31/10/2023

Ray (ABCD) Matrices

Electro-Optics & Applications

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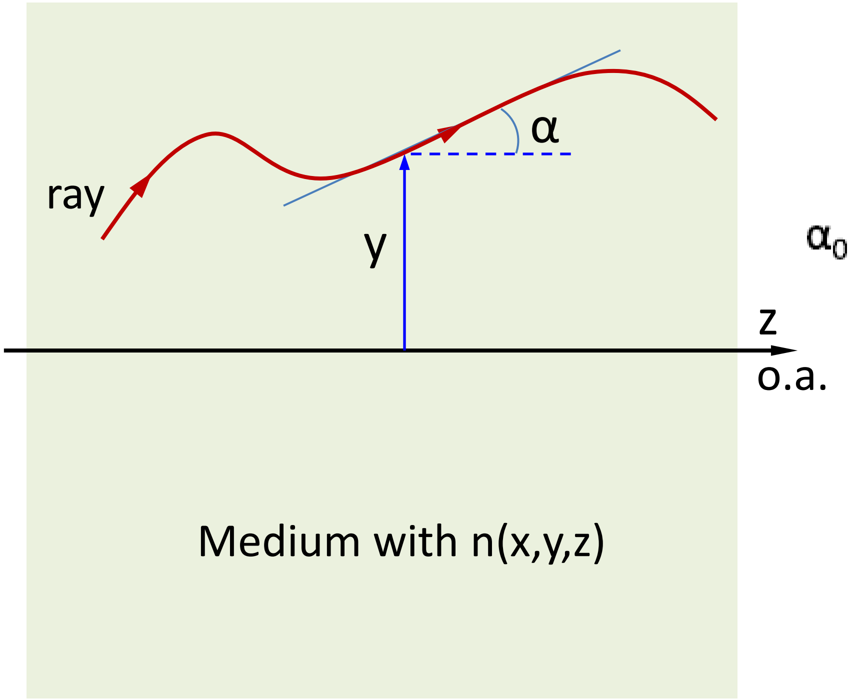
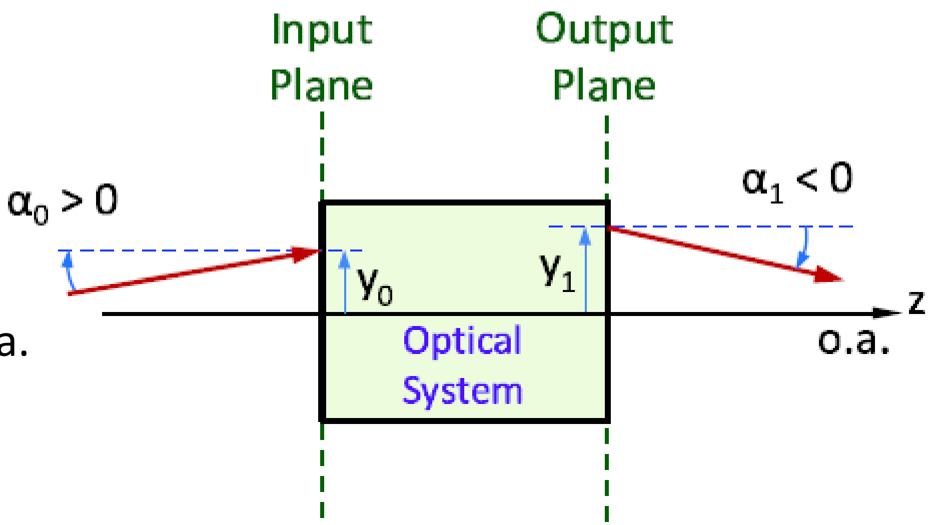
School of Electrical & Computer Engineering
National Technical University of Athens

Matrix Approach for Rays



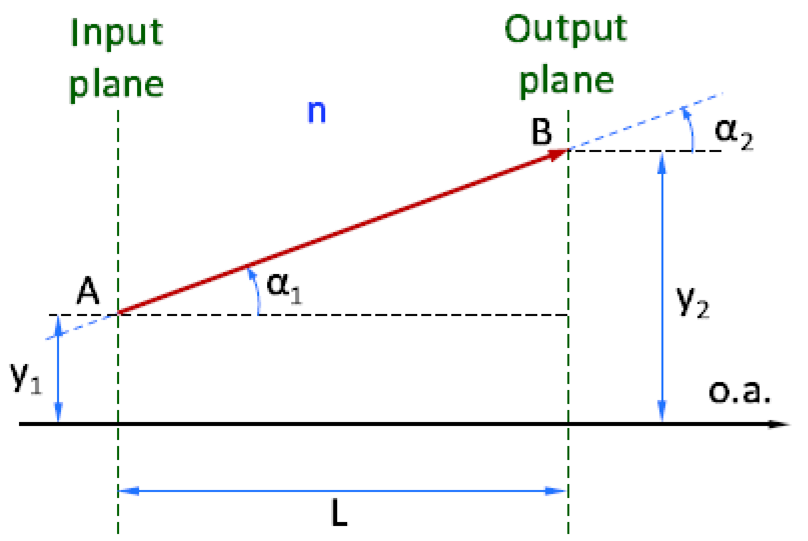


Ray description via y and α = dy/dz

$$\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = \tilde{M} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix}$$

Elementary ABCD Matrices

Translation Matrix



$$y_2 = y_1 + L\tan\alpha_1 \simeq y_1 + L\alpha_1,$$

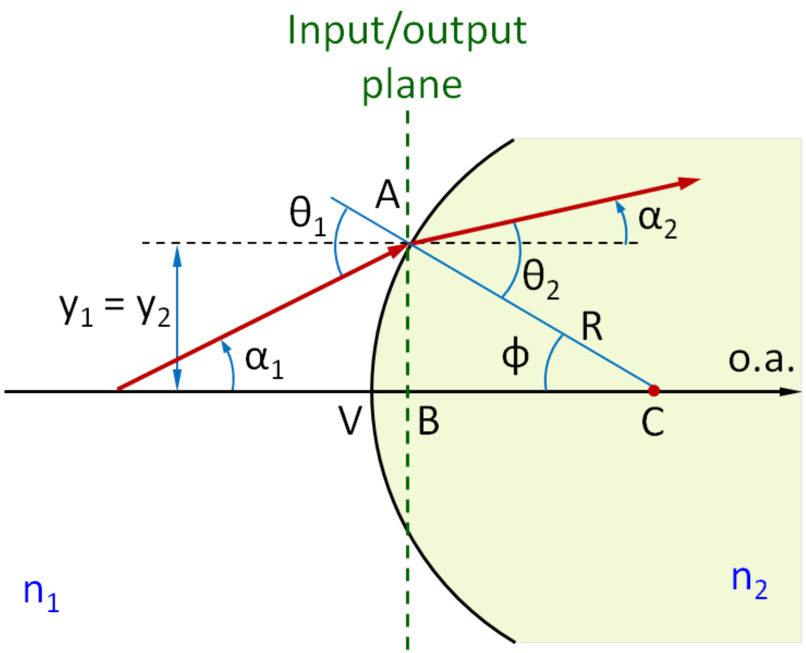
$$\alpha_2 = \alpha_1, \qquad \text{and the resulting } ABCD \text{ matrix is}$$

$$\tilde{M} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}.$$

Paraxial Approximation: $\sin\theta \simeq \theta$ $\tan\theta \simeq \theta$ $\cos\theta \simeq 1$

Elementary ABCD Matrices

Spherical Refraction Matrix

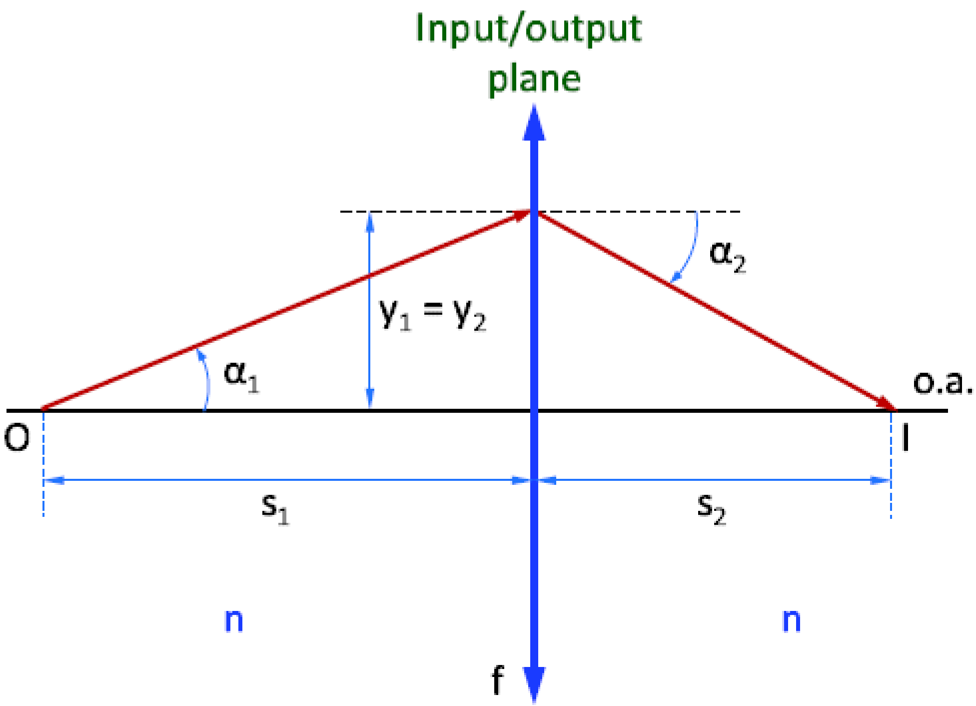


$$
\begin{aligned}
y_2 &= y_1 \\
n_1 \sin\theta_1 &= n_2 \sin\theta_2 \Longrightarrow n_1\theta_1 \simeq n_2\theta_2 \Longrightarrow \\
n_1(\alpha_1 + \phi) &= n_2(\alpha_2 + \phi) \Longrightarrow \\
\alpha_2 &= \left(\frac{n_1}{n_2} - 1\right)\phi + \frac{n_1}{n_2}\alpha_1 \Longrightarrow \\
\alpha_2 &= \left(\frac{n_1}{n_2} - 1\right)\frac{y_1}{R} + \frac{n_1}{n_2}\alpha_1 \qquad \text{resulting in} \\
\tilde{M} &= \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix}.
\end{aligned}
$$

Paraxial Approximation: $\sin\theta \simeq \theta \qquad \tan\theta \simeq \theta \qquad \cos\theta \simeq 1$

Elementary ABCD Matrices

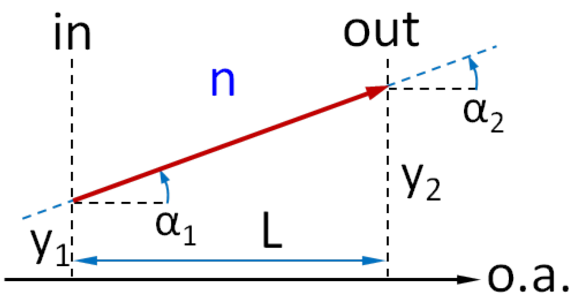
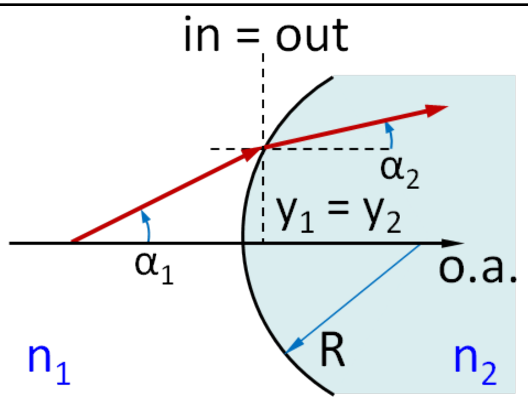
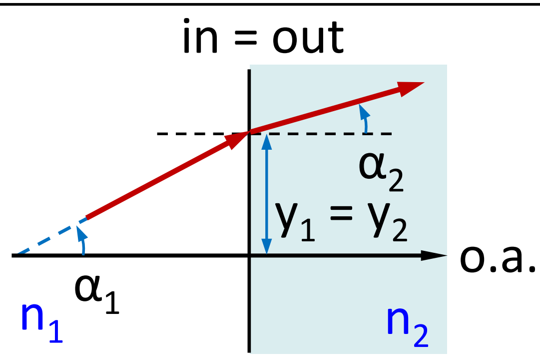
Spherical Refraction Matrix



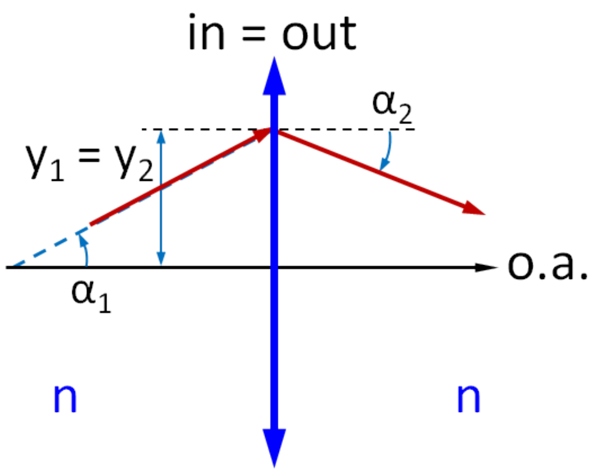
$$
\begin{aligned}
y_2 &= y_1 \\
|\alpha_2| &\simeq \frac{|y_1|}{s_2} = |y_1|\left(\frac{1}{f} - \frac{1}{s_1}\right) = \\
& \frac{|y_1|}{f} - \frac{|y_1|}{s_1} = \frac{|y_1|}{f} - \alpha_1 \Longrightarrow \\
\alpha_2 &= -\frac{y_1}{f} + \alpha_1, \qquad \text{resulting in} \\
\tilde{M} &= \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}.
\end{aligned}
$$

Paraxial Approximation: $\sin\theta \simeq \theta$ $\tan\theta \simeq \theta$ $\cos\theta \simeq 1$

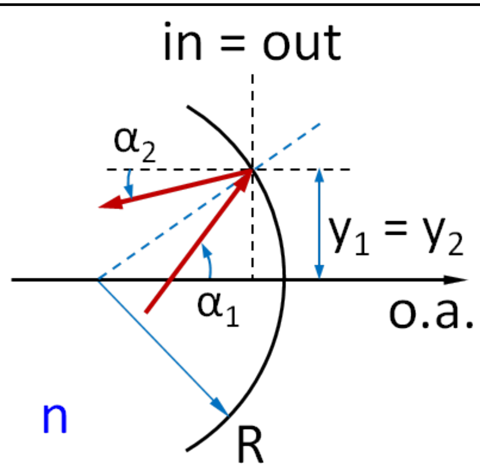
Elementary ABCD Matrices

Configuration	ABCD Matrix
	Translation Matrix $M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$
	Spherical Refraction Matrix $M = \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix}$ **R > 0** if convex interface and **R < 0** if concave interface (for light propagation from left to right)
	Planar Refraction Matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$

Elementary ABCD Matrices

<table>
<tr><th>Configuration</th><th>ABCD Matrix</th></tr>
<tr><td>


</td><td>
Thin-Lens Matrix

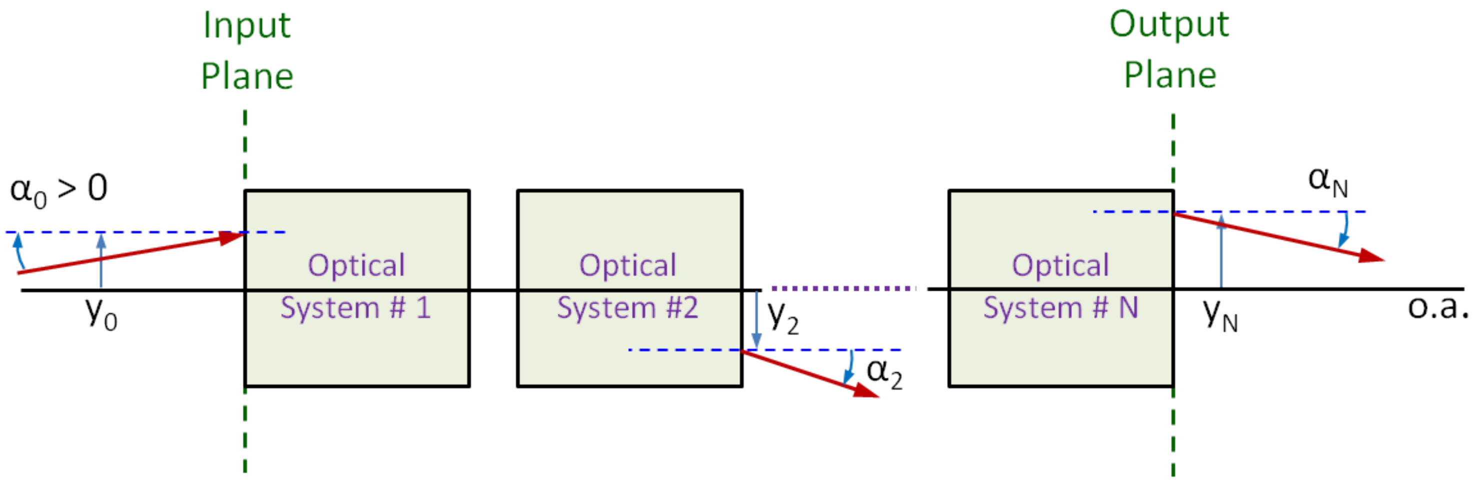
$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$
</td></tr>
<tr><td>


</td><td>
Spherical Mirror Matrix

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{bmatrix}$$

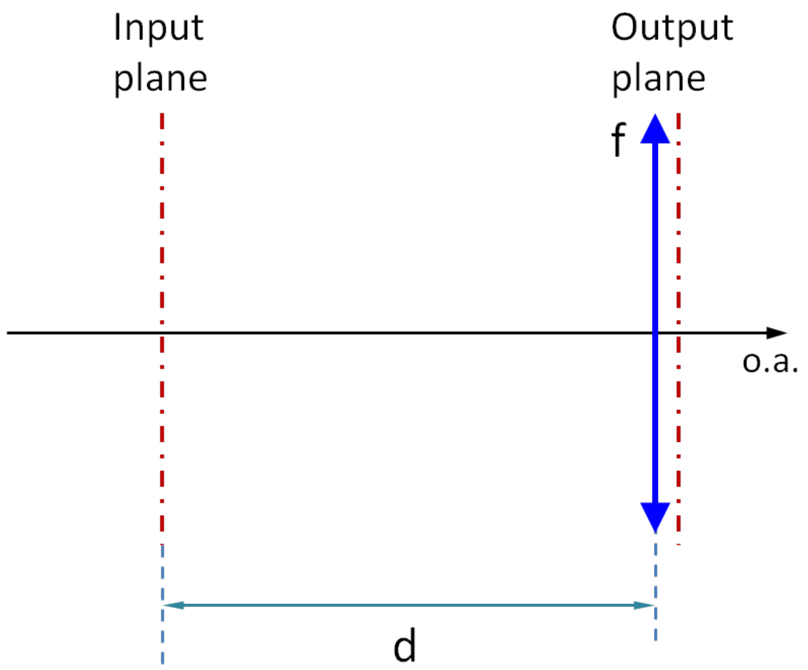
R > 0 if convex interface and **R < 0** if concave interface (for light propagation from left to right)
</td></tr>
</table>

Combining ABCD Matrices



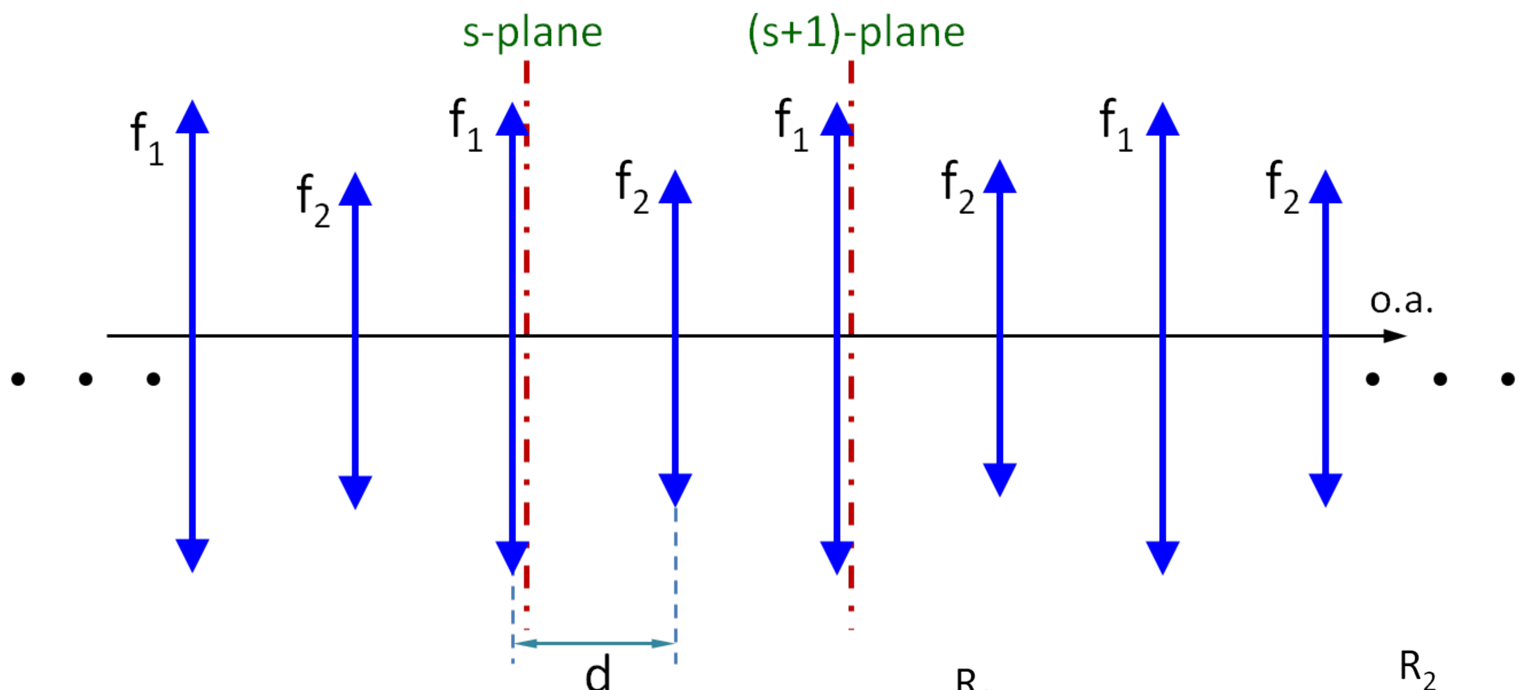
$$\begin{aligned}
\begin{bmatrix} y_N \\ \alpha_N \end{bmatrix} &= \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix} \begin{bmatrix} y_{N-1} \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix} \begin{bmatrix} A_{N-1} & B_{N-1} \\ C_{N-1} & D_{N-1} \end{bmatrix} \begin{bmatrix} y_{N-2} \\ \alpha_{N-2} \end{bmatrix} = \\
&= \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix} \begin{bmatrix} A_{N-1} & B_{N-1} \\ C_{N-1} & D_{N-1} \end{bmatrix} \cdots \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = \\
&= \left[\prod_{i=1}^{N} \tilde{M}_{N+1-i} \right] \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = \tilde{M}_{total} \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix},
\end{aligned}$$

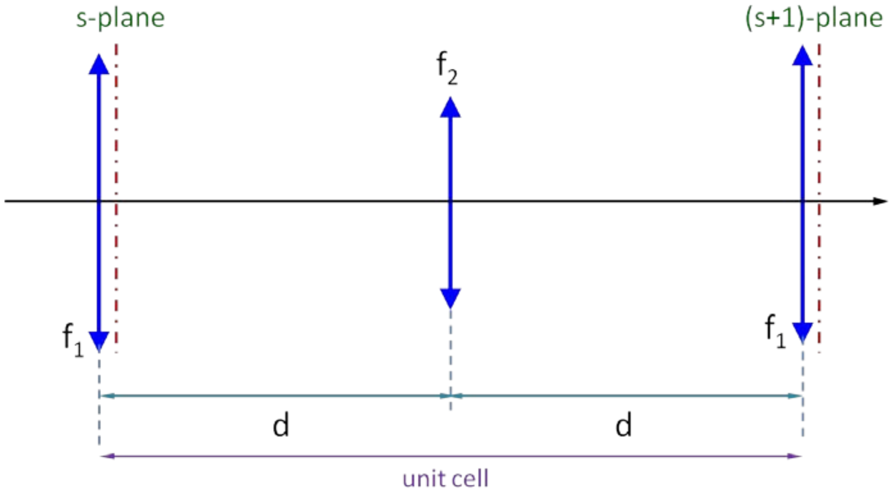
Simple Optical System

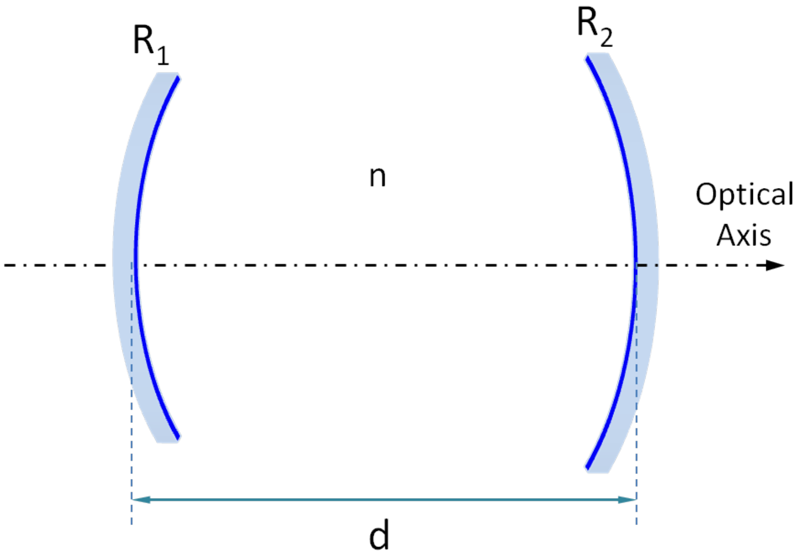


$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ -\dfrac{1}{f} & 1-\dfrac{d}{f} \end{bmatrix}$$

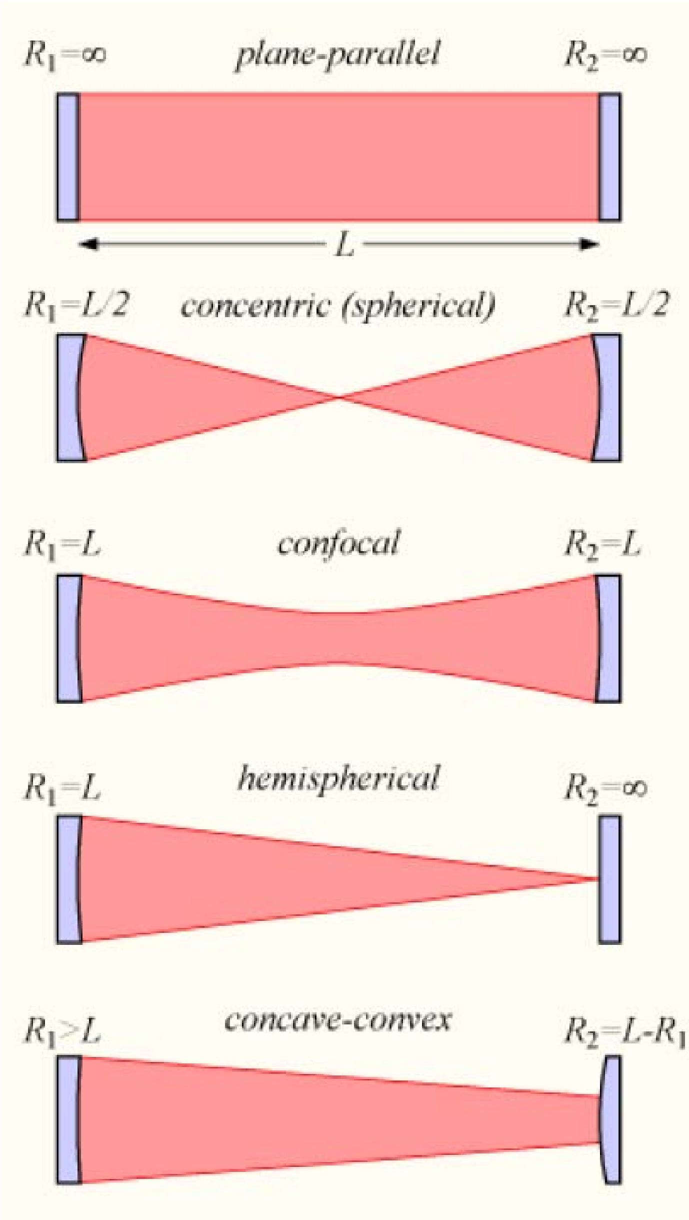
Biperiodic Optical System (Optical Cavity)

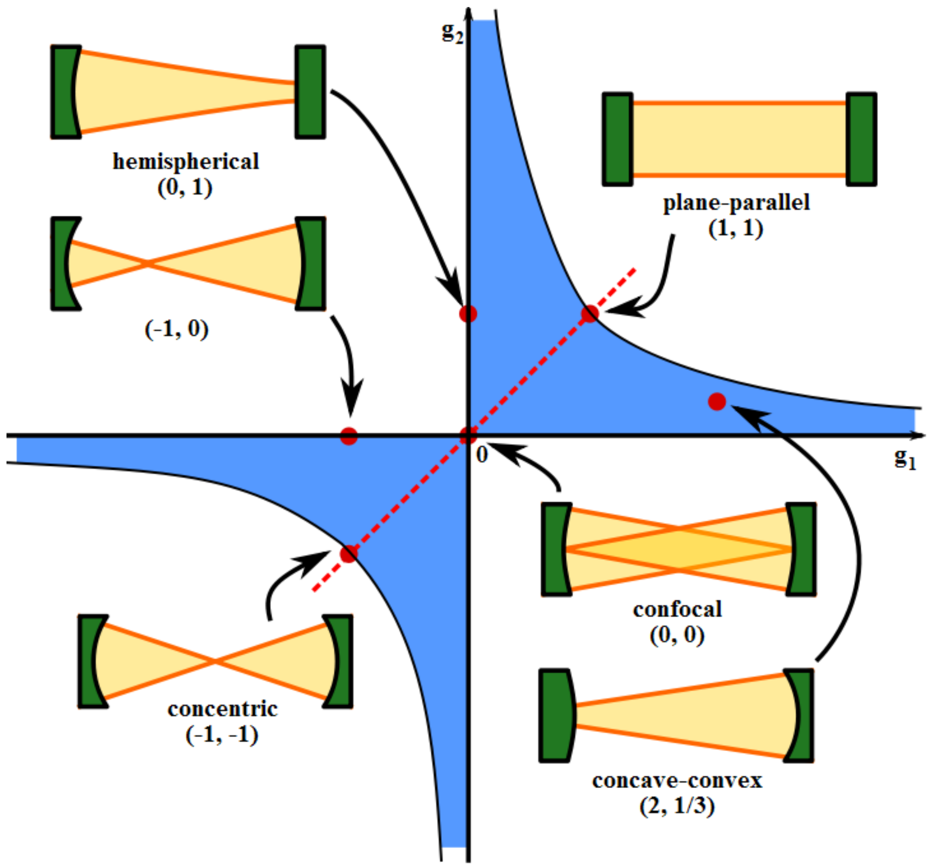






Cavity Resonators & Stability Diagram

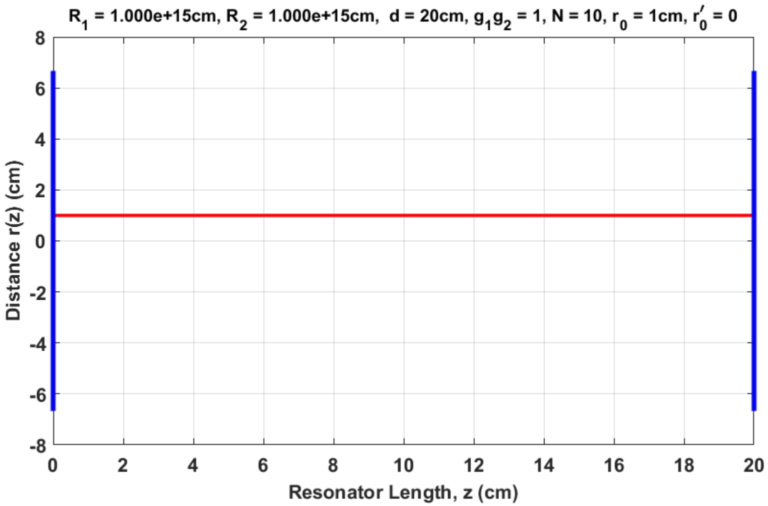




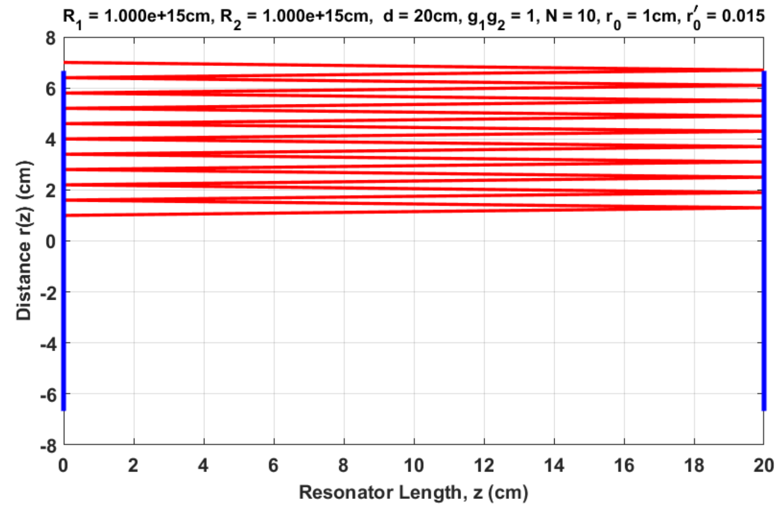
https://en.wikipedia.org/wiki/Optical_cavity

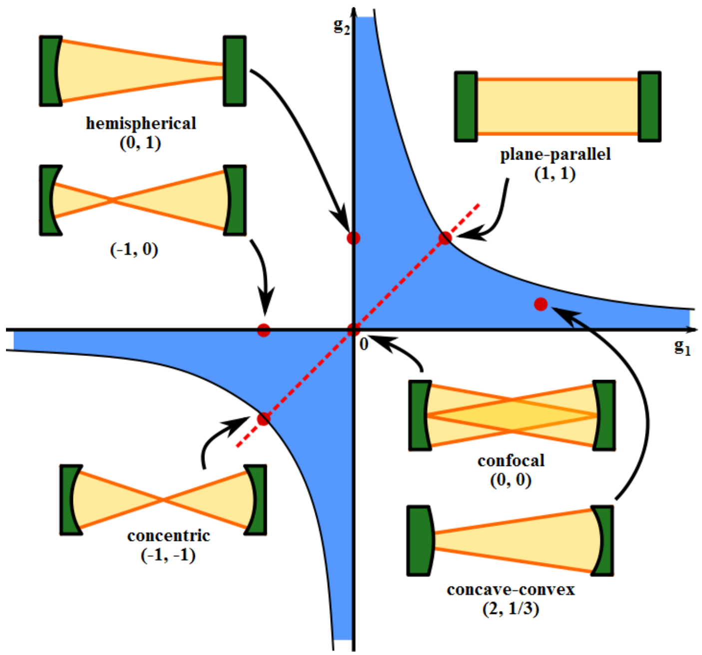
Cavity Resonators & Stability Diagram

Planar Resonator

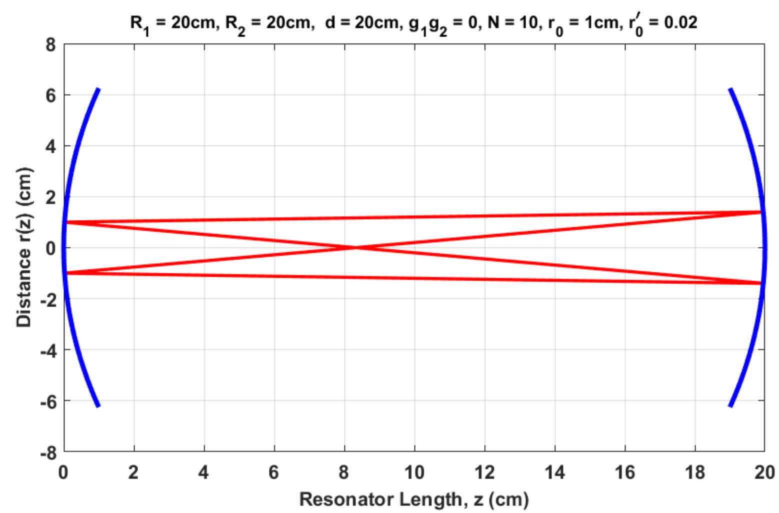


Planar Resonator



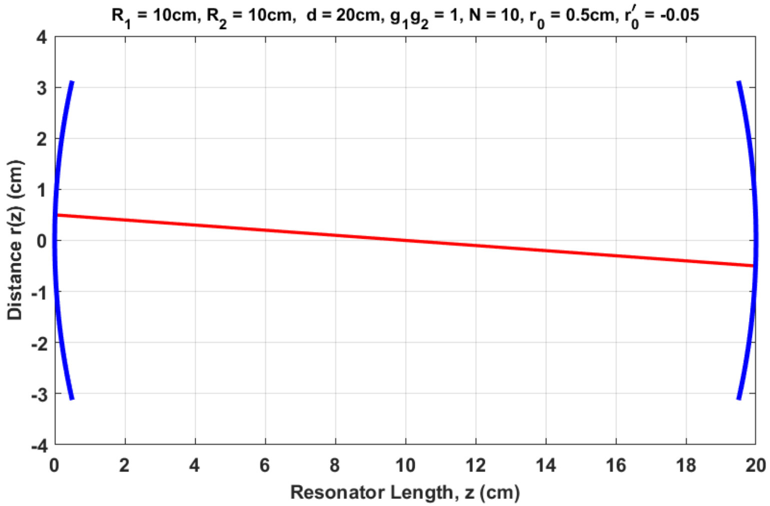


Confocal Resonator

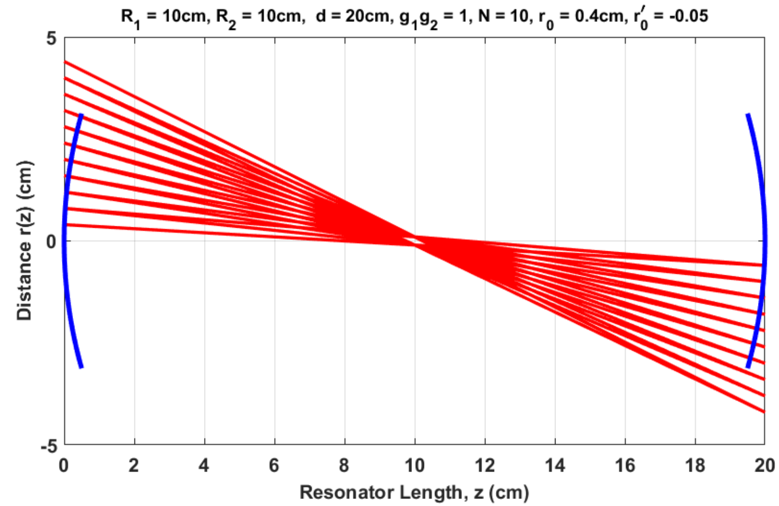


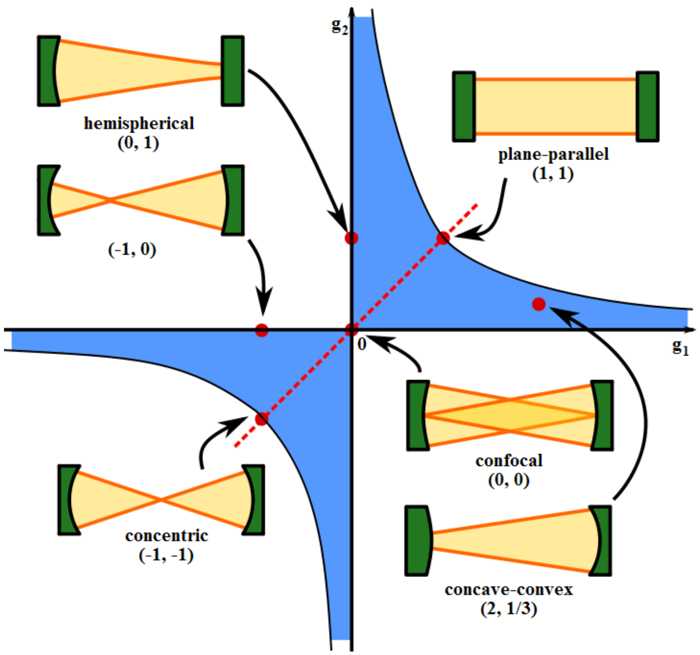
Cavity Resonators & Stability Diagram

Concentric Resonator

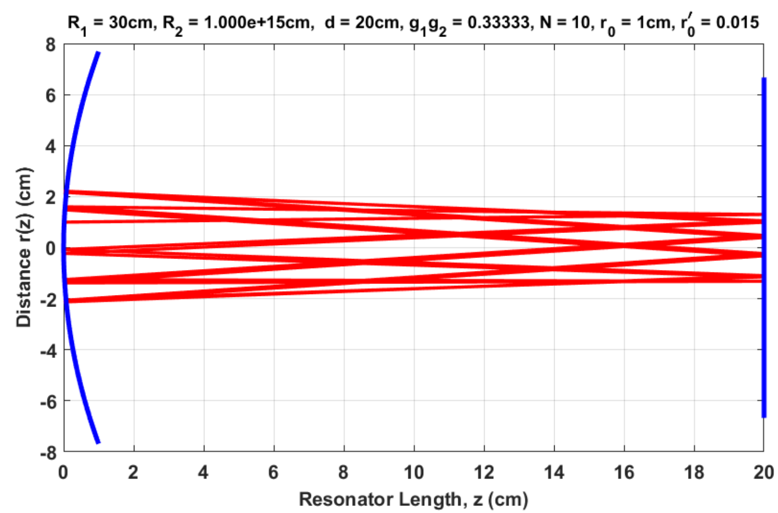


Concentric Resonator



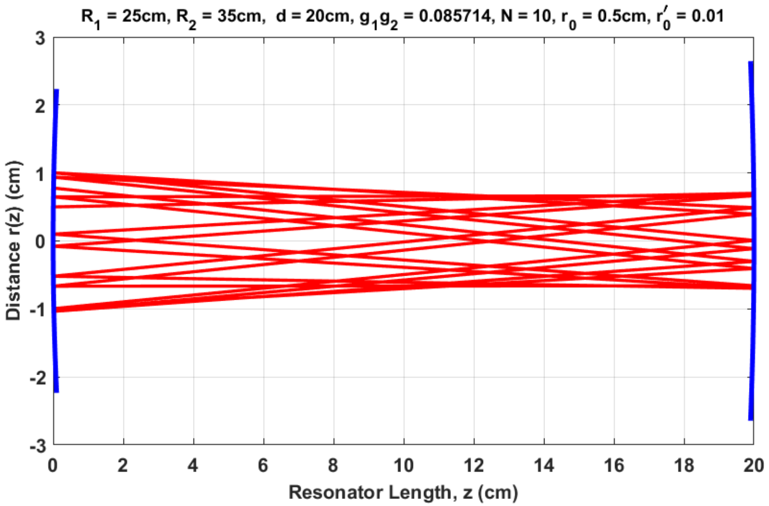


Hemispherical Resonator

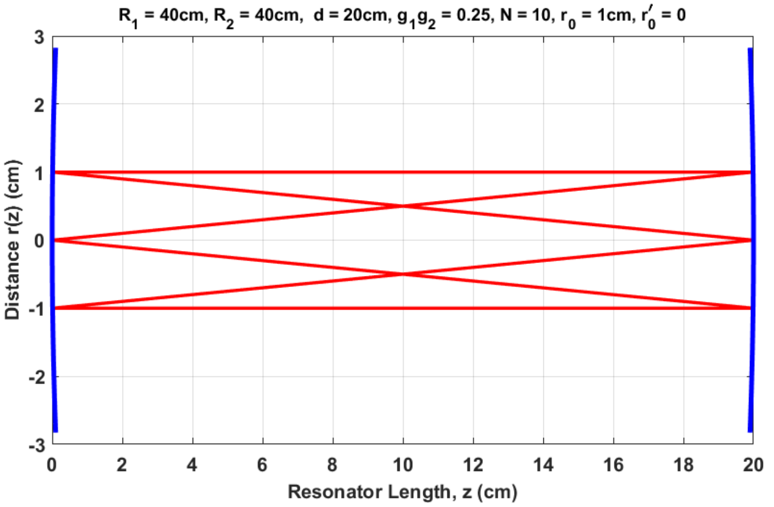


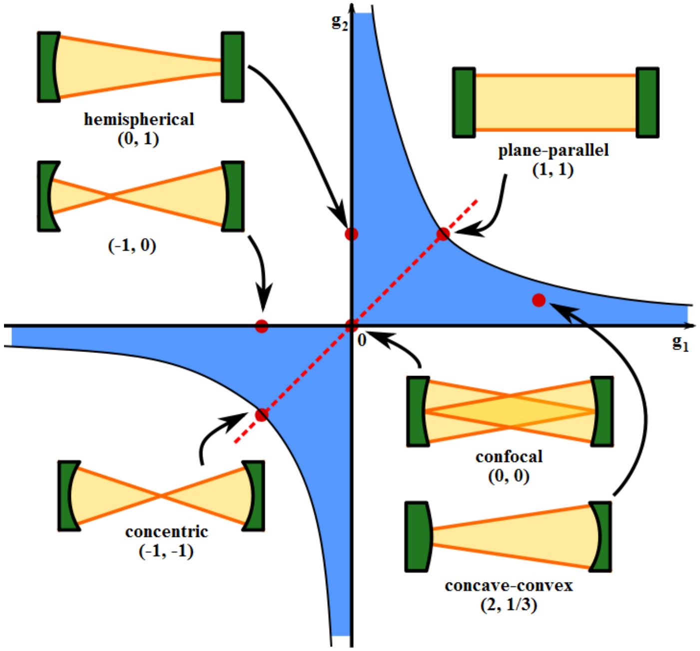
Cavity Resonators & Stability Diagram

Stable Resonator

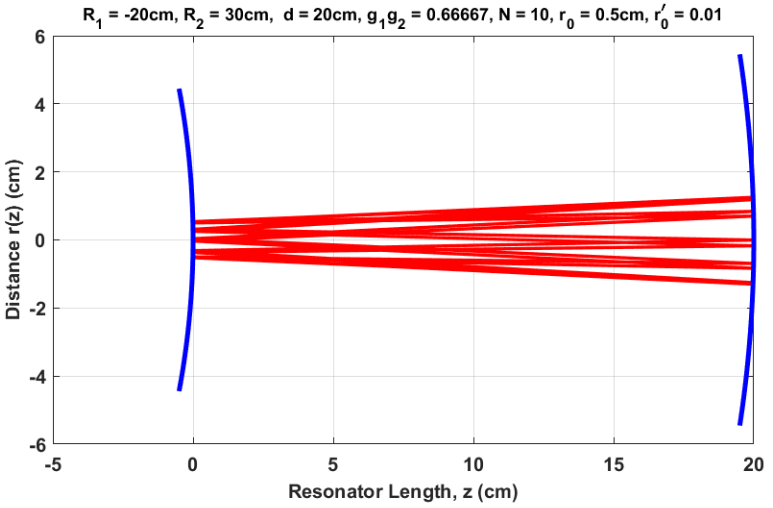


Stable Resonator



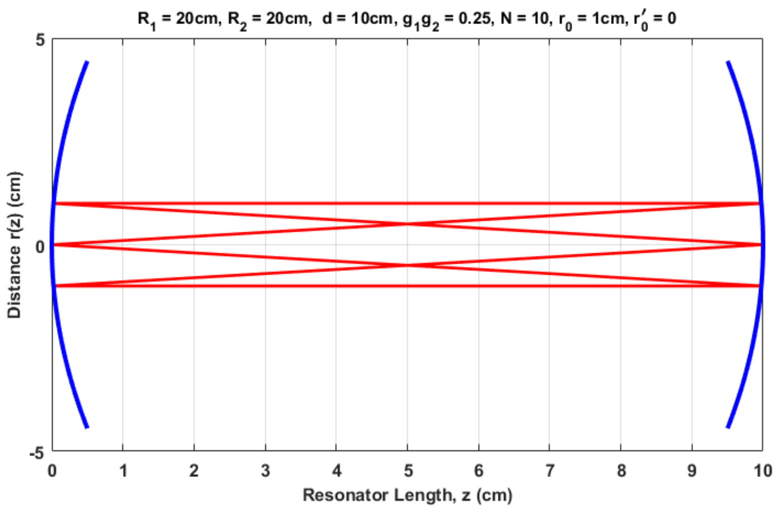


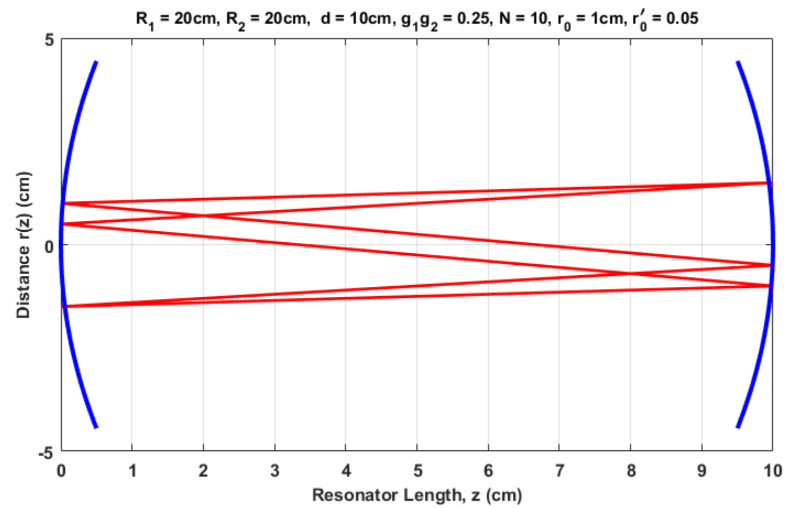
Concave-Convex Resonator

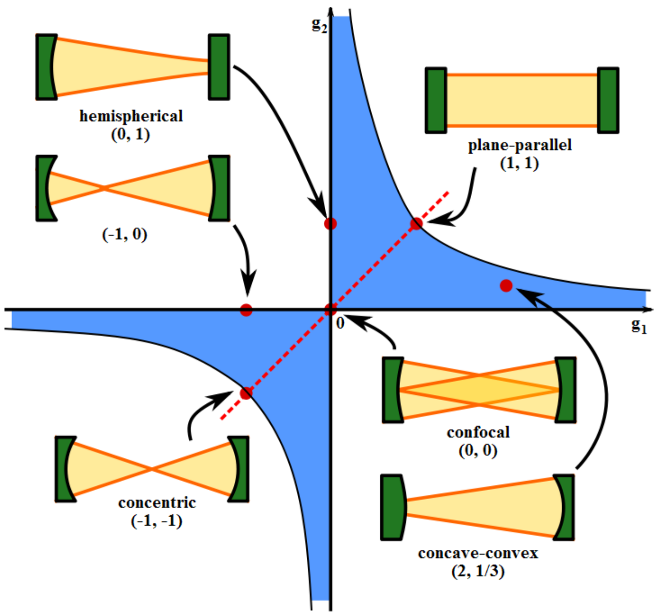


Cavity Resonators & Stability Diagram

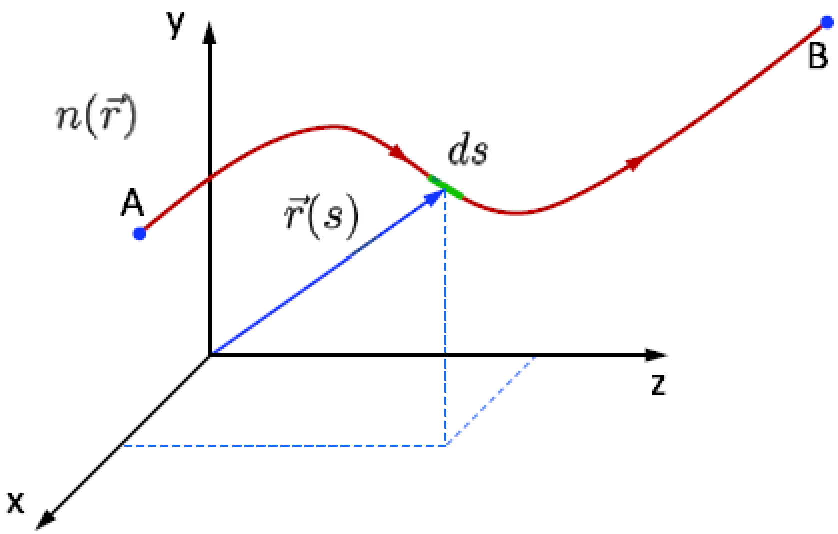
Focal Resonator







Ray Propagation in Inhomogeneous Medium



Fermat's Principle

$$I = \min\left\{\frac{1}{c}\int_A^B n(\vec{r})ds\right\}$$

$$\begin{aligned}
\vec{r} &= x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \\
ds &= \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt \\
I &= \min\left\{\frac{1}{c}\int_A^B \underbrace{n(x,y,z)\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}_{F(\vec{r},\dot{\vec{r}},t)}\,dt\right\}
\end{aligned}$$

Euler-Lagrange Equations

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \Longrightarrow \frac{d}{ds}\left(n\frac{dx}{ds}\right) = \frac{\partial n}{\partial x}$$

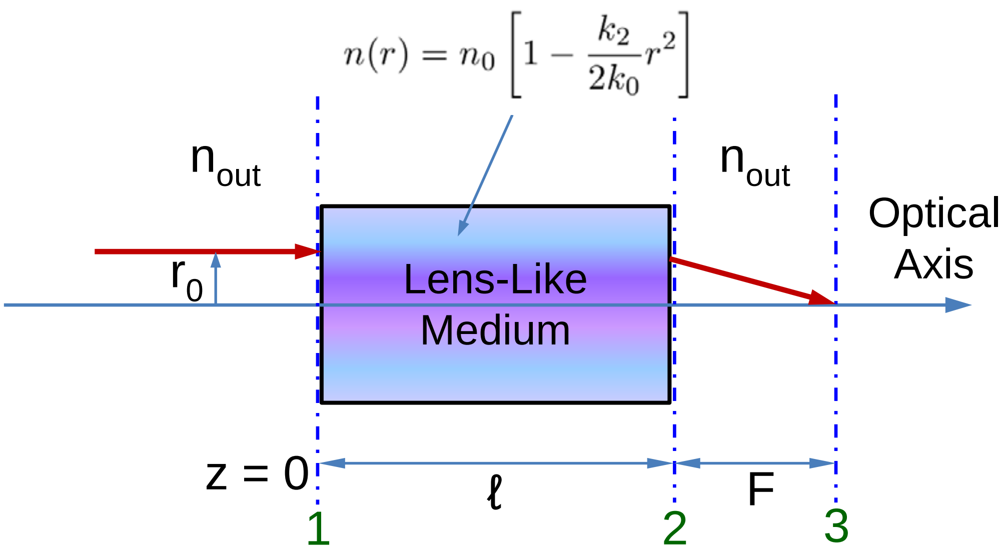
$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0 \Longrightarrow \frac{d}{ds}\left(n\frac{dy}{ds}\right) = \frac{\partial n}{\partial y}$$

$$\frac{\partial F}{\partial z} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}}\right) = 0 \Longrightarrow \frac{d}{ds}\left(n\frac{dz}{ds}\right) = \frac{\partial n}{\partial z}$$

Ray-Path Equation (Eikonal)

$$\boxed{\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \vec{\nabla} n}$$

Propagation in *Lens-Like Medium*



Ray-Path Equation (Eikonal)

$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \vec{\nabla} n.$$

Approximate Equation

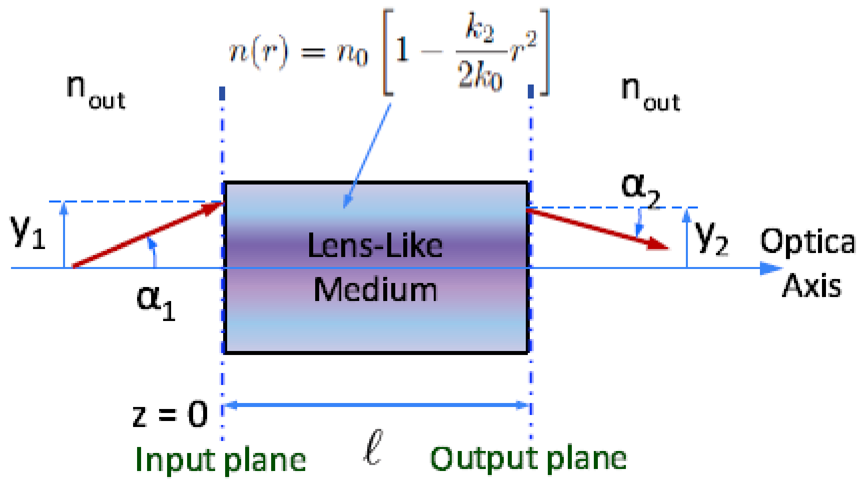
$$n_0 \frac{d^2 r}{dz^2} \simeq -n_0 \frac{k_2}{k_0} r$$

Transformation

$$t = \int ds/n$$

$$\frac{d^2\vec{r}}{dt^2} = n(\vec{r})\vec{\nabla} n$$

Propagation in Lens-Like Medium



$$g = \sqrt{k_2/k_0},\ r(z=\ell) = y_2 \text{ and } r'(z=\ell) = \alpha_2$$

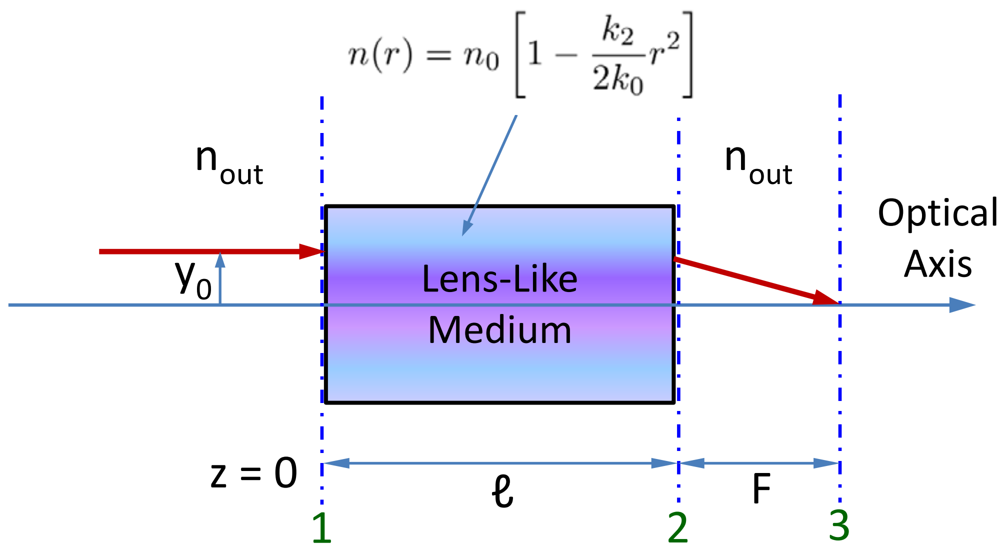
Approximate Equation

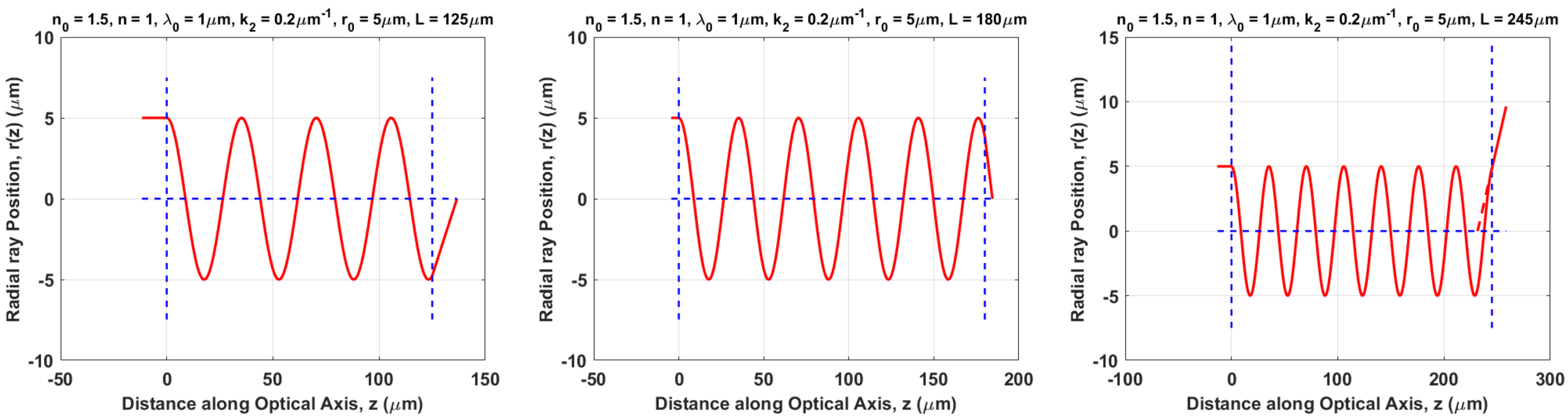
$$n_0\frac{d^2r}{dz^2} \simeq -n_0\frac{k_2}{k_0}r$$

$$\begin{bmatrix} r(z) \\ r'(z) \end{bmatrix} = \begin{bmatrix} \cos(gz) & \frac{1}{g}\sin(gz) \\ -g\sin(gz) & \cos(gz) \end{bmatrix}\begin{bmatrix} r_0 \\ r'_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} r_0 \\ r'_0 \end{bmatrix}$$

$$\begin{bmatrix} r(z) \\ r'(z) \end{bmatrix} = \begin{bmatrix} \cos\left(\sqrt{\frac{k_2}{k_0}}z\right) & \sqrt{\frac{k_0}{k_2}}\sin\left(\sqrt{\frac{k_2}{k_0}}z\right) \\ -\sqrt{\frac{k_2}{k_0}}\sin\left(\sqrt{\frac{k_2}{k_0}}z\right) & \cos\left(\sqrt{\frac{k_2}{k_0}}z\right) \end{bmatrix}\begin{bmatrix} r_0 \\ r'_0 \end{bmatrix}$$

Propagation in Lens-Like Medium





Propagation in Lens-Like Medium

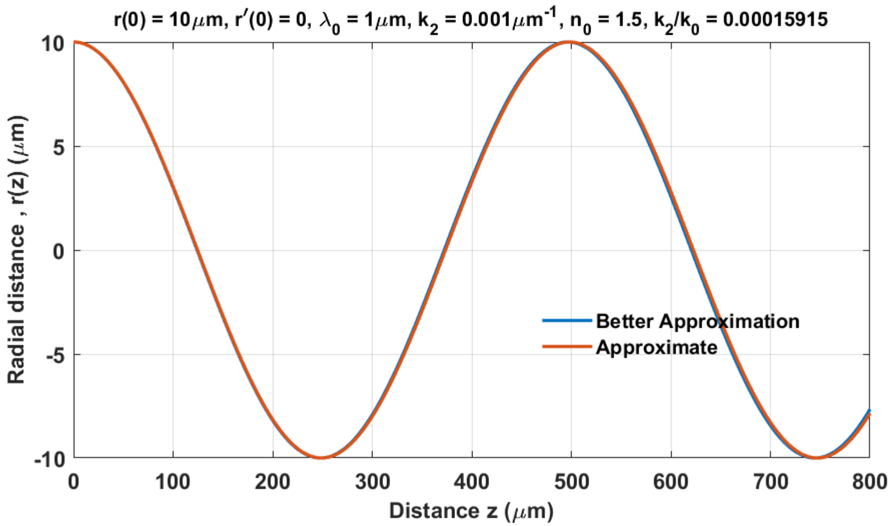
Comparison between Exact and Approximate Solutions

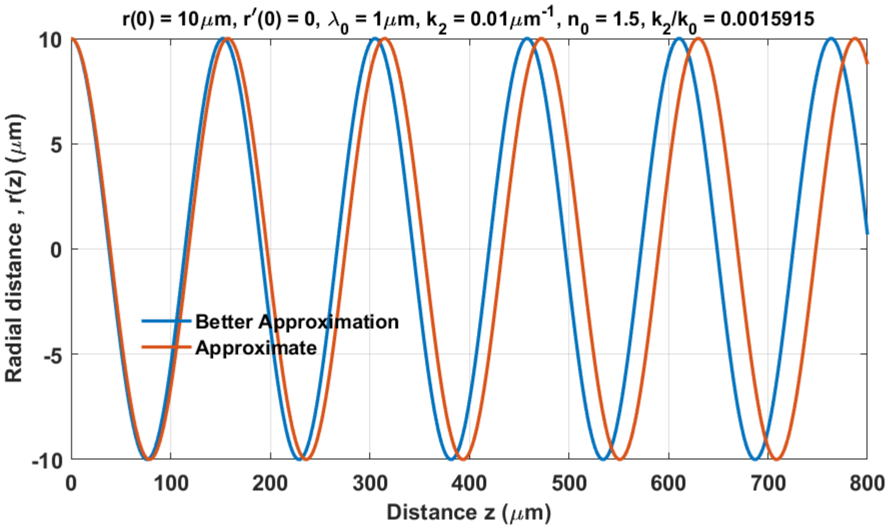
Better Approximation Equation

$$n_0 \left(1 - \frac{k_2}{2k_0} r^2\right) \frac{d^2 r}{dz^2} = -n_0 \frac{k_2}{k_0} r$$

Approximate Equation

$$n_0 \frac{d^2 r}{dz^2} \simeq -n_0 \frac{k_2}{k_0} r$$





Propagation in Lens-Like Medium

Comparison between Exact Eikonal and other Approximate Solutions

Exact Eikonal Equation

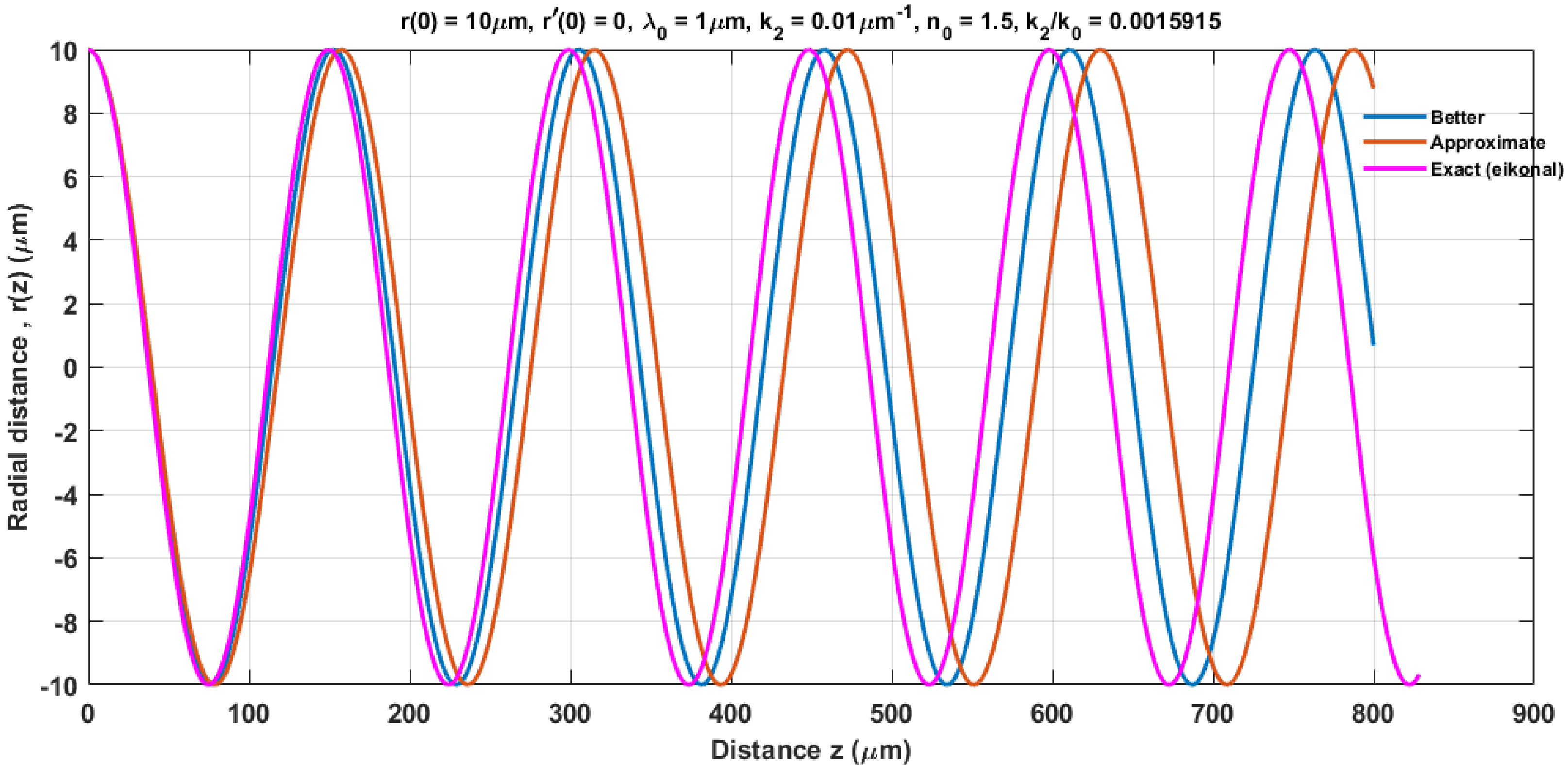
$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \vec{\nabla} n$$

Approximate Equation

$$n_0\frac{d^2r}{dz^2} \simeq -n_0\frac{k_2}{k_0}r$$

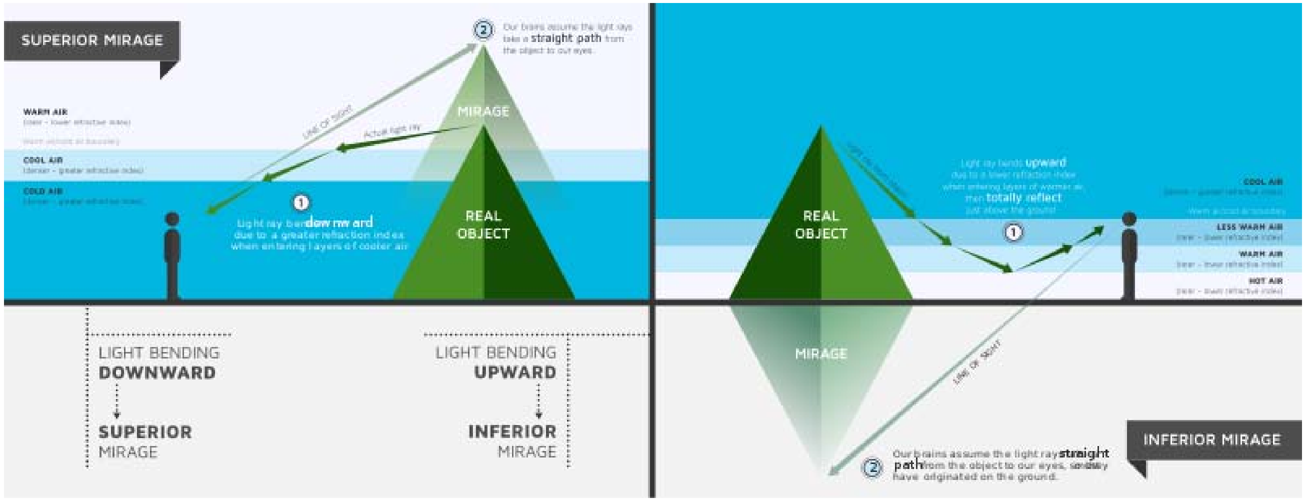
Better Approximation

$$n_0\left(1 - \frac{k_2}{2k_0}r^2\right)\frac{d^2r}{dz^2} = -n_0\frac{k_2}{k_0}r$$

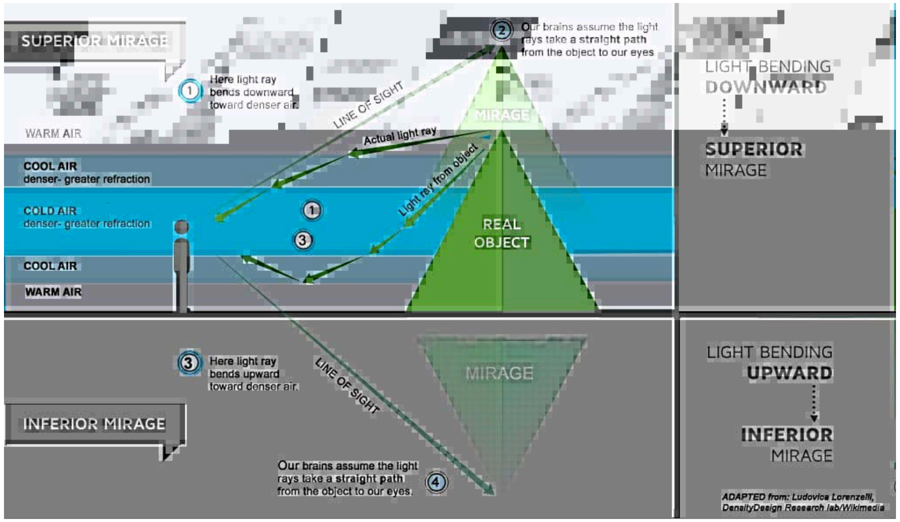


Example of Propagation in *Graded Index Medium*

Inferior and Superior Mirage Phenomenon



https://upload.wikimedia.org/wikipedia/commons/thumb/5/54/Superior_and_inferior_mirage.svg/1200pxSuperior_and_inferior_mirage.svg.png?20141201143416



https://www.friendslakeshorepreserve.com/mirage.html

Example of Propagation in *Graded Index Medium*

Inferior and Superior Mirage Phenomenon

http://www.astronomycafe.net/weird/lights/mirgal.htm

https://www.eoas.ubc.ca/courses/atsc113/sailing/met_concepts/10-met-local-conditions/10f-optical-phenomena/img-10f/10-superior-mirage.jpg

https://www.eoas.ubc.ca/courses/atsc113/sailing/met_concepts/10-met-local-conditions/10f-optical-phenomena/img-10f/10-inferior-mirage.jpg