ΗΛΕΚΤΡΟ-ΟΠΤΙΚΗ ΚΑΙ ΕΦΑΡΜΟΓΕΣ
(ELECTRO-OPTICS)

ΕΙΣΑΓΩΓΗ ΣΤΟΥΣ ΟΠΤΙΚΟΥΣ ΣΥΝΤΟΝΙΣΤΕΣ

(Introduction to Optical Resonators)

Σημειώσεις

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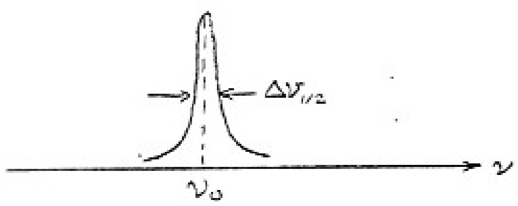
Optical Resonators:

Optical resonators are usually cavities comprised by two or more mirrors and are used to build up large field intensities with moderate input power. A universal measure of the quality of a resonator is its quality factor Q:

$$Q = \omega \left(\frac{\text{field energy stored by resonator}}{\text{power dissipated by resonator}} \right)$$

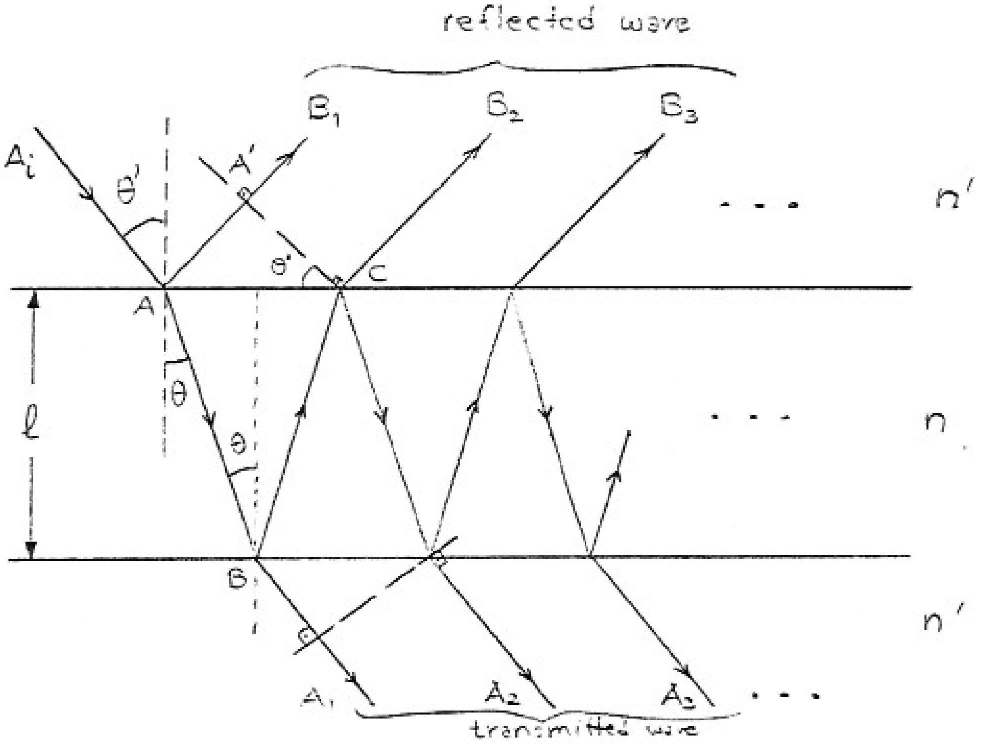
Another practical definition of Q is

$$Q = \frac{\nu_0}{\Delta\nu_{1/2}}$$



where ν_0 is the resonance frequency and $\Delta\nu_{1/2}$ is FWHM.

Fabry-Perot Etalon:



Assume a symmetric Fabry-Perot etalon as shown in the figure. Let's define r, t as the reflection and transmission coefficients from medium n' to n. Similarly, r', t' are the reflection and transmissi- coefficients from medium n to n'.

It can be easily shown that (Stokes Relations)

$$tt' + r^2 = 1 \quad \text{and} \quad r' = -r$$

The phase difference between B_1 and B_2 is

$$\delta = k_o n (AB + BC) - k_o n'(AA') =$$

$$= k_o [n \, 2\ell / \cos\theta] - k_o n' \, 2\ell \tan\theta \sin\theta' =$$

$$= k_o n \, 2\ell \left[\frac{1}{\cos\theta} - \tan\theta \sin\theta \right] = k_o n \, 2\ell \cos\theta$$

(where $n \sin\theta = n' \sin\theta'$ has been used (Snell's Law)).

Thus, if A_i is the amplitude of the incident wave:

$$B_1 = r A_i \quad , \quad B_2 = t r' t' e^{-j\delta} = tt' r' e^{-j\delta} A_i$$

$$B_3 = tt'(r')^3 e^{-j2\delta} A_i \quad \ldots$$

Then the complex amplitude of the total reflected field is

$$A_r = B_1 + B_2 + B_3 + \cdots =$$

$$= A_i \left\{ r + tt' r' e^{-j\delta} \left[1 + (r')^2 e^{-j\delta} + (r')^4 e^{-j2\delta} + \cdots \right] \right\} =$$

$$= A_i \frac{(1 - e^{-j\delta})\sqrt{R}}{1 - Re^{-j\delta}} \quad \text{where} \quad R = r^2 = r'^2$$

Similarly, the total transmitted wave is

$$A_t = A_1 + A_2 + A_3 + \cdots =$$

$$= tt' A_i + tt'(r')^2 e^{-j\delta} A_i + tt'(r')^4 e^{-j2\delta} A_i + \cdots =$$

$$= A_i\, tt' \left[1 + (r')^2 e^{-j\delta} + (r')^4 e^{-2j\delta} + \cdots \right] =$$

$$= A_i \frac{T}{1 - Re^{-j\delta}} \quad \text{where} \quad T = 1 - R = 1 - r^2 = tt'$$

The the fractions of the incident power reflected and trasmitted are

$$\frac{I_r}{I_i} = \frac{4R \sin^2(\frac{\delta}{2})}{(1-R)^2 + 4R \sin^2(\delta/2)} = \frac{|A_r|^2}{|A_i|^2}$$

$$\frac{I_t}{I_i} = \frac{(1-R)^2}{(1-R)^2 + 4R \sin^2(\delta/2)} = \frac{|A_t|^2}{|A_i|^2}$$

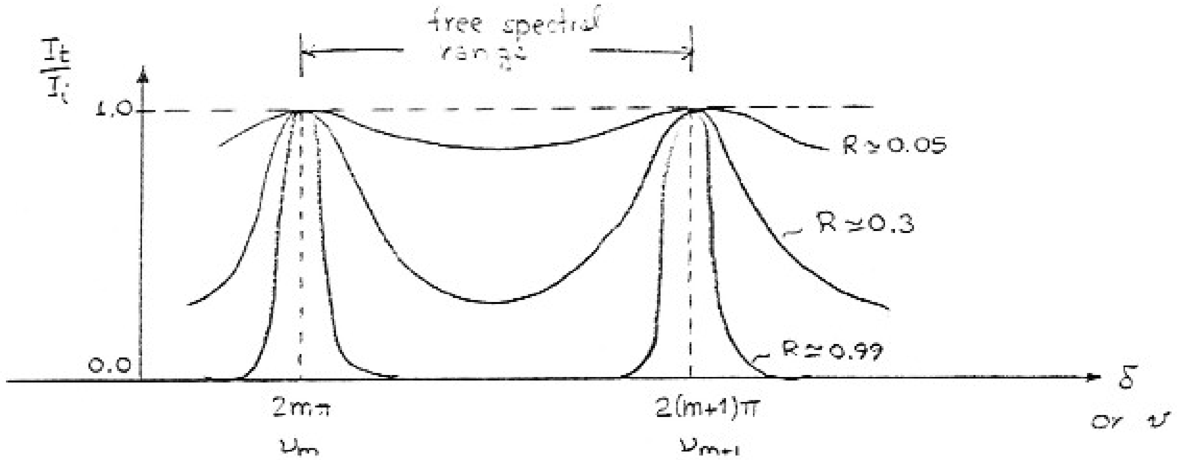
<u>Note:</u> In case of an asymmetric Fabry-Perot

$$\frac{I_r}{I_i} = \frac{(\sqrt{R_1} - \sqrt{R_2})^2 + 4\sqrt{R_1 R_2}\sin^2(\delta/2)}{(1 - \sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2}\sin^2(\delta/2)}$$

$$\frac{I_t}{I_i} = \frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2}\sin^2(\delta/2)}$$

where $R_1 = r_1^2 = (r_1')^2 = 1 - T_1 = 1 - t_1 t_1'$ (upper boundary)

$R_2 = r_2^2 = (r_2')^2 = 1 - T_2 = 1 - t_2 t_2'$ (lower boundary)



Maximum transmission occurs when $\delta = 2m\pi \Rightarrow \frac{I_t}{I_i} = 1$ and $\frac{I_r}{I_i} = 0$

The minimum transmission happens when $\frac{\delta}{2} = (2m+1)\frac{\pi}{2} \Rightarrow \delta = (2m+1)\pi$

and $\left(\frac{I_t}{I_i}\right)_{min} = \left(\frac{1-R}{1+R}\right)^2$.

The resonance condition is

$$\left.\begin{aligned} \delta &= k_o n \, 2\ell \cos\theta = 2m\pi \\ k_o &= \frac{\omega}{c} = \frac{2\pi\nu}{c} \end{aligned}\right\} \Rightarrow \nu_m = m\left(\frac{c/n}{2\ell\cos\theta}\right)$$

For fixed ℓ, and θ these are the transmission resonance freequencies of the etalon. The separation between two successive ν_m is called free spectral range.

$$\Delta\nu = \nu_{m+1} - \nu_m = \frac{c/n}{2\ell\cos\theta}$$

If the cavity includes losses then the maximum transmission drops to

$$\left(\frac{I_t}{I_i}\right)_{max} = \frac{(1-R)^2 A}{(1-RA)^2}$$

where $1-A$ is the fractional intensity loss per pass.

Fabry-Perot Etalons as Optical Spectrum Analyzers:

From the resonance condition of a Fabry-Perot etalon

$$\delta = k_o n \, 2\ell \cos\theta = 2m\pi \quad \text{or} \quad \nu_m = m \frac{c/n}{2\ell\cos\theta} \implies$$

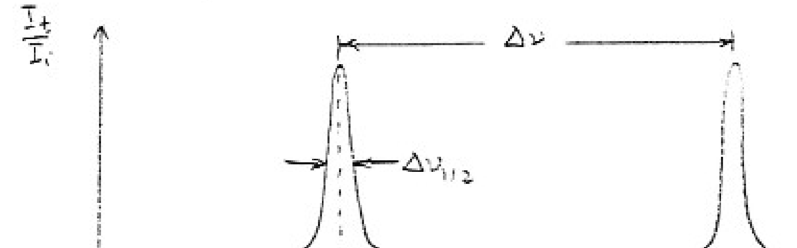
$$d\nu_m = m \frac{c/n}{2\cos\theta}\left(-\frac{1}{\ell^2}\right) d\ell \implies \frac{d\nu_m}{\Delta\nu} = -\frac{m}{\ell}\, d\ell$$

But for normal incidence $(\theta = 0°) \sim \quad \ell = m \dfrac{\lambda_{om}}{2n}$

Therefore,

$$\frac{d\nu_m}{\Delta\nu} = -\frac{d\ell}{(\lambda_{om}/2n)}$$

$\dfrac{\lambda_{om}}{2n}$ is half resonance wavelength inside the cavity of refractive index n. The above equation implies that the resonance frequency ν_m of the etalon can be changed by changing its length. Thus, by changing its length we can use the etalon as a spectrum analyzer. This of course is true when the transmission peak width is small in comparison to the spectral detail of the optical signal to be analyzed. In addition, the spectral width of the signal must be smaller than the intermode spacing $\Delta\nu$ of the etalon. Thus $d\ell < \dfrac{\lambda_{om}}{2n}$



$\Delta\nu_{1/2}$ is the FWHM and can be calculated to be

$$\Delta\nu_{1/2} \simeq \left(\frac{c/n}{2\ell\cos\theta}\right)\left(\frac{1-R}{\pi\sqrt{R}}\right) = \Delta\nu \cdot \frac{1}{F}$$

where $F = \pi\sqrt{R}/(1-R)$ = the etalon finesse.

Numerical Example:

Fabry - Perot etalon to study a He-Ne laser, where

$l_{Laser} = 100\text{cm}$, $\Delta\nu_{gain} = 1.5 \cdot 10^9$ Hz, $\lambda_0 = 0.6328\ \mu m$.

The free spectral range of the etalon $\Delta\nu \geq \Delta\nu_{gain}$ $\Rightarrow$

$\frac{c/n}{2\, l_{etalon}} \geqslant \Delta\nu_{gain}$ (assuming normal incidence).

or $2n\, l_{etalon} \leqslant \frac{c}{\Delta\nu_{gain}} = 0.2\text{ m} = 20\text{cm}$

Laser mode separation $\Delta\nu_{laser} \simeq \frac{c/n_{laser}}{2\, l_{laser}} = \frac{c}{2\, l_{laser}}$ $(n_{laser} \simeq 1.0)$

$$\Delta\nu_{laser} \simeq 1.5 \cdot 10^8 \text{ Hz}$$

Choose etalon resolution $\Delta\nu_{1/2} \leq 0.1\ \Delta\nu_{laser} \leq 1.5 \cdot 10^7$ Hz

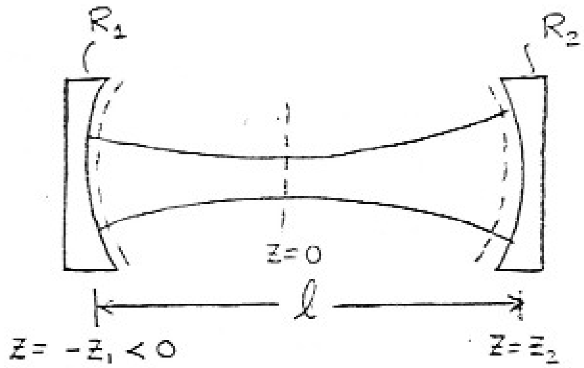
Then $\frac{\Delta\nu}{F} \leq 1.5 \cdot 10^7$ Hz $\Rightarrow$ $\frac{c/n}{2\, l_{etalon} F} \leq 1.5 \cdot 10^7$ $\Rightarrow$

$2n\, l_{etalon} F \geqslant \frac{c}{1.5 \cdot 10^7} = 2 \cdot 10\text{ m} = 2 \cdot 10^3\text{ cm}$

Thus, if $2n\, l_{etalon} = 20\text{cm}$, $F > 100$. $\sim$ $R \simeq 0.97$

Optical Resonators with spherical mirrors:

Optical resonators with spherical mirrors are designed in such a way in order the set of mirrors, their position and their curvature exactly match the surfaces of constant phase of the beam.
In all the following analysis Gaussian beams are considered.



$R_1, R_2 > \ell$

$R_1, R_2 > 0$

The above cavity is a stable resonator since $g_1 = 1 - \frac{\ell}{R_1} < 1$
$g_2 = 1 - \frac{\ell}{R_2} < 1 \rightarrow 0 < g_1 g_2 < 1 \sim$ stable resonator.
In order to match the curvature of the beam with the curvature of the spherical mirrors we require:

$$\left.\begin{aligned} z_2 + z_1 &= \ell \\ -R_1 &= -z_1\left[1+\left(\frac{z_0}{z_1}\right)^2\right] < 0 \\ R_2 &= z_2\left[1+\left(\frac{z_0}{z_2}\right)^2\right] > 0 \end{aligned}\right\}$$

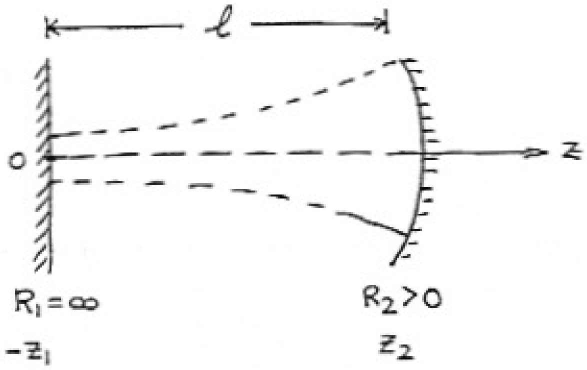
Given R_1, R_2, and ℓ we can find

$$z_0^2 = \frac{\ell[R_1-\ell][R_2-\ell][R_2+R_1-\ell]}{[R_2+R_1-2\ell]^2}$$

$$z_1 = \frac{\ell[R_2-\ell]}{R_1+R_2-2\ell} \qquad z_2 = \frac{\ell[R_1-\ell]}{R_1+R_2-2\ell}$$

Note that R_1, R_2 have their own sign (same with the corresponding focal length) but we permit the curvature to be negative for $z<0$.

Example:



$$z_1 = \lim_{R_1 \to \infty} \frac{\ell\,[R_2 - \ell]}{R_1 + R_2 - 2\ell} = \lim_{R_1 \to \infty} \frac{\ell\,(R_2 - \ell)}{R_1\left[1 + \frac{R_2 - 2\ell}{R_1}\right]} = 0$$

$$z_2 = \lim_{R_1 \to \infty} \frac{\ell\,[R_1 - \ell]}{R_1 + R_2 - 2\ell} = \lim_{R_1 \to \infty} \frac{R_1\left[1 - \ell/R_1\right]\ell}{R_1\left[1 + \frac{R_2 - 2\ell}{R_1}\right]} = \ell$$

$$z_0^2 = \lim_{R_1 \to \infty} \frac{\ell\,[R_1 - \ell][R_2 - \ell][R_1 + R_2 - \ell]}{[R_2 + R_1 - 2\ell]^2} =$$

$$= \lim_{R_1 \to \infty} \frac{\cancel{R_1^2}\,\ell\left[1 - \frac{\ell}{R_1}\right][R_2 - \ell]\left[1 + \frac{R_2 - \ell}{R_1}\right]}{\cancel{R_1^2}\left[1 + \frac{R_2 - 2\ell}{R_1}\right]^2} = \ell(R_2 - \ell)$$

$$w_0^2 = \frac{\lambda_0 z_0}{\pi n} = \frac{\lambda_0}{\pi n}\sqrt{\ell(R_2 - \ell)}$$

Application of ABCD Law to a Generalized Resonator:

A stable eigenmode of an optical resonator reproduces itself after a round trip. Consequently, choosing an arbitrary reference plane and the complex beam parameter q_s, we have after a round trip

$$q_{s+1} = \frac{A q_s + B}{C q_s + D} = q_s \quad \text{for stable steady-state mode}$$

where A, B, C, D is the matrix that describes the round trip.

The above equation can be written in the form:

$$B\left(\frac{1}{q}\right)^2 + 2\left(\frac{A-D}{2}\right)\left(\frac{1}{q}\right) - C = 0$$

$$\frac{1}{q} = -\frac{A-D}{2B} \pm \frac{1}{B}\left[\left(\frac{A-D}{2}\right)^2 + BC\right]^{1/2}$$

and using the fact that $AD - BC = 1$.

$$\frac{1}{q} = -\frac{A-D}{2B} - j\,\frac{1}{|B|}\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}$$

$$\frac{1}{q} = \frac{1}{R} - j\,\frac{\lambda_0}{\pi n w^2}$$

If q, R, w are defined at $z = z_1$ (arbitrary) then:

$$R(z_1) = -\frac{2B}{A-D}$$

$$\frac{\pi n w^2(z_1)}{\lambda_0} = \frac{B}{\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}}$$

where $\left|\frac{A+D}{2}\right| = |\cos\theta| < 1$ for stable cavities.

Therefore, the above approach applies for stable cavities only.

Resonance Frequencies of Optical Resonators:

For optical resonators with spherical mirrors the resonance condition can be written as

$$k_q d - (\ell+m+1)\left[\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right)\right] = q\pi$$

$$k_q = \frac{\omega_q}{c} \cdot n = \frac{2\pi n}{c}\nu_q \qquad \nu_q = q\left(\frac{c/n}{2d}\right)$$

If ℓ, m are constant:

$$k_{q+1} - k_q = \frac{\pi}{d} \Rightarrow \nu_{q+1} - \nu_q = \frac{c/n}{2d} = \Delta\nu$$

intermode spacing of longitudinal modes.

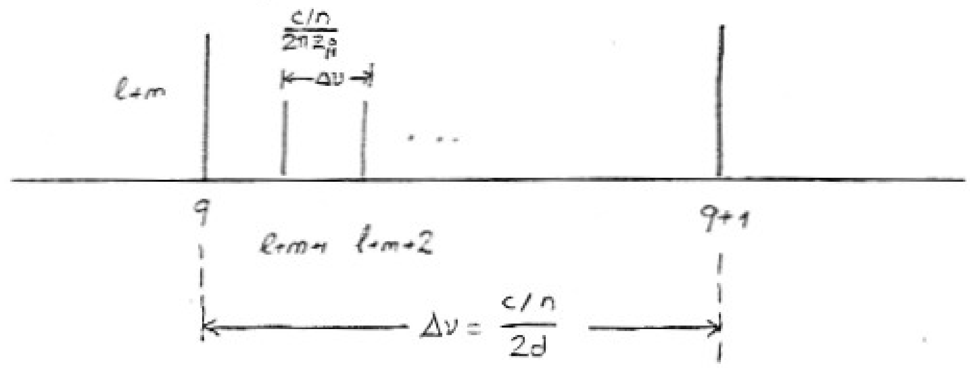
For transverse modes:

$$\Delta\nu = \frac{c/n}{2\pi d}\,\Delta(\ell+m)\left[\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right)\right]$$

For almost planar resonators $R_1, R_2 \gg d$.

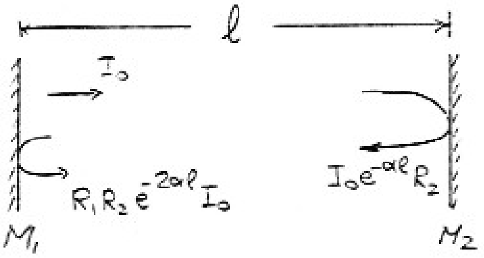
Then it can be shown that $|z_1|, z_2 \ll z_0$ and

$$\Delta\nu \simeq \frac{c/n}{2\pi z_0}\,\Delta(\ell+m)$$



Losses in Optical Resonators:

The decay lifetime (photon lifetime) t_c of a cavity mode can be calculated as follows:



After a round trip the intensity is $I_o R_1 R_2 e^{-2\alpha\ell}$ where α describes the scattering and absorption losses of the medium. After N round trips $I_o (R_1R_2)^N e^{-2\alpha\ell N}$ of the initial intensity will remain.

If $\dfrac{I_o (R_1R_2)^N e^{-2\alpha\ell N}}{I_o} = e^{-1}$ corresponds to a time

$$t_N = \frac{2\ell N}{c/n} = t_c$$

The above equation can be written as

$$e^{-2\alpha\ell N + 2N\ell\left[\frac{1}{\ell}\ln\sqrt{R_1R_2}\right]} = e^{-1} \Rightarrow$$

$$2\alpha\ell N - 2N\ell\left[\frac{1}{\ell}\ln\sqrt{R_1R_2}\right] = 1$$

$$2\ell N\left[\alpha - \left[\frac{1}{\ell}\ln\sqrt{R_1R_2}\right]\right] = 1$$

$$\frac{2\ell N}{c/n}\left[\alpha - \frac{1}{\ell}\ln\sqrt{R_1R_2}\right] = n/c \Rightarrow$$

$$t_c = \frac{n/c}{\alpha - \frac{1}{\ell}\ln\sqrt{R_1R_2}} = \frac{n\ell}{c\left[\alpha\ell - \ln(\sqrt{R_1R_2})\right]}$$

If $R_1R_2 \simeq 1$ $\quad t_c \simeq \dfrac{n\ell}{c\left[\alpha\ell + (1-\sqrt{R_1R_2})\right]}$

In terms of the Q-factor:

$$Q = \omega \frac{\mathcal{E}}{P} = \omega \frac{\mathcal{E}}{-d\mathcal{E}/dt} \qquad \text{where } -\frac{d\mathcal{E}}{dt} = \text{power dissipated.}$$

Then $\quad \frac{d\mathcal{E}}{dt} = -\frac{\omega}{Q}\mathcal{E} \Rightarrow \mathcal{E}(t) = \mathcal{E}(0)\, e^{-t/t_c} \sim t_c = \frac{Q}{\omega}$

We have also defined $Q = \frac{\nu}{\Delta\nu_{1/2}} \Rightarrow \Delta\nu_{1/2} = \frac{\nu}{Q} \Rightarrow$

$$\Rightarrow \Delta\nu_{1/2} = \frac{\omega}{2\pi}\frac{1}{Q} = \frac{1}{2\pi t_c} = \frac{1}{2\pi}\cdot\frac{c\,[\alpha \ell - \ln\sqrt{R_1R_2}\,]}{n\,\ell}$$

If other loss mechanisms are present they can be taken into account by

$$\frac{1}{\hat{t}_c} = \frac{1}{t_c} + \frac{1}{t_{other}}$$

where $\hat{t}_c$ is the composite cavity lifetime, t_{other} is the lifetime due to other mechanisms.

A more exact approach of the cavity lifetime can be found by determining the time evolution of the cavity photons, ϕ.

For the previous cavity we can write:

$$\frac{d\phi}{dt} = \left(\frac{R_1R_2e^{-2\alpha l}-1}{\tau_{RT}}\right)\phi = -\frac{1}{t_c}\phi$$

$\downarrow$

rate of change of cavity photons

τ_{RT} = round-trip transit time = $\frac{2l}{c/n}$

Then $\frac{d\phi}{dt} + \frac{1}{t_c}\phi = 0 \Rightarrow \phi(t) = \phi(0)e^{-t/t_c}$

and $t_c = \frac{\tau_{RT}}{1-R_1R_2e^{-2\alpha l}} = \frac{2ln}{c[1-R_1R_2e^{-2\alpha l}]}$

It is interesting to compare the two approaches: (if $\alpha l \ll 1$, $R_1R_2 \simeq 1$)

$$t_c = \frac{2ln}{c[1-R_1R_2e^{-2\alpha l}]} \simeq \frac{2ln}{c[1-R_1R_2(1-2\alpha l)]} = \frac{2ln}{c[1-R_1R_2+2\alpha l R_1R_2]} \simeq \frac{2ln}{c[2\alpha l + 1 - R_1R_2]}$$

In addition, from the previous formula,

$$t_c = \frac{nl}{c[\alpha l - \frac{1}{2}\ln R_1R_2]} = \frac{2nl}{c[2\alpha l - \ln R_1R_2]} \simeq \frac{2nl}{c[2\alpha l + 1 - R_1R_2]}$$

Therefore, the two equations are equivalent in the case of small loss and high reflectivity.