Cylindrical Dielectric Waveguides

Integrated Optics

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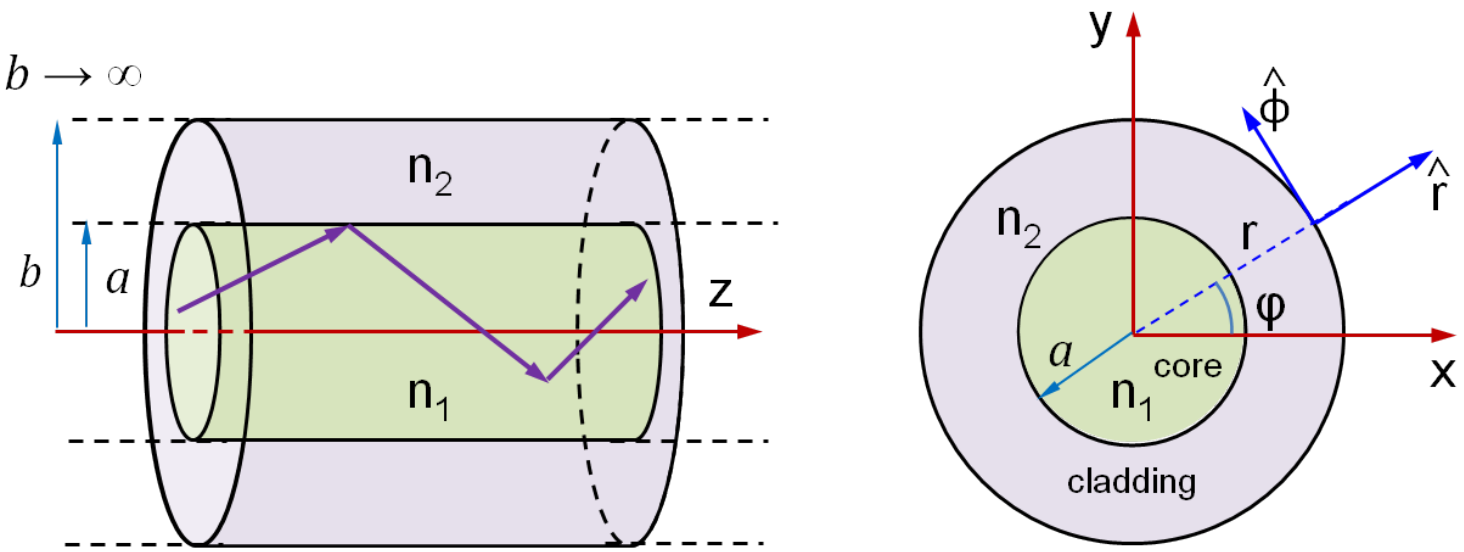
Geometry of a Single Core Layer Cylindrical Waveguide



Maxwell's Equations:

$$\vec{\nabla} \times \vec{E}_i = -j\omega\mu_0 \vec{H}_i,$$
$$\vec{\nabla} \times \vec{H}_i = +j\omega\epsilon_0 n_i^2 \vec{E}_i, \qquad i = 1, 2$$

Helmholtz's Equation:

$$\nabla^2 \vec{E}_i + k_0^2 n_i^2 \vec{E}_i = 0, \qquad (i = 1, 2)$$

$$\vec{E}_i = \vec{E}_{ti} + E_{zi}\hat{z} = \left[E_{ri}\hat{r} + E_{\phi i}\hat{\phi}\right] + E_{zi}\hat{z}, \quad (i = 1, 2)$$

Single Core Layer Cylindrical Waveguide

Helmholtz's Equations for Transverse and Longitudinal Components

$$\begin{aligned} \nabla^2 \vec{E}_{ti} + k_0^2 n_i^2 \vec{E}_{ti} &= 0, \\ \nabla^2 E_{zi} + k_0^2 n_i^2 E_{zi} &= 0, \qquad (i = 1, 2) \end{aligned}$$

$$\begin{aligned} \nabla^2 \vec{H}_{ti} + k_0^2 n_i^2 \vec{H}_{ti} &= 0, \\ \nabla^2 H_{zi} + k_0^2 n_i^2 H_{zi} &= 0, \qquad (i = 1, 2) \end{aligned}$$

Expressions for Longitudinal Components

$$\begin{aligned} E_{zi} &= E_{zi}(r, \phi) e^{-j\beta z}, \\ H_{zi} &= H_{zi}(r, \phi) e^{-j\beta z}, \qquad (i = 1, 2) \end{aligned}$$

Single Core Layer Cylindrical Waveguide

Relations of Transverse and Longitudinal Components
Maxwell's Curl Equations

$$
\begin{aligned}
\frac{1}{r}\frac{\partial E_{zi}}{\partial \phi} + j\beta E_{\phi i} &= -j\omega\mu_0 H_{ri}, \\
-j\beta E_{ri} - \frac{\partial E_{zi}}{\partial r} &= -j\omega\mu_0 H_{\phi i}, \\
\frac{1}{r}\frac{\partial (rE_{\phi i})}{\partial r} - \frac{1}{r}\frac{\partial E_{ri}}{\partial \phi} &= -j\omega\mu_0 H_{zi}, \\
\frac{1}{r}\frac{\partial H_{zi}}{\partial \phi} + j\beta H_{\phi i} &= +j\omega\epsilon_0 n_i^2 E_{ri}, \\
-j\beta H_{ri} - \frac{\partial H_{zi}}{\partial r} &= +j\omega\epsilon_0 n_i^2 E_{\phi i}, \\
\frac{1}{r}\frac{\partial (rH_{\phi i})}{\partial r} - \frac{1}{r}\frac{\partial H_{ri}}{\partial \phi} &= +j\omega\epsilon_0 n_i^2 E_{zi}.
\end{aligned}
$$

Single Core Layer Cylindrical Waveguide

Transverse Field Components as Functions of Longitudinal Field Components

$$E_{ri} = -\frac{j\beta}{k_0^2 n_i^2 - \beta^2}\left[\frac{\partial E_{zi}}{\partial r} + \frac{\omega\mu_0}{\beta}\frac{1}{r}\frac{\partial H_{zi}}{\partial \phi}\right],$$

$$E_{\phi i} = -\frac{j\beta}{k_0^2 n_i^2 - \beta^2}\left[\frac{1}{r}\frac{\partial E_{zi}}{\partial \phi} - \frac{\omega\mu_0}{\beta}\frac{\partial H_{zi}}{\partial r}\right],$$

$$H_{ri} = -\frac{j\beta}{k_0^2 n_i^2 - \beta^2}\left[\frac{\partial H_{zi}}{\partial r} - \frac{\omega\epsilon_0 n_i^2}{\beta}\frac{1}{r}\frac{\partial E_{zi}}{\partial \phi}\right],$$

$$H_{\phi i} = -\frac{j\beta}{k_0^2 n_i^2 - \beta^2}\left[\frac{1}{r}\frac{\partial H_{zi}}{\partial \phi} + \frac{\omega\epsilon_0 n_i^2}{\beta}\frac{\partial E_{zi}}{\partial r}\right].$$

Single Core Layer Cylindrical Waveguide

Longitudinal Electric Field Component

$$\frac{\partial^2 E_{zi}}{\partial r^2} + \frac{1}{r}\frac{\partial E_{zi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_{zi}}{\partial \phi^2} + (k_0^2 n_i^2 - \beta^2) E_{zi} = 0, \qquad (i = 1, 2)$$

Separation of Variables

$$E_{zi} = R_i(r)\Phi_i(\phi)$$

$$\begin{aligned} \frac{d^2\Phi_i}{d\phi^2} + \nu^2 \Phi_i &= 0, \\ \frac{d^2 R_i}{dr^2} + \frac{1}{r}\frac{dR_i}{dr} + \left(k_0^2 n_i^2 - \beta^2 - \frac{\nu^2}{r^2}\right) R_i &= 0, \qquad (i = 1, 2). \end{aligned}$$

Single Core Layer Cylindrical Waveguide Field Solutions

Azimuthal and Radial Solutions

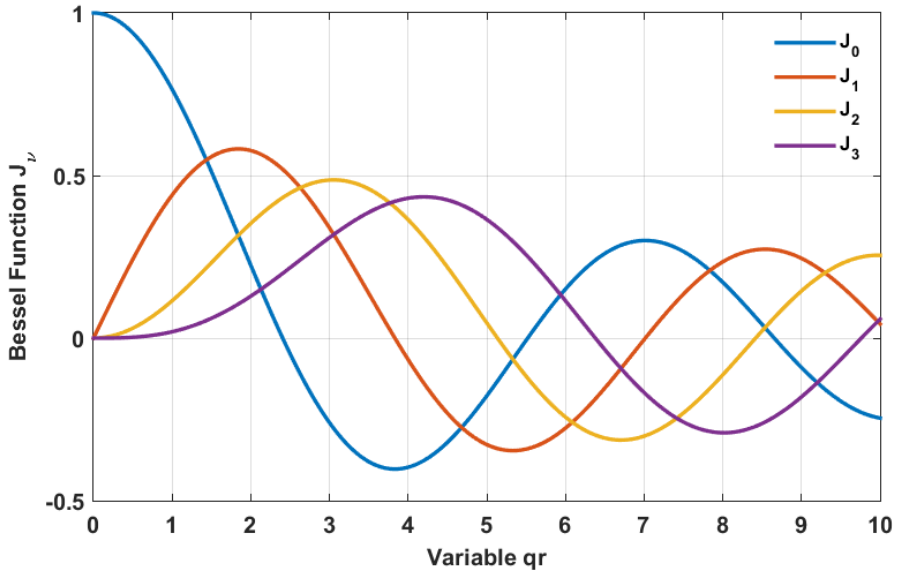
$$\Phi_i(\phi) = C_i e^{+j\nu\phi} + D_i e^{-j\nu\phi}, \qquad (i = 1, 2)$$

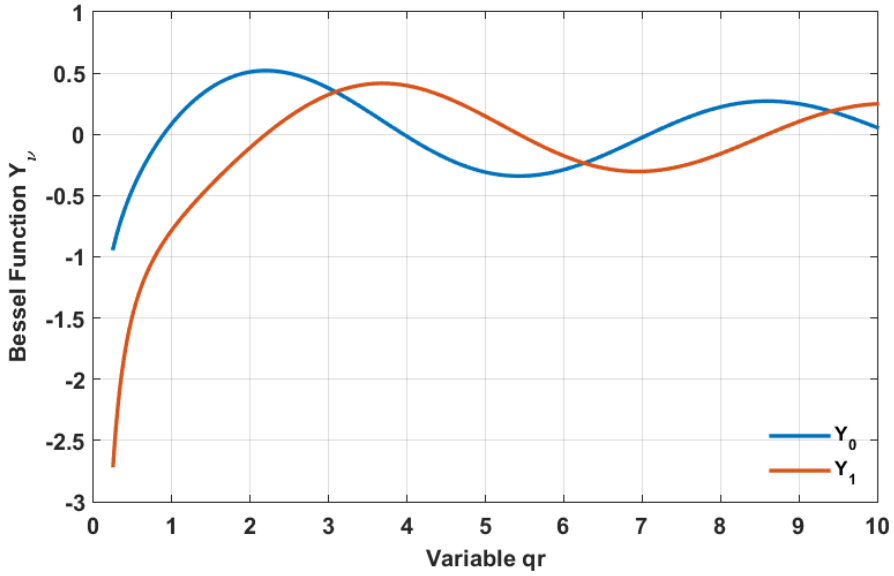
$$R_i(r) = \begin{cases} A_i J_\nu(qr) + B_i Y_\nu(qr), & q^2 = k_0^2 n_i^2 - \beta^2, \quad \text{if} \quad \beta < k_0 n_i, \\ A_i I_\nu(qr) + B_i K_\nu(qr), & q^2 = \beta^2 - k_0^2 n_i^2, \quad \text{if} \quad \beta > k_0 n_i, \end{cases}$$

Transverse Electric Field Component Solution

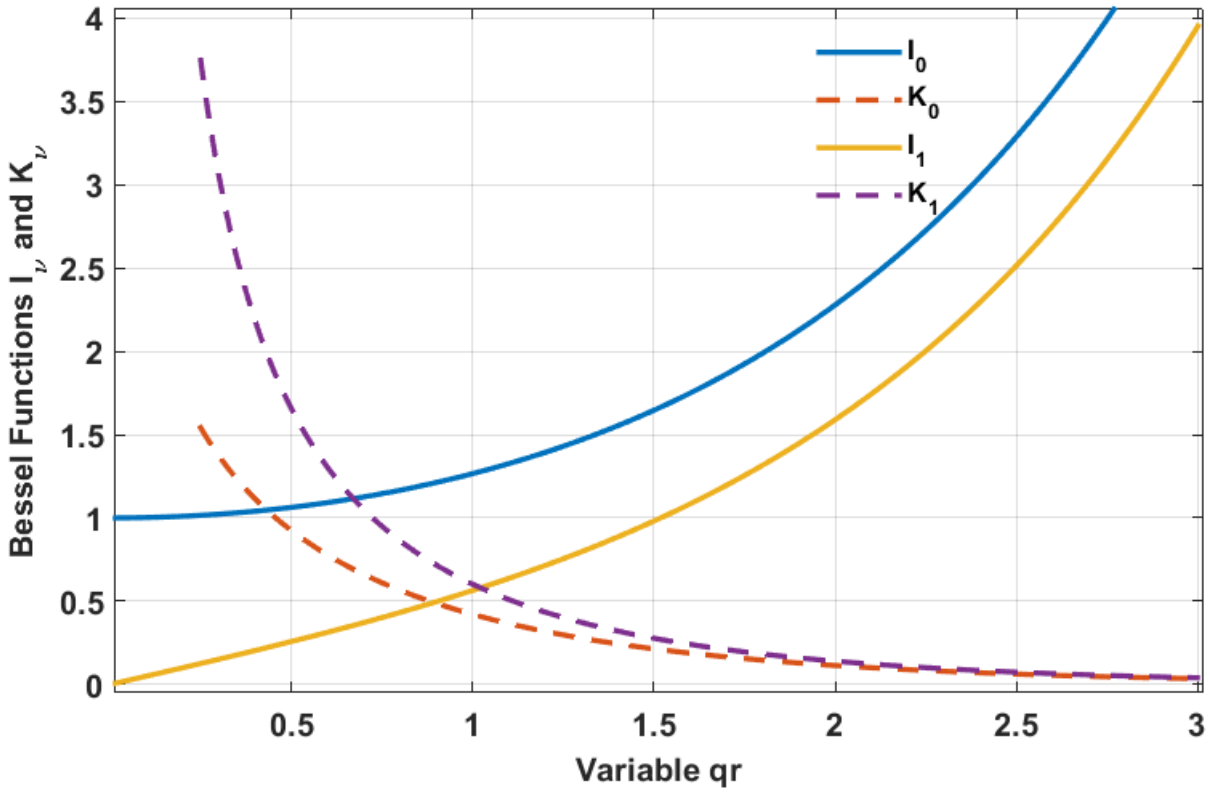
$$E_{zi} = \left[A_i \mathcal{C}_\nu(qr) + B_i \mathcal{D}_\nu(qr)\right] \left[C_i e^{+j\nu\phi} + D_i e^{-j\nu\phi}\right]$$

Oscillatory Bessel Functions Behavior (J_ν and Y_ν)





Modified Bessel Functions Behavior (I_v and K_v)



Single Core Layer Cylindrical Waveguide Field Solutions

Guided Modes Condition

$$0 < k_0 n_2 < \beta < k_0 n_1$$

Core-Region Longitudinal Field Components (r < a)

$$\begin{aligned} E_{z1} &= A_1 J_\nu(\kappa r)\left[C_{e1}e^{+j\nu\phi} + D_{e1}e^{-j\nu\phi}\right]\exp(-j\beta z), \\ H_{z1} &= F_1 J_\nu(\kappa r)\left[C_{h1}e^{+j\nu\phi} + D_{h1}e^{-j\nu\phi}\right]\exp(-j\beta z), \\ \kappa &= [k_0^2 n_1^2 - \beta^2]^{1/2}, \end{aligned}$$

Cladding-Region Longitudinal Field Components (r > a)

$$\begin{aligned} E_{z2} &= B_2 K_\nu(\gamma r)\left[C_{e2}e^{+j\nu\phi} + D_{e2}e^{-j\nu\phi}\right]\exp(-j\beta z), \\ H_{z2} &= G_2 K_\nu(\gamma r)\left[C_{h2}e^{+j\nu\phi} + D_{h2}e^{-j\nu\phi}\right]\exp(-j\beta z), \\ \gamma &= [\beta^2 - k_0^2 n_2^2]^{1/2}, \end{aligned}$$

Single Core Layer Cylindrical Waveguide Boundary Conditions

z-component of Electric field (E_z) at r = a

$$A_1 J_\nu(\kappa a)\left[C_{e1}e^{+j\nu\phi} + D_{e1}e^{-j\nu\phi}\right] = B_2 K_\nu(\gamma a)\left[C_{e2}e^{+j\nu\phi} + D_{e2}e^{-j\nu\phi}\right],$$

which implies,

$$A_1 J_\nu(\kappa a) = B_2 K_\nu(\gamma a),$$

$$C_{e1} = C_{e2} = C_e \quad \text{and} \quad D_{e1} = D_{e2} = D_e,$$

z-component of Magnetic field (H_z) at r = a

$$F_1 J_\nu(\kappa a)\left[C_{h1}e^{+j\nu\phi} + D_{h1}e^{-j\nu\phi}\right] = G_2 K_\nu(\gamma a)\left[C_{h2}e^{+j\nu\phi} + D_{h2}e^{-j\nu\phi}\right],$$

which implies,

$$F_1 J_\nu(\kappa a) = G_2 K_\nu(\gamma a),$$

$$C_{h1} = C_{h2} = C_h \quad \text{and} \quad D_{h1} = D_{h2} = D_h,$$

Single Core Layer Cylindrical Waveguide Boundary Conditions

ϕ-component of Electric field (E_ϕ) at r = a

$$\frac{-j\beta}{k_0^2 n_1^2 - \beta^2}\left\{A_1 \frac{j\nu}{a} J_\nu(\kappa a)\left[C_e e^{+j\nu\phi} - D_e e^{-j\nu\phi}\right] - F_1 \frac{\omega\mu_0}{\beta}\kappa J'_\nu(\kappa a)\left[C_h e^{+j\nu\phi} + D_h e^{-j\nu\phi}\right]\right\} =$$
$$\frac{-j\beta}{k_0^2 n_2^2 - \beta^2}\left\{B_2 \frac{j\nu}{a} K_\nu(\gamma a)\left[C_e e^{+j\nu\phi} - D_e e^{-j\nu\phi}\right] - G_2 \frac{\omega\mu_0}{\beta}\gamma K'_\nu(\gamma a)\left[C_h e^{+j\nu\phi} + D_h e^{-j\nu\phi}\right]\right\},$$

which implies,

$$\frac{-j\beta}{k_0^2 n_1^2 - \beta^2}\left\{A_1 \frac{j\nu}{a} J_\nu(\kappa a) - F_1 \frac{\omega\mu_0}{\beta}\kappa J'_\nu(\kappa a)\right\} =$$
$$\frac{-j\beta}{k_0^2 n_2^2 - \beta^2}\left\{B_2 \frac{j\nu}{a} K_\nu(\gamma a) - G_2 \frac{\omega\mu_0}{\beta}\gamma K'_\nu(\gamma a)\right\} \Longrightarrow$$

$$A_1 \frac{\beta\nu}{\kappa^2 a} J_\nu(\kappa a) + F_1 \frac{j\omega\mu_0}{\kappa} J'_\nu(\kappa a) = B_2 \frac{\beta\nu}{-\gamma^2 a} K_\nu(\gamma a) + G_2 \frac{j\omega\mu_0}{-\gamma} K'_\nu(\gamma a),$$
$$\frac{C_e}{C_h} = -\frac{D_e}{D_h} = \chi = 1,$$
$$C_e = C_h = C \quad \text{and} \quad D_e = -D_h = D,$$

Single Core Layer Cylindrical Waveguide Boundary Conditions

ϕ-component of Magnetic field (H_ϕ) at r = a

$$\frac{-j\beta}{k_0^2 n_1^2 - \beta^2}\left\{F_1 \frac{j\nu}{a} J_\nu(\kappa a)\left[C^{+j\nu\phi} + De^{-j\nu\phi}\right] - A_1 \frac{\omega\epsilon_0 n_1^2}{\beta}\kappa J'_\nu(\kappa a)\left[Ce^{+j\nu\phi} + D_e^{-j\nu\phi}\right]\right\} =$$
$$\frac{-j\beta}{k_0^2 n_2^2 - \beta^2}\left\{G_2 \frac{j\nu}{a} K_\nu(\gamma a)\left[Ce^{+j\nu\phi} + De^{-j\nu\phi}\right] - G_2 \frac{\omega\mu_0}{\beta}\gamma K'_\nu(\gamma a)\left[Ce^{+j\nu\phi} + De^{-j\nu\phi}\right]\right\},$$

which implies,

$$F_1 \frac{\beta\nu}{\kappa^2 a} J_\nu(\kappa a) - A_1 \frac{j\omega\epsilon_0 n_1^2}{\kappa} J'_\nu(\kappa a) = G_2 \frac{\beta\nu}{-\gamma^2 a} K_\nu(\gamma a) - B_2 \frac{j\omega\epsilon_0 n_2^2}{-\gamma} K'_\nu(\gamma a)$$

Single Core Layer Cylindrical Waveguide Dispersion Equation

Guided Modes Condition

$$0 < k_0 n_2 < \beta < k_0 n_1$$

$$\underbrace{\begin{bmatrix} J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) & 0 \\ 0 & J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) \\ \frac{\beta\nu}{\kappa^2 a} J_\nu(\kappa a) & \frac{j\omega\mu_0}{\kappa} J'_\nu(\kappa a) & \frac{\beta\nu}{\gamma^2 a} K_\nu(\gamma a) & \frac{j\omega\mu_0}{\beta}\gamma K'_\nu(\gamma a) \\ -\frac{j\omega\epsilon_0 n_1^2}{\kappa} J'_\nu(\kappa a) & \frac{\beta\nu}{\kappa^2 a} J_\nu(\kappa a) & -\frac{j\omega\epsilon_0 n_2^2}{\gamma} K'_\nu(\gamma a) & \frac{\beta\nu}{\gamma^2 a} K_\nu(\gamma a) \end{bmatrix}}_{\tilde{\mathcal{A}}(\beta,\nu)} \begin{bmatrix} A_1 \\ F_1 \\ B_2 \\ G_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$det\left\{\tilde{\mathcal{A}}(\beta,\nu)\right\} = 0$$

Single Core Layer Cylindrical Waveguide Dispersion Equation

$$\left[\frac{J'_\nu(\kappa a)}{\kappa a J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma a K_\nu(\gamma a)}\right]\left[n_1^2\frac{J'_\nu(\kappa a)}{\kappa a J_\nu(\kappa a)}+n_2^2\frac{K'_\nu(\gamma a)}{\gamma a K_\nu(\gamma a)}\right]=\frac{\beta^2\nu^2}{k_0^2}\left[\frac{1}{\kappa^2a^2}+\frac{1}{\gamma^2a^2}\right]^2$$

Relations between coefficients

$$B_2 = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)}A_1,$$

$$G_2 = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)}F_1,$$

$$F_1 = \frac{j\beta\nu}{\omega\mu_0}\left(\frac{1}{\kappa^2a^2}+\frac{1}{\gamma^2a^2}\right)\left[\frac{J'_\nu(\kappa a)}{\kappa a J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma a K_\nu(\gamma a)}\right]^{-1}A_1,$$

$$F_1 = \frac{j\omega\epsilon_0}{\beta\nu}\left[n_1^2\frac{J'_\nu(\kappa a)}{\kappa a J_\nu(\kappa a)}+n_2^2\frac{K'_\nu(\gamma a)}{\gamma a K_\nu(\gamma a)}\right]\left(\frac{1}{\kappa^2a^2}+\frac{1}{\gamma^2a^2}\right)^{-1}A_1,$$

Single Core Layer Cylindrical Waveguide Dispersion Equation

Case of ν = 0: TE_{0m} and TM_{0m} Guided Modes

$$\underbrace{\left[\frac{J_0'(\kappa a)}{\kappa a J_0(\kappa a)}+\frac{K_0'(\gamma a)}{\gamma a K_0(\gamma a)}\right]}_{\mathcal{T}_1}\underbrace{\left[n_1^2\frac{J_0'(\kappa a)}{\kappa a J_0(\kappa a)}+n_2^2\frac{K_0'(\gamma a)}{\gamma a K_0(\gamma a)}\right]}_{\mathcal{T}_2}=0$$

Relations between coefficients

$$\begin{aligned}
B_2-\frac{J_0(\kappa a)}{K_0(\gamma a)}A_1 &= 0,\\
G_2-\frac{J_0(\kappa a)}{K_0(\gamma a)}F_1 &= 0,\\
F_1\left[\frac{J_0'(\kappa a)}{\kappa a J_0(\kappa a)}+\frac{K_0'(\gamma a)}{\gamma a K_0(\gamma a)}\right] &= 0,\\
A_1\left[n_1^2\frac{J_0'(\kappa a)}{\kappa a J_0(\kappa a)}+n_2^2\frac{K_0'(\gamma a)}{\gamma a K_0(\gamma a)}\right] &= 0.
\end{aligned}$$

Single Core Layer Cylindrical Waveguide Dispersion Equation

Case of ν = 0: TE_{0m} and TM_{0m} Guided Modes

$$\frac{J_1(\kappa a)}{\kappa a J_0(\kappa a)} + \frac{K_1(\gamma a)}{\gamma a K_0(\gamma a)} = 0, \qquad \text{for } TE_{0m},$$

$$n_1^2 \frac{J_1(\kappa a)}{\kappa a J_0(\kappa a)} + n_2^2 \frac{K_1(\gamma a)}{\gamma a K_0(\gamma a)} = 0, \qquad \text{for } TM_{0m}.$$

Single Core Layer Cylindrical Waveguide Dispersion Equation

Case of ν ≠ 0: EH_{vm} and HE_{vm} Guided Modes

Dispersion Equation for EH_{vm} Guided Modes

$$\frac{J_{\nu+1}(\kappa a)}{\kappa a J_\nu(\kappa a)} = \frac{n_1^2+n_2^2}{2n_1^2}\frac{K'_\nu(\gamma a)}{\gamma a K_\nu(\gamma a)} + \left(\frac{\nu}{\kappa^2 a^2} - \mathcal{R}\right)$$

Dispersion Equation for HE_{vm} Guided Modes

$$\frac{J_{\nu-1}(\kappa a)}{\kappa a J_\nu(\kappa a)} = -\frac{n_1^2+n_2^2}{2n_1^2}\frac{K'_\nu(\gamma a)}{\gamma a K_\nu(\gamma a)} + \left(\frac{\nu}{\kappa^2 a^2} - \mathcal{R}\right),$$

$$\text{with} \quad \mathcal{R} = \left[\left(\frac{n_1^2-n_2^2}{2n_1^2}\right)^2\left(\frac{K'_\nu(\gamma a)}{\gamma a K_\nu(\gamma a)}\right)^2 + \left(\frac{\nu\beta}{n_1 k_0}\right)^2\left(\frac{1}{\kappa^2 a^2}+\frac{1}{\gamma^2 a^2}\right)^2\right]^{1/2}$$

TE_{0m} and TM_{0m} Guided Modes
Electric & Magnetic Fields

TE_{0m} Guided Modes

$$
\begin{aligned}
E_z &= 0, \qquad \text{for} \quad 0 \le r < \infty, \\
E_r &= 0, \qquad \text{for} \quad 0 \le r < \infty, \\
E_\phi &= \begin{cases} F_1 Z_0 \left(\dfrac{+jk_0}{\kappa} \right) J_0'(\kappa r) e^{-j\beta_{0m}^{TE} z}, & 0 \le r \le a, \\ G_2 Z_0 \left(\dfrac{-jk_0}{\gamma} \right) K_0'(\gamma r) e^{-j\beta_{0m}^{TE} z}, & r \ge a \end{cases} \\
H_z &= \begin{cases} F_1 J_0(\kappa r) e^{-j\beta_{0m}^{TE} z}, & 0 \le r \le a, \\ G_2 K_0(\gamma r) e^{-j\beta_{0m}^{TE} z}, & r \ge a \end{cases} \\
H_r &= \begin{cases} F_1 \left(\dfrac{-j\beta}{\kappa} \right) J_0'(\kappa r) e^{-j\beta_{0m}^{TE} z}, & 0 \le r \le a, \\ G_2 \left(\dfrac{+j\beta}{\gamma} \right) K_0'(\gamma r) e^{-j\beta_{0m}^{TE} z}, & r \ge a \end{cases} \\
H_\phi &= 0, \qquad \text{for} \quad 0 \le r < \infty,
\end{aligned}
$$

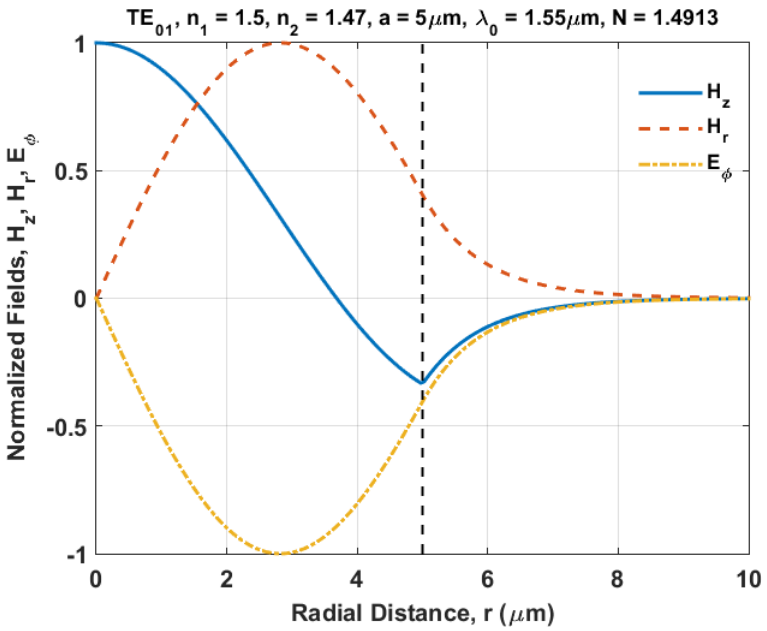
TE_{0m} and TM_{0m} Guided Modes
Electric & Magnetic Fields

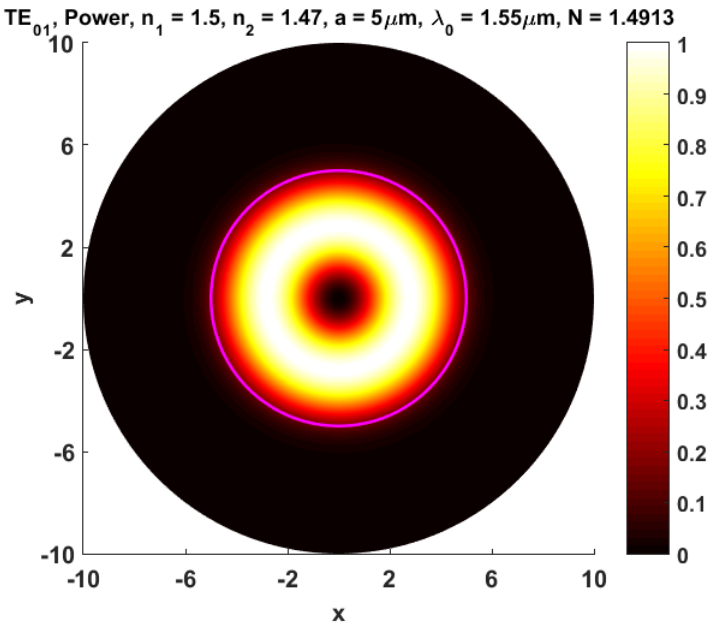
TM_{0m} Guided Modes

$$
\begin{aligned}
E_z &= \begin{cases} A_1 J_0(\kappa r) e^{-j\beta_{0m}^{TM} z}, & 0 \le r \le a, \\ B_2 K_0(\gamma r) e^{-j\beta_{0m}^{TM} z}, & r \ge a \end{cases} \\
E_r &= \begin{cases} A_1 \left(\dfrac{-j\beta}{\kappa}\right) J_0'(\kappa r) e^{-j\beta_{0m}^{TM} z}, & 0 \le r \le a, \\ B_2 \left(\dfrac{+j\beta}{\gamma}\right) K_0'(\gamma r) e^{-j\beta_{0m}^{TM} z}, & r \ge a \end{cases} \\
E_\phi &= 0, \qquad \text{for} \quad 0 \le r < \infty, \\
H_z &= 0, \qquad \text{for} \quad 0 \le r < \infty, \\
H_r &= 0, \qquad \text{for} \quad 0 \le r < \infty, \\
H_\phi &= \begin{cases} \dfrac{A_1}{Z_0} \left(\dfrac{-jk_0 n_1^2}{\kappa}\right) J_0'(\kappa r) e^{-j\beta_{0m}^{TM} z}, & 0 \le r \le a, \\ \dfrac{B_2}{Z_0} \left(\dfrac{+jk_0 n_2^2}{\gamma Z_0}\right) K_0'(\gamma r) e^{-j\beta_{0m}^{TM} z}, & r \ge a \end{cases}
\end{aligned}
$$

TE_{0m} and TM_{0m} Guided Modes
Electric & Magnetic Fields

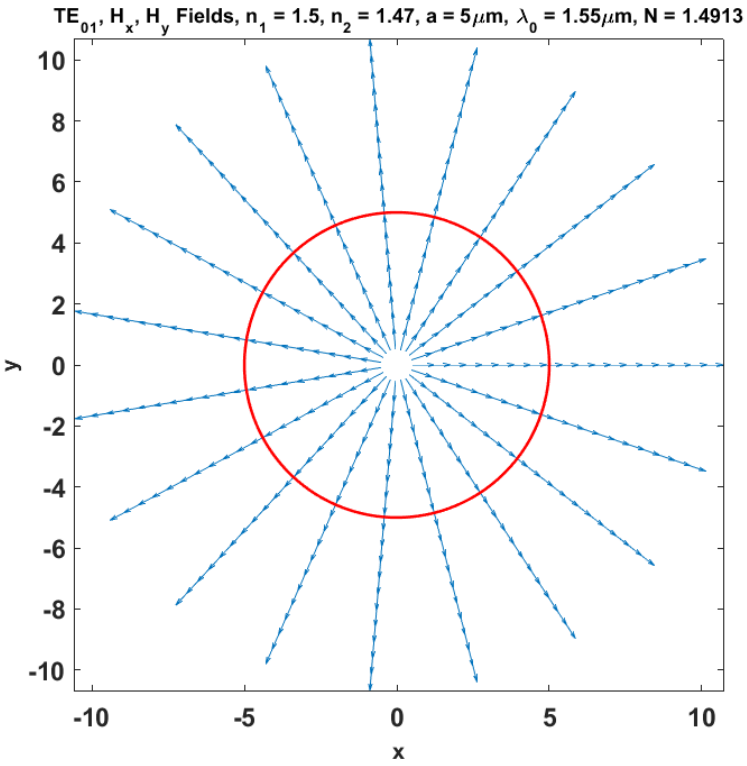
TE_{01} Guided Mode: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

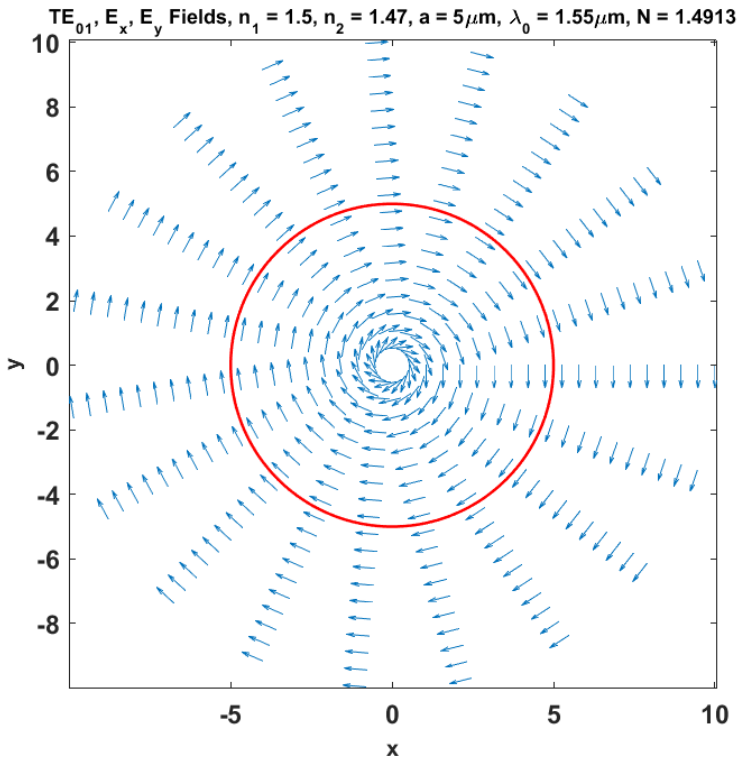




TE_{0m} and TM_{0m} Guided Modes
Electric & Magnetic Fields

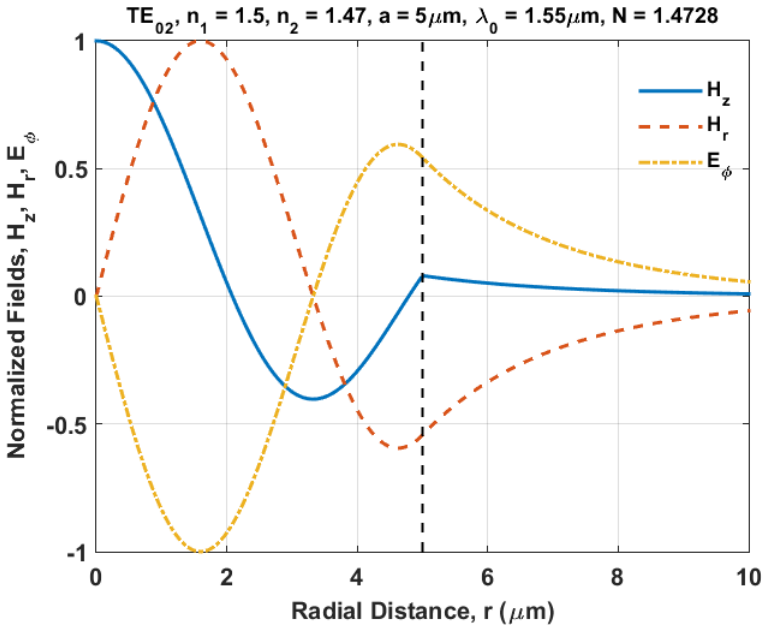
TE_{01} Guided Mode: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

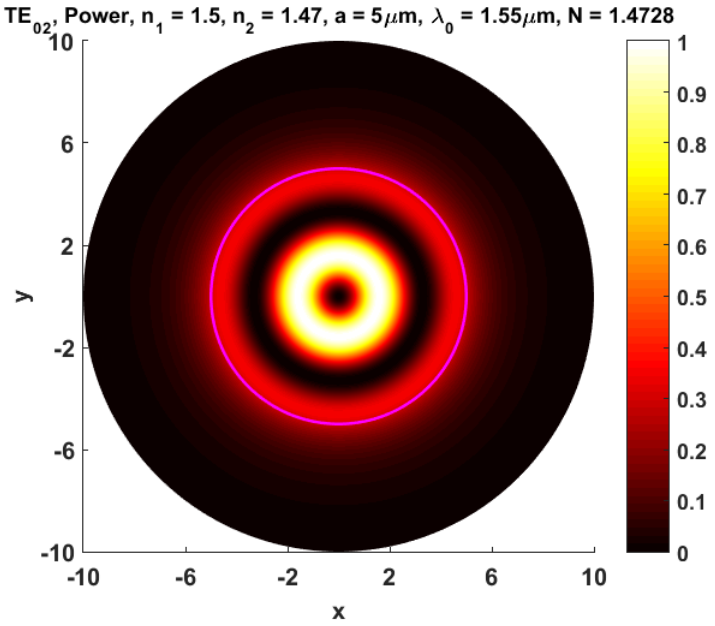




TE_{0m} and TM_{0m} Guided Modes
Electric & Magnetic Fields

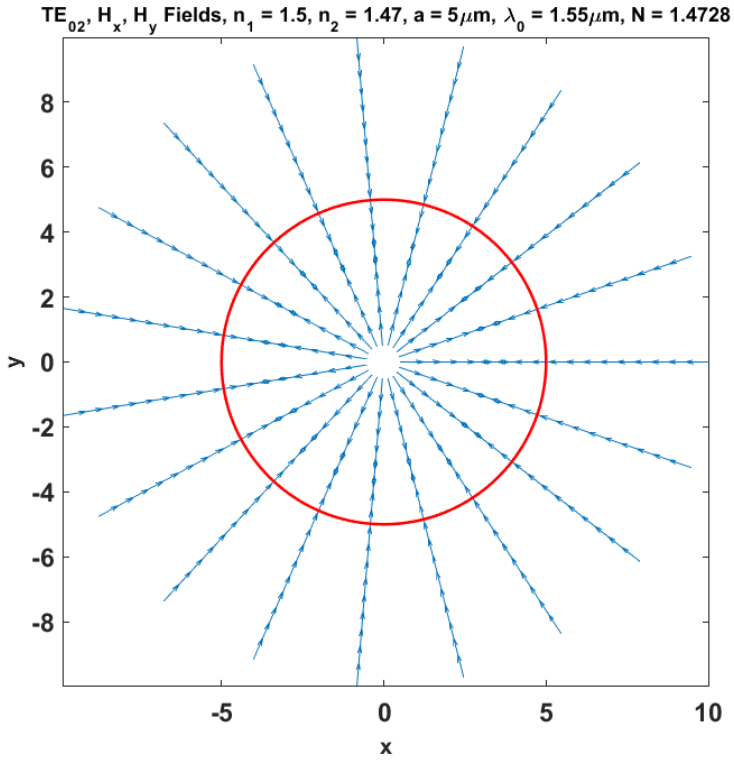
TE_{02} Guided Mode: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

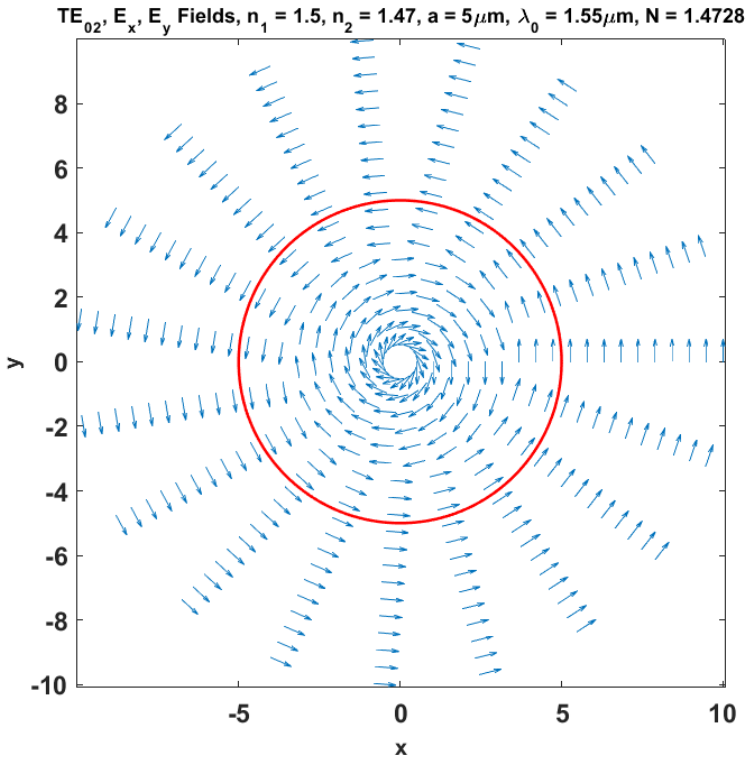




TE_{0m} and TM_{0m} Guided Modes
Electric & Magnetic Fields

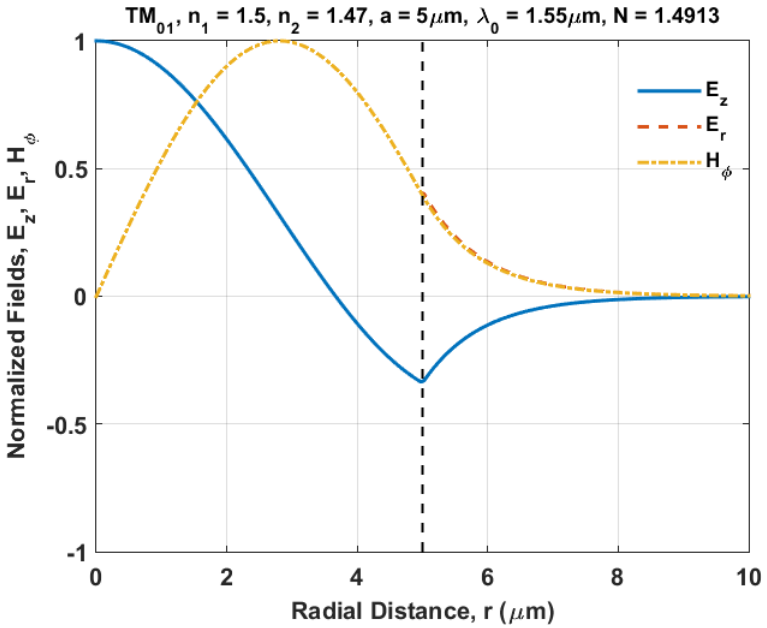
TE_{02} Guided Mode: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

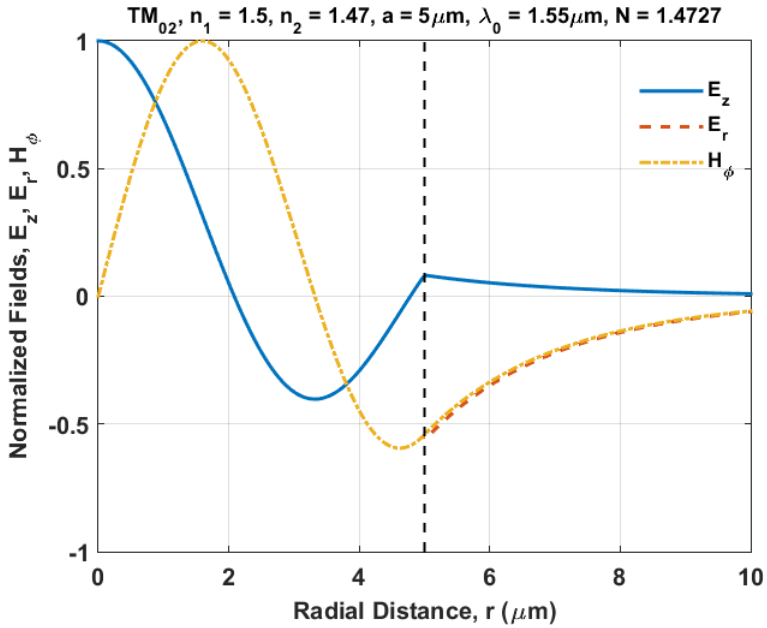




TE_{0m} and TM_{0m} Guided Modes
Electric & Magnetic Fields

TM_{01} & TM_{02} Guided Modes: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$





EH_{vm} and HE_{vm} Guided Modes
Electric Fields

$$E_z = \begin{cases} A_1 J_\nu(\kappa r) f_e(\phi) e^{-j\beta_{\nu m} z}, & 0 \le r \le a, \\ B_2 K_\nu(\gamma r) f_e(\phi) e^{-j\beta_{\nu m} z}, & r \ge a \end{cases}$$

$$E_r = \begin{cases} \left[-jA_1 \dfrac{\beta}{\kappa} J'_\nu(\kappa r) + Z_0 F_1 \dfrac{\nu k_0}{\kappa} \dfrac{J_\nu(\kappa r)}{\kappa r} \right] f_e(\phi) e^{-j\beta_{\nu m} z}, & 0 \le r \le a, \\ \left[jB_2 \dfrac{\beta}{\gamma} K'_\nu(\gamma r) - Z_0 G_2 \dfrac{\nu k_0}{\gamma} \dfrac{K_\nu(\gamma r)}{\gamma r} \right] f_e(\phi) e^{-j\beta_{\nu m} z}, & r \ge a \end{cases}$$

$$E_\phi = \begin{cases} \left[A_1 \dfrac{\nu\beta}{\kappa} \dfrac{J_\nu(\kappa r)}{\kappa r} + jZ_0 F_1 \dfrac{k_0}{\kappa} J'_\nu(\kappa r) \right] f_h(\phi) e^{-j\beta_{\nu m} z}, & 0 \le r \le a, \\ \left[-B_2 \dfrac{\nu\beta}{\gamma} \dfrac{K_\nu(\gamma r)}{\gamma r} - jZ_0 G_2 \dfrac{k_0}{\gamma} K'_\nu(\gamma r) \right] f_h(\phi) e^{-j\beta_{\nu m} z}, & r \ge a \end{cases}$$

$EH_{\nu m}$ and $HE_{\nu m}$ Guided Modes
Magnetic Fields

$$H_z = \begin{cases} F_1 J_\nu(\kappa r) f_h(\phi) e^{-j\beta_{\nu m} z}, & 0 \le r \le a, \\ G_2 K_\nu(\gamma r) f_h(\phi) e^{-j\beta_{\nu m} z}, & r \ge a \end{cases}$$

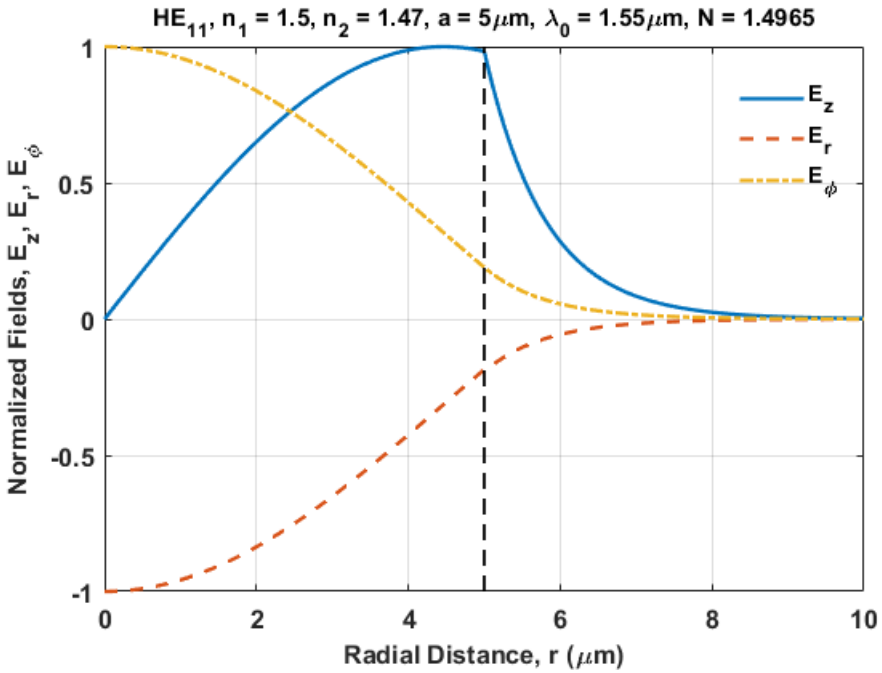
$$H_r = \begin{cases} \left[-jF_1 \dfrac{\beta}{\kappa} J'_\nu(\kappa r) - \dfrac{A_1}{Z_0} \dfrac{\nu k_0 n_1^2}{\kappa} \dfrac{J_\nu(\kappa r)}{\kappa r} \right] f_h(\phi) e^{-j\beta_{\nu m} z}, & 0 \le r \le a, \\ \left[jG_2 \dfrac{\beta}{\gamma} K'_\nu(\gamma r) + \dfrac{B_2}{Z_0} \dfrac{\nu k_0 n_2^2}{\gamma} \dfrac{K_\nu(\gamma r)}{\gamma r} \right] f_h(\phi) e^{-j\beta_{\nu m} z}, & r \ge a \end{cases}$$

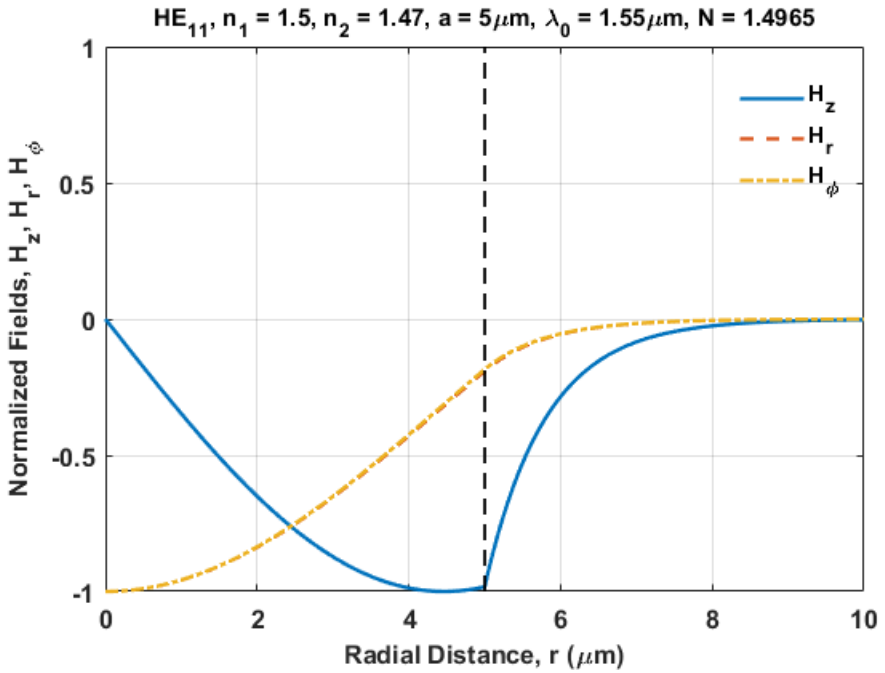
$$H_\phi = \begin{cases} \left[F_1 \dfrac{\nu\beta}{\kappa} \dfrac{J_\nu(\kappa r)}{\kappa r} - j \dfrac{A_1}{Z_0} \dfrac{k_0 n_1^2}{\kappa} J'_\nu(\kappa r) \right] f_e(\phi) e^{-j\beta_{\nu m} z}, & 0 \le r \le a, \\ \left[-G_2 \dfrac{\nu\beta}{\gamma} \dfrac{K_\nu(\gamma r)}{\gamma r} + j \dfrac{B_2}{Z_0} \dfrac{k_0 n_2^2}{\gamma} K'_\nu(\gamma r) \right] f_e(\phi) e^{-j\beta_{\nu m} z}, & r \ge a \end{cases}$$

$$f_e(\phi) = \begin{pmatrix} \cos(\nu\phi) \\ \sin(\nu\phi) \end{pmatrix} \qquad f_h(\phi) = \begin{pmatrix} j\sin(\nu\phi) \\ -j\cos(\nu\phi) \end{pmatrix}$$

HE_{11} Guided Mode Electric & Magnetic Fields

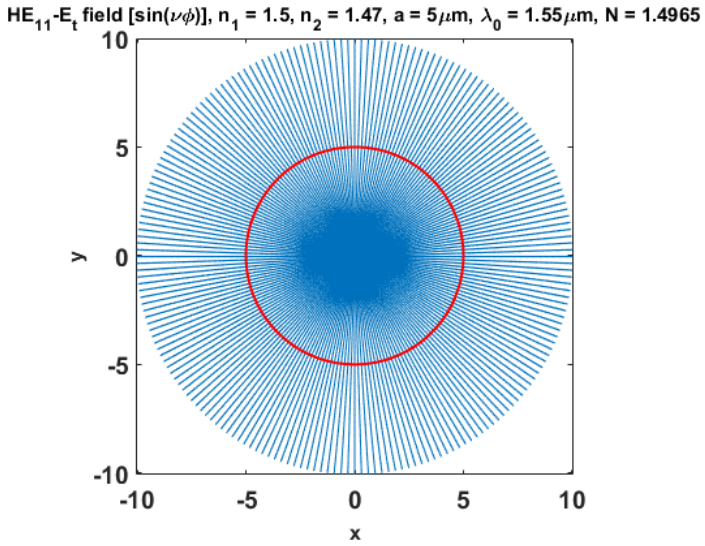
$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

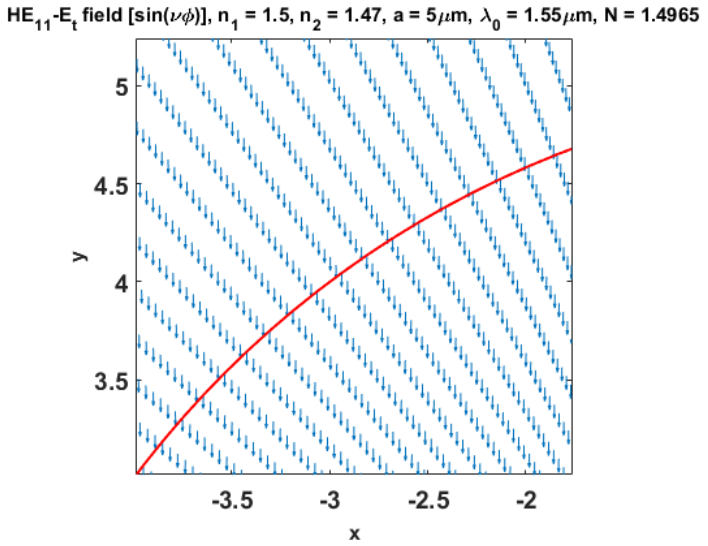


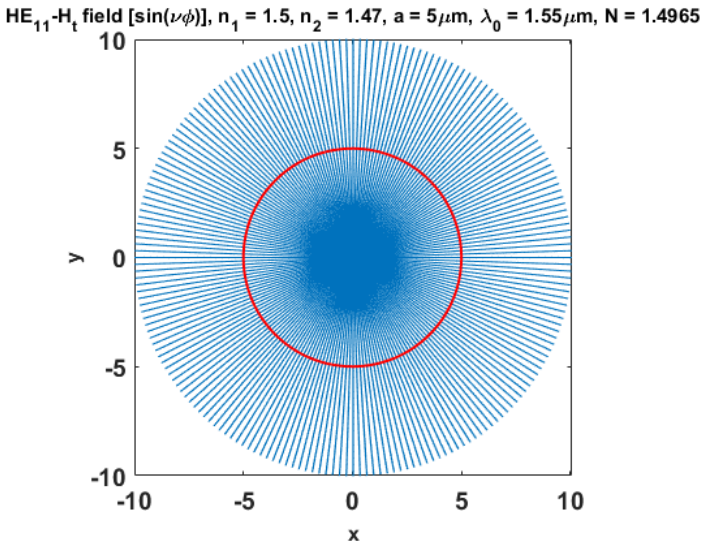


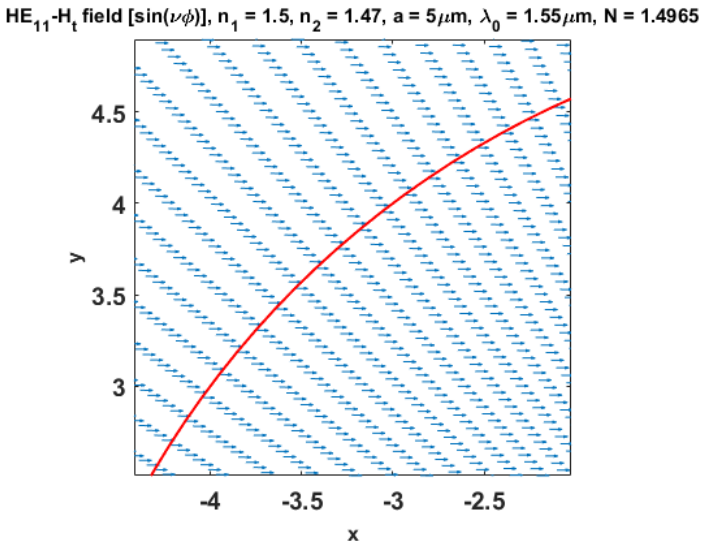
HE_{11} Guided Mode Electric & Magnetic Fields

$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$



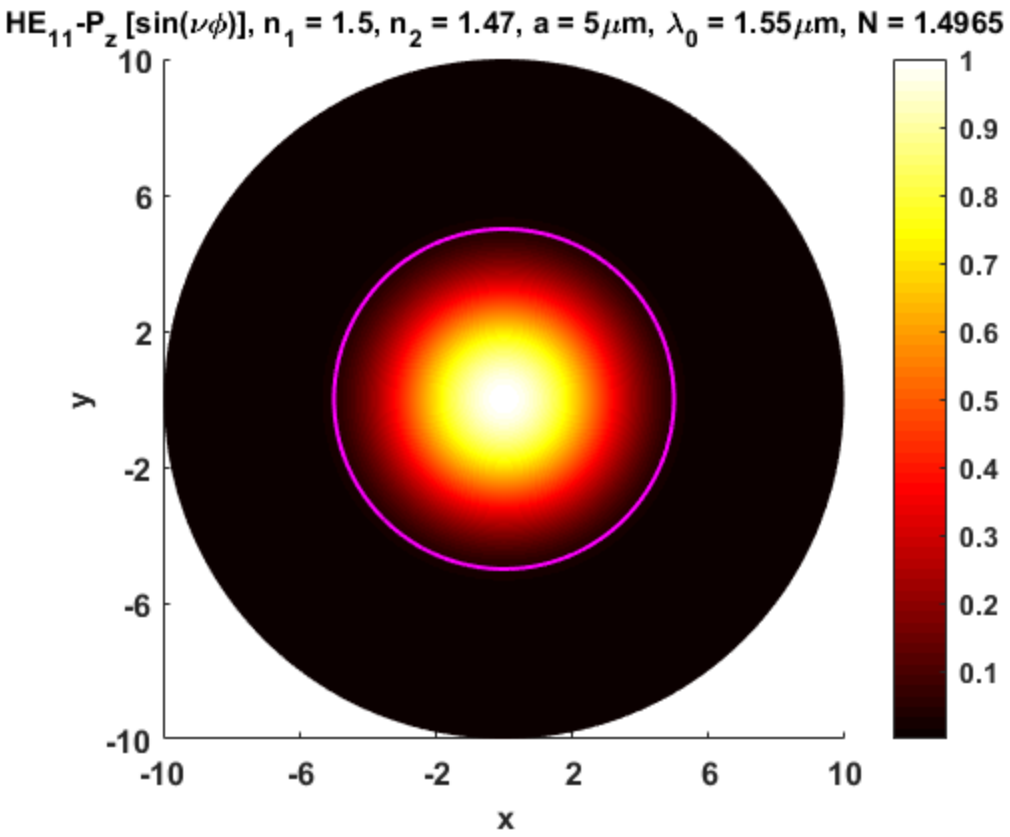






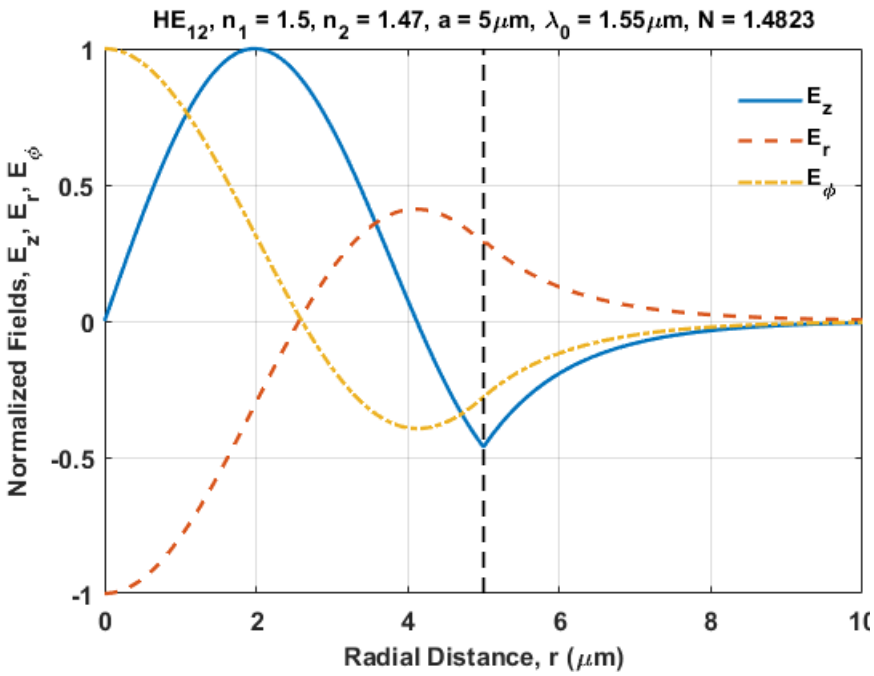
HE_{11} Guided Mode Poynting Vector

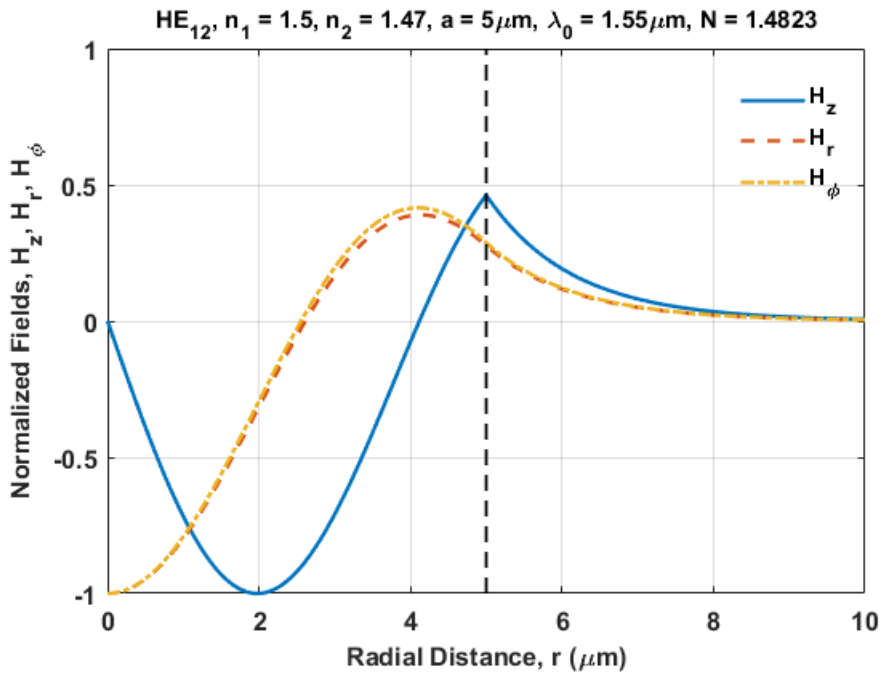
$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$



HE_{12} Guided Mode Electric & Magnetic Fields

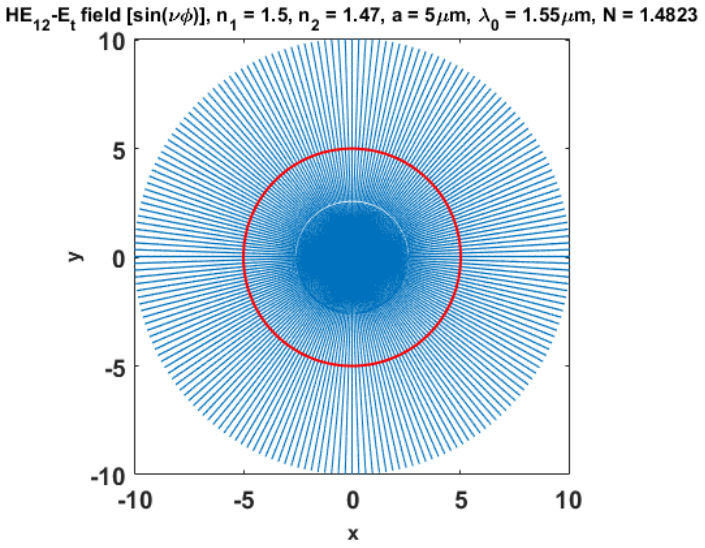
$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

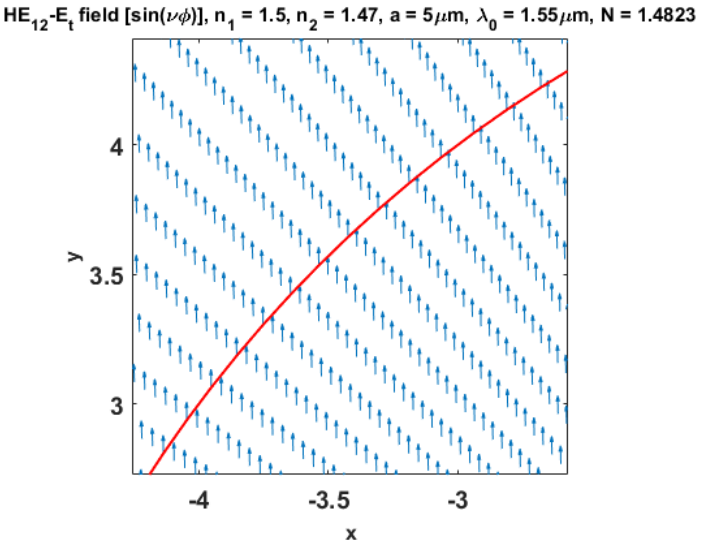


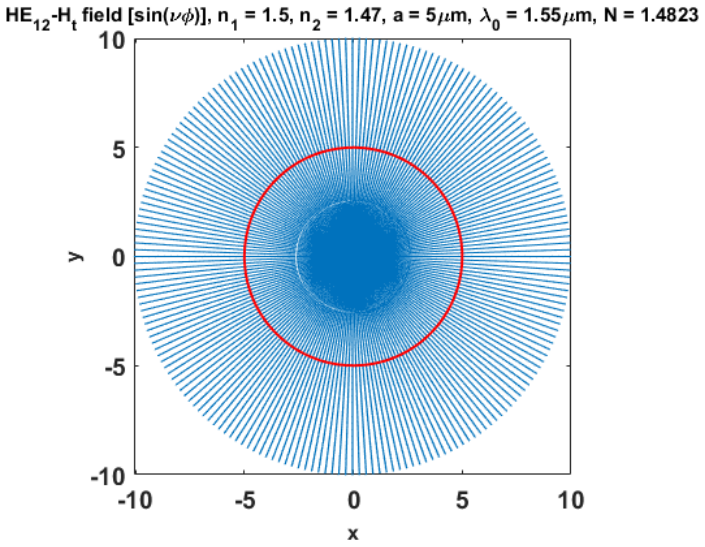


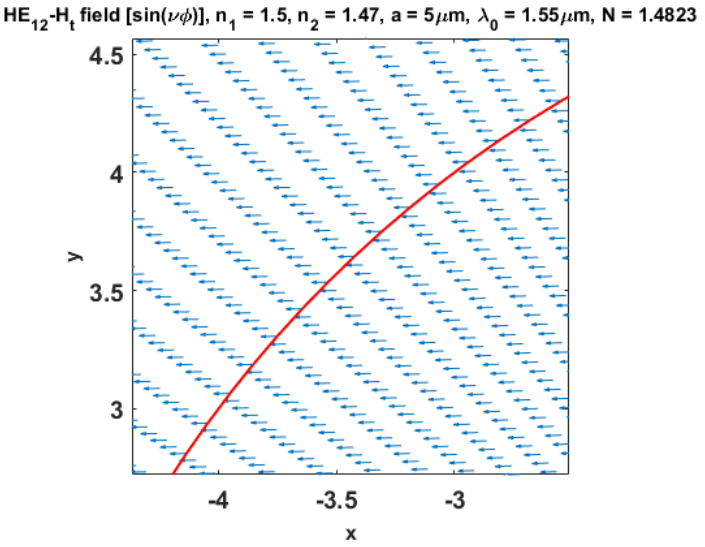
HE_{12} Guided Mode Electric & Magnetic Fields

$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$



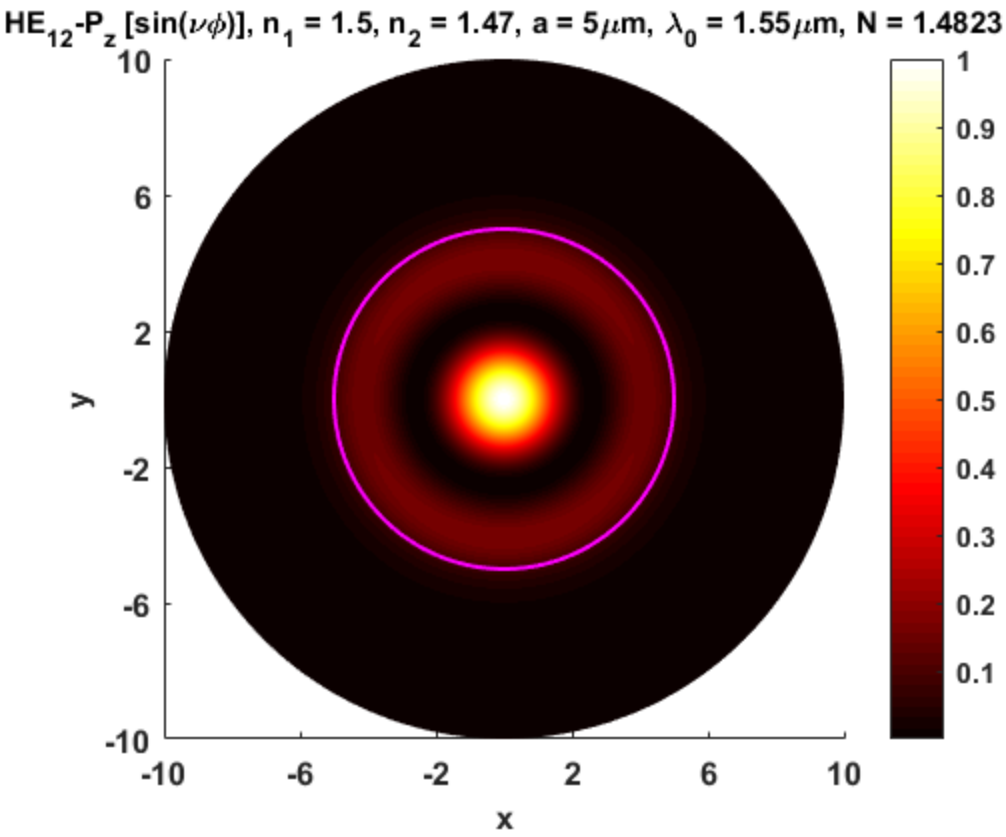






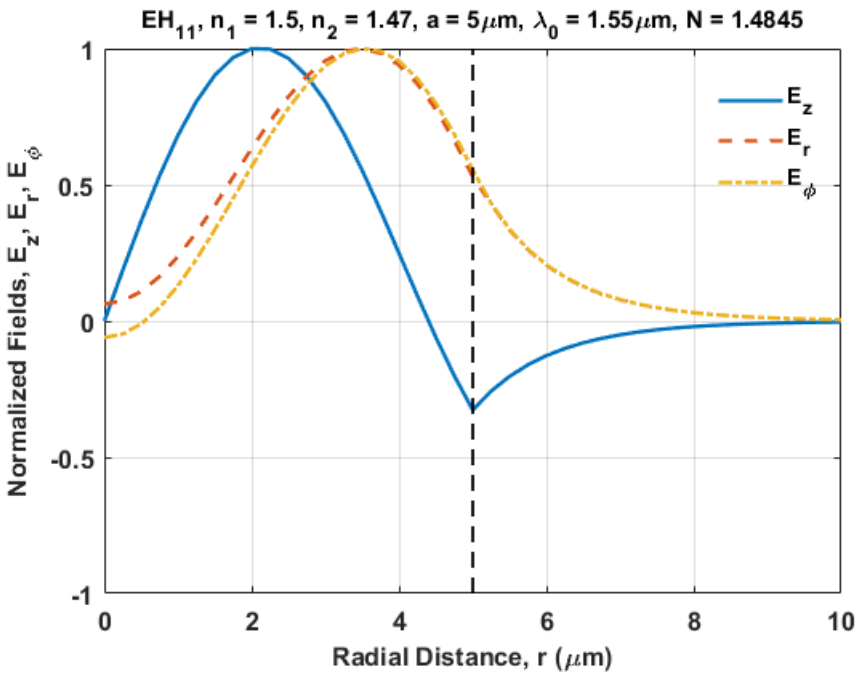
HE_{12} Guided Mode Poynting Vector

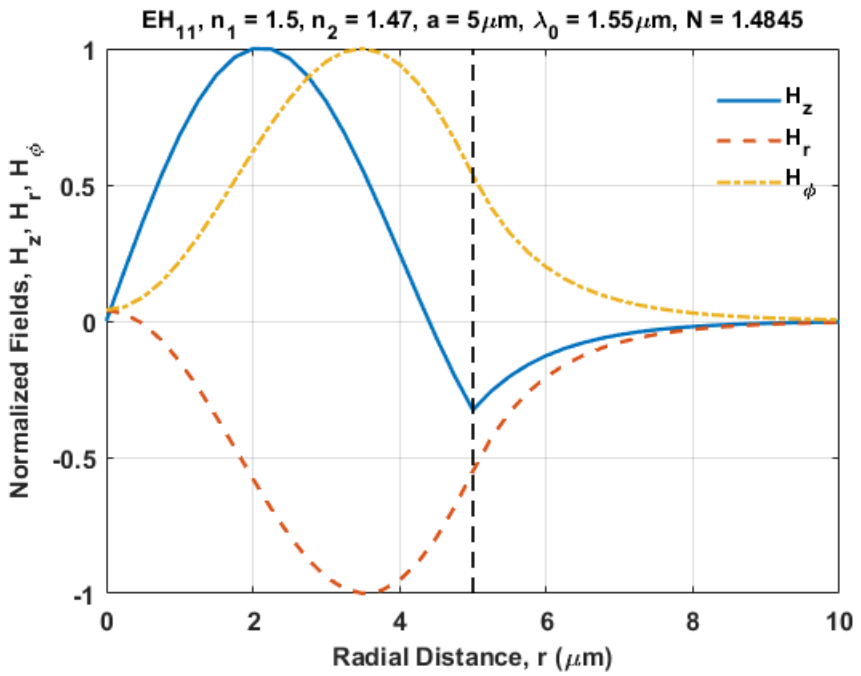
$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$



EH_{11} Guided Mode Electric & Magnetic Fields

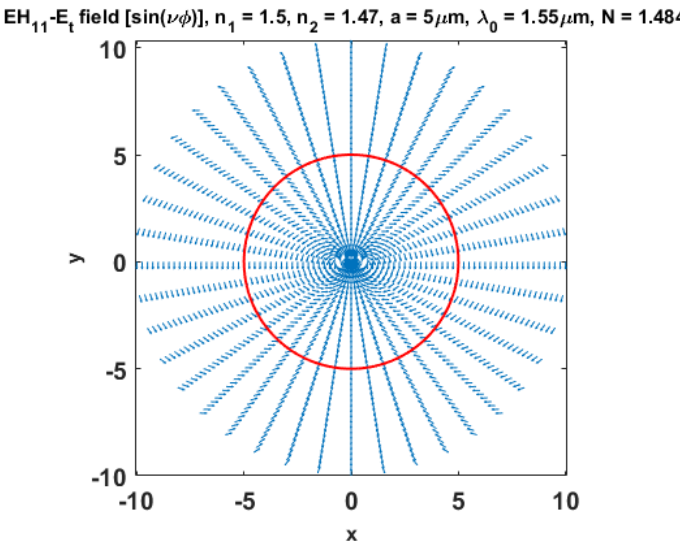
$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

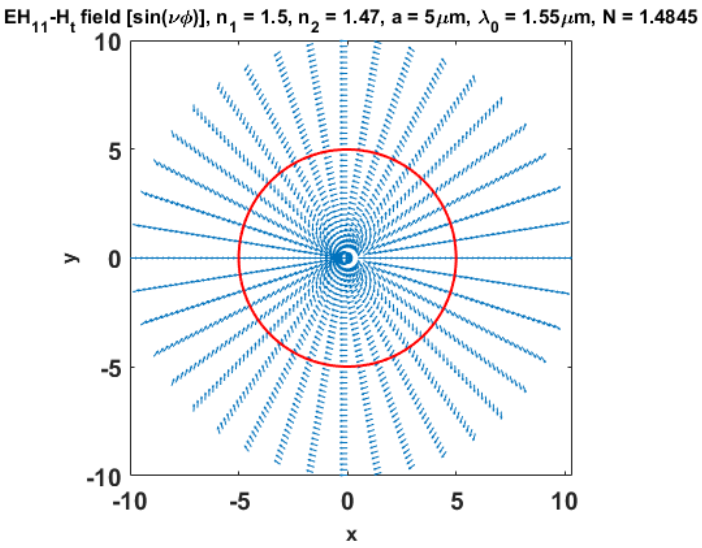


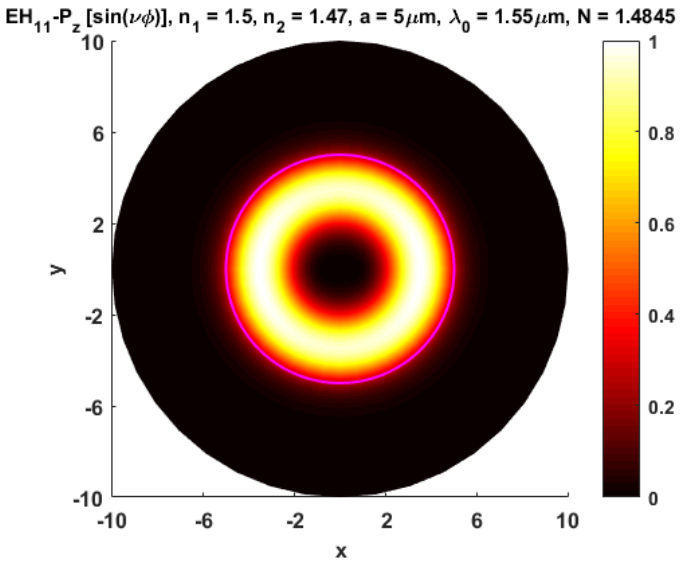


EH_{11} Guided Mode Electric & Magnetic Fields

$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

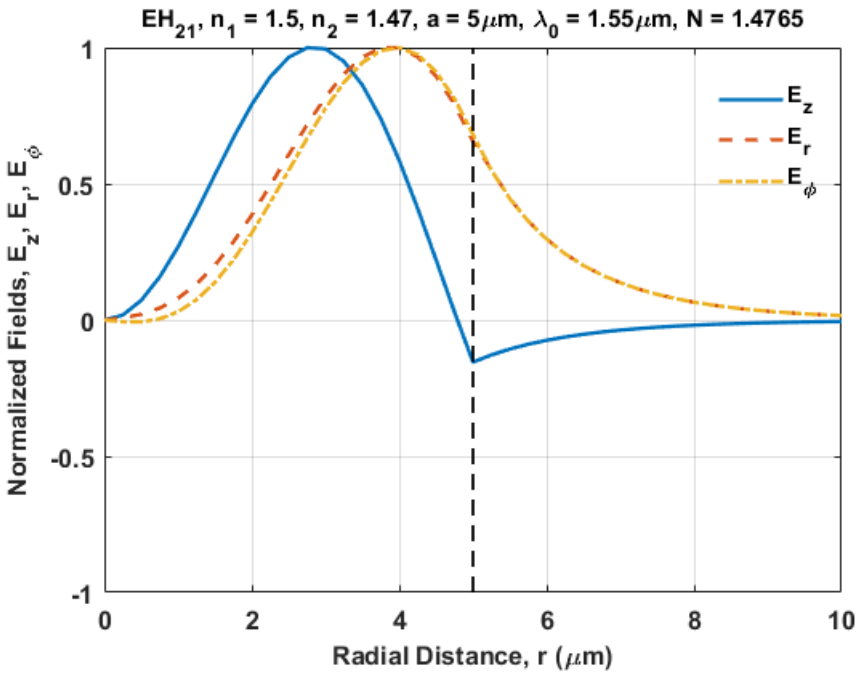


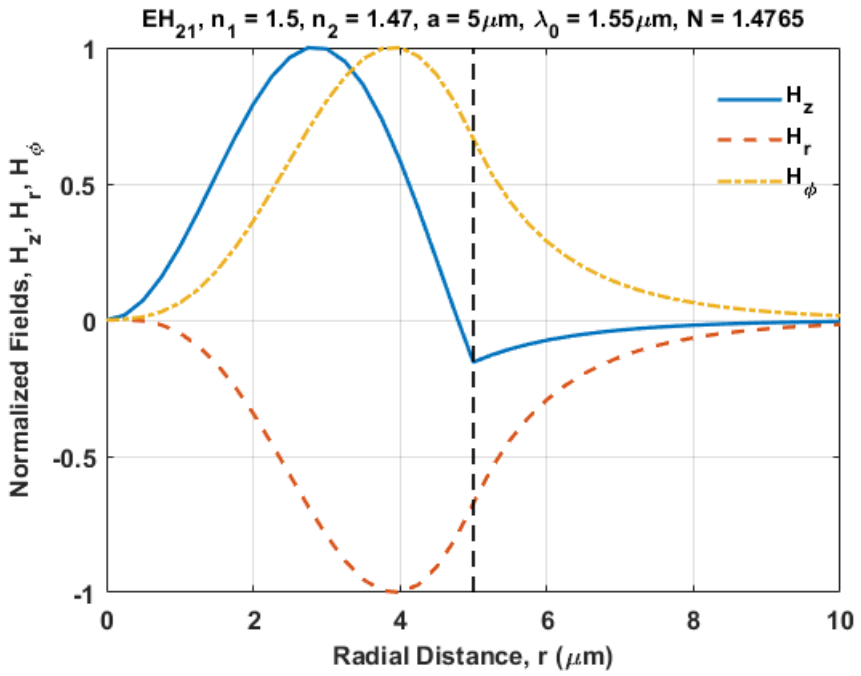




EH_{21} Guided Mode Electric & Magnetic Fields

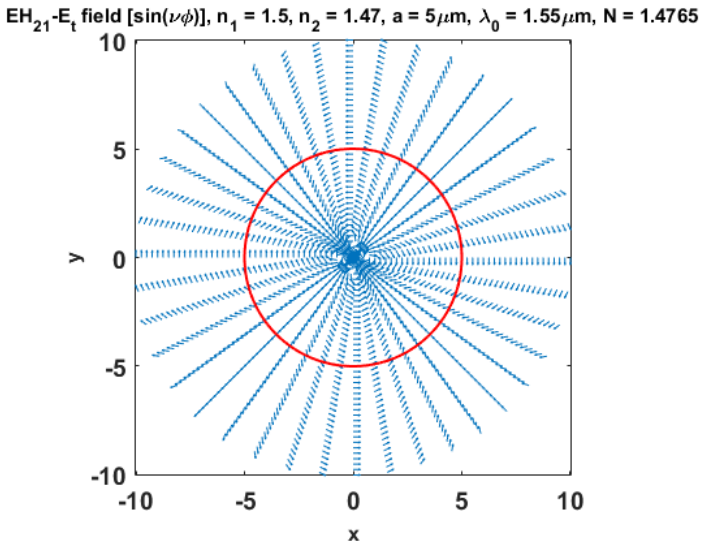
$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

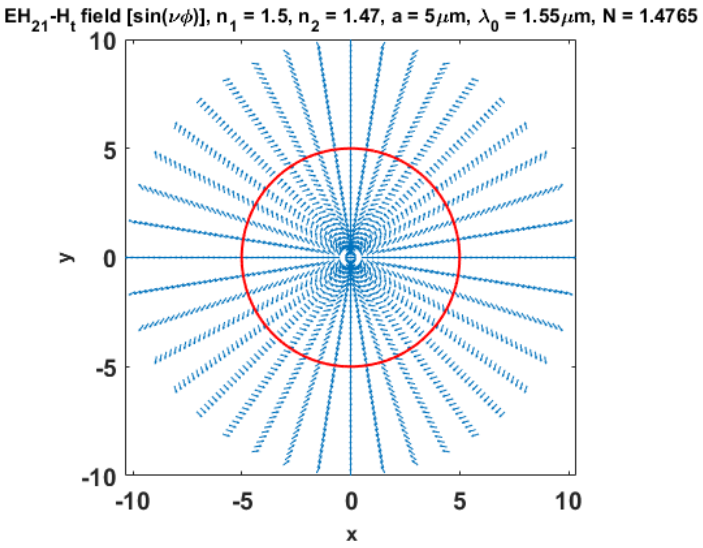


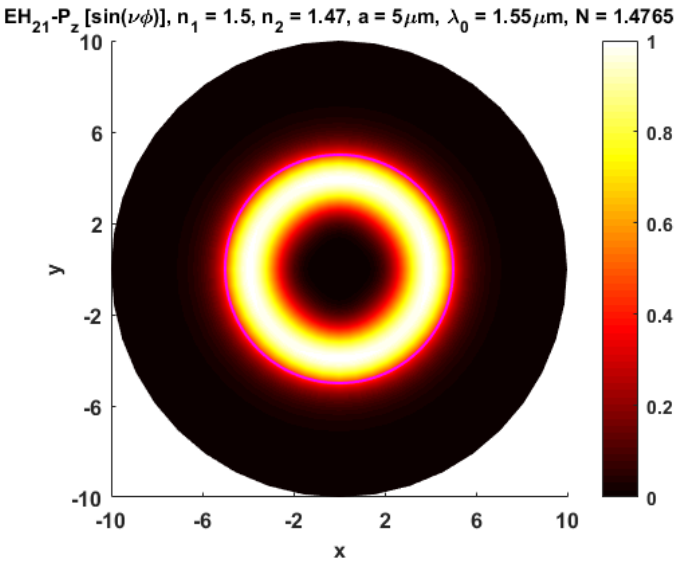


EH_{21} Guided Mode Electric & Magnetic Fields

$n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$







TE_{0m} , TM_{0m} , $Eh_{\nu m}$, and $HE_{\nu m}$ Guided Modes

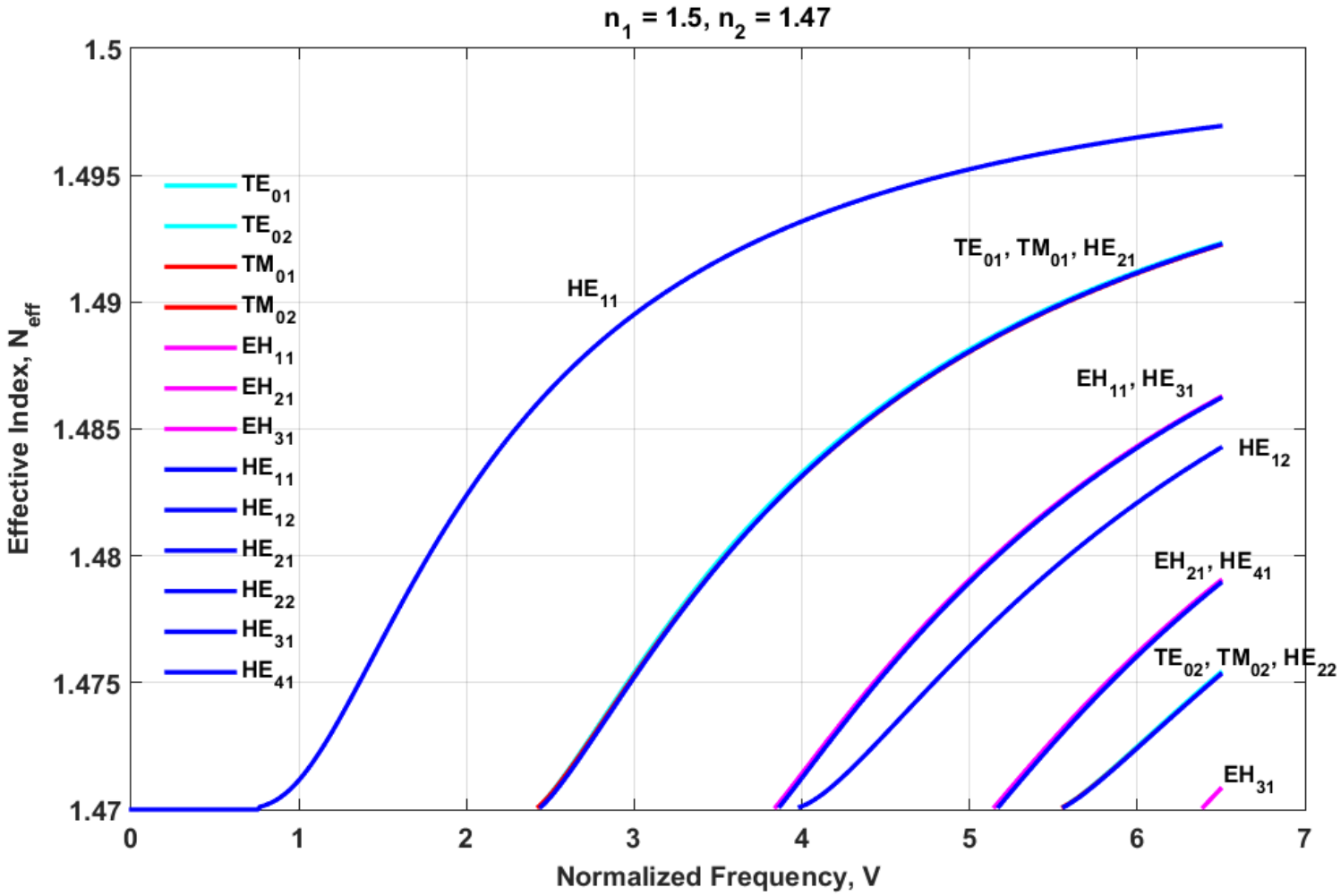
Test Case Parameters: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

Effective Indices

	TE_{0m}		TM_{0m}	
ν	$m=1$	$m=2$	$m=1$	$m=2$
0	1.49133190	1.47278703	1.49125723	1.47272794
	$EH_{\nu m}$		$HE_{\nu m}$	
ν	$m=1$	$m=2$	$m=1$	$m=2$
1	1.48453284	–	1.49654129	1.48232990
2	1.47648180	–	1.49127863	1.47271676
3	–	–	1.48447568	–
4	–	–	1.47635769	–

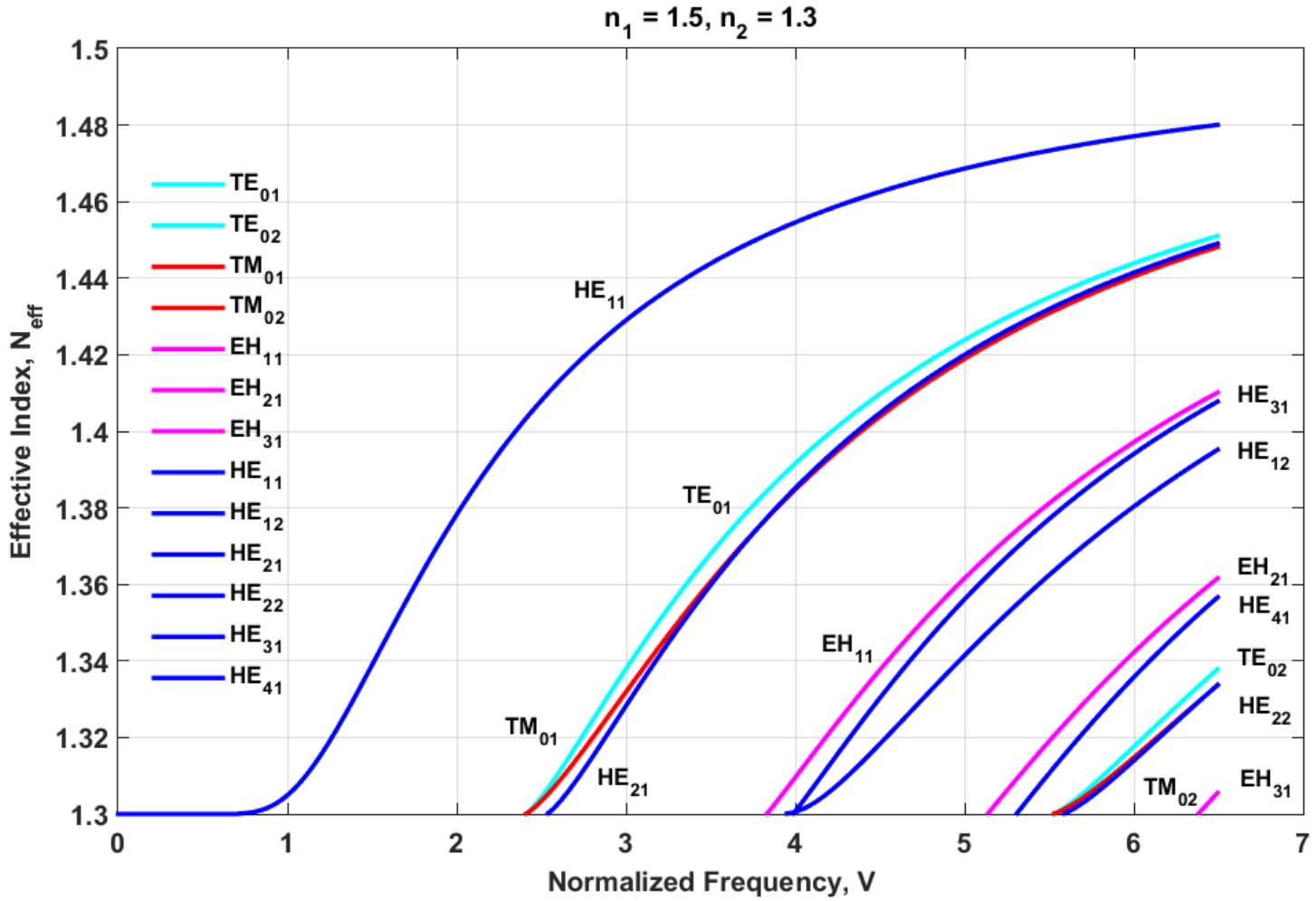
Mode Effective Index vs Normalized Frequency

Small $\Delta n = n_1 - n_2 = 0.03$

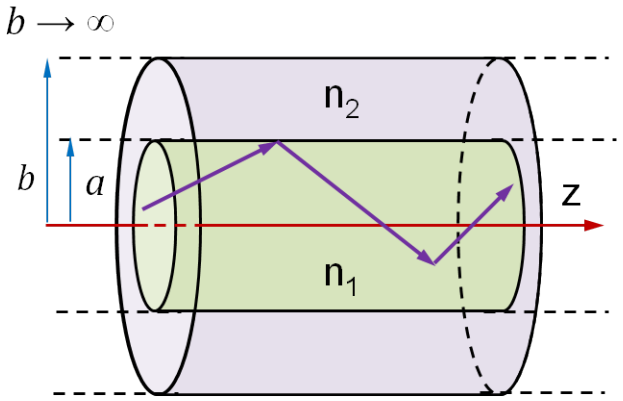


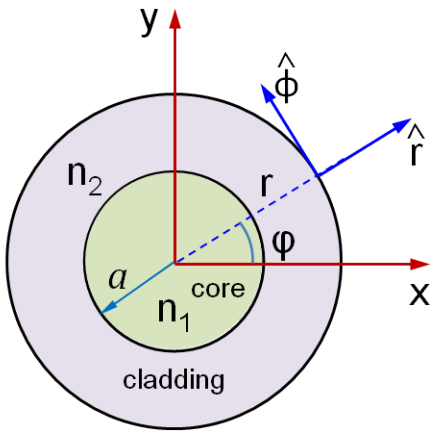
Mode Effective Index vs Normalized Frequency

Large $\Delta n = n_1 - n_2 = 0.20$



Weakly Guided Approximation





$$n_1 \simeq n_2 \longrightarrow \frac{n_1^2}{n_2^2} \simeq 1$$

Assume only transverse fields – Scalar Wave Equation

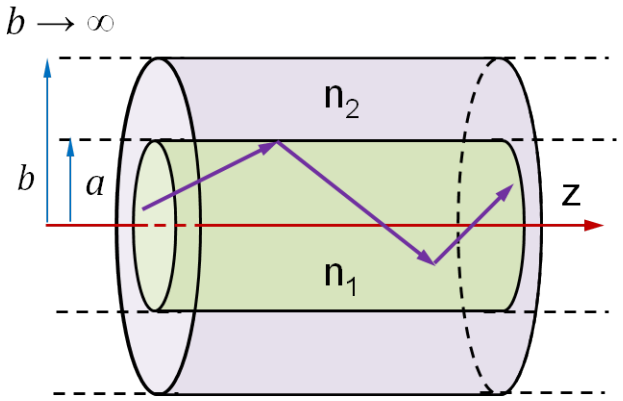
$$\psi = \psi(r,\phi)e^{-j\beta z}$$

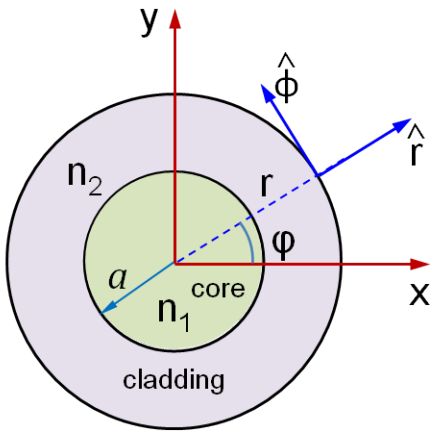
$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + (k_0 n^2 - \beta^2)\psi = 0$$

$$\begin{aligned} \psi(r,\phi) &= [Ac_\nu(qr) + Bd_\nu(qr)]\left[Ce^{j\nu\phi} + De^{-j\nu\phi}\right] \\ q^2 &= k_0^2 n^2 - \beta^2 \\ c_\nu &= J_\nu \text{ or } I_\nu \\ d_\nu &= Y_\nu \text{ or } K_\nu \end{aligned}$$

$$\psi(r,\phi) = \begin{cases} A_1 J_\nu(\kappa r)\left[Ce^{j\nu\phi} + De^{-j\nu\phi}\right]e^{-j\beta z} & r \le a \\ \kappa = \sqrt{k_0^2 n_1^2 - \beta^2} & \\ B_2 K_\nu(\gamma r)\left[Ce^{j\nu\phi} + De^{-j\nu\phi}\right]e^{-j\beta z} & r \ge a \\ \gamma = \sqrt{\beta^2 - k_0^2 n_2^2} & \end{cases}$$

Weakly Guided Approximation





$$n_1 \simeq n_2 \longrightarrow \frac{n_1^2}{n_2^2} \simeq 1$$

Dispersion Equation for Guided Modes

$$\kappa a \frac{J_{\ell-1}(\kappa a)}{J_\ell(\kappa a)} = -\gamma a \frac{K_{\ell-1}(\gamma a)}{K_\ell(\gamma a)}$$

$$\ell = \begin{cases} 1 & \text{for} \quad TE_{0m},\ TM_{0m} \\ \nu+1 & \text{for} \quad EH_{\nu m} \\ \nu-1 & \text{for} \quad HE_{\nu m} \end{cases}$$

Linearly Polarized Modes (LP-modes)

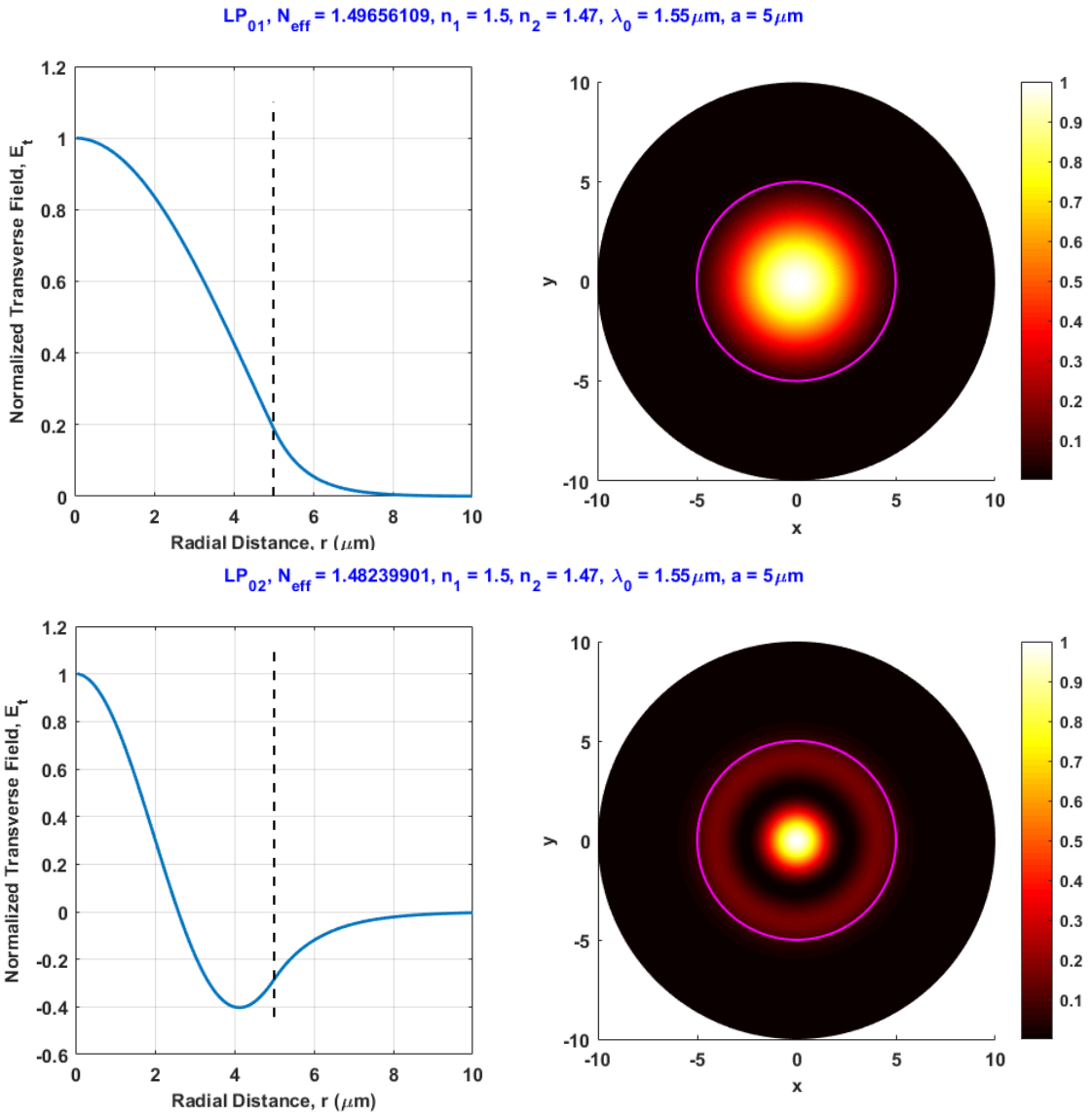
$$LP_{1m} \longrightarrow \text{sum of } TE_{0m},\ TM_{0m},\ HE_{2m}$$

$$LP_{\nu m} \longrightarrow \text{sum of } HE_{\nu+1,m},\ EH_{\nu-1,m} \quad (\nu \geq 2)$$

$$LP_{0m} \longrightarrow HE_{1m}$$

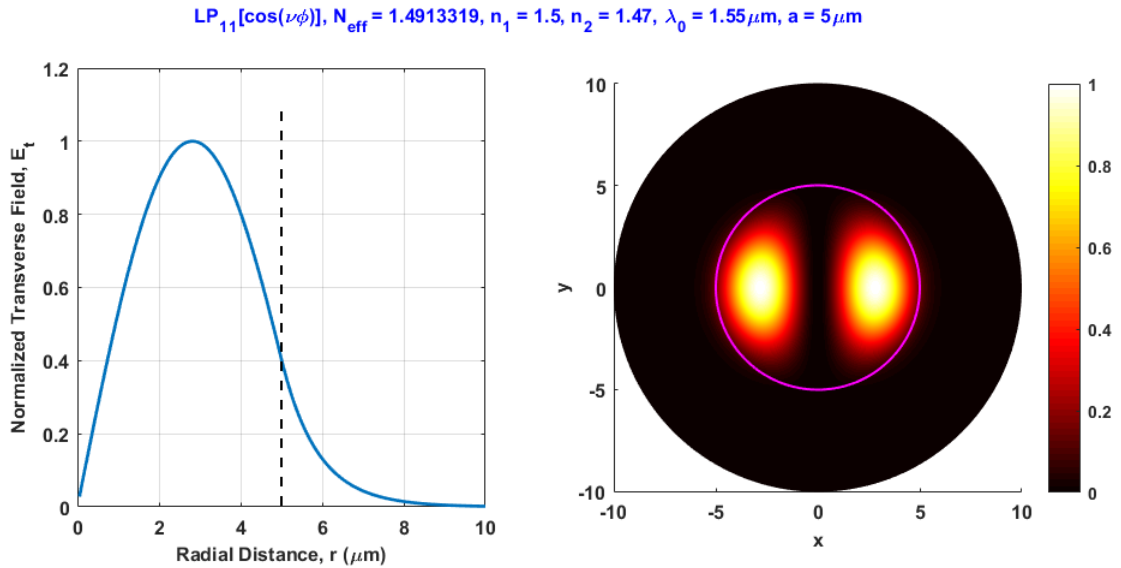
LP-Modes Examples

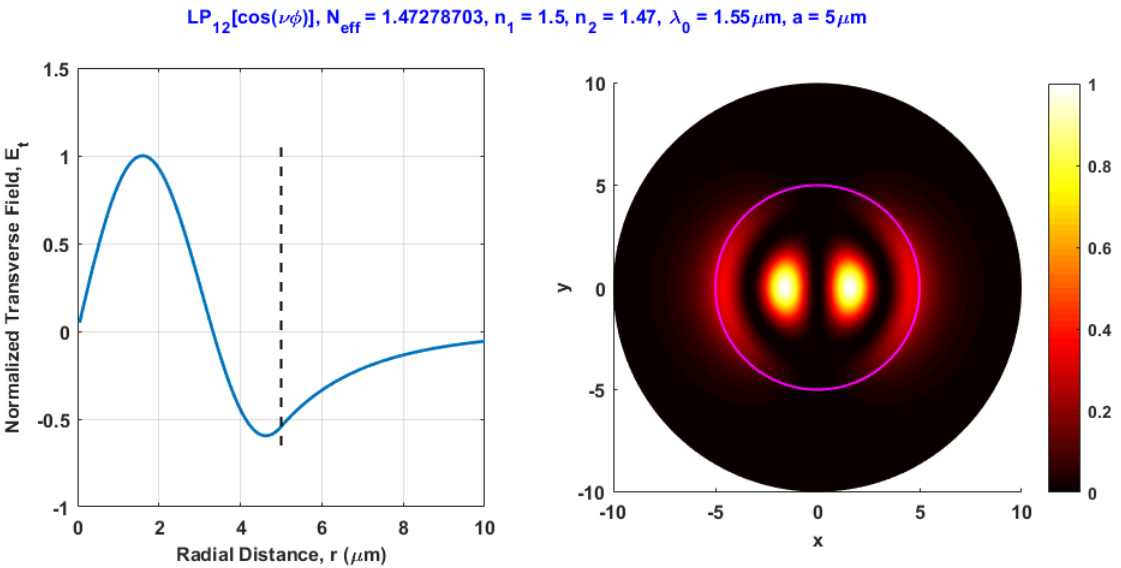
Test Case Parameters: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$



LP-Modes Examples

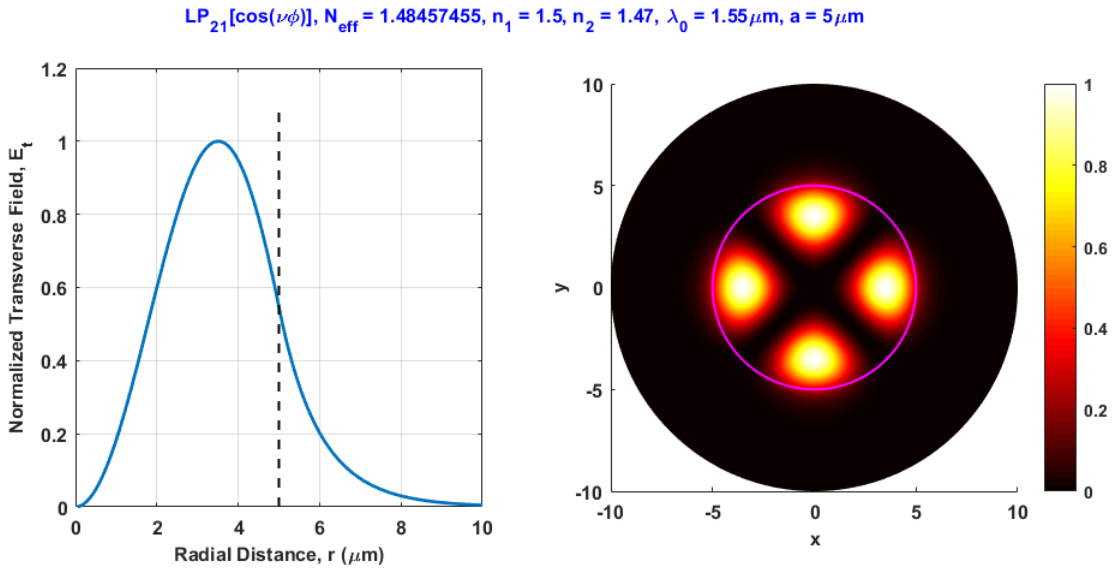
Test Case Parameters: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

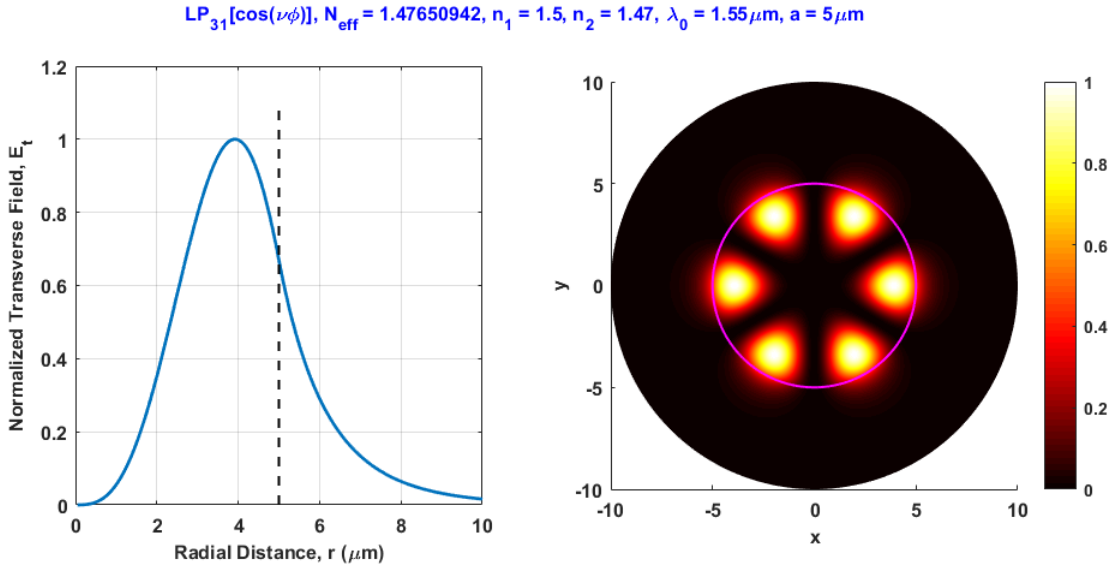




LP-Modes Examples

Test Case Parameters: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$





LP-Modes Examples

Test Case Parameters: $n_1 = 1.50$, $n_2 = 1.47$, $a = 5\mu m$, $\lambda_0 = 1.55\mu m$

Hybrid Modes

	TE_{0m}		TM_{0m}	
ν	$m=1$	$m=2$	$m=1$	$m=2$
0	1.49133190	1.47278703	1.49125723	1.47272794
	$EH_{\nu m}$		$HE_{\nu m}$	
ν	$m=1$	$m=2$	$m=1$	$m=2$
1	1.48453284	–	1.49654129	1.48232990
2	1.47648180	–	1.49127863	1.47271676
3	–	–	1.48447568	–
4	–	–	1.47635769	–

LP Modes

$LP_{\nu m}$			
ν	m	N_{eff}	Degenerate Hybrid Modes
0	1	1.49656109	HE_{11}
0	2	1.48239901	HE_{12}
1	1	1.49133190	$TE_{01}, TM_{01}, HE_{21}$
1	2	1.47278703	$TE_{02}, TM_{02}, HE_{22}$
2	1	1.48457455	HE_{31}, EH_{11}
3	1	1.47650942	HE_{41}, EH_{21}

Cutoff Conditions
(under weakly-guiding approximation)

$$\gamma = (\beta^2 - k_0^2 n_2^2)^{1/2} = 0, \qquad V = k_0 a (n_1^2 - n_2^2)^{1/2}$$

$$\frac{V_c J_{\ell-1}(V_c)}{J_\ell(V_c)} = 0 \qquad \ell = \begin{cases} 1 & \text{for} \quad TE_{0m},\ TM_{0m} \\ \nu + 1 & \text{for} \quad EH_{\nu m} \\ \nu - 1 & \text{for} \quad HE_{\nu m} \end{cases}$$

$$J_0(V_{cm}) = 0 \qquad \text{for} \quad TE_{0m},\ TM_{0m}$$

$$V_c = 0 \qquad \text{for} \quad HE_{11}$$

$$J_1(V_{cm}) = 0 \qquad \text{for} \quad HE_{1m}\ (m \geq 2)$$

$$J_{\nu-2}(V_{cm}) = 0 \qquad \text{for} \quad HE_{\nu m}\ (\nu \geq 2)$$

$$\text{or} \quad \left(\frac{n_1^2}{n_2^2} + 1\right) J_{\nu-1}(V_{cm}) = \frac{V_{cm}}{\nu - 1} J_\nu(V_{cm}) \qquad \text{for} \quad HE_{\nu m}\ (\nu \geq 2)$$

$$J_\nu(V_{cm}) = 0 \qquad \text{for} \quad EH_{\nu m}\ (\nu \geq 1)$$

$$J_1(V_{cm}) = 0 \qquad \text{for} \quad LP_{0m}$$

$$J_0(V_{cm}) = 0 \qquad \text{for} \quad LP_{1m}$$

$$J_{\nu-1}(V_{cm}) = 0 \qquad \text{for} \quad LP_{\nu m}$$

Cutoff Conditions
LP-Modes (from J. A. Buck, "Fiber Optics")

Table 3.2 Cutoff Conditions and Designations of the First 12 LP Modes in a Step Index Fiber

V_c	Bessel Function	l	Degenerate Modes	LP Designation
0	—	0	HE_{11}	LP_{01}
2.405	J_0	1	$TE_{01}, TM_{01}, HE_{21}$	LP_{11}
3.832	J_1	2	EH_{11}, HE_{31}	LP_{21}
3.832	J_{-1}	0	HE_{12}	LP_{02}
5.136	J_2	3	EH_{21}, HE_{41}	LP_{31}
5.520	J_0	1	$TE_{02}, TM_{02}, HE_{22}$	LP_{12}
6.380	J_3	4	EH_{31}, HE_{51}	LP_{41}
7.016	J_1	2	EH_{12}, HE_{32}	LP_{22}
7.016	J_{-1}	0	HE_{13}	LP_{03}
7.588	J_4	5	EH_{41}, HE_{61}	LP_{51}
8.417	J_2	3	EH_{22}, HE_{42}	LP_{32}
8.654	J_0	1	$TE_{03}, TM_{03}, HE_{23}$	LP_{13}

Cutoff Conditions
LP-Modes (from A. Ghatak and K. Thyagarajan, "Introduction to Fiber Optics")

Table 8.2. *Cutoff frequencies of various LP_{lm} modes in a step index fiber*

$l = 0$ modes	$(J_1(V_c) = 0)$	$l = 1$ modes	$(J_0(V_c) = 0)$
Mode	V_c	Mode	V_c
LP_{01}	0	LP_{11}	2.4048
LP_{02}	3.8317	LP_{12}	5.5201
LP_{03}	7.0156	LP_{13}	8.6537
LP_{04}	10.1735	LP_{14}	11.7915
$l = 2$ modes	$(J_1(V_c) = 0; V_c \neq 0)$	$l = 3$ modes	$(J_0(V_c) = 0; V_c \neq 0)$
Mode	V_c	Mode	V_c
LP_{21}	3.8317	LP_{31}	5.1356
LP_{22}	7.0156	LP_{32}	8.4172
LP_{23}	10.1735	LP_{33}	11.6198
LP_{24}	13.3237	LP_{34}	14.7960

Power Considerations (LP Modes)

The Poynting vector along the z-direction is given by: (for a mode)

$$S_z = \frac{1}{2}\,\mathrm{Re}\left\{\vec{E}\times\vec{H}^*\right\}_z = \frac{1}{2}\,\mathrm{Re}\left\{E_x H_y^* - E_y H_x^*\right\} = \frac{1}{2}\,\mathrm{Re}\left\{E_r H_\phi^* - E_\phi H_r^*\right\}$$

Then the power in the core and cladding can be found from:

$$P_{core} = \int_{\phi=0}^{2\pi}\int_{r=0}^{a} S_z\, r\,dr\,d\phi$$

$$P_{clad} = \int_{\phi=0}^{2\pi}\int_{r=a}^{\infty} S_z\, r\,dr\,d\phi$$

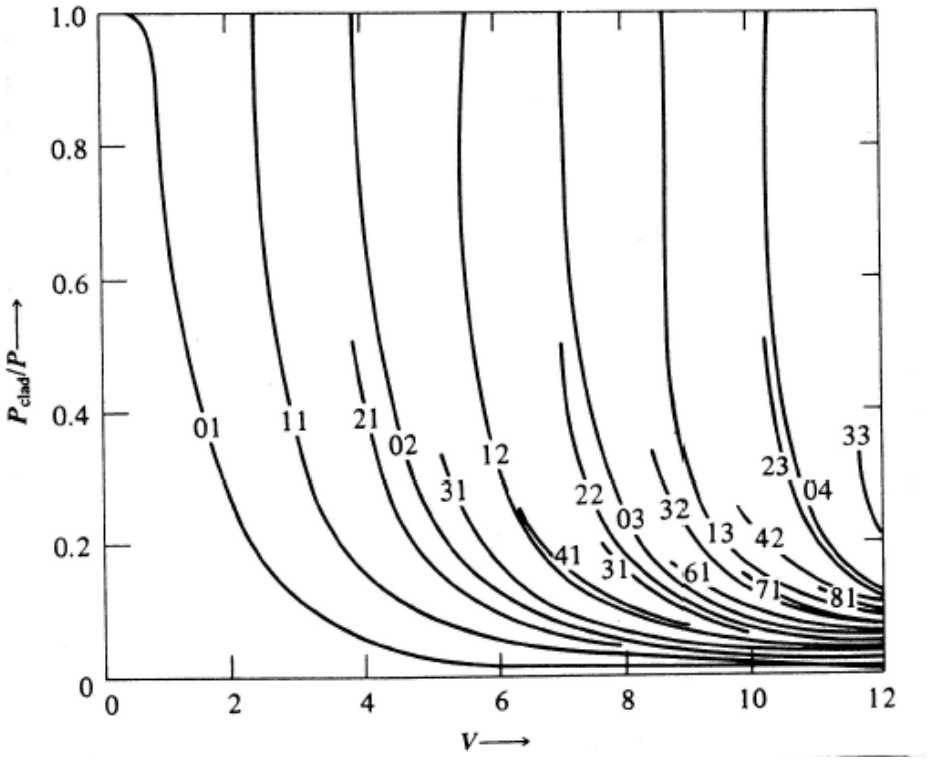
For $LP_{\nu m}$ modes (weakly guiding approximation) it can be shown that

$$P_{core} = \frac{\beta_\nu}{2\omega\mu_0}\pi a^2 |A_1|^2 \left[J_\nu^2(\varkappa a) - J_{\nu-1}(\varkappa a) J_{\nu+1}(\varkappa a)\right]$$

$$P_{clad} = \frac{\beta_\nu}{2\omega\mu_0}\pi a^2 |A_1|^2 \left[-J_\nu^2(\varkappa a) - \left(\frac{\varkappa}{\gamma}\right)^2 J_{\nu-1}(\varkappa a) J_{\nu+1}(\varkappa a)\right]$$

$$P = P_{clad} + P_{core} = \frac{\beta_\nu}{2\omega\mu_0}\pi a^2 |A_1|^2 \left(1 + \left(\frac{\varkappa}{\gamma}\right)^2\right)\left(-J_{\nu-1}(\varkappa a) J_{\nu+1}(\varkappa a)\right)$$

Power Considerations (LP Modes)

$$\Gamma_{cl} = \frac{P_{clad}}{P} = \frac{1}{V^2}\left[(\kappa a)^2 + \frac{(\gamma a)^2 J_\nu^2(\kappa a)}{J_{\nu-1}(\kappa a)\, J_{\nu+1}(\kappa a)}\right] = \left(\frac{\kappa a}{V}\right)^2\left(1 - \frac{K_\nu^2(\gamma a)}{K_{\nu-1}(\gamma a) K_{\nu+1}(\gamma a)}\right)$$

$$\Gamma_{co} = \frac{P_{core}}{P} = 1 - \Gamma_{cl} = 1 - \left(\frac{\kappa a}{V}\right)^2\left[1 - \frac{K_\nu^2(\gamma a)}{K_{\nu-1}(\gamma a) K_{\nu+1}(\gamma a)}\right]$$