ΗΛΕΚΤΡΟ-ΟΠΤΙΚΗ ΚΑΙ ΕΦΑΡΜΟΓΕΣ
(ELECTRO-OPTICS)

ΔΙΑΔΟΣΗ ΑΚΤΙΝΩΝ

(Propagation of Rays and Beams)

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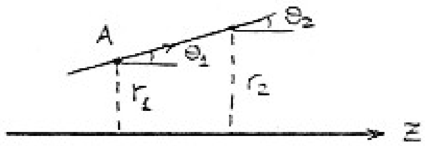
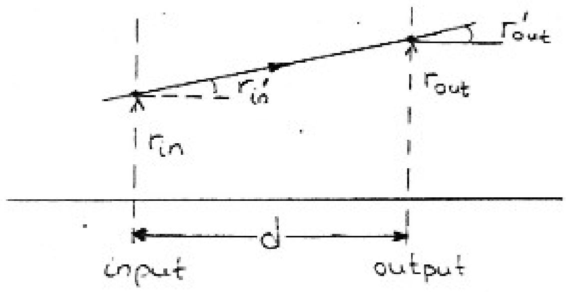
Propagation of Rays & Beams:

Matrix Approach:



We can describe a ray passing through point A with its distance from the z-axis (optic axis) and its slope. Consequently, r_1 and θ_1 (or equivalently $r_1' = \left.\frac{dr}{dz}\right|_A$) can be used to describe the ray at point A. In the following analysis it is assumed that the rays are paraxial, i.e. their slope with respect to z-axis is small in order to use the approximation $\tan\theta \simeq \sin\theta \simeq \theta$ for every paraxial ray.

In order to illustrate the matrix method let's apply it to describe the propagation of a ray through a homegeneous dielectric.

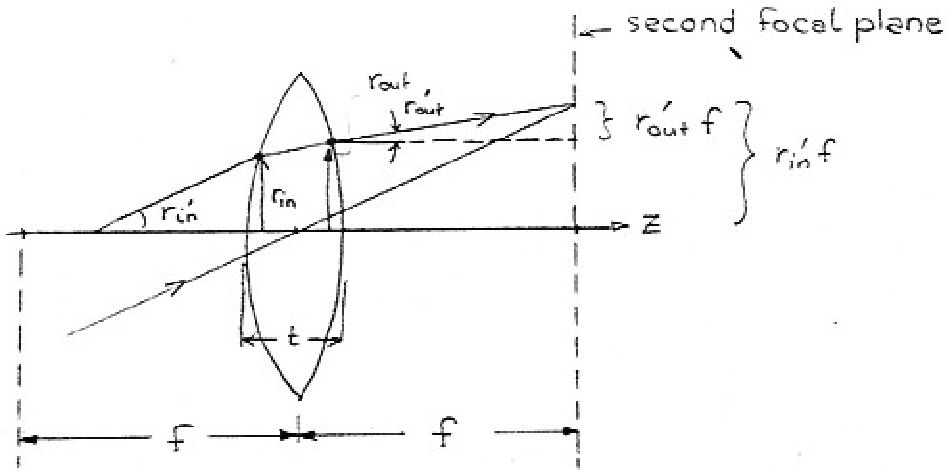


$$\left.\begin{aligned} r_{out} &= r_{in} + \tan(r_{in}')\, d \simeq r_{in} + d\, r_{in}' \\ r_{out}' &= r_{in}' \end{aligned}\right\} \Rightarrow$$

$$\begin{bmatrix} r_{out} \\ r_{out}' \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_{in} \\ r_{in}' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ r_{in}' \end{bmatrix}$$

Notice that $AD - BC = 1 \cdot 1 - d \cdot \emptyset = 1$. This will be a property of the ABCD matrix in general.

Now let's try to find the ABCD matrix for a thin lens.



For thin lens $t \simeq 0$ and any displacement of the ray will be neglected. Then,

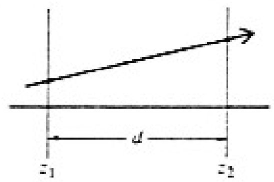
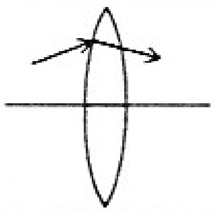
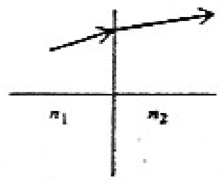
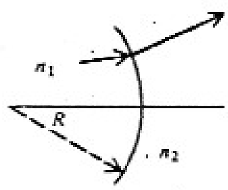
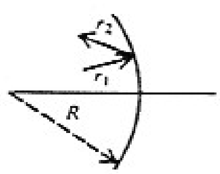
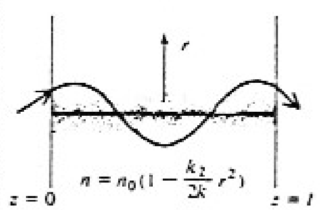
$$\left.\begin{aligned} r_{out} &\simeq r_{in} \\ r'_{out} f &= r'_{in} f - r_{in} \Rightarrow r'_{out} = -\frac{r_{in}}{f} + r'_{in} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}$$

Notice again that $AD - BC = 1 - (-\frac{1}{f}) \cdot \emptyset = 1$

Similarly, ABCD matrices can be found for other optical elements as shown in the next table (from A. Yariv).

Table 2-1 Ray Matrices for Some Common Optical Elements and Media

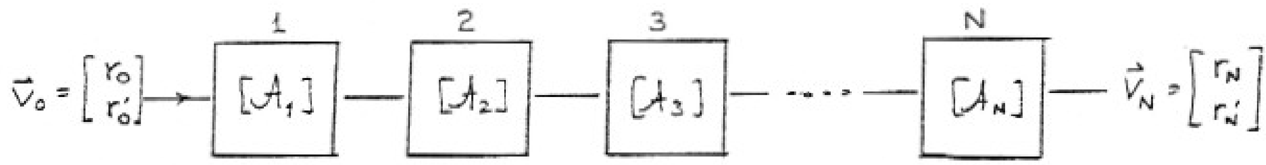
(1) Straight Section: Length d		$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
(2) Thin Lens: Focal length f ($f > 0$, converging; $f < 0$, diverging)		$\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$
(3) Dielectric Interface: Refractive indices n_1, n_2		$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$
(4) Spherical Dielectric Interface: Radius R		$\begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$
(5) Spherical Mirror: Radius of curvature R		$\begin{bmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{bmatrix}$
(6) A medium with a quadratic index profile		$\begin{bmatrix} \cos\left(\sqrt{\frac{k_2}{k}}\, l\right) & \sqrt{\frac{k}{k_2}} \sin\left(\sqrt{\frac{k_2}{k}}\, l\right) \\ -\sqrt{\frac{k_2}{k}} \sin\left(\sqrt{\frac{k_2}{k}}\, l\right) & \cos\left(\sqrt{\frac{k_2}{k}}\, l\right) \end{bmatrix}$

$R < 0$ if convex 

$R > 0$ if concave 

For propagation of light from left to right

If there are N cascaded optical elements that can be described by ABCD matrices then the input and output of this optical system are related as follows:



$$[\mathcal{A}_i] = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \qquad i = 1, 2, \ldots, N$$

$$\begin{bmatrix} r_N \\ r_N' \end{bmatrix} = \vec{V}_N = [\mathcal{A}_N]\,\vec{V}_{N-1} = [\mathcal{A}_N][\mathcal{A}_{N-1}]\,\vec{V}_{N-2} = \cdots =$$

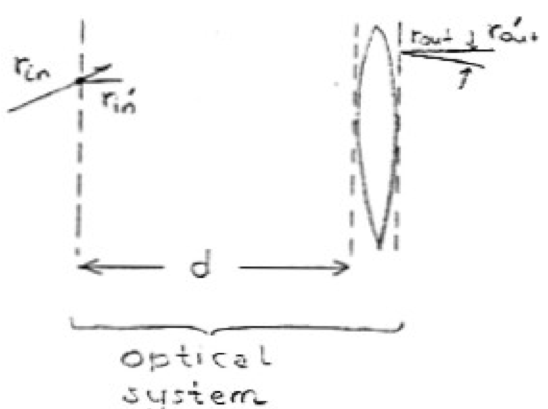
$$= [\mathcal{A}_N][\mathcal{A}_{N-1}]\cdots[\mathcal{A}_1]\,\vec{V}_0 \qquad \text{or}$$

$$\begin{bmatrix} r_N \\ r_N' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_0 \\ r_0' \end{bmatrix}$$

and $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\mathcal{A}_N][\mathcal{A}_{N-1}]\cdots[\mathcal{A}_1]$$

Notice again that $AD - BC = 1$. [if $n_0 = n_N$ (Liouville's Theorem)]

Let's apply the above result to a simple optical system which is comprised of a thin lens and a free space.

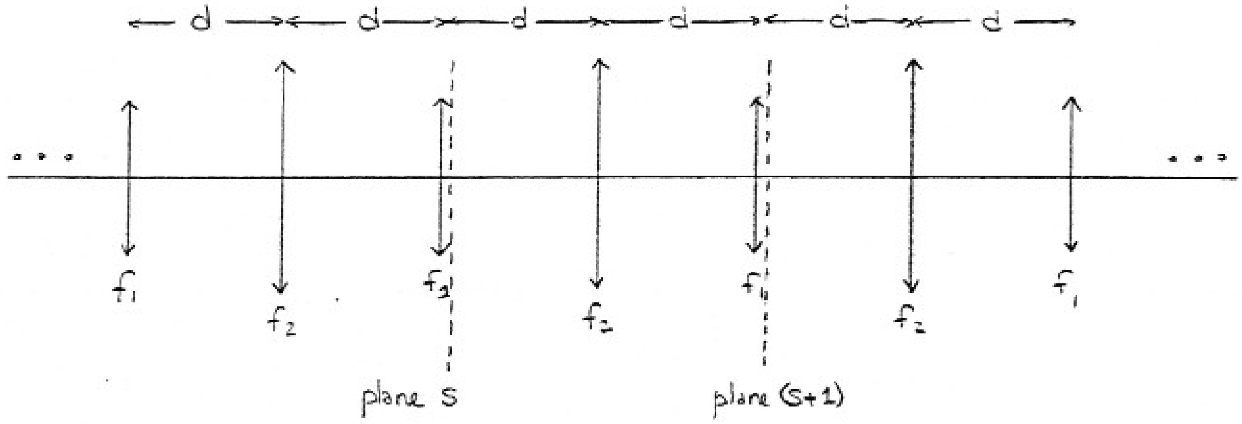


For the optical system shown in the figure:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & d \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix}$$

Now assume that we want to study a biperiodic lens system consisting of thin lenses of focal lengths f_1 and f_2 separated by distance d.



A unit cell of the structure is shown between plane s and s+1. The ABCD matrix from s to s+1 can be easily written as

$$[T] = \begin{bmatrix} 1 & d \\ -\frac{1}{f_1} & \left(1-\frac{d}{f_1}\right) \end{bmatrix} \begin{bmatrix} 1 & d \\ -\frac{1}{f_2} & \left(1-\frac{d}{f_2}\right) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$A = 1 - d/f_2$$

$$B = d\left(2 - \frac{d}{f_2}\right)$$

$$C = -\left[\frac{1}{f_1} + \frac{1}{f_2}\left(1-\frac{d}{f_1}\right)\right]$$

$$D = -\left[\frac{d}{f_1} - \left(1-\frac{d}{f_1}\right)\left(1-\frac{d}{f_2}\right)\right]$$

Consequently, we can write:

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_s \\ r'_s \end{bmatrix} \quad \text{or} \quad \begin{matrix} r_{s+1} = A r_s + B r'_s \\ r'_{s+1} = C r_s + D r'_s \end{matrix}$$

$$r'_s = \frac{1}{B}[r_{s+1} - A r_s] \quad \text{or} \quad r'_{s+1} = \frac{1}{B}[r_{s+2} - A r_{s+1}]$$

but

$$\left.\begin{matrix} r'_{s+1} = \frac{1}{B}[r_{s+2} - A r_{s+1}] \\ r'_{s+1} = C r_s + D \frac{1}{B}[r_{s+1} - A r_s] \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow r_{s+2} - 2\left(\frac{A+D}{2}\right) r_{s+1} + r_s = 0$$ (taking into account AD-BC=1).

Equation $r_{s+2} - 2\left(\frac{A+D}{2}\right) r_{s+1} + r_s = 0$ is a difference equation. Let's seek bounded solutions of the above equation. We can either look for solutions of the form $r_0 e^{jsq}$ or we can use the z-transform technique.

Define $b = (A+D)/2$. Then the characteristic equation of the difference equation is $z^2 - 2bz + 1 = 0$

$$z_{1,2} = \frac{1}{2}\left[2b \pm \sqrt{4b^2 - 4}\,\right] = b \pm \sqrt{b^2-1}$$

Then the general solution will be of the form:

$$r_s = a_1 z_1^s + a_2 z_2^s$$

For bound solutions $b^2 - 1 < 0 \implies b < 1$

Then $z_{1,2} = b \pm j\sqrt{1-b^2} = \cos\theta \pm j\sin\theta = e^{\pm j\theta}$

where $\cos\theta = b$.

The general bounded solution then looks like

$r_s = c_1 e^{js\theta} + c_2 e^{-js\theta}$, and since r_s should be real $c_1^* = c_2$ and the solution can be written as

$$r_s = r_{max} \sin[s\theta + \alpha]$$

where r_{max}, α are constants that can be determined from the initial conditions.

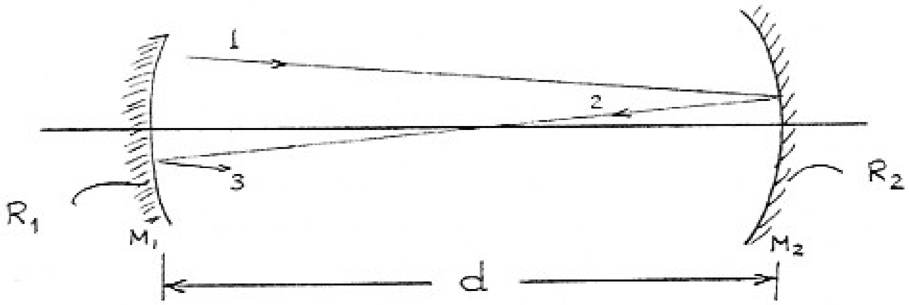
The condition to get solutions of this form was

$$\left.\begin{aligned} &b = \cos\theta \rightarrow |b| \le 1 \rightarrow -1 \le b \le +1 \\ &b = \frac{A+D}{2} = 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2f_1 f_2} \end{aligned}\right\} \implies$$

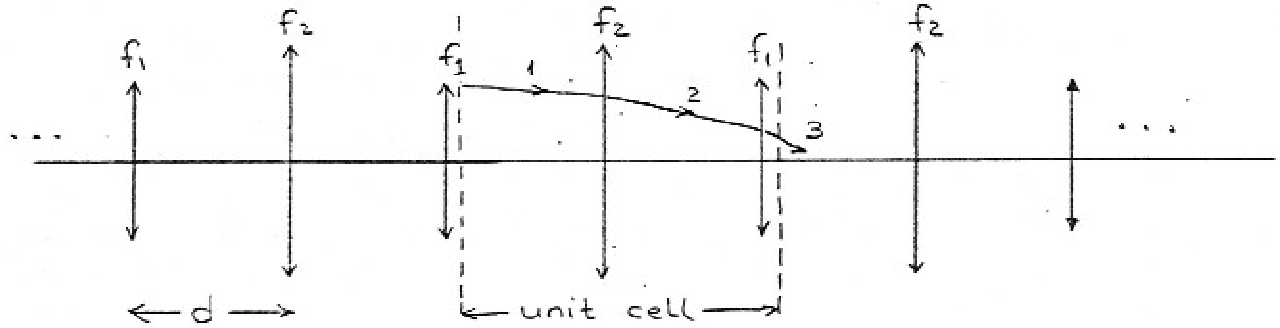
$$0 \le \left[1 - \frac{d}{2f_1}\right]\left[1 - \frac{d}{2f_2}\right] \le 1$$

If $b > 1$ then $|z_1, z_2| > 1 \rightarrow$ the r_s solution diverges.

Application of the Lens Waveguide to Optical Cavities:



The resonator cavity above is equivalent to the biperiodic waveguide that we studied previously. I.e. the above cavity is equivalent to the following optical system:



where $2f_1 = R_1$ and $2f_2 = R_2$. Here the sign convention remains as it was presented previously. Thus,
$R_1 > 0$ if center of mirror 1 is in the direction of 2. Similarly, R_2 is defined as positive or negative.

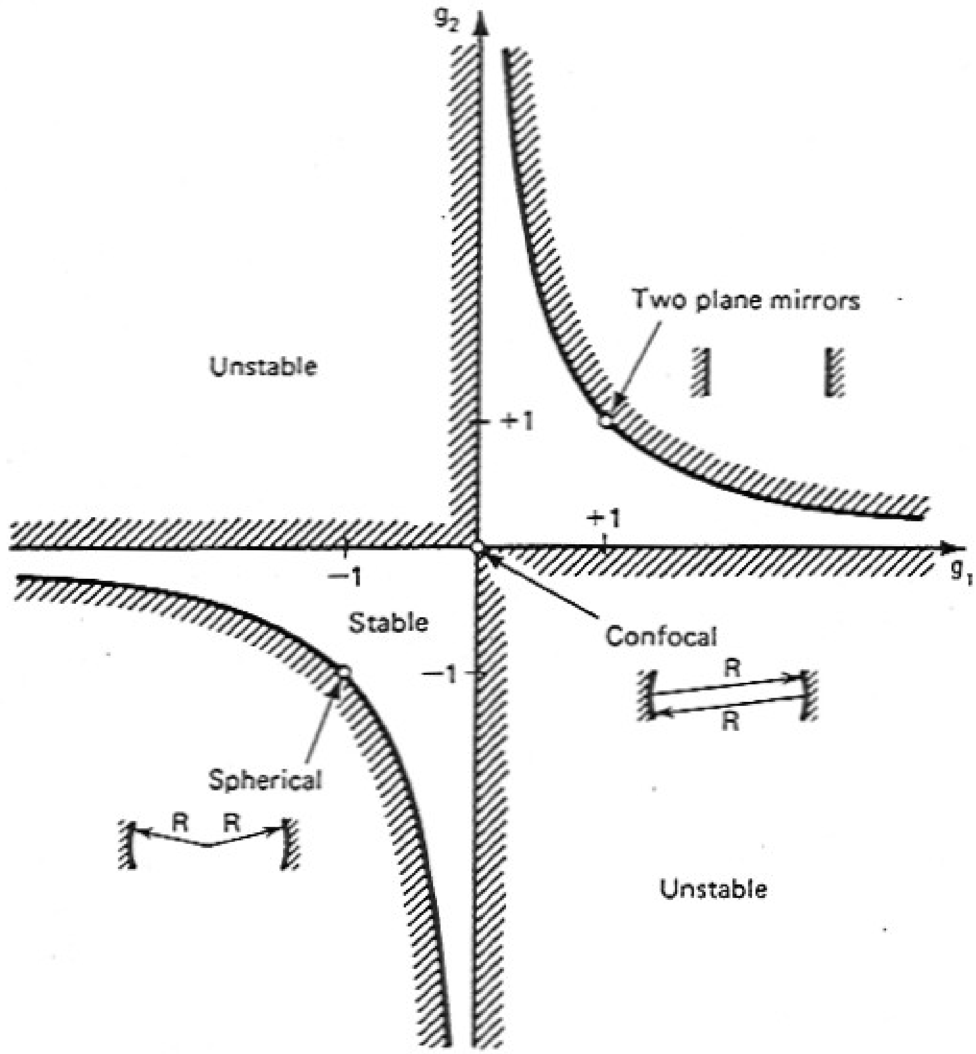
Consequently, all the previous analysis applies to the optical resonator case. For stable solutions we showed that

$$0 \leq \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) \leq 1 \quad \text{or}$$

$$0 \leq \left(1 - \frac{d}{R_1}\right)\left(1 - \frac{d}{R_2}\right) \leq 1 \quad \text{or}$$

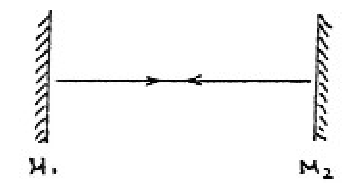
$$0 \leq g_1 g_2 \leq 1 \qquad g_{1,2} = 1 - \frac{d}{R_{1,2}}$$

STABILITY REGIONS



RESONATOR TYPES

Parallel Plane:

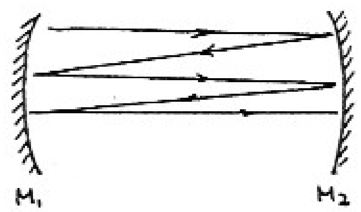


$R_1 = R_2 = \infty$

$g_1 \cdot g_2 = 1$ $\quad$ $g_1 \cdot g_2 = 1$

Marginally Stable system

Slightly Concave:

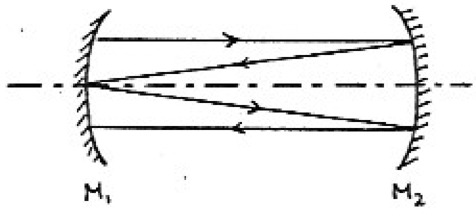


$R_1, R_2 > 0$ and $R_1, R_2 >> d$

$g_1 = g_2 \lesssim 1$ $\quad$ $g_1 g_2 \lesssim 1$

Stable system

Focal:



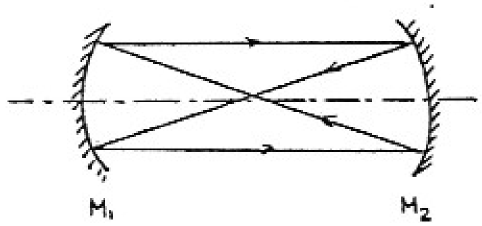
$R_1, R_2 > 0$

$R_1 = R_2 = 2d$

$g_1 = g_2 = \frac{1}{2}$ $\quad$ $g_1 g_2 = \frac{1}{4} < 1$

Stable system

Confocal:

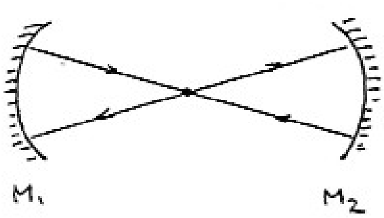


$R_1 = R_2 = d$

$g_1 = g_2 = 0$ $\quad$ $g_1 g_2 = 0$

Marginally Stable

Concentric:



$R_1 = R_2 = \frac{d}{2}$

$g_1 = g_2 = -1$ $\quad$ $g_1 g_2 = 1$

Marginally stable

Initial conditions

A. Stable Cavities:

For stable cavities $|b| \leq 1 \rightsquigarrow \quad b = \cos\theta = \frac{A+D}{2} \implies$

$\theta = \cos^{-1}\left(\frac{A+D}{2}\right)$.

Let's assume that the initial conditions are: $r_0 = \alpha$, $r_0' = m$.

The general solution $r_s = r_{max} \sin[s\theta + \alpha]$

For $s=0$:

$$r_0 = \alpha = r_{max} \sin(\alpha) \qquad (*)$$

For $s=1$:

$$r_1 = A r_0 + B r_0' = r_{max} \sin(\theta + \alpha) \implies$$

$$A\alpha + Bm = r_{max} \sin(\theta + \alpha) \qquad (**)$$

r_{max} and α can be determined from Eqs (*), (**), taking into account that $\cos\theta = \frac{A+D}{2}$, $\sin\theta = \pm\sqrt{1-\cos^2\theta}$

$$\alpha = \tan^{-1}\left\{ \frac{\pm\alpha\left[1-\left(\frac{A+D}{2}\right)^2\right]^{1/2}}{\alpha\left(\frac{A-D}{2}\right) + Bm} \right\}$$

Then $\quad r_{max} = \dfrac{\alpha}{\sin(\alpha)}$

B. Unstable Cavities:

For $b^2 - 1 > 0$ the solution for r_s is unstable and can be written in the form :

$$r_s = a_1 z_1^s + a_2 z_2^s$$

$$z_1 = b + \sqrt{b^2-1} \quad \text{and} \quad z_2 = b - \sqrt{b^2-1}$$

Depending on $b = \left(\frac{A+D}{2}\right)$ one of the two real z_1, z_2 will be greater than 1 and the other will be less than 1.

For example for $b > 1 \sim z_1 > 1 > z_2$.

Now let's assume again initial conditions $r_0 = \alpha$ and $r_0' = m$.

For $s=0$:

$$r_0 = \alpha = a_1 z_1^0 + a_2 z_2^0 = a_1 + a_2 \qquad (*)$$

For $s=1$:

$$r_1 = \alpha A + B m = a_1 z_1 + a_2 z_2 \qquad (**)$$

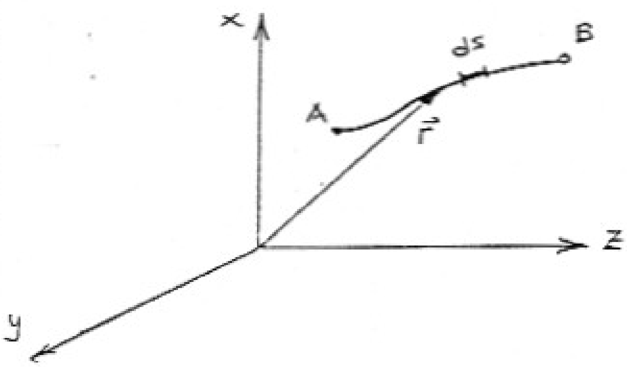
a_1, a_2 can be determined from the above Eqs $(*)$, $(**)$:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{z_2 - z_1} \cdot \begin{bmatrix} \alpha(z_2 - A) - Bm \\ \alpha(A - z_1) + Bm \end{bmatrix}$$

Rays in Lenslike Media:

In order to determine the trajectories of light rays in an inhomogeneous medium with a refractive index $n(\vec{r})$ we can use Fermat's principle, that light travels from point A to B following the path of shortest time. This is equivalent that rays follows the path that is the minimum optical path. This can be written as a variational problem:

$$\min \left\{ \int_A^B n(\vec{r})\, ds \right\}$$



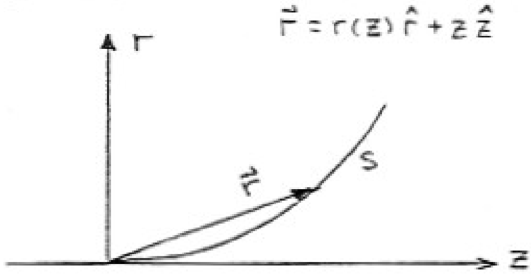
The solution to the above variational problem is given by

$$\frac{d}{ds}\left(n(\vec{r}) \frac{d\vec{r}}{ds} \right) = \vec{\nabla} n$$

where $\vec{r} = \vec{r}(s)$ specifies each point of the ray.

For paraxial rays $s \simeq z$ and the above equation can be written as

$$\vec{r} = r(z)\hat{r} + z\hat{z}$$



$$\frac{d}{dz}\left[n(r) \frac{d\vec{r}}{dz} \right] = \hat{r} \frac{dn}{dr}$$

$$n(r) \frac{d^2 r}{dz^2} = \frac{dn}{dr}$$

A "lens-like" medium has a refractive index variation $n(x,y) = n_0 \left[1 - \frac{k_2}{2k}(x^2+y^2) \right]$ with k_2, k constants. The reason that this medium is called a lens-like medium is because its refractive index variation causes a phase shift similar to a lens. For comparison the phase shift of an ideal thin lens is

$$t(x,y) = \exp\left\{ j k_0 \frac{x^2+y^2}{2f} \right\}$$

where $t(x,y)$ is the lens amplitude transmittance, $k_0 = \frac{2\pi}{\lambda_0}$, and f its focal length.

For a lens-like medium then $n(r) = n_0 \left[1 - \frac{k_2}{2k} r^2\right]$ $\quad (k, k_2 > 0)$

where $\left|\frac{k_2}{2k} r^2\right| \ll 1$.

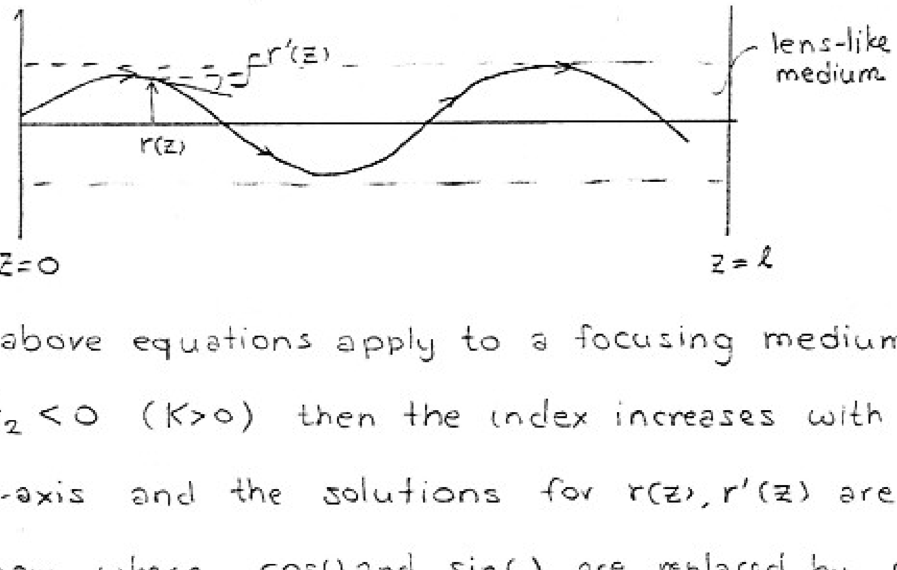
In this case the Fermat's principle gives:

$$n_0 \left[1 - \frac{k_2}{2k} r^2\right] \frac{d^2 r}{dz^2} = -n_0 \frac{k_2}{k} r \quad \Longrightarrow$$

$$n_0 \frac{d^2 r}{dz^2} \simeq -n_0 \frac{k_2}{k} r \quad \Rightarrow \quad \frac{d^2 r}{dz^2} + \frac{k_2}{k} r \simeq 0$$

If at $z=0$ the ray has radius r_0 and slope r_0' then

$$\begin{bmatrix} r(z) \\ r'(z) \end{bmatrix} = \begin{bmatrix} \cos\left(\sqrt{\frac{k_2}{k}}\, z\right) & \sqrt{\frac{k}{k_2}} \sin\left(\sqrt{\frac{k_2}{k}}\, z\right) \\ -\sqrt{\frac{k_2}{k}} \sin\left(\sqrt{\frac{k_2}{k}}\, z\right) & \cos\left(\sqrt{\frac{k_2}{k}}\, z\right) \end{bmatrix} \begin{bmatrix} r_0 \\ r_0' \end{bmatrix}$$



The above equations apply to a focusing medium with $k_2 > 0$. If $k_2 < 0$ $(K>0)$ then the index increases with the distance from the z-axis and the solutions for $r(z), r'(z)$ are similar to the one above where cos() and sin() are replaced by cosh(), and sinh() and $k_2 \rightarrow |k_2|$. This medium behaves like a negative lens.

Quadratic Refractive Index Distributions:

Thermally induced refractive index changes:

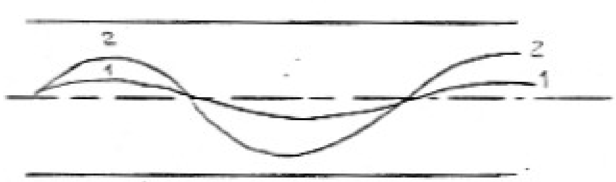
$$n = n_0 + \frac{dn}{dT} \Delta T$$

If $\frac{dn}{dT} < 0$, $\Delta T > 0$ defocusing as in propagation of Gaussian beams in slightly absorbing media.

If $\frac{dn}{dT} > 0$, $\Delta T > 0$ the beams are focused.

Graded-index Fiber waveguides:

$n = n_0 - n_1 r^2 \qquad (n_1 > 0)$



Both rays 1 and 2 arrive at the same time.

Can eliminate differential delays.