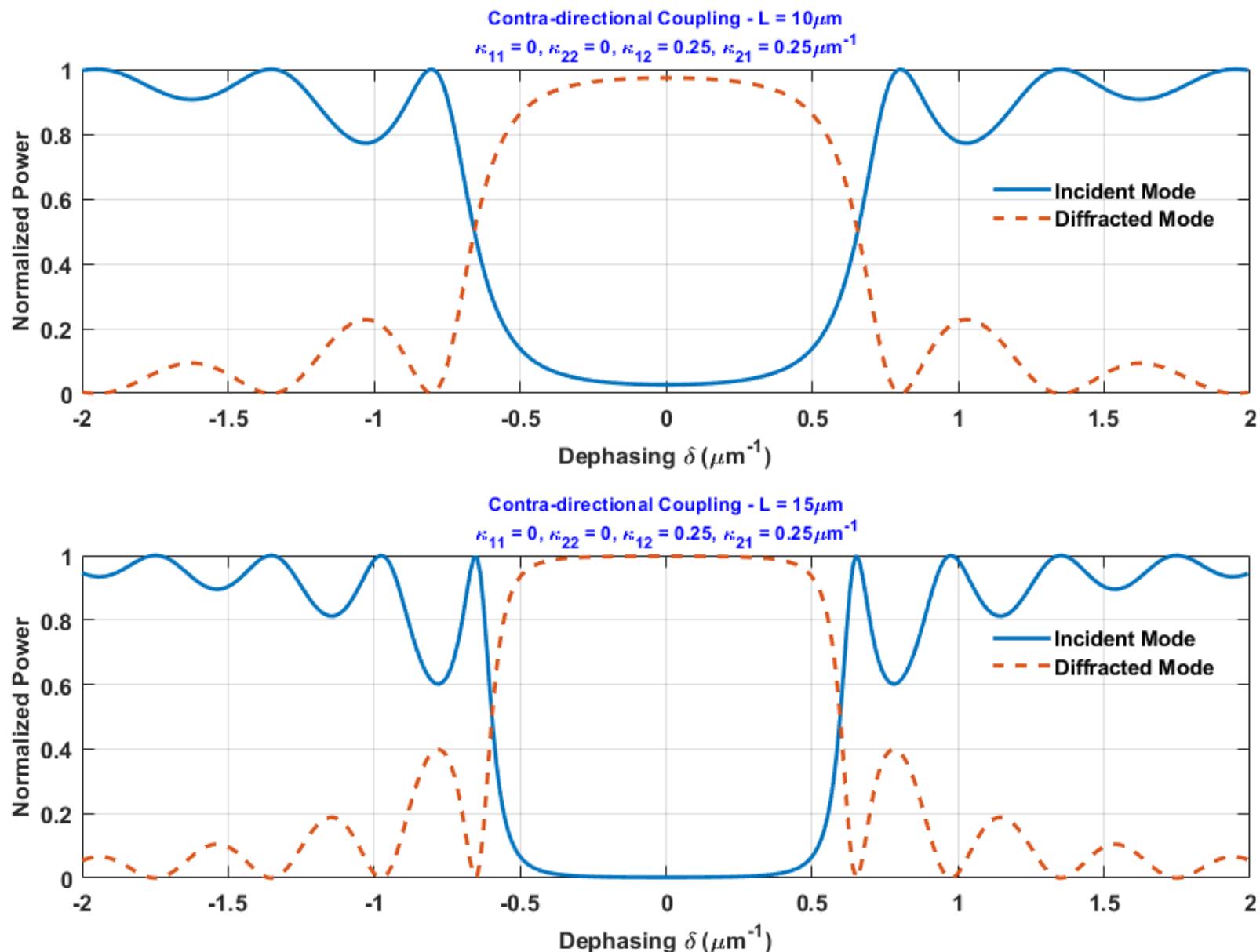
$$\kappa_{11} = 0, \kappa_{22} = 0, \kappa_{12} = 0.25, \kappa_{21} = 0.25 \mu\text{m}^{-1}$$



Contradirectional Coupling



Contradirectional Coupling



Contradirectional Coupling

