30/11/2022

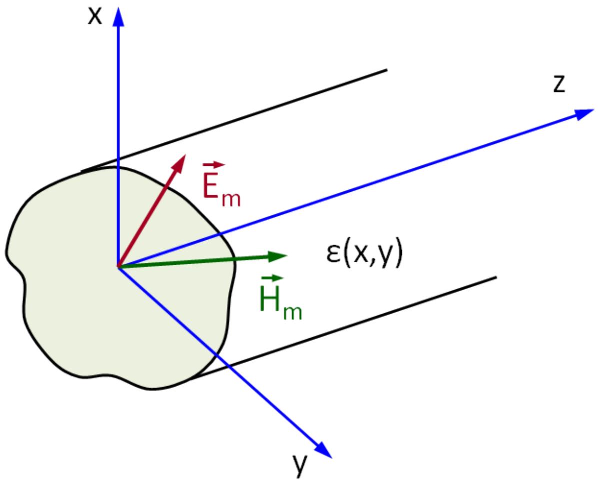
Coupled-Mode Theory

Integrated Optics

Prof. Elias N. Glytsis

School of Electrical & Computer Engineering
National Technical University of Athens

Coupled-Mode Theory



Waveguide Modes (m-th and n-th):

$$\begin{bmatrix} \vec{E}_m \\ \vec{H}_m \end{bmatrix} = \begin{bmatrix} \vec{\mathcal{E}}_m(x,y) \\ \vec{\mathcal{H}}_m(x,y) \end{bmatrix} e^{-j\beta_m z}, \qquad \begin{bmatrix} \vec{E}_n \\ \vec{H}_n \end{bmatrix} = \begin{bmatrix} \vec{\mathcal{E}}_n(x,y) \\ \vec{\mathcal{H}}_n(x,y) \end{bmatrix} e^{-j\beta_n z}$$

Orthogonality Conditions

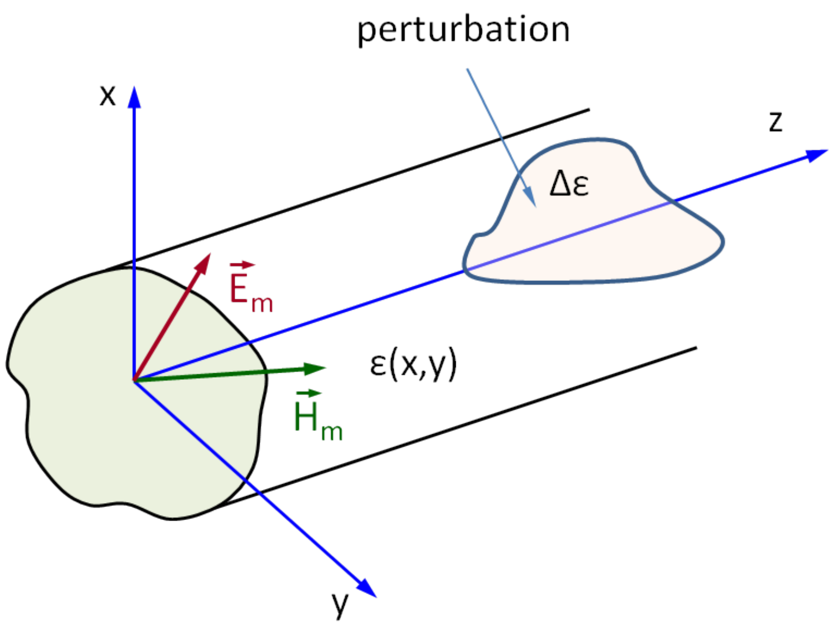
Guided Modes

$$\langle \vec{\mathcal{E}}_m, \vec{\mathcal{H}}_n \rangle = \frac{1}{2}\iint_S \left[\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n^*\right] \cdot \hat{z}dxdy = \frac{1}{2}\iint_S \left[\vec{\mathcal{E}}_{t,m} \times \vec{\mathcal{H}}_{t,n}^*\right] \cdot \hat{z}dxdy = \frac{1}{2}\iint_S \mathrm{Re}\left\{\vec{\mathcal{E}}_{t,m} \times \vec{\mathcal{H}}_{t,n}^*\right\} \cdot \hat{z}dxdy = \frac{\beta_m}{|\beta_m|} P_m \delta_{|m||n|},$$

Radiation Modes

$$\langle \vec{\mathcal{E}}_\beta, \vec{\mathcal{H}}_{\beta'} \rangle = \frac{1}{2}\iint_S \left[\vec{\mathcal{E}}_\beta \times \vec{\mathcal{H}}_{\beta'}^*\right] \cdot \hat{z}dxdy = \frac{1}{2}\iint_S \left[\vec{\mathcal{E}}_{t,\beta} \times \vec{\mathcal{H}}_{t,\beta'}^*\right] \cdot \hat{z}dxdy = \frac{1}{2}\iint_S \mathrm{Re}\left\{\vec{\mathcal{E}}_{t,\beta} \times \vec{\mathcal{H}}_{t,\beta'}^*\right\} \cdot \hat{z}dxdy = \frac{\beta}{|\beta|} P_\beta \delta(\beta - \beta').$$

Perturbation along the Waveguide



$$\epsilon'(x,y,z) = \epsilon(x,y) + \Delta\epsilon(x,y,z)$$

$$\vec{E}' = \vec{E}'_t + \hat{z}E'_z$$

$$\vec{H}' = \vec{H}'_t + \hat{z}H'_z$$

$$\hat{z}E'_z = \frac{1}{j\omega\epsilon'}\left(\vec{\nabla}_t \times \vec{H}'_t\right)$$

$$\hat{z}H'_z = -\frac{1}{j\omega\mu_0}\left(\vec{\nabla}_t \times \vec{E}'_t\right)$$

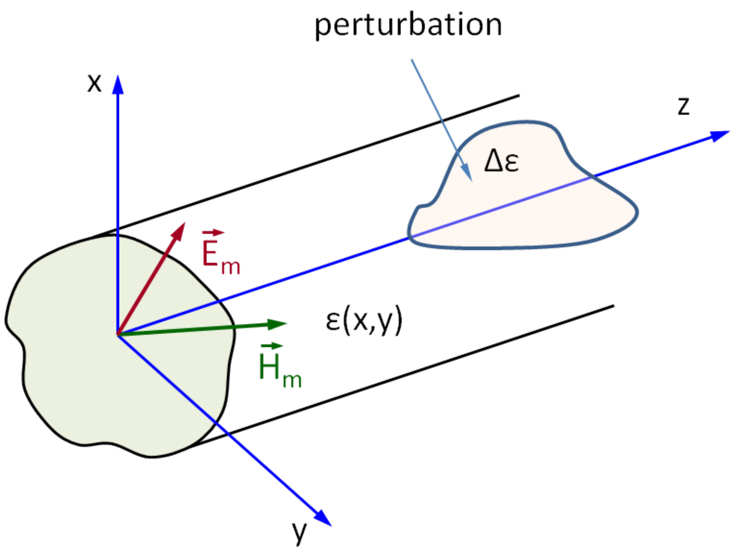
Transverse Field Expansion for the Perturbed Waveguide

$$\begin{bmatrix} \vec{E}'_t \\ \vec{H}'_t \end{bmatrix} = \sum_m a_m(z) \begin{bmatrix} \vec{\mathcal{E}}_{tm}(x,y) \\ \vec{\mathcal{H}}_{tm}(x,y) \end{bmatrix} \exp(-j\beta_m z) + \int_\beta q(z;\beta) \begin{bmatrix} \vec{\mathcal{E}}_{t\beta}(x,y) \\ \vec{\mathcal{H}}_{t\beta}(x,y) \end{bmatrix} \exp(-j\beta z) d\beta$$

$$\begin{bmatrix} E'_z \\ H'_z \end{bmatrix} = \sum_m a_m(z) \begin{bmatrix} \frac{\epsilon}{\epsilon'}\mathcal{E}_{zm}(x,y) \\ \mathcal{H}_{zm}(x,y) \end{bmatrix} \exp(-j\beta_m z) + \int_\beta q(z;\beta) \begin{bmatrix} \frac{\epsilon}{\epsilon'}\mathcal{E}_{z\beta}(x,y) \\ \mathcal{H}_{z\beta}(x,y) \end{bmatrix} \exp(-j\beta z) d\beta$$

D. L. Lee, Electromagnetic Principles of Integrated Optics, John Wiley & Sons, 1986

Perturbation along the Waveguide

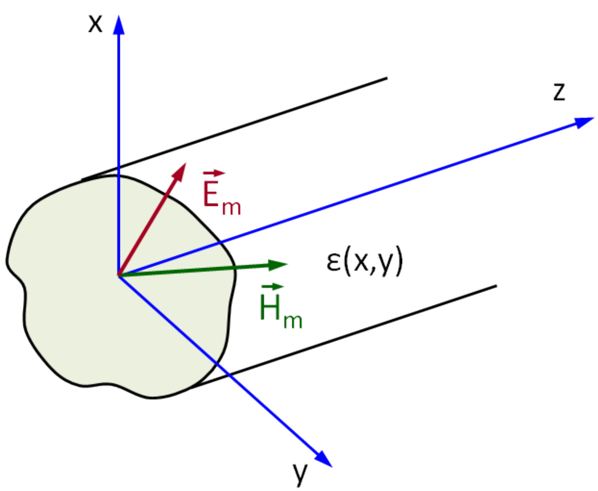


Neglect Radiation Modes

(for problems that light remains mostly guided)

$$\begin{bmatrix} \vec{E}'_t \\ \vec{H}'_t \end{bmatrix} \simeq \sum_m a_m(z) \begin{bmatrix} \vec{\mathcal{E}}_{tm}(x,y) \\ \vec{\mathcal{H}}_{tm}(x,y) \end{bmatrix} \exp(-j\beta_m z)$$

$$\begin{bmatrix} E'_z \\ H'_z \end{bmatrix} \simeq \sum_m a_m(z) \begin{bmatrix} \frac{\epsilon}{\epsilon'}\mathcal{E}_{zm}(x,y) \\ \mathcal{H}_{zm}(x,y) \end{bmatrix} \exp(-j\beta_m z)$$

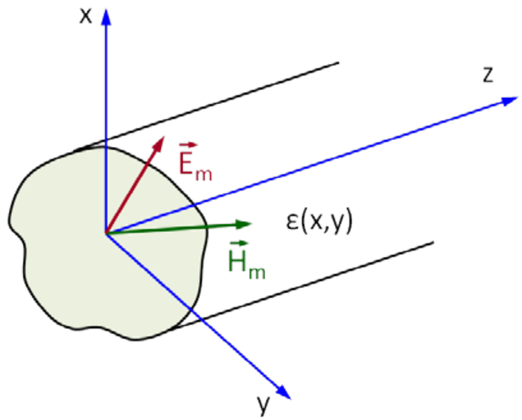


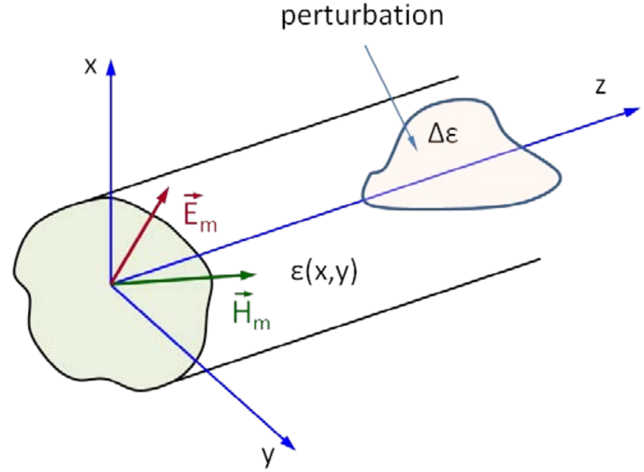
$$\begin{aligned} \vec{\nabla} \times \vec{E}' &= -j\omega\mu_0 \vec{H}' \\ \vec{\nabla} \times \vec{H}' &= +j\omega\epsilon' \vec{E}' \end{aligned}$$

$$\begin{aligned} \vec{\nabla} \times \vec{E}_m &= -j\omega\mu_0 \vec{H}_m \\ \vec{\nabla} \times \vec{H}_m &= +j\omega\epsilon \vec{E}_m \end{aligned}$$

D. L. Lee, Electromagnetic Principles of Integrated Optics, John Wiley & Sons, 1986

Lorentz Reciprocity Theorem



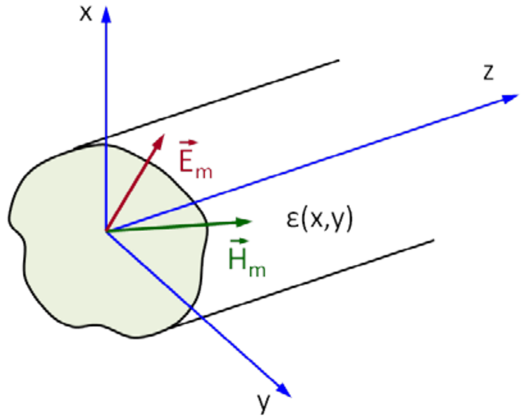


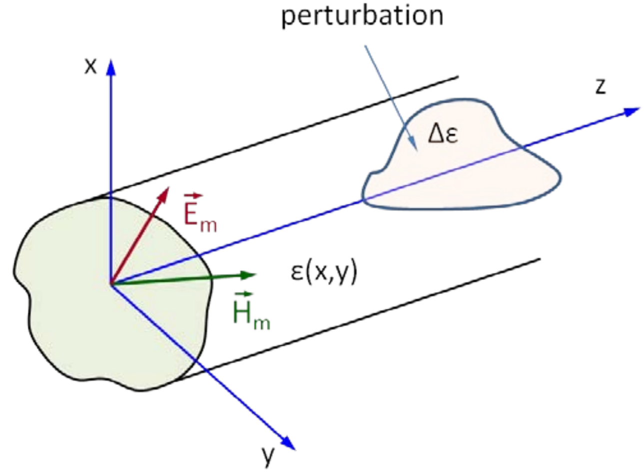
$$\underbrace{\iint_S \frac{\partial}{\partial z}\left[\vec{E}^*_{tm}\times\vec{H}'_t+\vec{E}'_t\times\vec{H}^*_{tm}\right]\cdot\hat{z}\;dxdy}_{LHS} \;=\; \underbrace{-j\omega\iint_S(\epsilon'-\epsilon)\,\vec{E}^*_m\cdot\vec{E}'\;dxdy}_{RHS}$$

$$\begin{aligned}
LHS &= \iint_S \frac{\partial}{\partial z}\left[\vec{E}^*_{tm}\times\left(\sum_n a_n(z)\vec{\mathcal{H}}_{tn}e^{-j\beta_n z}\right)+\left(\sum_n a_n(z)\vec{\mathcal{E}}_{tn}e^{-j\beta_n z}\right)\times\vec{H}^*_{tm}\right]\cdot\hat{z}dxdy \\
&= \iint_S \frac{\partial}{\partial z}\left[\vec{\mathcal{E}}^*_{tm}e^{+j\beta_m z}\times\left(\sum_n a_n(z)\vec{\mathcal{H}}_{tn}e^{-j\beta_n z}\right)+\left(\sum_n a_n(z)\vec{\mathcal{E}}_{tn}e^{-j\beta_n z}\right)\times\vec{\mathcal{H}}^*_{tm}e^{+j\beta_m z}\right]\cdot\hat{z}dxdy \\
&= \sum_n \frac{\partial}{\partial z}\left[a_n(z)e^{j(\beta_m-\beta_n)z}\left\{\iint_S \vec{\mathcal{E}}^*_{tm}\times\vec{\mathcal{H}}_{tn}\cdot\hat{z}dxdy+\vec{\mathcal{E}}_{tn}\times\vec{\mathcal{H}}^*_{tm}\cdot\hat{z}dxdy\right\}\right] \\
&= \sum_n \frac{\partial}{\partial z}\left[a_n(z)e^{j(\beta_m-\beta_n)z}\left\{4P_m\delta_{|m||n|}\frac{\beta_m}{|\beta_m|}\right\}\right] \;=\; 4P_m\frac{\beta_m}{|\beta_m|}\frac{da_m}{dz}
\end{aligned}$$

D. L. Lee, Electromagnetic Principles of Integrated Optics, John Wiley & Sons, 1986

Lorentz Reciprocity Theorem





$$
\begin{aligned}
RHS &= -j\omega \iint_S (\epsilon' - \epsilon)\, \vec{\mathcal{E}}_m^* e^{j\beta_m z} \cdot \vec{E}'\; dxdy \\
&= -j\omega \iint_S (\epsilon' - \epsilon) \left(\vec{\mathcal{E}}_{tm}^* + \hat{z}\mathcal{E}_{zm} \right) e^{j\beta_m z} \cdot \left(\vec{E}_t' + \hat{z} E_z' \right)\; dxdy \\
&= -j\omega \iint_S (\epsilon' - \epsilon) \left(\vec{\mathcal{E}}_{tm}^* + \hat{z}\mathcal{E}_{zm} \right) e^{j\beta_m z} \cdot \left(\sum_n a_n(z) \vec{\mathcal{E}}_{tn} e^{-j\beta_n z} + \hat{z} \frac{\epsilon}{\epsilon'} \sum_n a_n(z) \mathcal{E}_{zn} e^{-j\beta_n z} \right)\; dxdy \\
&= -j\omega \left[\sum_n a_n(z) e^{j(\beta_m - \beta_n) z} \left\{ \iint_S (\epsilon' - \epsilon) \left(\vec{\mathcal{E}}_{tm}^* \cdot \vec{\mathcal{E}}_{tn} + \frac{\epsilon}{\epsilon'} \mathcal{E}_{zm}^* \mathcal{E}_{zn} \right) dxdy \right\} \right]
\end{aligned}
$$

$$
LHS \;=\; RHS \;\Longrightarrow
$$

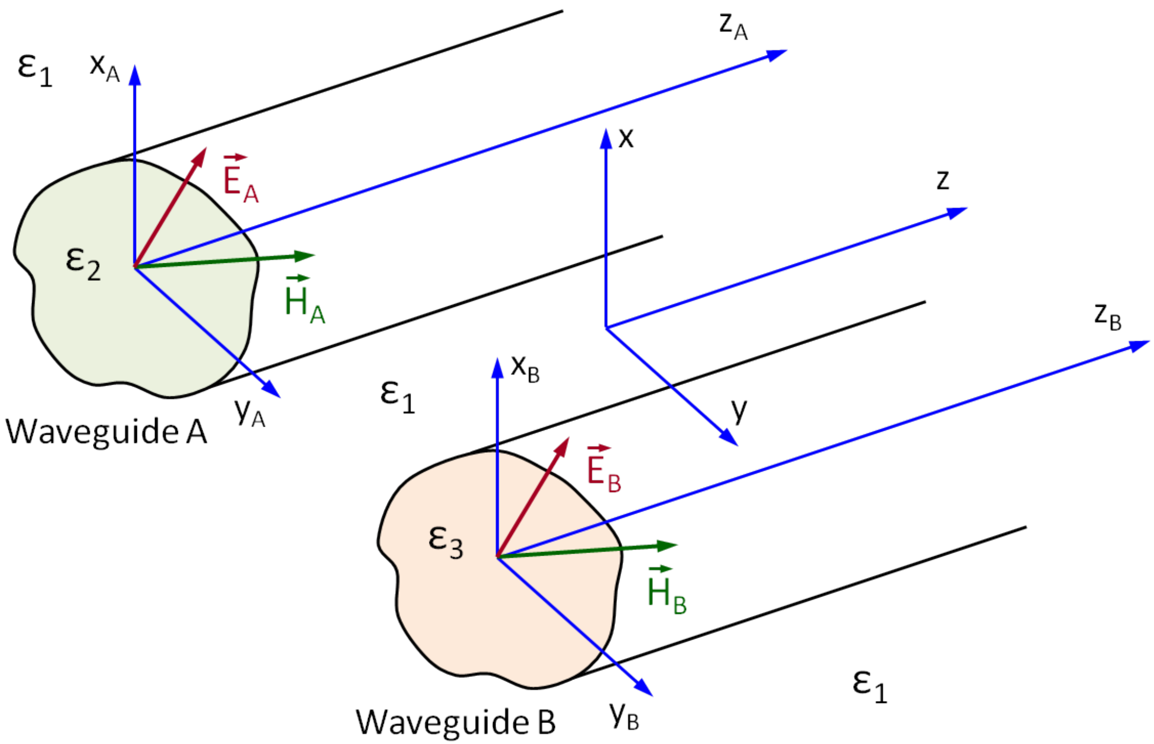
Coupled-Mode Equations

$$
\begin{aligned}
\frac{da_m}{dz} &= -j \sum_n C_{nm} a_n(z) e^{j\beta_m - \beta_n)z} \\
C_{nm} &= C_{nm}^t + C_{nm}^z \\
C_{nm}^t &= \frac{\omega}{4P_m} \frac{\beta_m}{|\beta_m|} \iint_S (\epsilon' - \epsilon) \left(\vec{\mathcal{E}}_{tm}^* \cdot \vec{\mathcal{E}}_{tn} \right) dxdy \\
C_{nm}^z &= \frac{\omega}{4P_m} \frac{\beta_m}{|\beta_m|} \iint_S (\epsilon' - \epsilon) \frac{\epsilon}{\epsilon'} \mathcal{E}_{zm}^* \mathcal{E}_{zn} \; dxdy
\end{aligned}
$$

Usually all modes (guided) are normalized to unit power $P_m = 1W$

Usually perturbation is small ($\epsilon/\epsilon' \approx 1$)

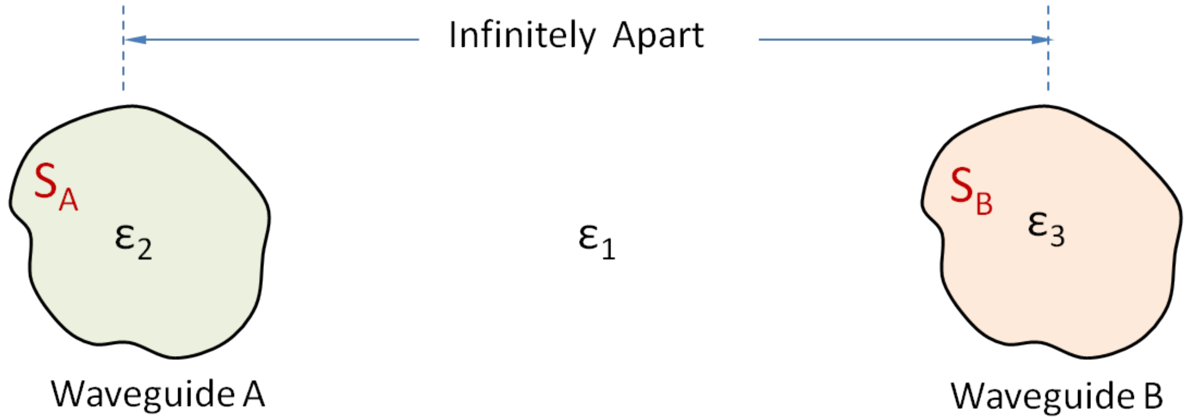
Application of Coupled-Mode Equations to two-coupled waveguides



Two waveguides (A and B) in proximity

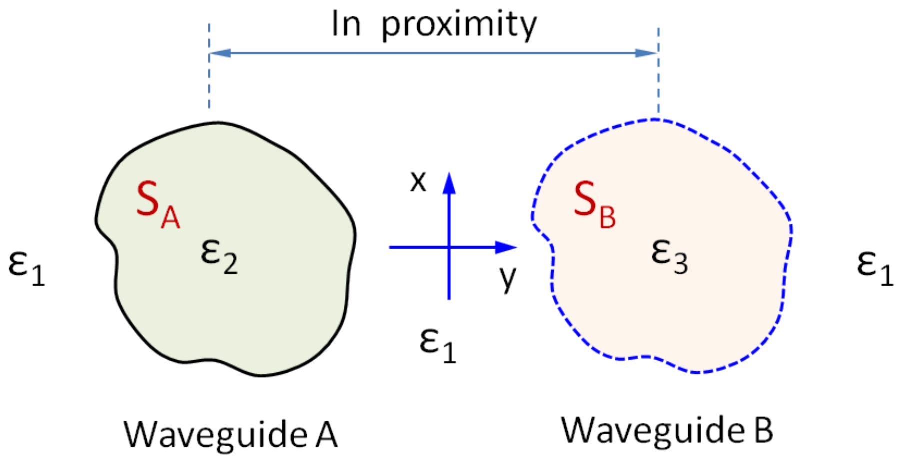
The presence of each consists a perturbation to its neighbor

Application of Coupled-Mode Equations to two-coupled waveguides



$$\boxed{\vec{E}_A = \vec{\mathcal{E}}_A(x,y)e^{-j\beta_A z}}$$

$$\boxed{\vec{E}_B = \vec{\mathcal{E}}_B(x,y)e^{-j\beta_B z}}$$

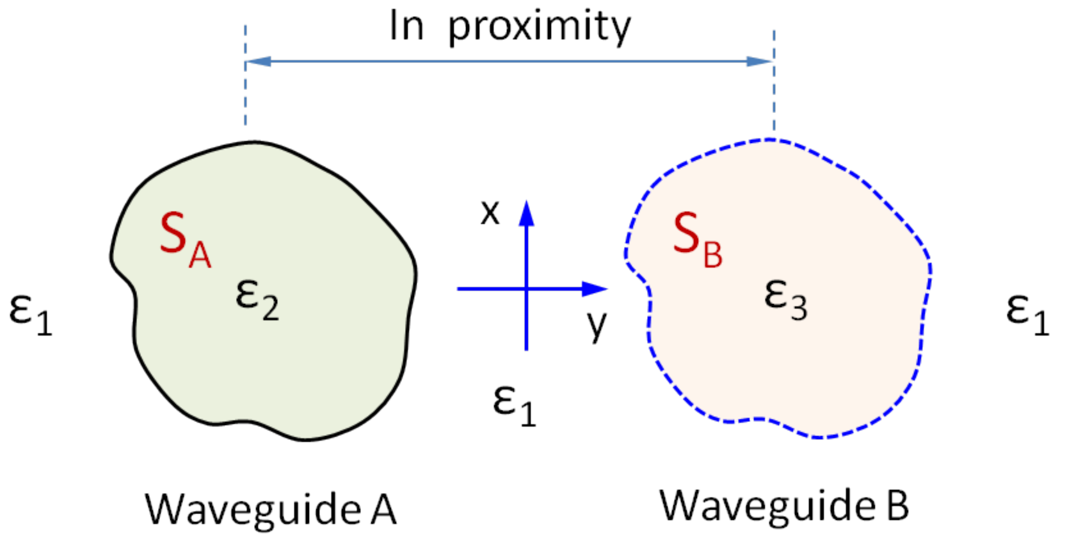


Waveguide B consists a perturbation for waveguide A

$$\epsilon' - \epsilon = \begin{cases} \epsilon_3 - \epsilon_1 & \vec{r} \in S_B \\ 0 & \text{otherwise} \end{cases}$$

Assume that both waveguides A and B are single (guided)-mode waveguides.

Application of Coupled-Mode Equations to two-coupled waveguides



$$\begin{bmatrix} \vec{E}_A \\ \vec{H}_A \end{bmatrix} = \begin{bmatrix} \vec{\mathcal{E}}_A(x,y) \\ \vec{\mathcal{H}}_A(x,y) \end{bmatrix} e^{-j\beta_A z} \qquad \begin{bmatrix} \vec{E}_B \\ \vec{H}_B \end{bmatrix} = \begin{bmatrix} \vec{\mathcal{E}}_B(x,y) \\ \vec{\mathcal{H}}_B(x,y) \end{bmatrix} e^{-j\beta_B z}$$

Assume that: $\iint_S \left(\vec{\mathcal{E}}_A \times \vec{\mathcal{H}}_B^*\right) \cdot \hat{z}\,dxdy \simeq 0$ (approximate orthogonality condition)

$$\boxed{\begin{bmatrix} \vec{E}' \\ \vec{H}' \end{bmatrix} \simeq a_A(z) \begin{bmatrix} \vec{\mathcal{E}}_A(x,y) \\ \vec{\mathcal{H}}_A(x,y) \end{bmatrix} e^{-j\beta_A z} + a_B(z) \begin{bmatrix} \vec{\mathcal{E}}_B(x,y) \\ \vec{\mathcal{H}}_B(x,y) \end{bmatrix} e^{-j\beta_B z}}$$

Application of Coupled-Mode Equations to two-coupled waveguides

$$\frac{da_A}{dz} = -j\left[a_A(z)C_{AA} + a_B(z)C_{BA}e^{j(\beta_A-\beta_B)z}\right]$$

$$\frac{da_B}{dz} = -j\left[a_A(z)C_{AB}e^{j(\beta_B-\beta_A)z} + a_B(z)C_{BB}\right]$$

Coupling Coefficients

$$C_{AA} \simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\iint_{S_B}(\epsilon'-\epsilon)\left(\vec{\mathcal{E}}_A^*\cdot\vec{\mathcal{E}}_A\right)dxdy$$

$$C_{BB} \simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\iint_{S_A}(\epsilon'-\epsilon)\left(\vec{\mathcal{E}}_B^*\cdot\vec{\mathcal{E}}_B\right)dxdy$$

$$C_{BA} \simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\iint_{S_B}(\epsilon'-\epsilon)\left(\vec{\mathcal{E}}_A^*\cdot\vec{\mathcal{E}}_B\right)dxdy$$

$$C_{AB} \simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\iint_{S_A}(\epsilon'-\epsilon)\left(\vec{\mathcal{E}}_B^*\cdot\vec{\mathcal{E}}_A\right)dxdy$$

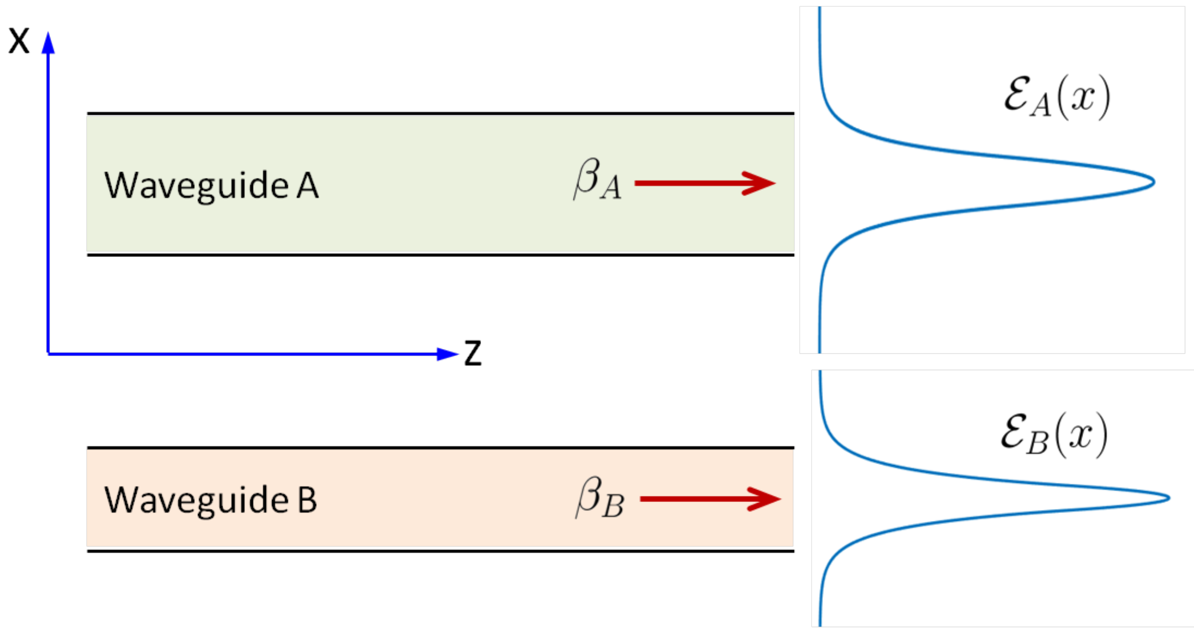
However: $C_{AA} \simeq 0 \quad \& \quad C_{BB} \simeq 0$

Application of Coupled-Mode Equations to two-coupled waveguides

$$\begin{aligned}
\frac{da_A}{dz} &= -ja_B(z)C_{BA}e^{+j(\beta_A-\beta_B)z} \\
\frac{da_B}{dz} &= -ja_A(z)C_{AB}e^{-j(\beta_A-\beta_B)z}
\end{aligned}$$

$$\begin{aligned}
C_{BA} &\simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\iint_{S_B}(\epsilon'-\epsilon)\left(\vec{\mathcal{E}}_A^*\cdot\vec{\mathcal{E}}_B\right)dxdy \\
C_{AB} &\simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\iint_{S_A}(\epsilon'-\epsilon)\left(\vec{\mathcal{E}}_B^*\cdot\vec{\mathcal{E}}_A\right)dxdy
\end{aligned}$$

Application of Coupled-Mode Equations to two-coupled planar waveguides (directional coupler)



$$\Delta\beta = \beta_A - \beta_B$$

$$\frac{\beta_A}{|\beta_A|} = \frac{\beta_B}{|\beta_B|} = 1$$

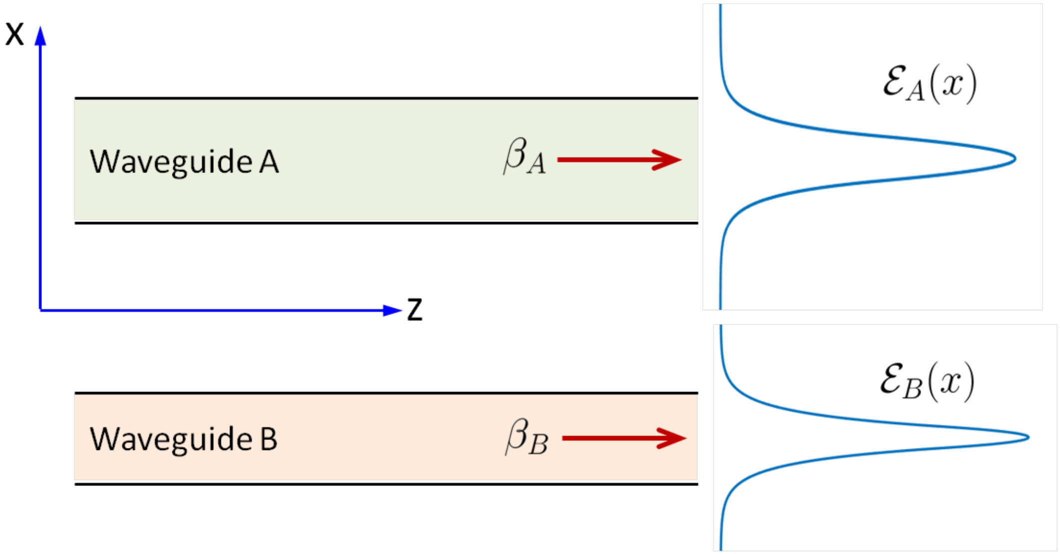
$$\frac{da_A}{dz} = -j a_B(z) C_{BA} e^{+j\Delta\beta z}$$

$$\frac{da_B}{dz} = -j a_A(z) C_{AB} e^{-j\Delta\beta z}$$

$$C_{BA} \simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\int_{h_B} (\epsilon' - \epsilon)\left(\vec{\mathcal{E}}_A^* \cdot \vec{\mathcal{E}}_B\right) dx$$

$$C_{AB} \simeq \frac{\omega}{4}\frac{\beta_A}{|\beta_A|}\int_{h_A} (\epsilon' - \epsilon)\left(\vec{\mathcal{E}}_B^* \cdot \vec{\mathcal{E}}_A\right) dx$$

Application of Coupled-Mode Equations to two-coupled planar waveguides (directional coupler)



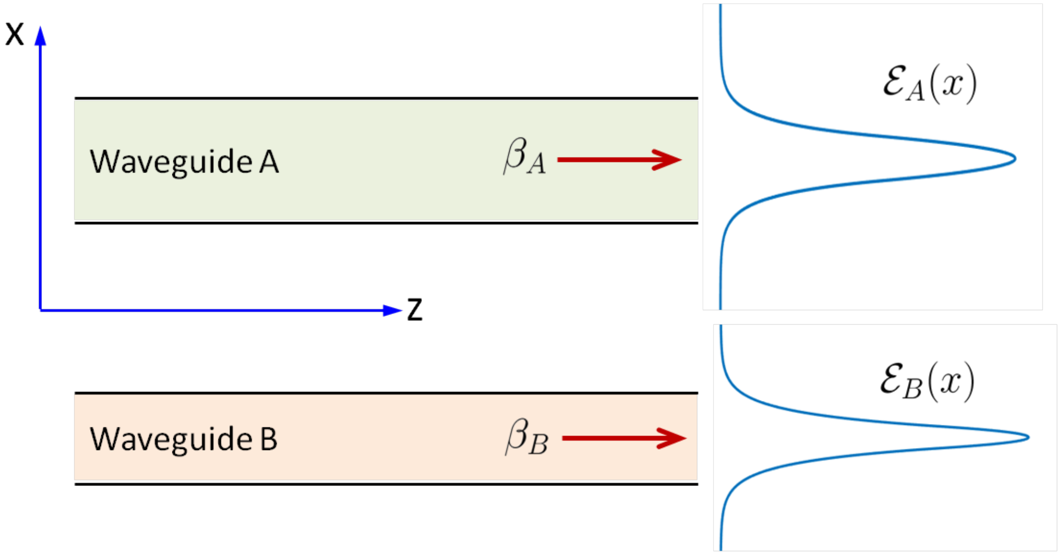
Assume solutions of the form:

$$a_A(z) = A_0 e^{-j\gamma_A z}$$
$$a_B(z) = B_0 e^{-j\gamma_B z}$$

$$-j\gamma_A A_0 e^{-j\gamma_A z} = -jC_{BA} e^{j\Delta\beta z} B_0 e^{-j\gamma_B z} \Longrightarrow \gamma_A A_0 = C_{BA} B_0 e^{j(\Delta\beta + \gamma_A - \gamma_B)z}$$
$$-j\gamma_B B_0 e^{-j\gamma_B z} = -jC_{AB} e^{-j\Delta\beta z} A_0 e^{-j\gamma_A z} \Longrightarrow \gamma_B B_0 = C_{AB} A_0 e^{-j(\Delta\beta + \gamma_A - \gamma_B)z}$$

$$\boxed{\begin{aligned} \Delta\beta + \gamma_A - \gamma_B &= 0 \\ \gamma_A &= -\frac{\Delta\beta}{2} \pm \left[\left(\frac{\Delta\beta}{2}\right)^2 + C_{AB}C_{BA}\right]^{1/2} \\ \gamma_B &= +\frac{\Delta\beta}{2} \pm \left[\left(\frac{\Delta\beta}{2}\right)^2 + C_{AB}C_{BA}\right]^{1/2} \end{aligned}}$$

Application of Coupled-Mode Equations to two-coupled planar waveguides (directional coupler)



$$
\begin{aligned}
a_A(z) &= e^{j(\Delta\beta/2)z}\left[A_1 e^{jSz} + A_2 e^{-jSz}\right] \\
S &= \left[\left(\frac{\Delta\beta}{2}\right)^2 + C_{AB}C_{BA}\right]^{1/2} \\
a_B(z) &= j\frac{1}{C_{BA}} e^{-j(\Delta\beta/2)z}\frac{da_A(z)}{dz}
\end{aligned}
$$

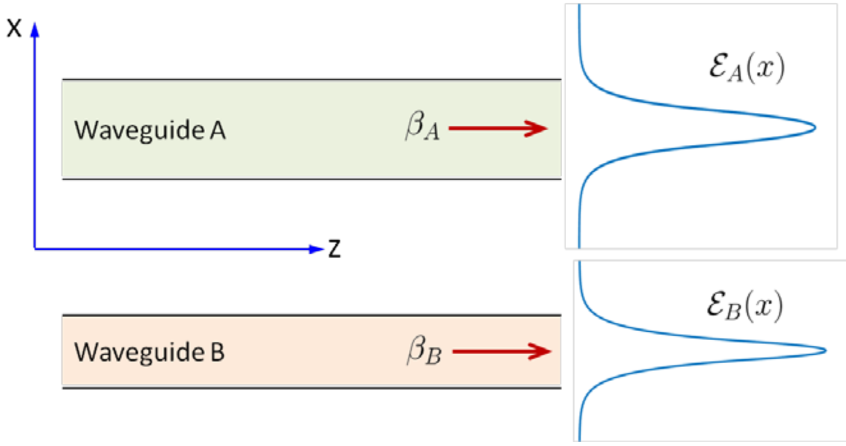
Assume power transfer from B to A: $a_A(z=0) = 0 \longrightarrow A_1 = -A_2$

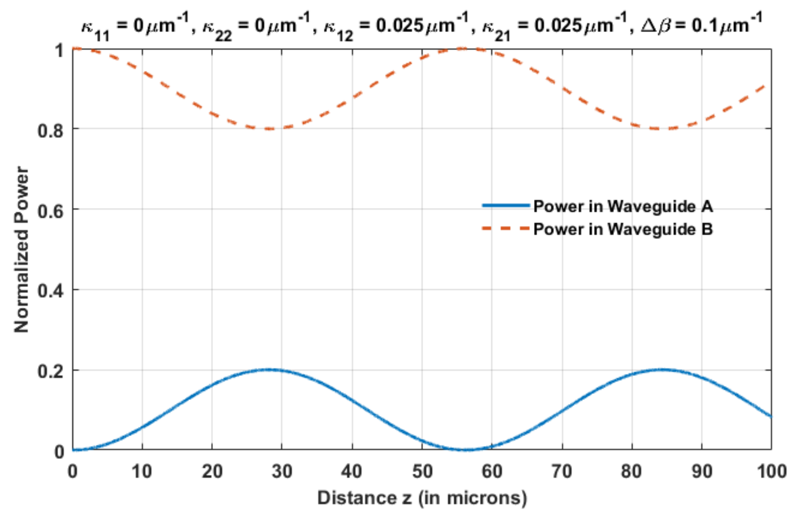
$$
\boxed{
\begin{aligned}
a_A(z) &= 2jA_1 e^{j(\Delta\beta/2)z}\sin(Sz) \\
a_B(z) &= j\frac{2A_1}{C_{BA}} e^{-j(\Delta\beta/2)z}\left[-\frac{\Delta\beta}{2}\sin(Sz) + jS\cos(Sz)\right]
\end{aligned}
}
$$

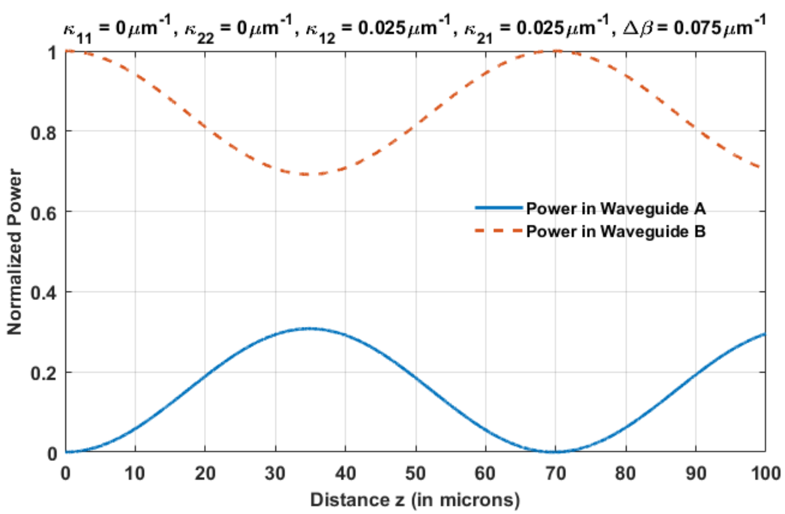
Waveguide Powers:

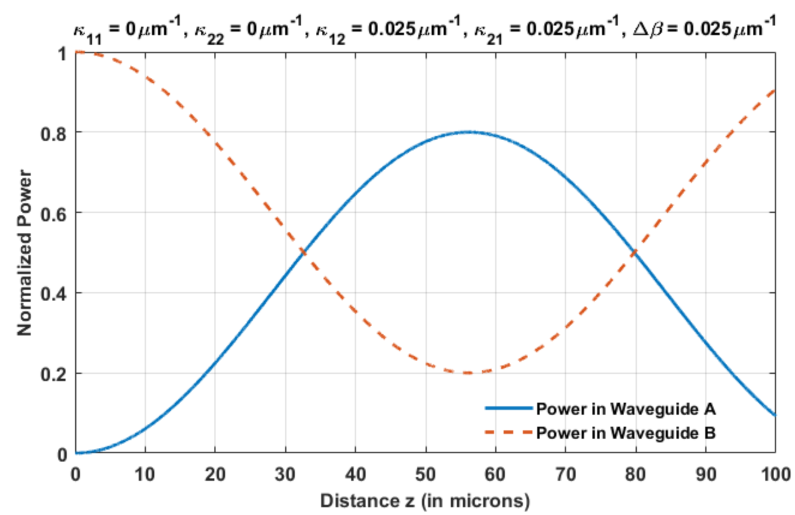
$$
\begin{aligned}
P_A(z) &= P_0\frac{|C_{BA}|^2}{S^2}\sin^2(Sz) \\
P_B(z) &= P_0\left[\left(\frac{\Delta\beta}{2S}\right)^2\sin^2(Sz) + \cos^2(Sz)\right]
\end{aligned}
$$

Application of Coupled-Mode Equations to two-coupled planar waveguides (directional coupler)







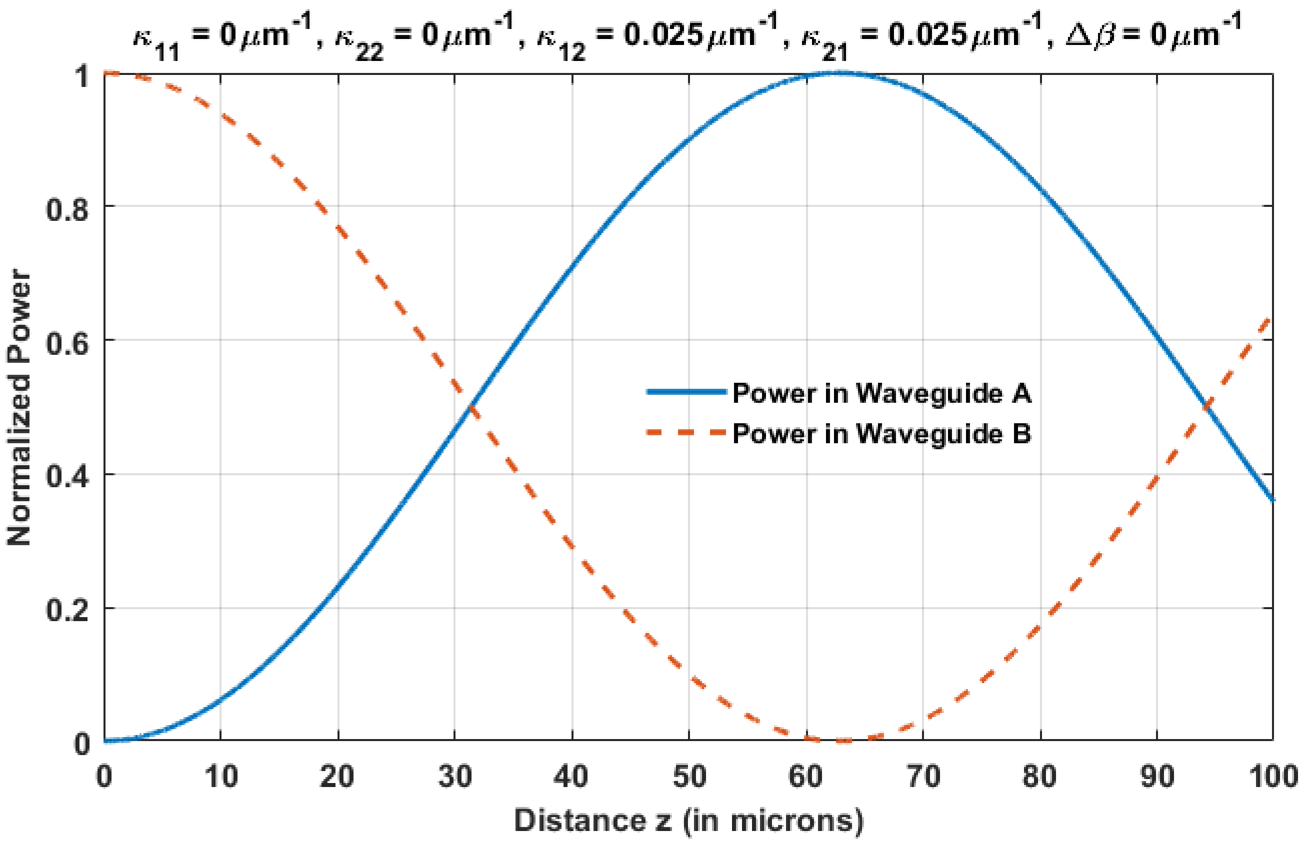


Application of Coupled-Mode Equations to two-coupled planar waveguides (directional coupler)

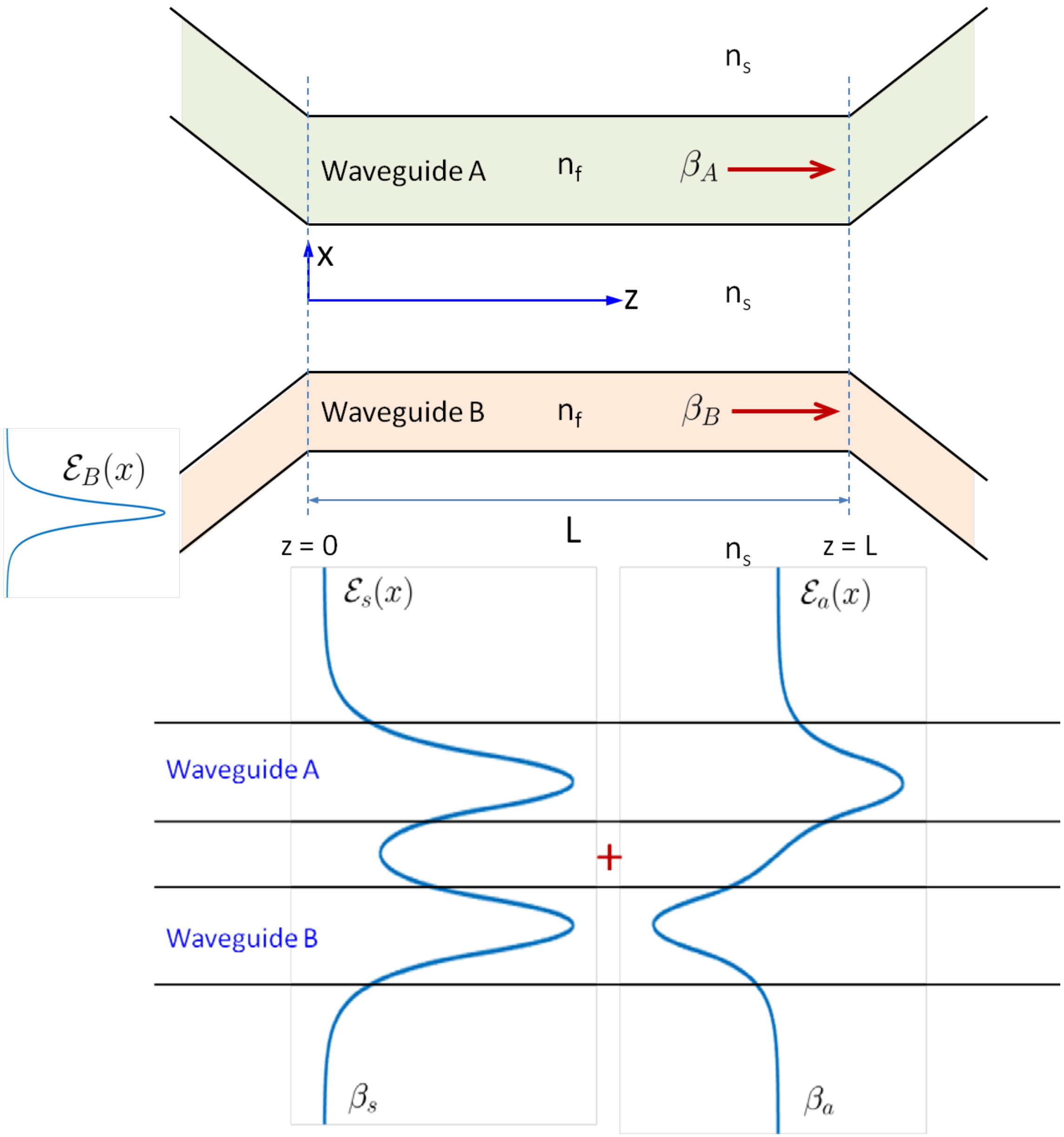
$$\Delta\beta = 0, \qquad C_{AB} = C_{BA} = C$$

$$\boxed{\begin{aligned} P_A(z) &= P_0 \sin^2(Cz) \\ P_B(z) &= P_0 \cos^2(Cz) \end{aligned}}$$

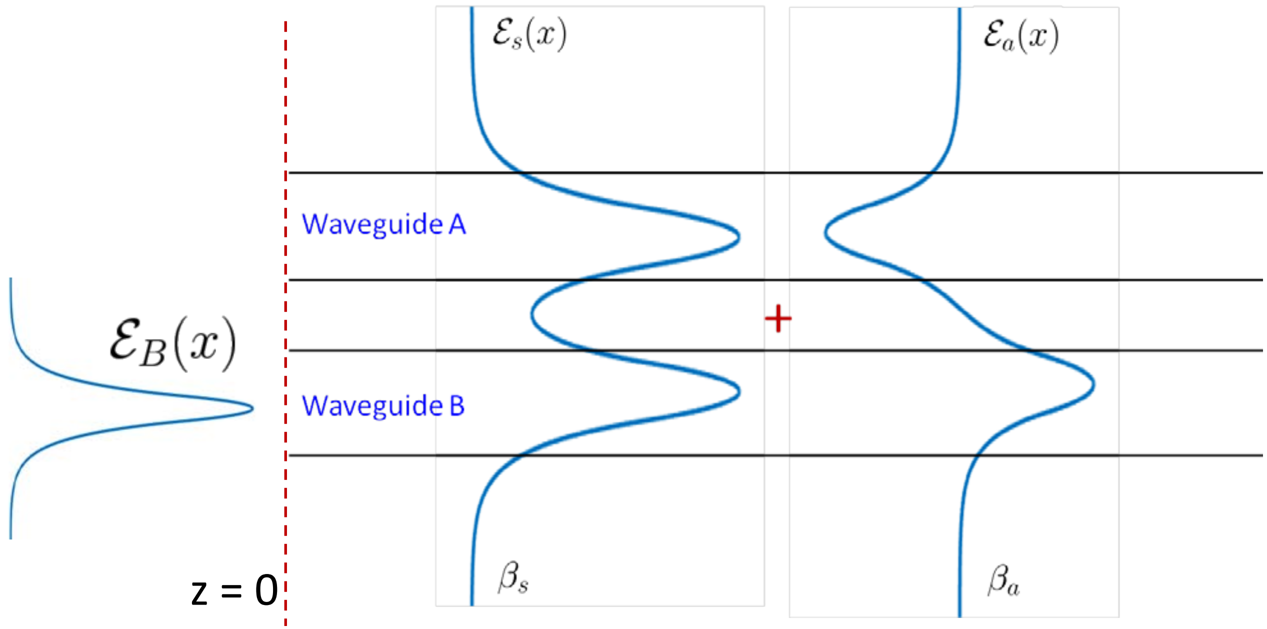
$$CL = \frac{\pi}{2} \implies L = \frac{\pi}{2C}$$



Supermode Analysis of Directional Coupler



Supermode Analysis of Directional Coupler



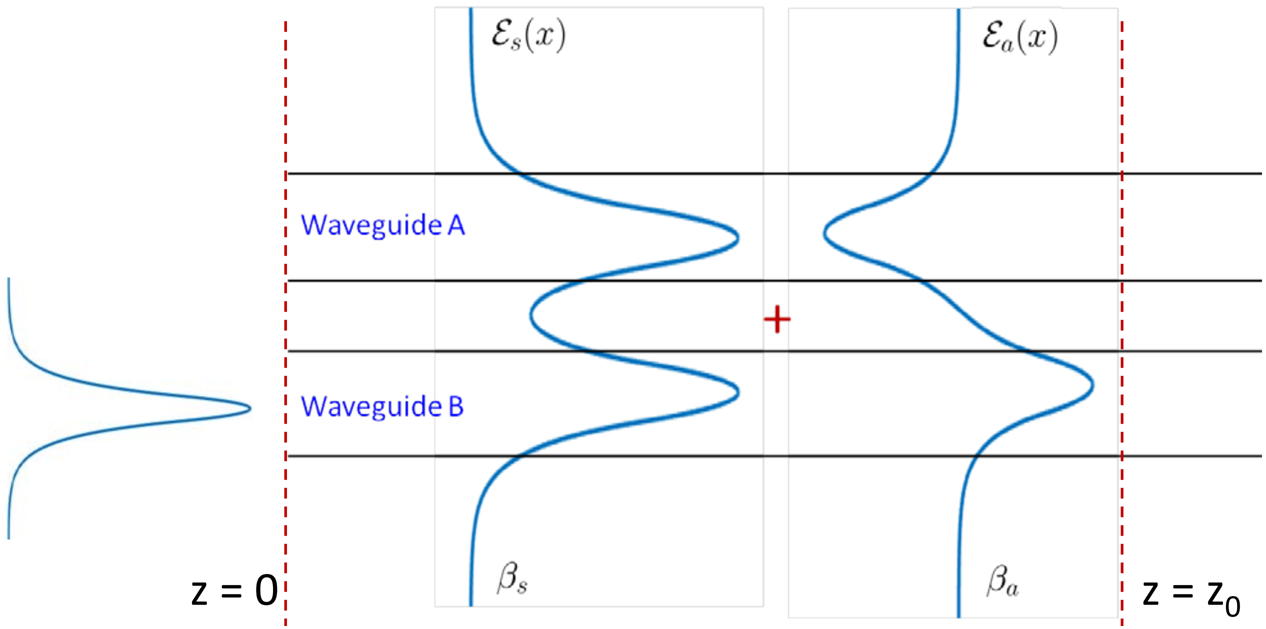
Assume TE modes:

$$
\begin{aligned}
E(x,z) &= \hat{y}\left[c_s\mathcal{E}_s(x)e^{-j\beta_s z} + c_a\mathcal{E}_a(x)e^{-j\beta_a z}\right] \\
E_{inc}(x,z) &= \hat{y}\mathcal{E}_B(x)e^{-j\beta_B z}
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 \implies E_{inc}(x,z=0) = E(x,z=0) \\
c_s &= \langle \mathcal{E}_B(x), \mathcal{E}_s(x)\rangle = \frac{1}{P}\frac{\beta_s}{2\omega\mu_0}\int_{-\infty}^{+\infty}\mathcal{E}_B(x)\mathcal{E}_s^*(x)dx \\
c_a &= \langle \mathcal{E}_B(x), \mathcal{E}_a(x)\rangle = \frac{1}{P}\frac{\beta_a}{2\omega\mu_0}\int_{-\infty}^{+\infty}\mathcal{E}_B(x)\mathcal{E}_a^*(x)dx
\end{aligned}
$$

$$\boxed{c_s \simeq c_a = c}$$

Supermode Analysis of Directional Coupler



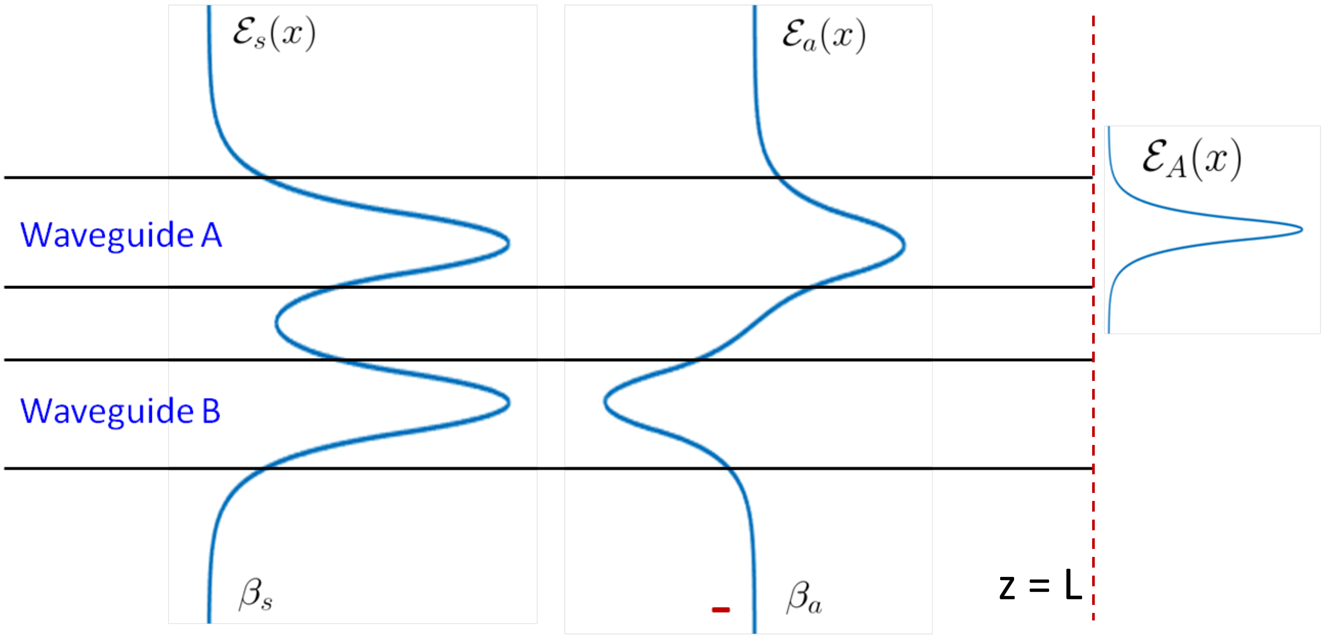
$$E(x,z_0) = \hat{y}\left[c_s\mathcal{E}_s(x)e^{-j\beta_s z_0} + c_a\mathcal{E}_a(x)e^{-j\beta_a z_0}\right]$$

$$\simeq c\mathcal{E}_s(x)e^{-j\beta_s z_0}\left[1 + \frac{\mathcal{E}_a(x)}{\mathcal{E}_s(x)}e^{j(\beta_s-\beta_a)z_0}\right]$$

$$(\beta_s - \beta_a)z_0 = \pi \implies z_0 = L = \frac{\pi}{\beta_s - \beta_a}$$

$$E(x,z_0) = \hat{y}ce^{-j\beta_s z_0}\left[\mathcal{E}_s(x) - \mathcal{E}_a(x)\right]$$

Supermode Analysis of Directional Coupler



$$(\beta_s - \beta_a) z_0 = \pi \implies z_0 = L = \frac{\pi}{\beta_s - \beta_a}$$

$$E(x, z_0) = \hat{y} c e^{-j\beta_s z_0} \left[\mathcal{E}_s(x) - \mathcal{E}_a(x) \right]$$