ΗΛΕΚΤΡΟ-ΟΠΤΙΚΗ ΚΑΙ ΕΦΑΡΜΟΓΕΣ
(ELECTRO-OPTICS)

ΑΛΛΗΛΕΠΙΔΡΑΣΗ ΑΚΤΙΝΟΒΟΛΙΑΣ ΚΑΙ ΑΤΟΜΙΚΩΝ ΣΥΣΤΗΜΑΤΩΝ

(Interaction of Radiation and Atomic Systems)

Σημειώσεις

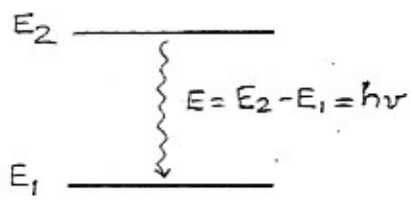
Καθ. Ηλία Ν. Γλύτση

Radiative Processes:

Two of most important concepts of quantum mechanics are:

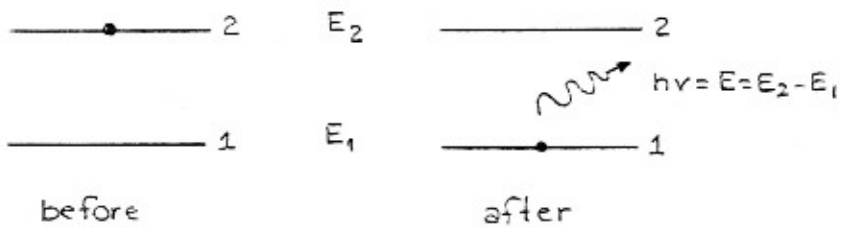
(a) There are discrete energy levels in an atomic (or molecular, or solid, or semiconductor) system.

(b) The system can make transitions between any two energy states E_2, E_1, $(E_2 > E_1)$ by the emission of a photon of energy $E = E_2 - E_1 = h\nu$, where $h = 6.626 \cdot 10^{-34}$ J·sec (Planck's constant) and ν is the frequency of the photon.



Einstein identified 3 radiative processes that affect the concentration of atoms in states 2 and 1. These are:

(1) Spontaneous Emission: An atom in state 2 can undergo a transition to state 1 through the emission of a photon.

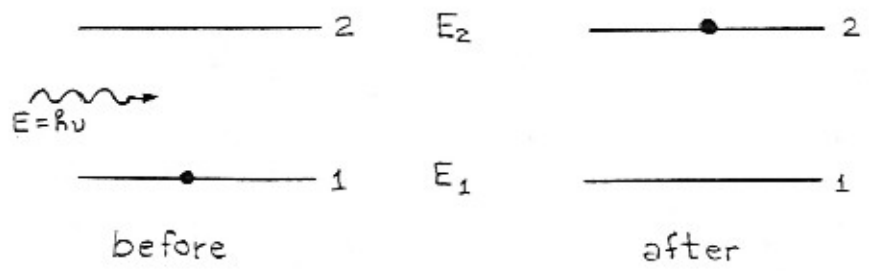


If N_2 atoms are in state 2 at $t = 0$ then the average number undergoing transition to state 1 is

$$\left.\frac{dN_2}{dt}\right|_{Sp.Em.} = -A_{21} N_2 = -\left.\frac{dN_1}{dt}\right|_{sp.em.}$$

where $A_{21} = \frac{1}{t_{sp.em}}$ is called the spontaneous transition rate, and $t_{sp.em.}$ is called the spontaneous lifetime.

(2) Absorption: An atom in state 1 can absorb a photon of energy $E = E_2 - E_1 = h\nu$ and makes a transition to state 2.



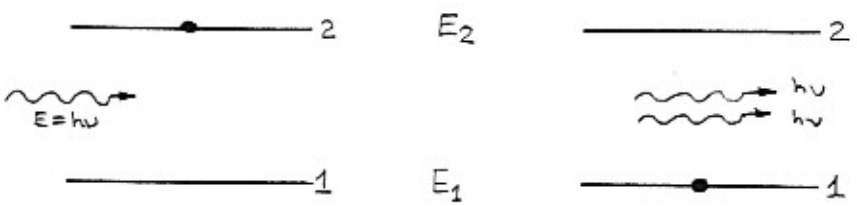
The rate of the process is

$$\left.\frac{dN_2}{dt}\right|_{abs.} = + B_{12} N_1 \rho(\nu) = - \left.\frac{dN_1}{dt}\right|_{abs.}$$

where B_{12} is a constant and

$\rho(\nu) = \frac{8\pi n^3 h \nu^3}{c^3} \cdot \frac{1}{e^{h\nu/k_B T} - 1}$ is the energy density per unit frequency ($J/m^3 Hz$). [n: refr. index, k_B = Boltzmann's constant = $1.38 \cdot 10^{-23} J/K$]

(3) Stimulated Emission: This process is the inverse of absorption. An atom at state 2 undergoes a transition to state 1, in the presence of a photon $E = h\nu$, through an emission of another photon of the same frequency, same phase, same polarization, and same direction of propagation with the initial photon.



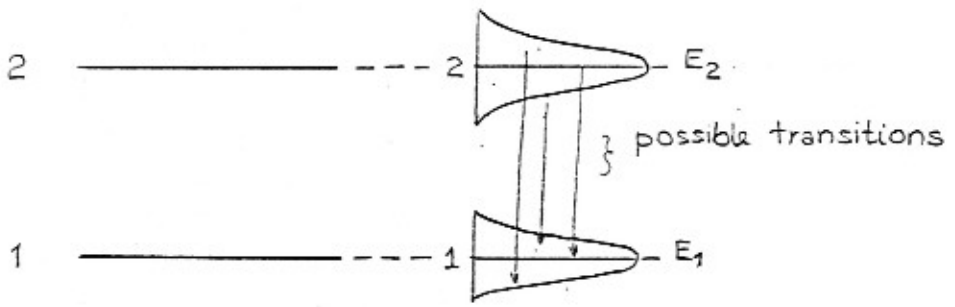
The rate equation of this process is

$$\left.\frac{dN_2}{dt}\right|_{st.em.} = - B_{21} \rho(\nu) N_2 = - \left.\frac{dN_1}{dt}\right|_{st.em}$$

where B_{21} a constant.

Lineshape Function:

The energy states that are involved in the radiative processes can not be perfectly sharp. If they were, then according to the uncertainty principle they would have an infinite uncertainty in the time that the atoms remain in these states. In other words, the atoms once excited to such a state would remain there forever! Therefore, the energy states 1 & 2 (as were defined previously) and the corresponding transitions have an energy smearing.

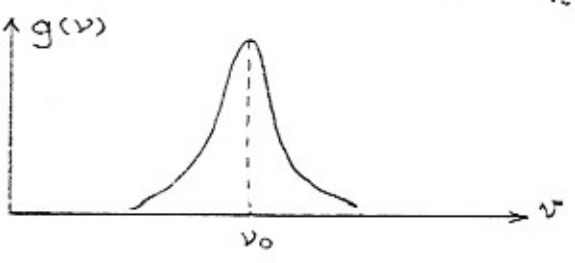


This effect is taken into account with the definition of the lineshape function $g(\nu)$. Then $g(\nu)d\nu$ is the probability of emission of a photon with frequency between ν and $\nu+d\nu$. Then it is obvious that

$$\int_0^\infty g(\nu)\,d\nu = 1$$

Similarly, $g(\nu)d\nu$ is the probability that a photon of frequency between ν and $\nu+d\nu$ will get absorbed.

Usually $g(\nu)$ is peaked around $\nu_0 = \frac{E_2 - E_1}{h}$



Homogeneous Broadening:

Consider the radiated field of a spontaneous emission process. This field can be written as

$$e(t) = E_0 e^{-t/\tau} \cos(\omega_0 t) =$$

$$= \frac{E_0}{2}\left[e^{j(\omega + j\frac{\sigma}{2})t} + e^{-j(\omega - j\frac{\sigma}{2})t} \right]$$

where $\frac{\sigma}{2} = \tau^{-1}$ is the field decay rate. If we compute the Fourier transform of the field, $E(\omega)$, we can find the spectral density of the spontaneous emission process since it is proportional to $|E(\omega)|^2$. It can be shown that

$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0)^2 + (\frac{\sigma}{2})^2}$$

This type of curve is called Lorentzian and they characterize the response of dumped resonant systems.

The FWHM of the Lorentzian is called linewidth.

$$\Delta\nu = \frac{\sigma}{2\pi} = \frac{1}{\pi\tau}$$

In the case of atomic transitions between two levels 1 & 2 the field can be interrupted by the finite lifetime of the states (τ_1, τ_2) or by elastic collisions that erase phase memory. (τ_{c1}, τ_{c2}). Then it can be shown that

$$\Delta\nu = \frac{1}{\pi\tau} = \frac{1}{\pi}\left[\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_{c1}} + \frac{1}{\tau_{c2}} \right]$$

Then the lineshape function $g(\nu)$ (normalized Lorentzian) is given by:

$$g(\nu) = \frac{\Delta\nu}{2\pi} \cdot \frac{1}{(\nu-\nu_0)^2 + (\frac{\Delta\nu}{2})^2}$$

This type of energy broadening is called homogeneous broadening. In this type of broadening all atoms are indistinguishable. The most common mechanisms of homogeneous broadening are:

(a) Spontaneous emission lifetimes

(b) Collisions with other atoms or with the crystal lattice (phonons).

(c) Pressure broadening of atoms in a gas.

Inhomogeneous Broadening:

In many situations, there are different atoms constituting a medium, that have differenent lineshape functions.

Then,

$$g(\nu) = \langle g_i(\nu) \rangle$$

where $g_i(\nu)$ is the lineshape of atomic population i, and $\langle\ \rangle$ denotes average. This type of broadening is called inhomogeneous broadening.

Another mechanism of inhomogeneous broadening is the Doppler broadening. This broadening is due to the Doppler effect since atoms are in relative motion relative to an observer. Then the emitted radiation is shifted in frequency and the observed frequency is:

$$\nu = \nu_0 \frac{1 \pm \frac{v_x}{c}}{\left[1 - \left(\frac{v_x}{c}\right)^2\right]^{1/2}} \simeq \nu_0 \left[1 \pm \frac{v_x}{c}\right] \quad \left(\frac{v_x}{c} << 1\right)$$

the "+" sign is for the atom moving towards the observer, and the "−" sign is for the atom moving away from the observer. Let's assume that v_x is the component of the velocity along

the line connecting the atom with the observer. In this case it can be shown that the lineshape $g(\nu)$ is approximately given by

$$g(\nu) = \frac{c}{\nu_0}\left(\frac{M}{2\pi k_B T}\right)^{1/2} e^{-\left(\frac{M}{2k_B T}\right)\left(\frac{c}{\nu_0}\right)^2(\nu-\nu_0)^2}$$

This is the normalized Doppler-broadened line shape. The functional dependence is Gaussian. In the above equation:

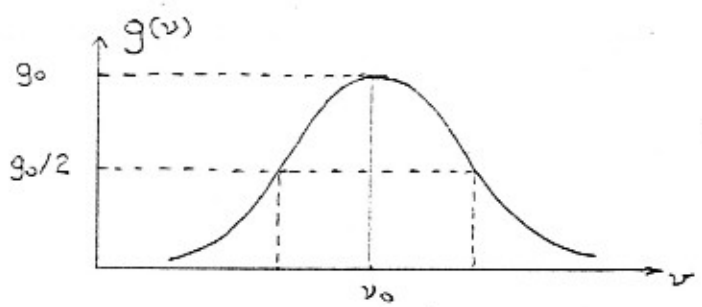
M : atomic mass of the gas atoms

k_B : Boltzmann's constant $= 1.38 \cdot 10^{-23}$ J/°K

T : temperature in degrees Kelvin (°K)

c : the velocity of light in freespace

ν_0 : the center frequency of the transition.



The full-width-at-half-maximum (FWHM) is

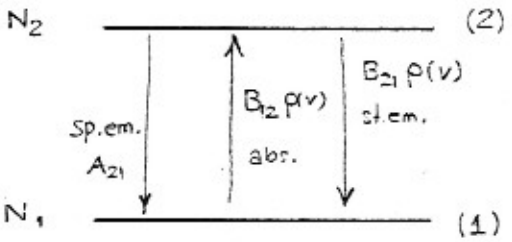
$$\Delta\nu_D = 2\nu_0\left[\frac{2k_B T}{Mc^2}\ln 2\right]^{1/2}$$

Then $g(\nu)$ can be also expressed as

$$g(\nu) = \frac{2(\ln 2)^{1/2}}{\sqrt{\pi}\,\Delta\nu_D} e^{-(4\ln 2)\left[\frac{\nu-\nu_0}{\Delta\nu_D}\right]^2}$$

The above expression for $g(\nu)$ is valid when the Doppler width $\Delta\nu_D$ is much larger than the homogeneous width.

Relationship between A_{21}, B_{12}, and B_{21} coefficients:



Using the three radiative processes previously described (spontaneous emission, absorption, and stimulated emission) we can write the following rate equations for the populations N_1, N_2 of states 1, 2 respectively.

$$\frac{dN_2}{dt} = -A_{21} N_2 + B_{12}\rho(\nu) N_1 - B_{21}\rho(\nu) N_2 = -\frac{dN_1}{dt}$$

At equilibrium $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0 \Rightarrow$

$$\frac{N_2}{N_1} = \frac{B_{12}\rho(\nu)}{A_{21} + B_{21}\rho(\nu)}$$

In addition, according to Boltzmann statistics

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/k_B T} \quad \text{where} \quad h\nu = E_2 - E_1 \rightsquigarrow \rho(\nu) = \frac{A_{21}}{B_{21}} \frac{1}{\frac{B_{12} g_1}{B_{21} g_2} e^{h\nu/k_B T} - 1}$$

and g_2, g_1 are the degeneracies of states 2 & 1 respectively.

Using, the above equations and the $\rho(\nu) = \frac{8\pi n^3 h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$ we can show:

$$g_2 B_{21} = g_1 B_{12}$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi n^3 h\nu^3}{c^3}$$

(usually it is assumed that $g_1 = g_2 \rightsquigarrow B_{12} = B_{21}$).

These are the relationships between the Einstein coefficients.

The induced due to a field rate from $1 \rightarrow 2$ (absorption) is $W_i' = B_{12}\rho(\nu) = \frac{A_{21} c^3}{8\pi n^3 h\nu^3}\rho(\nu) = \frac{c^3}{8\pi n^3 h\nu^3 t_{spont}}\rho(\nu)$

where $g_1 = g_2$ was assumed for simplicity. This transition rate is due to a uniform spectrum with spectral density $\rho(\nu)$. In order to modify the rate due to the presence of a monochromatic field we can write:

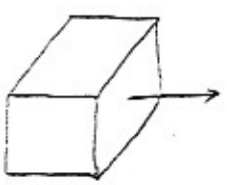
$$W_i(\nu) = \frac{c^3 \rho_\nu}{8\pi n^3 h \nu^3 t_{spont}} g(\nu) \qquad (A_{21}^{-1} = t_{spont}).$$

where ρ_ν is the energy density (J/m^3) of the electromagnetic field. Using the intensity of the electromagnetic wave we can re-write $W_i(\nu)$ in the form: $\left[I_\nu = \frac{c}{n} \rho_\nu \right]$

$$W_i(\nu) = \frac{c^2 I_\nu}{8\pi n^2 h \nu^3 t_{spont}} \cdot g(\nu) =$$

$$= \frac{\lambda_0^2 I_\nu}{8\pi n^2 h \nu \, t_{spont}} \cdot g(\nu)$$

Note:



$$\left.\begin{aligned} \rho_\nu = w_e &= \frac{1}{2} \vec{E} \cdot \vec{D} = \frac{1}{2} \epsilon |\vec{E}|^2 \\ I_\nu &= \frac{1}{2} \mathrm{Re}(\vec{E} \times \vec{H}^*) = \frac{1}{2\eta} |\vec{E}|^2 \end{aligned}\right\} \Rightarrow$$

$$\rho_\nu = \frac{1}{2} 2\eta I_\nu \epsilon = \sqrt{\frac{\mu}{\epsilon}} \epsilon I_\nu = \frac{I_\nu}{1/\sqrt{\mu\epsilon}} = \frac{I_\nu}{c/n}$$

Absorption and Amplification:

Consider a monochromatic wave, of frequency $\nu = (E_2 - E_1)/h$, propagating through a medium in which N_2 atoms are in state 2 and N_1 atoms are in state 1. The net power generated per unit volume is:

$$\frac{P}{\text{volume}} = \underbrace{W_i(\nu) N_2 h\nu}_{2 \to 1 \text{ transitions (stimulated emission)}} - \underbrace{W_i(\nu) N_1 h\nu}_{1 \to 2 \text{ transitions (absorption)}} = (N_2 - N_1) W_i(\nu) h\nu$$

The above equation can be written as

$$\frac{dI_\nu}{dz} = \left\{ (N_2 - N_1) \frac{c^2 g(\nu)}{8\pi n^2 \nu^2 t_{spont}} \right\} I_\nu = \gamma(\nu) I_\nu$$

$$\Rightarrow \quad I_\nu(z) = I_\nu(0)\, e^{\gamma(\nu) z}$$

Thus, I_ν increases if $\gamma(\nu) > 0$ or $N_2 > N_1$. On the other hand I_ν decreases if $\gamma(\nu) < 0$ or $N_2 < N_1$.

Under thermal equilibrium $N_2/N_1 = e^{-h\nu/k_0 T} < 1 \Rightarrow N_2 < N_1$, I.e. absorption prevails in systems under thermal equilibrium.

The condition for amplification is $N_2 > N_1$ ~ population inversion.

Numerical Example:

For a ruby laser: $N_2 - N_1 = 5 \cdot 10^{17}$ /cm^3

$\Delta\nu \simeq 2 \cdot 10^{11}$ Hz, $\nu_0 = 4.326 \cdot 10^{14}$ Hz ($\lambda_0 = 0.693\,\mu$m)

$g(\nu) \simeq g(\nu_0) \simeq \frac{1}{\Delta\nu}$ (approximate with the average)

$t_{spont} = 3 \cdot 10^{-3}$ sec, $\frac{c}{n} \simeq 1.69 \cdot 10^{10}$ cm/sec ($n \simeq 1.78$)

Then $\gamma(\nu) \simeq 5 \cdot 10^{-2}$ cm^{-1}. Thus the intensity of the beam $I_\nu(z) = I_\nu(0) e^{\gamma(\nu) z} \simeq I_\nu(0)[1 + \gamma(\nu) z] \Big|_{z=1\text{cm}} \simeq 1.05\, I_\nu(0)$

~ 5% amplification /cm.

Alternate Gain Definition:

Another definition of the gain is the following:

$$\gamma(\nu) \equiv \frac{1}{I_\nu}\cdot\frac{dI_\nu}{dz} = \frac{\text{net power emitted per unit volume } (W/m^3)}{\text{power per unit area traversing that volume } (W/m^2)}$$

Then according to the above definition:

$$\gamma(\nu) = \frac{1}{I_\nu}\; h\nu\left\{\left.\frac{dN_2}{dt}\right|_{net}\right\} = \frac{1}{I_\nu}\; h\nu\left\{\underbrace{B_{21}\rho_\nu g(\nu) N_2}_{\text{stimulated emission}} - \underbrace{B_{12}\rho_\nu g(\nu) N_1}_{\text{absorption}}\right\}$$

But $B_{12} = \frac{g_2}{g_1} B_{21}$ (keeping degeneracy factors).

$$B_{21} = A_{21}\frac{c^3}{8\pi n^3 h\nu^3}$$

} Einstein's relations

Then the gain can be written as:

$$\gamma(\nu) = \frac{1}{I_\nu}\, h\nu\left\{ A_{21}\frac{c^3}{8\pi n^2 h\nu^3}\underbrace{\left(\frac{I_\nu}{c/n}\right)}_{\rho_\nu} g(\nu)\left[N_2 - \frac{g_2}{g_1}N_1\right]\right\} =$$

$$= \frac{1}{I_\nu}\, h\nu\left\{ A_{21}\frac{c^2}{8\pi n^2\nu^2} g(\nu)\,\frac{I_\nu}{h\nu}\left[N_2 - \frac{g_2}{g_1}N_1\right].\right\} =$$

$$= \left\{A_{21}\frac{c^2}{8\pi n^2\nu^2} g(\nu)\right\}\left[N_2 - \frac{g_2}{g_1}N_1\right]$$

The term $A_{21}\frac{c^2}{8\pi n^2\nu^2} g(\nu) = A_{21}\frac{\lambda_0^2}{8\pi n^2} g(\nu) = \sigma(\nu) =$ stimulated emission cross-section (m^2).

Replacing A_{21} with $1/t_{spont}$ ang assuming that $g_1 = g_2$ (equal degeneracies) the previous definition of the gain is obtained.

$$\gamma(\nu) = \left(\frac{c^2}{8\pi n^2\nu^2}\cdot\frac{1}{t_{spont}}\cdot g(\nu)\right)(N_2 - N_1).$$

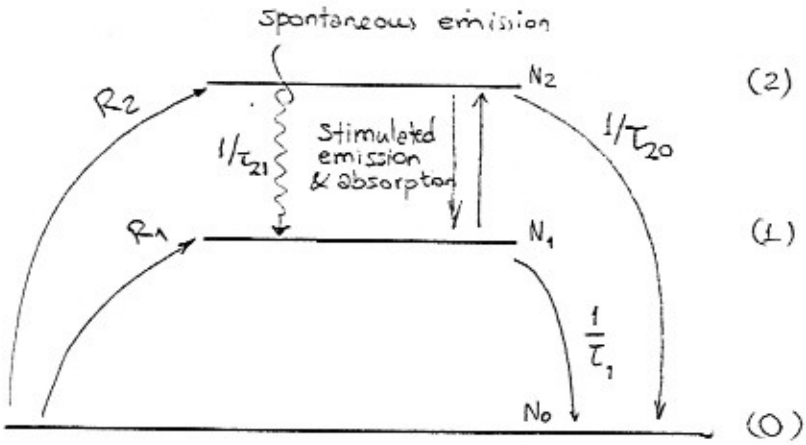
Gain Saturation in homogeneous laser media:

We have shown before that the gain in an active medium is

$$\gamma(\nu) = (N_2 - N_1)\,\frac{c^2\, g(\nu)}{8\pi n^2 \nu^2\, t_{spont}},$$

and depends on the population inversion $N_2 - N_1$. However, the light intensity does not increase forever. For each photon added to the field $N_2 - N_1$ decreases by 2 ($N_2 \downarrow$ by 1 and $N_1 \uparrow$ by one). Therefore, the gain decreases since $N_2 - N_1$ decreases. When the stimulating field is so large that causes the excited atoms to give up their energy as fast as they are being pumped into the excited state, we have reached an equilibrium. In this case the gain of the medium saturates to a smaller value. This phenomenon is called gain saturation.

Consider the following transition diagram:



Assume that states 1, 2 are pumped at rates R_1, R_2 respectively. The rate of spontaneous emission from $2 \to 1$ (and any other collision process) is $1/\tau_{21}$ ($= 1/t_{spont} = A_{21}$). The rate of transitions from $2 \to 0$ is $1/\tau_{20}$ and from $1 \to 0$ is $1/\tau_1$. Let's assume that $N_1 + N_2 \ll N_0$ in order not to worry about the conservation of mass.

The rate equations for populations N_1, N_2 are:

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - W_i N_2 + W_i N_1 \qquad \left(\frac{1}{\tau_2} = \frac{1}{\tau_{20}} + \frac{1}{\tau_{21}} \right)$$

↑ pump — ↑ rate of decrease of state 2 due to any other proces exept stim.em. — ↑ stim. em. — ↑ absorption

$$\frac{dN_1}{dt} = R_1 + \frac{N_2}{\tau_{21}} - \frac{N_1}{\tau_1} + W_i N_2 - W_i N_1$$

↑ spont.em. — ↑ rate of decrease of state 1 — ↑ stim. em. — ↑ absorption

where $W_i = \dfrac{c^3 g_\nu}{8\pi n^3 h\nu^3 t_{spont}} g(\nu) = \dfrac{c^2 g(\nu)}{8\pi n^2 h\nu^3 t_{spont}} \cdot I_\nu =$

$$= \frac{\sigma(\nu)}{h\nu} I_\nu$$

and $\sigma(\nu) = \dfrac{\lambda_0^2 / n^2}{8\pi\, t_{spont}} \cdot g(\nu)$ = stimulated emission cross section (m^2).

Then the above equations become:

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - \frac{\sigma I_\nu}{h\nu} (N_2 - N_1)$$

$$\frac{dN_1}{dt} = R_1 + \frac{N_2}{\tau_{21}} - \frac{N_1}{\tau_1} + \frac{\sigma I_\nu}{h\nu} (N_2 - N_1)$$

The steady-state solution $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$ is

$$N_2 = \left[R_2 \left(\frac{1}{\tau_1} + \frac{\sigma I_\nu}{h\nu} \right) + R_1 \left(\frac{\sigma I_\nu}{h\nu} \right) \right] / \Delta$$

$$N_1 = \left[R_1 \left(\frac{1}{\tau_2} + \frac{\sigma I_\nu}{h\nu} \right) + R_2 \left(\frac{1}{\tau_{21}} + \frac{\sigma I_\nu}{h\nu} \right) \right] / \Delta$$

$$\Delta = \frac{1}{\tau_1 \tau_2} \left[1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}} \right) \left(\frac{\sigma I_\nu}{h\nu} \right) \right]$$

where it is remirded that $\tau_{21} \simeq t_{spont}$

Special Cases:

Assume no electromagnetic field. ($I_\nu = 0$) and no pumping of state 1 ($R_1 = 0$). Then,

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} \quad \text{and} \quad R_2 = R_{20}\, u(t) \quad \text{where } u(t) \text{ is the step function.}$$

The solution for $N_2(t)$ is :

$$N_2(t) = R_{20}\,\tau_2 \left(1 - e^{-t/\tau_2}\right)$$

The equation for N_1 is

$$\frac{dN_1}{dt} = \frac{N_2}{\tau_{21}} - \frac{N_1}{\tau_1} \quad \Rightarrow \quad \text{(using Laplace's transform).}$$

$$N_1(t) = \phi_{21}\, R_{20}\,\tau_1 \left[1 + \frac{\tau_1/\tau_2}{1-\tau_1/\tau_2}\, e^{-t/\tau_1} - \frac{1}{1-\tau_1/\tau_2}\, e^{-t/\tau_2}\right]$$

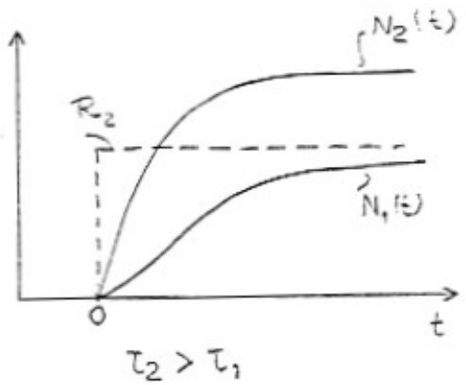
where $\phi_{21} = \dfrac{1/\tau_{21}}{1/\tau_2} = \dfrac{\tau_2}{\tau_{21}}$ = branching ratio.

The steady-state populations are:

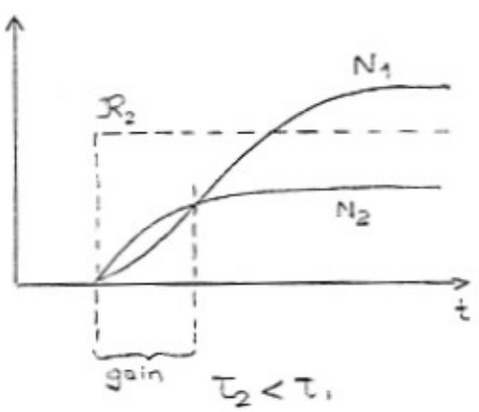
$$N_1(t \to \infty) = \phi_{21}\, R_{20}\,\tau_1$$

$$N_2(t \to \infty) = \tau_2\, R_{20}$$

For simplicity assume $\phi_{21} \simeq 1$ (favorable branching ratio). Then $\dfrac{N_2}{N_1} = \dfrac{\tau_2}{\tau_1}$. If $\dfrac{\tau_2}{\tau_1} > 1$ (favorable lifetime ratio) then $N_2 > N_1$ and invertion will happen for every t. If $\tau_2/\tau_1 < 1$ (unfavorable lifetime ratio) then inversion is possible for a short initial interval.



$\tau_2 > \tau_1$



$\tau_2 < \tau_1$

Now let's include a finite value of the electromagnetic field ($I_\nu \neq 0$). For simplicity assume $\tau_1 = 0$ and $R_1 = 0$. (ideal pumping and lifetime ratio). Then,

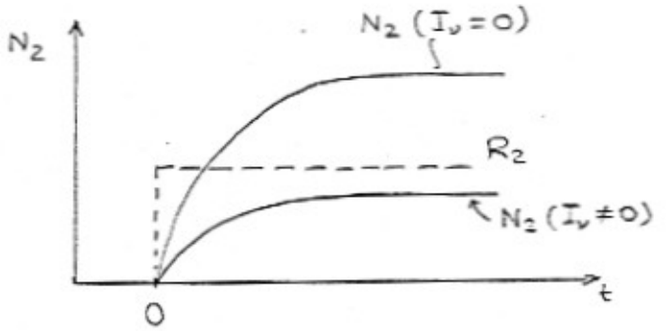
$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2}\left[1 + \frac{\sigma\tau_2}{h\nu} I_\nu\right] \Rightarrow$$

$$\frac{dN_2}{dt} + \frac{1}{\tau_2}\left[1 + \frac{I_\nu}{I_s}\right] N_2 = R_{20}\, u(t)$$

where $I_s = \frac{h\nu}{\sigma\tau_2}$ = saturation intensity.

Then,

$$N_2(t) = \frac{R_{20}\tau_2}{1 + (I_\nu/I_s)} \left\{1 - \exp\left[-\frac{t}{\tau_2}\left(1 + \frac{I_\nu}{I_s}\right)\right]\right\}$$



Effectively, the gain ($\propto N_2 - N_1 = 0$) or the lifetime of state 2 are reduced as I_ν increases. This is gain saturation.

The steady-state population inversion $N_2 - N_1$ is

$$N_2 - N_1 = \frac{R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{12}}\right) - R_1 \tau_1}{1 + \left(\tau_1 + \tau_2 - \frac{\tau_1 \tau_2}{\tau_{21}}\right) \frac{\sigma I_\nu}{h\nu}}$$

Initially $I_\nu \simeq 0$ and the initial gain is

$$\gamma_0(\nu) = (N_2 - N_1)\,\sigma(\nu) = \sigma(\nu)\left[R_2 \tau_2 \left(1 - \frac{\tau_1}{\tau_{12}}\right) - R_1 \tau_1\right]$$

and the steady-state gain is

$$\gamma(\nu) = \frac{\gamma_0(\nu)}{1 + \left(\frac{I_\nu}{I_S(\nu)}\right)}$$

where

$$I_S(\nu) = \frac{h\nu}{\sigma(\nu)\tau_2} \frac{1}{1 + \frac{\tau_1}{\tau_2}\left(1 - \frac{\tau_2}{\tau_{21}}\right)} = \text{saturation intensity.}$$

Usually $\tau_1 \ll \tau_2$ or $\tau_2 \simeq \tau_{21}$ and

$$I_S(\nu) \simeq \frac{h\nu}{\sigma(\nu)\tau_2}$$

For inhomogeneous media the gain saturation due to the Doppler broadening gives the following gain dependence:

$$\gamma(\nu) \simeq \frac{\gamma_0(\nu)}{\left(1 + \frac{I_\nu}{I_S(\nu)}\right)^{1/2}} \quad \text{for not very high intensities.}$$

Consequently, for inhomogeneously broadened the gain saturation is slower dependent on the intensity.

Electron Oscillator Model :

Using the classical electron oscillator model the deviation $x(t)$ of an electron (around the nucleus) from its equilibrium position under the application of an electric field, can be described by :

$$\frac{d^2x}{dt^2} + \sigma \frac{dx}{dt} + \frac{k}{m} x = -\frac{e}{m} e(t)$$

where σ is the damping coefficient, kx is the restoring force, m is the electron mass and $e(t)$ is the applied electric field.

Using phasors and defining the resonance frequency $\omega_o = \sqrt{\frac{k}{m}}$ the above equation can be transformed into:

$$-\omega^2 X(\omega) + j\sigma\omega X(\omega) + \omega_o^2 X(\omega) = -\frac{e}{m} \cdot E(\omega) \quad \Rightarrow$$

$$\Rightarrow \quad X(\omega) = \frac{-(e/m)\, E(\omega)}{(\omega_o^2 - \omega^2) + j\omega\sigma}$$

For frequencies near the resonance $\omega \simeq \omega_o$ the above expression reduces to $X(\omega) \simeq \dfrac{-(e/m)\, E(\omega \simeq \omega_o)}{2\omega_o(\omega_o - \omega) + j\omega_o\sigma}$

The corresponding electric dipole moment is (microscopic)

$$\mu(t) = -e\, x(t) \rightsquigarrow \mathcal{M}(\omega) = -e\, X(\omega)$$

and the macroscopic electric polarization

$$p(t) \simeq [-e\, x(t)]\, N \qquad (N \text{ oscillators per unit volume})$$

or $P(\omega) = -Ne\, X(\omega) \Rightarrow$

$$P(\omega) \simeq \frac{-j(Ne^2/m\omega_o\sigma)}{1 + j\left(\frac{2(\omega - \omega_o)}{\sigma}\right)} E(\omega) \qquad (\omega \simeq \omega_o)$$

But $P(\omega) = \epsilon_0 \chi(\omega) E(\omega) = \epsilon_0 (\chi'(\omega) - j\chi''(\omega)) E(\omega)$

$$P(\omega) = \epsilon_0 \frac{-j\left[\frac{Ne^2}{m\omega_0\epsilon_0\sigma}\right]\left(1 - j\frac{2(\omega-\omega_0)}{\sigma}\right)}{1 + \frac{4(\omega-\omega_0)^2}{\sigma^2}} E(\omega)$$

Therefore,

$$\chi'(\omega) = \left(\frac{Ne^2}{m\omega_0\epsilon_0\sigma}\right) \frac{2(\omega_0-\omega)/\sigma}{1 + 4(\omega-\omega_0)^2/\sigma^2}$$

$$\chi''(\omega) = \left(\frac{Ne^2}{m\omega_0\epsilon_0\sigma}\right) \frac{1}{1 + 4(\omega-\omega_0)^2/\sigma^2}$$

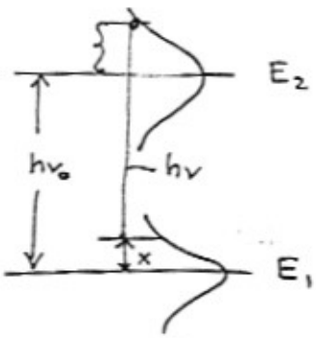
If ω is replaced with $\nu = \omega/2\pi$ and $\Delta\nu = \frac{\sigma}{2\pi}$ then

$$\chi''(\nu) = \frac{Ne^2}{16\pi^2 m \nu_0 \epsilon_0} \frac{\Delta\nu}{(\nu-\nu_0)^2 + (\Delta\nu/2)^2}$$

$$\chi'(\nu) = \frac{2(\nu_0-\nu)}{\Delta\nu}\chi''(\nu) = \frac{Ne^2}{8\pi^2 m \nu_0 \epsilon_0} \frac{\nu_0-\nu}{(\nu-\nu_0)^2 + (\Delta\nu/2)^2}$$

1: $A_1 e^{-\frac{(E-E_1)^2}{2\Delta_1^2}}$

2: $A_2 e^{-\frac{(E-E_2)^2}{2\Delta_2^2}}$



$$g(E) = \int_{-\infty}^{+\infty} A_1 e^{-\frac{(E_1+x-E_1)^2}{2\Delta_1^2}} A_2 e^{-\frac{(E_1+h\nu+x-E_2)^2}{2\Delta_2^2}} dx$$

$$= A_1 A_2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\Delta_1^2}} e^{-\frac{(\delta+x)^2}{2\Delta_2^2}} dx$$

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2+\delta^2+x^2+2x\delta}{2(\Delta_1^2+\Delta_2^2)}} dx =$$

$x^2 +$

$$\int e^{(-(ax^2+bx+c)/2d}$$

$$\int_{-\infty}^{+\infty} e^{-p^2x^2-qx} dx = e^{\frac{q^2}{4p^2}} \cdot \frac{\sqrt{\pi}}{|p|}$$

e

$$A \int_{-\infty}^{+\infty} e^{\frac{-2x^2}{2(\Delta_1^2+\Delta_2^2)} - \frac{2\delta x}{2(\Delta_1^2+\Delta_2^2)}} e^{-\delta^2/2(\Delta_1^2+\Delta_2^2)} dx$$

$$p^2 = \frac{1}{\Delta_1^2+\Delta_2^2}$$

$$q = \frac{\delta}{(\Delta_1^2+\Delta_2^2)}$$

$$I = e^{\frac{\frac{\delta^2}{(\Delta_1^2+\Delta_2^2)^2}}{4\cdot\frac{1}{(\Delta_1^2+\Delta_2^2)}}} = e^{\frac{\delta^2}{4(\Delta_1^2+\Delta_2^2)}}$$

$$g(h\nu) = e^{\frac{\delta^2}{4(\Delta_1^2+\Delta_2^2)} - \frac{\delta^2}{2(\Delta_1^2+\Delta_2^2)}}$$

$$= e^{-\frac{\delta^2}{2(\Delta_1^2+\Delta_2^2)}}$$

$$g(h\nu) = A\, e^{-\frac{(h\nu-(E_2-E_1))^2}{2\left[\sqrt{\Delta E_1^2+\Delta E_2^2}\right]^2}}$$

for Gaussian lineshapes

$$\boxed{\Delta E = \sqrt{\Delta E_1^2 + \Delta E_2^2}}$$

while for Lorentzians,

$$\boxed{\Delta E = \Delta E_1 + \Delta E_2}$$